Splitting, Squeezing and Diluting: Policy Moderation when Candidacy is Endogenous

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Some proponents of electoral reform claim that giving every voter multiple votes to cast for the different candidates would lead to more moderate policies. Their argument relies on the squeezing effect, whereby centrist candidates are squeezed between the left and right candidates who capture all the leftist and rightist votes and leave the centrist candidates with the sole centrist votes. I revisit this claim in a setting where candidacy is endogenous. In this context, I show that giving every voter multiple votes to cast can actually lead to more extreme, instead of more moderate, policies! I argue that this turnaround follows because of two features: (1) the inability to deter spoiling and duplicate candidacies; and (2) a greater multiplicity of voting profiles, which helps deter candidate defections and new candidate entries. Finally, I identify a set of voting procedures which, under some restrictions, are not subject to these two features and would therefore lead to the adoption of the most moderate policies.

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1. INTRODUCTION

Several countries (e.g., Canada, USA) elect their policy-makers by means of Plurality Voting. Plurality Voting is a voting procedure in which a voter can vote for one candidate, and the candidate who receives the most votes wins the election. Some proponents of electoral reform have claimed that giving every voter multiple votes to cast for the different candidates would improve the electoral prospects of the centrist candidates and thus yield policy moderation compared to Plurality Voting.[1] Policy moderation is defined as the adoption of policies that are preferred by the median voter. Obviously, policy moderation can have important economic implications by, for example, limiting the magnitude of policy changes each time a new government assumes office.

Several scholars have provided a theoretical underpinning for this claim. Their contributions are of two types. First, there are those contributions that assume voting to be sincere, i.e., voters cast ballots that reflect their true preferences for

[1] This claim has been made by the advocates of Approval Voting, among others. Approval Voting is a voting procedure in which a voter can vote for as many candidates as she wishes, and the candidate who receives the most votes wins the election. For example, Cox (1985; 112) writes: "Advocates of approval voting have argued that centrist candidates are favored in multicandidate elections held under approval voting, whereas extremist candidates are sometimes favored under the plurality rule." For more details on Approval Voting, see Brams (2007).

the candidates (e.g., Cox 1987, 1990). The argument in these contributions relies on the *squeezing effect*. It is most easily understood through the following (simplistic) example.[2] Consider an election with three candidates, say, a left candidate, a centrist candidate and a right candidate. Call leftist (centrist, rightist, resp.) every voter whose preferred candidate is the left candidate (centrist candidate, right candidate, resp.). If the election is held under Plurality Voting, then every voter will vote for the candidate she prefers. The left and right candidates will thus capture all the leftist and rightist votes, thereby leaving the centrist candidate with the sole centrist votes. As long as the positions of the left and right candidates are not too polarized, the centrist candidate will thus be unable to garner enough votes to win the election. In this case, the centrist candidate is said to be squeezed between the left and right candidates. Now, suppose that every voter is given a second vote to cast for another candidate. Then, every leftist will vote for both the left candidate and the centrist candidate, every rightist will vote for both the right candidate and the centrist candidate and, finally, every centrist will vote for both the centrist candidate and either the left candidate or the right candidate (whichever she prefers). It follows that the centrist candidate will receive a vote from every voter and will thus be elected outright. In other words, the centrist candidate will no longer be squeezed between the left and right candidates.[3] Hence the policy moderation.

Second, there are those contributions that assume voting to be strategic, i.e., every voter casts a ballot that maximizes her expected utility given the voting decisions of all other voters (e.g., Myerson and Weber 1993). The argument in these contributions relies on the *wasting-the-vote effect*. As above, this argument is most easily understood through the simple example of a three-candidate election. When voting is strategic, the centrist candidate need not be elected under Plurality Voting, even if a majority of voters are centrists. This can happen because of self-fulfilling prophecies. Specifically, if voters begin to believe that the centrist candidate is unlikely to win the election and that the race is actually between the left and right candidates, then voters will fear wasting their only vote on the centrist candidate. This fear will induce voters to desert the centrist candidate and cast their only vote either for the left candidate or for the right candidate (whichever they prefer), thereby confirming the underdog status of the centrist candidate. Now, suppose that every voter is given a second vote to cast for another candidate. In this case, voters will arguably be less reluctant to cast a vote for a candidate they perceive to be trailing in the race. People have claimed that this will benefit mainly the centrist candidate since a centrist candidate is arguably considered acceptable by a large fraction of the electorate who will therefore cast one of their votes for this candidate. Hence the policy moderation.

Key to note is that all these contributions adopt the Downsian approach to electoral competition. Specifically, they assume an exogenous set of purely office-motivated candidates who compete for office by choosing a platform along the line. However, in many elections, especially political elections, candidacy is endogenous and candidates are policy-motivated. If one is serious about studying the implications of giving every voter multiple votes to cast, one must then incorporate these

[2] It is worth mentioning that although simpler than the actual argument, the example captures the key features of the argument.

[3] For example, Brams and Straffin (1982; 194-195) write: "...with approval voting, a third candidate can always position himself in the middle between the left and right candidates—however small the distance that separates them—and win by getting many second-place approval votes from most of their supporters."

two features into the analysis. Endogenizing candidacy has indeed received both theoretical and empirical justification. On the theoretical side, Dutta et al. (2001) establish that any non-dictatorial voting procedure (that satisfies a mild unanimity axiom) is subject to strategic candidacy decisions. On the empirical side, world elections provide ample evidence that voting procedures yield different incentives for candidates to stand for election (e.g., Lijphart 1994, Gallagher and Mitchell 2005). Taking the policy-motivation of candidates into account has also received both theoretical and empirical justification. On the theoretical side, the extent of policy moderation in two-candidate elections depends on whether or not candidates can credibly commit to campaign promises and, therefore, on whether or not candidates are policy-motivated (Alesina 1988). On the empirical side, evidence suggests that policy-makers' own preferences are the primary determinant of their policy decisions (e.g., Levitt 1996).

In this paper, I revisit the argument that relies on the squeezing effect in a setting where candidacy is endogenous and candidates are policy-motivated.[4] For this purpose, I adopt the citizen-candidate approach to electoral competition.[5] In this approach, a community must elect a representative to select and implement a policy (e.g., a tax rate, the location of a public facility). The policy-making process has three stages. At the first stage, every potential candidate decides whether or not to stand for election. Standing for election entails a sunk cost. Candidates are policy-motivated and cannot commit to campaign promises. At the second stage, an election is held to select one among the self-declared candidates to become the policy-maker. At the third stage, the elected candidate chooses and implements policy.

There exist countless voting procedures that give every voter multiple votes to cast. In this paper, I focus on two classes of such voting procedures, namely, the Multiple Voting rules and the Ordinal Voting rules. These two classes include many of the most commonly used and studied voting procedures. A Multiple Voting rule is a voting procedure in which every voter is given q equally-weighed votes to cast for the different candidates, and the candidate who receives the most votes is elected. Multiple Voting rules differ in the number of votes, q. By contrast, an Ordinal Voting rule is a voting procedure in which every voter rank-orders the candidates. Ordinal Voting rules differ in the way voters' rankings are transformed into votes for the different candidates.

The present analysis provides qualified support for the claim that giving every voter multiple votes to cast for the different candidates would yield policy moderation compared to Plurality Voting. Specifically, a Multiple Voting rule is shown to always yield policy moderation compared to Plurality Voting only if: (1) the Multiple Voting rule is either Negative Voting or Approval Voting;[6] and (2) attention is restricted to the sole serious equilibria. (An equilibrium is said to be serious if in equilibrium, every candidate is elected with a positive probability.) Moreover, in the case of Approval Voting, a further restriction must be imposed on the voting behavior.

[4] In a companion paper, Dellis (2009a), I adopt a similar framework and revisit the argument that relies on the wasting-the-vote effect. When candidacy is endogenous, the two arguments yield very different conclusions.

[5] The citizen-candidate approach was pioneered by Osborne and Slivinski (1996) and Besley and Coate (1997). The present paper is closer in spirit to Osborne and Slivinski (1996) who, as in this paper, assume voting to be sincere.

[6] Negative Voting is the Multiple Voting rule in which a voter must vote for all except one of the candidates (or, equivalently, must vote against one of the candidates).

That no Multiple Voting rule other than Negative and Approval Voting can always yield policy moderation compared to Plurality Voting follows because Multiple Voting rules cannot deter duplicate candidacies. Specifically, no two candidates are standing at the same position when the election is held under Plurality Voting; this is because they would split their votes, thereby helping the election of rival candidates. This is the so-called splitting-the-vote effect. By contrast, Multiple Voting rules are not subject to this splitting-the-vote effect, and can therefore accomodate multiple candidates standing at the same position. These duplicate candidacies are shown to trigger a strengthening of the squeezing effect, which can therefore lead to the adoption of policies that are even more extreme than under Plurality Voting. That duplicate candidacies are not an issue under Negative Voting and Approval Voting follows because these two Multiple Voting rules share a common feature, namely, that every voter is always given one more vote to cast each time a new candidate enters the race.

That attention must be restricted to the sole serious equilibria follows because the presence of spoilers—i.e., candidates who enter the race not to be elected but because their candidacy will trigger an election outcome they prefer—helps deter candidate defections and new candidate entries, thereby rendering the elimination of the squeezing effect irrelevant. Multiple Voting rules can then support equilibria that are extreme compared to every Plurality Voting equilibrium.

Finally, Approval Voting admits a greater multiplicity of voting profiles compared to Plurality Voting. A restriction must then be imposed on the voting behavior so that an extremist does cast a vote for a centrist candidate whenever she prefers this candidate to the winning lottery.

The present analysis also identifies two Ordinal Voting rules—namely, the Borda Count and Coombs Voting—that yield policy moderation compared to Plurality Voting and under which the extent of policy moderation is maximal.[7] The Borda Count is shown to yield policy moderation compared to Plurality Voting because it is subject to an *inverse squeezing effect*, whereby a candidate gains from having other candidates standing on either side of his platform. By contrast, Coombs Voting is shown to yield policy moderation compared to Plurality Voting because it always elects the Condorcet winner (i.e., the candidate who defeats any other candidate in a pairwise contest). This implies that Coombs Voting is not only insulated from the squeezing effect, but that it can also deter spoiling candidacies.

The remainder of the paper is organized as follows. Section 2 reviews the related literature. Section 3 presents the model. Sections 4 and 5 study policy moderation in Multiple Voting and Ordinal Voting elections. Section 6 concludes. All proofs are in the appendix.

2. RELATED LITERATURE

The present work is related to a number of papers that study polarization in the Downsian framework. In the prototypical Downsian model, two candidates

[7] The Borda Count is the voting procedure in which every voter must rank-order the candidates and the candidate who is ranked last on the ballot receives no point, the candidate who is ranked just above receives one point, the next candidate receives two points, and so on. The election winner is the candidate with the most points. By contrast, under Coombs Voting, a candidate is elected if he is ranked first by a majority of voters. If there is no such candidate, then the candidate who is ranked last by a plurality of voters is eliminated and the ballots (from which the name of the eliminated candidate has been removed) are then re-counted. The process continues until a candidate is finally ranked first on a majority of ballots.

compete for office by choosing a platform along the line. Both candidates care only about winning the election and can commit to carry out their platform in case they are elected. Voters' preferences over policies are single-peaked. In this context, both candidates are shown to adopt the platform preferred by the median voter. This is the so-called median voter theorem. Subsequent contributions have shown that the median voter theorem is not robust to relaxing the different assumptions underlying the Downsian model.[8]

More closely related to the present paper are those contributions that, within the Downsian framework, study polarization in multi-candidate elections under alternative voting procedures. Cox (1987, 1990) assume the voting behavior to be sincere and restrict attention to convergent equilibria, i.e., equilibria in which all candidates adopt the same platform. No convergent equilibrium is shown to exist when the election is held under Plurality Voting. Cox then shows that increasing the number of votes each voter casts triggers centripetal forces. In particular, he finds that both Approval Voting and the Borda Count support a unique convergent equilibrium in which all candidates stand at the median voter's ideal policy. In contrast to Cox, Myerson and Weber (1993) assume the voting behavior to be strategic. In this context, they show that Plurality Voting puts few restrictions on the location of the serious contenders. This is because of the wasting-the-vote effect. By contrast, in any Approval Voting equilibrium all serious contenders are shown to be standing at the median voter's ideal policy. The present paper focuses on the squeezing effect and shows that the above results must be qualified when one takes candidacy decisions into account.[9]

The present paper is also related to the citizen-candidate literature that was initiated by Osborne and Slivinski (1996) and Besley and Coate (1997). In contrast to the canonical Downsian model, the citizen-candidate approach to political competition assumes candidacy to be endogenous and candidates to care about the policy that will be implemented. The latter implies that candidates are unable to pre-commit to the policy they will implement if elected. In this context, equilibria are shown to exist where in Plurality Voting elections two candidates enter the race and are standing on polarized platforms. Several works in this literature have compared policy outcomes under alternative voting procedures. Osborne and Slivinski (1996) assume the voting behavior to be sincere. They show that in equilibrium, candidates' positions are less polarized when the election is held under Plurality Runoff than under Plurality Voting. Dellis (2009a) assumes the voting behavior to be strategic. He identifies conditions under which letting people vote for multiple candidates would yield policy moderation compared to Plurality Voting. His focus is on the wasting-the-vote effect. By contrast, the present paper focuses on the squeezing effect. Moreover, the present paper considers several voting procedures

[8]For example, candidates are shown to choose polarized platforms if: (1) candidates care about the policy that will be implemented and are uncertain about the distribution of preferences in the electorate (Wittman 1983, Calvert 1985, Roemer 1997); (2) candidates are policy-motivated and are unable to pre-commit to the policy they will implement if elected (Alesina 1988); (3) one of the candidates is an incumbent (Bernhardt and Ingberman 1985); (4) policymaking is divided (Ortuno-Ortin 1997, Alesina and Rosenthal 2000); (5) special interest groups provide candidates with campaign contributions (Baron 1994); or (6) the two candidates compete under the threat of entry by a third candidate (Palfrey 1984, Weber 1992 and 1998, Callander 2005). For a review of this literature see, for example, Grofman (2004).

[9]It is worth mentioning that several contributions have endogenized candidacy within the Downsian framework (e.g., Feddersen et al. 1990, Osborne 1993 and 2000, Sengupta and Sengupta 2008). By contrast, the present paper endogenizes candidacy within the citizen-candidate framework and considers alternative voting procedures.

that were not considered in Dellis (2009a) (e.g., the Alternative Vote, Coombs Voting).[10]

3. MODEL

Consider a community $\mathcal{N}$ that must elect a representative to choose and implement a policy. The set of policy alternatives X is an interval on the real line, with typical element x.[11]

The community has a continuum of citizens of unit mass.[12] Every citizen $\ell \in \mathcal{N}$ has preferences on X that can be represented by a concave utility function $u_\ell : X \to \mathbb{R}$. To simplify the analysis, I assume that $u_\ell(x) = u(|x - x_\ell|)$ for every alternative $x \in X$, where $x_\ell = \arg\max_{x \in X} u_\ell(x)$ is citizen ℓ's unique ideal policy.[13] Without loss of generality, I normalize $u_\ell(x_\ell) = 0$. Citizens' ideal policies are distributed on X according to a cumulative distribution function $F : X \to [0, 1]$. I assume that F has full support and is continuous and strictly increasing on X. I further assume that F is symmetric around the median $m \equiv F^{-1}(1/2)$.[14] Throughout the analysis, I shall abuse notation and let m denote the median citizen and the median ideal policy.

Let $\mathcal{P} \subset \mathcal{N}$ be the finite set of potential candidates, with typical element i. Potential candidates are those citizens who can stand for election. Their ideal policies are distributed on X in such a way that every interval of length $\varepsilon > 0$ contains the ideal policy of at least one potential candidate. I shall consider the limit case where ε goes to zero.

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[10] While most of the works in the citizen-candidate literature are focused on Plurality Voting elections, several contributions have studied other voting procedures. Examples are Hamlin and Hjortlund (2000) and Morelli (2004) who study elections that are held under Proportional Representation, and Dellis and Oak (2006) who study Approval Voting elections.

[11] The assumption that X is unidimensional is made to facilitate comparison with the previous contributions, where the policy space is assumed to be an interval on the real line. Besides, some scholars (e.g., Cox 1990) have argued that this assumption need not be restrictive since empirical evidence suggests that, at least in the USA, voters' preferences tend to be correlated over the different issues (e.g., Hinich and Munger 1994, Bowler et al. 2009). This was most clearly stated by Alesina and Rosenthal (1995; 19) when they wrote: "...[T]his assumption does not do too much injustice to the American political system."

[12] Assuming that there is a continuum, rather than a finite number, of citizens is made to be consistent with the sincere voting assumption. Indeed, with a finite number of citizens, a single vote can be pivotal, and a sincere voting strategy need not therefore be a best reponse to the voting strategies of the other citizens. This problem does not arise when there are infinitely many citizens since no single vote is then pivotal.

[13] It is worth mentioning that the symmetry of preferences around the ideal policy is not a restrictive assumption. Indeed, most of the results in the paper do not depend on this assumption. For the other results, the symmetry of preferences is a necessary condition for the voting procedure to always yield policy moderation compared to Plurality Voting. Thus, relaxing this symmetry assumption would actually strengthen the main conclusion of the paper, namely, that when candidacy is endogenous, giving every voter multiple votes to cast for the different candidates need not yield policy moderation compared to Plurality Voting.

[14] It is worth mentioning that most of the results in the paper do not depend on this symmetry assumption. For those results that depend on it, the assumption is made to rule out situations where in a race between a left candidate L and a right candidate R, a potential candidate i with $x_i \in (x_L, x_R)$ enters the race not to win the election but because he (correctly) anticipates that he will capture more votes from the candidate he likes least than from the candidate he likes most. In cases where voters are not forced to cast all their votes, this could create 'holes' in the set of equilibrium outcomes, thereby complicating in a significant manner the comparison between voting procedures.

Following the citizen-candidate approach, the policy-making process is modeled as a three-stage game. At the first stage, every potential candidate decides whether to stand for election at a utility cost $\delta > 0$. Candidacy decisions are made simultaneously and non-cooperatively. A central premise of the citizen-candidate approach is that candidates cannot commit to campaign promises.[15] At the second stage, there is an election to select one of the self-declared candidates to become the policy-maker. At the third stage, the elected candidate chooses and implements policy. In case nobody stands for election, a default policy $x_0 \in X$ is implemented. I assume $-u_m(x_0) > \delta$, i.e., the median citizen prefers standing for election than letting the default policy be implemented.[16] I now analyze these stages in reverse order.

Policy selection stage. As candidates cannot commit to the policy they will implement if elected, the candidate who wins the election selects his ideal policy.

Election stage. Given a non-empty set of candidates $\mathcal{C} \subseteq \mathcal{P}$, let $b_\ell(\mathcal{C}) = \left(b_\ell^1, b_\ell^2, ..., b_\ell^{\#\mathcal{C}}\right)$ denote citizen ℓ's ballot, where $b_\ell^i \in \{1, 2, ..., \#\mathcal{C}\}$ is the position at which citizen ℓ ranks candidate i on her ballot and $b_\ell^i = \emptyset$ if citizen ℓ does not rank candidate i. A ballot must be such that no two candidates are ranked at the same position (i.e., $b_\ell^i \neq \emptyset \Rightarrow b_\ell^j \neq b_\ell^i$ for every candidate $j \in \mathcal{C}$, $j \neq i$) and there are no holes in the ranking (i.e., $b_\ell^i = n \Rightarrow$ for every $k \in \{1, ..., n-1\}$, $b_\ell^j = k$ for some $j \in \mathcal{C}$). In this paper, I consider two types of ballots: (1) the completely-filled ballots, where $b_\ell : \mathcal{C} \to \{1, 2, ..., \#\mathcal{C}\}$ is a bijection (i.e., citizens must rank every candidate); and (2) the truncated ballots, where $b_\ell(\mathcal{C})$ is any permutation of any of the vectors $(1, \emptyset, ..., \emptyset)$, $(1, 2, \emptyset, ..., \emptyset)$, ..., $(1, 2, ..., \#\mathcal{C})$ (i.e., citizens can rank as many of the candidates as they wish).[17]

Let $\alpha_\ell(\mathcal{C})$ denote citizen ℓ's voting strategy, where $b_\ell(\mathcal{C}) \in \alpha_\ell(\mathcal{C})$ if and only if citizen ℓ casts the ballot $b_\ell(\mathcal{C})$ with a strictly positive probability. If citizen ℓ is indifferent among several candidates, then she randomizes among these candidates.[18] The profile of voting strategies is denoted by $\alpha(\mathcal{C})$. I sometimes write $\alpha(\mathcal{C}) = (\alpha_\ell(\mathcal{C}), \alpha_{-\ell}(\mathcal{C}))$, where $\alpha_{-\ell}(\mathcal{C})$ denotes the voting profile of all citizens other than citizen ℓ.

In order to capture the squeezing effect of Plurality Voting, I assume that voting is sincere. The definition of sincere voting is borrowed from Cox (1990).

Definition 1 (Sincere voting). A ballot $b_\ell(\mathcal{C})$ is sincere for citizen ℓ if for every

[15] It is worth mentioning that this assumption is unimportant for the main conclusion of the paper, namely, that giving every voter multiple votes to cast need not yield policy moderation compared to Plurality Voting. Indeed, Brusco and Roy (2008) consider a model that is similar to the present one, except that they allow candidates to commit to campaign promises. They restrict attention to Plurality Voting elections, and show that the only equilibria are with a single candidate entering at a position close to the median m. The extent of policy moderation is thus maximal under Plurality Voting. In this context, it is then trivial that no other voting procedure can yield policy moderation compared to Plurality Voting.

[16] This extremely mild assumption is made to guarantee the existence of an equilibrium in pure strategies.

[17] Notice that I do not allow for citizens to abstain from voting. This is because voting is here costless and information is complete and perfect, in which case there is no reason for a citizen to abstain from voting. Besides, this assumption is made to facilitate comparison with the previous literature, where vote abstention is not allowed.

[18] Formally, if $u_\ell(x_i) = u_\ell(x_j)$ for some candidates $i, j \in \mathcal{C}$, $i \neq j$, then for every ballot $b_\ell(\mathcal{C}) \in \alpha_\ell(\mathcal{C})$, there exists another ballot $\widetilde{b}_\ell(\mathcal{C}) \in \alpha_\ell(\mathcal{C})$ such that $\widetilde{b}_\ell^i = b_\ell^j$, $\widetilde{b}_\ell^j = b_\ell^i$ and $\widetilde{b}_\ell^k = b_\ell^k$ for every candidate $k \in \mathcal{C}$, $k \neq i, j$.

pair of candidates $i, j \in \mathcal{C}$ with $u_\ell(x_i) > u_\ell(x_j)$,

$$b_\ell^j \neq \emptyset \Rightarrow b_\ell^j > b_\ell^i \geq 1.$$

A voting strategy $\alpha_\ell(\mathcal{C})$ is sincere for citizen ℓ if every ballot $b_\ell(\mathcal{C}) \in \alpha_\ell(\mathcal{C})$ is sincere.‖

Thus, voting is sincere for a citizen if whenever she ranks a candidate j, she ranks above candidate j every candidate she strictly prefers to j.

In the case where citizens are allowed to submit truncated ballots, I shall follow Cox (1987, 1990) and require voting strategies to be weakly undominated.[19] The elimination of weakly dominated strategies is a standard refinement in the voting literature. In the present context, it requires that a citizen ranks all the candidates she prefers most and does not rank any of the candidates she likes least.

For some voting procedures, sincere voting and weak undominance put relatively little restriction on the voting behavior. This is best examplified by Approval Voting, where a citizen can then vote only for the candidate(s) she likes the most, or she can vote for every candidate except the one(s) she likes the least, or she can draw the line anywhere in between. In view of this, I introduce another, more restrictive notion of sincere voting behavior, namely, Relative Sincerity. This notion of sincere voting was proposed in Dellis and Oak (2006). It provides an intuitive condition on where a citizen draws the line between those candidates she votes for and those candidates she does not vote for. Formally,

Definition 2 (Relative sincerity). Given a non-empty set of candidates $\mathcal{C}$, a sincere ballot $b_\ell(\mathcal{C})$ for citizen ℓ is sincere relative to others' voting strategies $\alpha_{-\ell}(\mathcal{C})$ if for every candidate $i \in \mathcal{C}$,

$$b_\ell^i \neq \emptyset \Leftrightarrow u_\ell(x_i) \geq \sum_{j \in \mathcal{C}} p_j(\mathcal{C}; b_\ell(\mathcal{C}), \alpha_{-\ell}(\mathcal{C}))\, u_\ell(x_j)$$

where $p_j(\mathcal{C}; b_\ell(\mathcal{C}), \alpha_{-\ell}(\mathcal{C}))$ denotes the probability that candidate j is elected.[20]

A voting strategy $\alpha_\ell(\mathcal{C})$ for citizen ℓ is sincere relative to others' voting strategies $\alpha_{-\ell}(\mathcal{C})$ if every ballot $b_\ell(\mathcal{C}) \in \alpha_\ell(\mathcal{C})$ is sincere relative to $\alpha_{-\ell}(\mathcal{C})$.‖

Thus, Relative Sincerity requires that a citizen puts on her ballot every candidate she prefers to the winning lottery and does not put on her ballot any of the candidates to whom she strictly prefers the winning lottery.

Candidacy stage. Let $e_i \in \{0, 1\}$ be a (pure) candidacy strategy for potential candidate i, where $e_i = 1$ if potential candidate i enters the race. The candidacy profile is denoted by $e = (e_i)_{i \in \mathcal{P}}$. I sometimes write $e = (e_i, e_{-i})$, where e_{-i} denotes the candidacy profile of all potential candidates other than i. In case truncated ballots are permissible, I assume that all potential candidates anticipate the same voting profile α when making their candidacy decision.

[19] For a characterization of weakly undominated voting strategies under alternative voting procedures, see Dellis (2009b).

[20] Notice that to be consistent with weak undominance, the inequality must be replaced with a strict inequality whenever $\sum_{j \in \mathcal{C}} p_j(\mathcal{C}; b_\ell(\mathcal{C}), \alpha_{-\ell}(\mathcal{C}))\, u_\ell(x_j) = \min_{k \in \mathcal{C}} u_\ell(x_k)$.

Given a candidacy profile e, define $\mathcal{C}(e) \equiv \{i \in \mathcal{P} : e_i = 1\}$ as the set of candidates. Potential candidate i's expected utility is given by

$$U_i(e, \alpha) = \begin{cases} \sum_{j \in \mathcal{P}} p_j(\mathcal{C}(e), \alpha(\mathcal{C}(e)))\, u_i(x_j) - \delta\, e_i & \text{if } \mathcal{C}(e) \neq \emptyset \\ u_i(x_0) & \text{if } \mathcal{C}(e) = \emptyset. \end{cases}$$

I now define a candidacy equilibrium.[21]

DEFINITION 3 (Candidacy equilibrium). Given a voting profile α, a candidacy profile e^* is a candidacy equilibrium if for every potential candidate $i \in \mathcal{P}$, e_i^* is such that

$$U_i\left(e_i^*, e_{-i}^*; \alpha\right) \geq U_i\left(e_i, e_{-i}^*; \alpha\right)$$

for every $e_i \in \{0, 1\}$.[22]‖

Political equilibrium. A political equilibrium (henceforth equilibrium) is a pair (e^*, α^*) where: (1) $\alpha^*(\mathcal{C})$ is a sincere voting profile for every non-empty set of candidates $\mathcal{C}$; and (2) e^* is a candidacy equilibrium given α^*.

I shall sometimes partition the equilibrium set into two subsets, one subset consisting of the serious equilibria and the other subset consisting of the spoiler equilibria. An equilibrium is said to be serious if every candidate is elected with a positive probability.[23] By contrast, an equilibrium is said to be spoiler if one or more candidates are elected with probability zero.

Policy moderation. It remains to define the concept of policy moderation. I start by introducing notions that will be instrumental in the definition. For every equilibrium E, let p_x denote the probability that alternative x is implemented. I write $x \in E$ if and only if $p_x > 0$.

DEFINITION 4. Let E and $\widetilde{E}$ be two distinct equilibria. Then,

1. E and $\widetilde{E}$ are *equivalent* if $p_x = \widetilde{p}_x$ for every alternative $x \in X$.
2. E is *moderate* compared to $\widetilde{E}$ if for every $x \in \left(E \backslash \widetilde{E}\right)$, $y \in \left(E \cap \widetilde{E}\right)$ and $z \in \left(\widetilde{E} \backslash E\right)$,

   $$u_m(x) \geq u_m(y) \geq u_m(z)$$

   with at least one strict inequality for one triple $\{x, y, z\}$.
3. E is *extreme* compared to $\widetilde{E}$ if $\widetilde{E}$ is moderate compared to E.‖

Thus, two equilibria are said to be equivalent if they yield the same lottery over policy outcomes. An equilibrium E is said to be moderate compared to another equilibrium $\widetilde{E}$ if the median citizen (i) prefers every policy outcome in E to any of the policy outcomes specific to $\widetilde{E}$, and (ii) prefers any of the policy outcomes specific to E to every policy outcome in $\widetilde{E}$. Extremism is the converse of moderation.

I am now ready to define the concept of policy moderation.

[21] As previous contributions in the citizen-candidate literature (e.g., Besley and Coate 1997, Felli and Merlo 2006), I restrict attention to candidacy equilibria in pure strategies.

[22] Without loss of generality, I shall assume that a potential candidate chooses not to stand for election when he is indifferent whether or not to do so.

[23] Formally, (e^*, α^*) is a serious equilibrium if $p_i(\mathcal{C}(e^*), \alpha^*(\mathcal{C}(e^*))) > 0$ for every potential candidate $i \in \mathcal{P}$ with $e_i^* = 1$.

Definition 5 (Policy moderation). Let V and $\widetilde{V}$ be any two voting procedures. V is said to yield policy moderation compared to $\widetilde{V}$ if:

1. for every equilibrium E under V, either there exists an equivalent equilibrium under $\widetilde{V}$ or equilibrium E is moderate compared to every equilibrium under $\widetilde{V}$; and
2. for every equilibrium $\widetilde{E}$ under $\widetilde{V}$, either there exists an equivalent equilibrium under V or equilibrium $\widetilde{E}$ is extreme compared to every equilibrium under V.‖

4. MULTIPLE VOTING RULES

In this section, I consider elections that are held under Multiple Voting rules. A Multiple Voting rule is a voting procedure in which every citizen is given $q \in \mathbb{N} \cup \{+\infty\}$ votes to cast for the candidates—or $c-1$ votes if the number of candidates $c \leq q$—and the election winner is the candidate who receives the most votes. Ties are broken randomly. Multiple Voting rules differ in the number of votes q. Examples of Multiple Voting rules are: (1) Plurality Voting, in which $q = 1$; (2) Dual Voting, in which $q = 2$; and (3) Negative Voting and Approval Voting, in which $q = +\infty$ (i.e., every citizen is given $c - 1$ votes when there are $c > 1$ candidates standing for election).[24] Throughout the rest of the analysis, I shall assume $q \geq 2$ whenever I refer to a Multiple Voting rule.

The following proposition identifies conditions under which a Multiple Voting rule always yields policy moderation compared to Plurality Voting.

Proposition 1. *A Multiple Voting rule always yields policy moderation compared to Plurality Voting if*

1. *the Multiple Voting rule is Negative Voting or Approval Voting (i.e., $q = +\infty$);*
2. *attention is restricted to the sole serious equilibria; and*
3. *in the case of Approval Voting, the voting behavior is Relatively Sincere.*‖

Thus, Proposition 1 provides (qualified) support for the claim that giving every citizen multiple votes to cast for the different candidates would yield policy moderation compared to Plurality Voting. However, if any one of the three conditions is not satisfied, then the Multiple Voting rule can actually lead to more extreme, rather than more moderate, policies as compared to Plurality Voting! Key to note is that the first condition is imposed on the voting procedure and, therefore, is a choice in the design of the constitution. That is not the case however for the other two conditions since they are imposed on the candidacy and voting behaviors. Whether these two conditions would be satisfied in actual elections is thus an empirical question.

I now argue that the result stated in Proposition 1 follows because of the following two features: (1) in contrast to Plurality Voting, Multiple Voting rules are unable to deter spoiling and duplicate candidacies; and (2) Multiple Voting rules that permit truncated ballots admit a greater multiplicity of voting profiles, which helps deter candidate defections and new candidate entries. To make this clear, I now partition the equilibrium set into four subsets: (1) the set of one-position serious equilibria, in which exactly one candidate stands for election; (2) the set of

[24] Negative Voting and Approval Voting differ in the type of ballot they permit. Specifically, Approval Voting permits truncated ballots, whereas Negative Voting permits only completely-filled ballots.

two-position serious equilibria, in which candidates are standing at two positions and every candidate ties for the first place; (3) the set of multi-position serious equilibria, in which candidates are standing at three or more positions and every candidate ties for the first place; and, finally, (4) the set of spoiler equilibria.

I start by characterizing the equilibrium set under Plurality Voting. (See Lemmas 1-3 in the Appendix for a formal characterization.) First, in every one-position serious equilibrium, exactly one candidate stands for election. In consequence, the one-position serious equilibria are equivalent under every voting procedure. Moreover, the one-position serious equilibria are the most moderate equilibria. This is because the sole candidate must be standing at a position which is sufficiently close to the median m so that no other potential candidate with ideal policy closer to m would want to enter the race (and win outright).[25]

In any two-position serious equilibrium under Plurality Voting, exactly two candidates are standing for election, one at a position x_L on the left of the median, and the other one at a position x_R on the right of the median. That there is only one candidate at each position follows because of the splitting-the-vote effect, whereby two candidates standing at the same position would split their votes and, therefore, help the election of the candidate(s) standing at the other position. Also, the symmetry of preferences implies that x_L and x_R must be equidistant from the median m, so that the votes are equally divided between the two candidates. It must also be that x_L and x_R are sufficiently polarized so that neither of the two candidates would be better off defecting and saving on the candidacy cost. At the same time, x_L and x_R must not be too polarized so that every potential candidate i with ideal policy $x_i \in (x_L, x_R)$ would be squeezed if he were to enter the race.

Finally, no multi-position serious equilibrium and no spoiler equilibrium exist under Plurality Voting. To see this, suppose a serious equilibrium with three positions, say, x_L, x_C and x_R, with $x_L < x_C < x_R$. Let $\overline{x} \in (x_L, x_R)$ denote the expected winning policy. Without loss of generality, suppose that $x_C \leq \overline{x}$. Because of the splitting-the-vote effect described above, there is exactly one candidate at each position. Now, suppose that the candidate at x_L were to defect from the race. Then, his votes would be transferred to the candidate at x_C, who would then be elected outright. Given that $x_C \leq \overline{x}$ and that the utility function $u(.)$ is concave, the candidate at x_L (weakly) prefers the outright election of the candidate at x_C to the lottery over x_L, x_C and x_R. Thus, the candidate at x_L is better off defecting since he then saves on the candidacy cost and gets a more preferred election outcome. Hence, this cannot be an equilibrium. By a similar argument, one can show that there is no spoiler equilibrium under Plurality Voting.[26]

4.1. Completely-filled ballots

I start by examining the case where only completely-filled ballots are permissible. In this case, Proposition 1 establishes that a Multiple Voting rule always yields policy moderation compared to Plurality Voting *if and only if* (1) the Multiple Voting rule is Negative Voting (i.e., $q = +\infty$), and (2) attention is restricted to the sole serious equilibria. To make the intuition clear, I proceed in several steps.

[25] Notice that if the candidacy cost δ is sufficiently close to zero, then in every one-position serious equilibrium the sole candidate will be standing at the median m. In the present framework, the utilitarian social welfare reaches its maximum when a candidate at the median is elected.

[26] It is worth mentioning that in a similar setting, Osborne and Slivinski (1996) establish that multi-position serious equilibria and spoiler equilibria can exist under Plurality Voting, but only if candidates are strongly office-motivated (i.e., care significantly about the spoils from office).

Multiple Voting rules with finite q. Let V be a Multiple Voting rule with finite number of votes q. That V does not always yield policy moderation compared to Plurality Voting follows because its two-position serious equilibrium set is an extreme superset of the two-position serious equilibrium set under Plurality Voting. To see this, consider a two-position serious equilibrium with candidates' positions x_L and x_R, $x_L < x_R$. Key for the result is that there are exactly q candidates at each position. This follows because of two things. First, the splitting-the-vote effect puts an upper-bound equal to q on the number of candidates standing at each position. Indeed, more than q candidates standing at one position would split their votes. One of these candidates would therefore be better off defecting from the race since he would save on the candidacy cost and would improve the prospects that his ideal policy is implemented. Second, the restriction to completely-filled ballots puts a lower-bound equal to q on the number of candidates standing at each position. Indeed, less than q candidates at, say, x_L would let some leftist votes go to the candidates at x_R. One more potential candidate at x_L would then be better off entering the race since he would capture some of those votes and would therefore improve the prospects that x_L is implemented.

Now, suppose that a potential candidate i with ideal policy $x_i \in (x_L, x_R)$ were to enter the race. If the election is held under Plurality Voting, then every leftist will vote for the candidate at x_L, every centrist will vote for candidate i, and every rightist will vote for the candidate at x_R.[27] By contrast, if the election is held under V, then every leftist will cast her q votes for the q candidates at x_L, every rightist will cast her q votes for the q candidates at x_R, and, finally, every centrist will cast one vote for candidate i and will cast her $q-1$ other votes either for the candidates at x_L or for the candidates at x_R (whichever she prefers). Key to note is that the vote total of candidate i is the same whether the election is held under Plurality Voting or under V. This is because in both cases, the candidates at x_L and x_R capture all the leftist and rightist votes and leave candidate i with the sole centrist votes. By contrast, the vote totals of the candidates at x_L and x_R are bigger under V than under Plurality Voting. This is because under V, a candidate at x_L (x_R, resp.) receives a vote from every leftist (rightist, resp.), as under Plurality Voting, plus a vote from some of the centrists. The vote share of candidate i is thus smaller under V than under Plurality Voting, whereas the reverse is true for the candidates at x_L and x_R. This implies that if candidate i is squeezed under Plurality Voting, then he is squeezed under V as well. However, the converse is not true. The set of two-position serious equilibria under V is thus a superset of the set of two-position serious equilibria under Plurality Voting.[28] Moreover, those equilibria that are specific to V are the most extreme equilibria. This is because it is the more difficult to deter a moderate from entering the race the more polarized x_L and x_R are. Hence the possibility that a Multiple Voting rule with finite q leads to more extreme policies compared to Plurality Voting since the most extreme equilibrium under Plurality Voting is a two-position serious equilibrium.[29]

[27] A leftist is a citizen who prefers x_L to both x_i and x_R. A centrist is a citizen who prefers x_i to both x_L and x_R. Finally, a rightist is a citizen who prefers x_R to both x_L and x_i.

[28] Whether a pair $\{x_L, x_R\}$ can be supported as a two-position serious equilibrium depends solely on whether moderates can be deterred from entering the race and, thus, on whether a moderate would be squeezed if he were to enter the race. This follows because Multiple Voting rules can always deter extremists—i.e., potential candidates with ideal policy $x \notin [x_L, x_R]$—from entering the race. This is true whether truncated ballots or only completely-filled ballots are permissible.

[29] It is worth mentioning that when only completely-filled ballots are permissible, no multi-position serious equilibrium exists under any Multiple Voting rule. A formal proof is available

This argument is illustrated in the following example.

Example 1. Consider a community that must elect a representative to choose a tax rate. The set of possible tax rates is $X = [0, 1]$. Let the default tax rate be $x_0 = 0$.[30] The utility of a citizen $\ell \in \mathcal{N}$ is given by $u_\ell(x) = -|x - x_\ell|$ if the implemented tax rate is $x \in X$. Ideal tax rates are uniformly distributed over X, with the median tax rate $m = 1/2$. Finally, the candidacy cost is $\delta = 1/5$.

If the election is held under Plurality Voting, then the set of two-position serious equilibria will consist of $\{x, 1-x\}$ for $x \in \left[\frac{1}{6}, \frac{3}{10}\right)$. In these equilibria, one potential candidate at x and another potential candidate at $1-x$ stand for election. Every citizen ℓ with $x_\ell < 1/2$ votes for the candidate at x, and every citizen ℓ with $x_\ell > 1/2$ votes for the candidate at $1-x$.[31] Thus, both candidates tie for the first place. The restriction that $x < \frac{3}{10}$ follows so that neither candidate wants to defect from the race. The restriction that $x \geq \frac{1}{6}$ follows so that every potential candidate i with $x_i \in (x, 1-x)$ would be squeezed if he were to enter the race.[32]

By contrast, if the election is held under Dual Voting ($q = 2$) and only completely-filled ballots are permissible, then the two-position serious equilibrium set will consist of $\{x, 1-x\}$ for $x \in \left[\frac{1}{10}, \frac{3}{10}\right)$. Notice that this set is an extreme superset of the two-position serious equilibrium set under Plurality Voting. To understand why this is the case, suppose that two candidates are standing for election, one at $1/8$ and another one at $7/8$. Clearly, both candidates tie for the first place. However, this cannot be an equilibrium since a second potential candidate at $1/8$ would then want to enter the race. Indeed, every citizen ℓ with $x_\ell < 1/2$ would then cast her two votes for the two candidates at $1/8$, whereas every citizen ℓ with $x_\ell > 1/2$ would cast one vote for the sole candidate at $7/8$ and would cast her other vote for one of the candidates at $1/8$. A candidate at $1/8$ would thus be elected the policy-maker. The probability that $1/8$ is implemented would thus increase from $1/2$ to 1. A second potential candidate at $1/8$ would thus be better off entering the race since his utility gain would be equal to $3/8$, which exceeds the candidacy cost. A second potential candidate at $7/8$ would likewise enter the race. With two candidates at each position, the splitting-the-vote effect would then kick-start and deter any further entry at these positions.

Now, suppose that a potential candidate at $1/2$ (or close to $1/2$) enters the race. If the election is held under Plurality Voting, then this candidate will receive a plurality of votes and be elected outright. Indeed, his vote total will be equal to $6/16$, whereas the vote total of every candidate at $1/8$ and $7/8$ will be equal to $5/16$. In other words, $1/8$ and $7/8$ are too polarized for a candidate at $1/2$ to be squeezed. Anticipating this, a potential candidate at $1/2$ will enter the race. This implies that $\left\{\frac{1}{8}, \frac{7}{8}\right\}$ cannot be a two-position serious equilibrium under Plurality Voting.

If instead the election is held under Dual Voting, then every citizen ℓ with $x_\ell < \frac{5}{16}$ ($x_\ell > \frac{11}{16}$, resp.) will cast her two votes for the two candidates at $1/8$ ($7/8$,

from the author upon request.

[30] It is worth emphasizing that the choice of the default tax rate is unimportant for the present purpose. This is because it matters only for the one-position serious equilibria and that these equilibria are equivalent under every voting procedure and are the most moderate equilibria.

[31] The citizens at $1/2$ are indifferent between the two candidates and thus randomize. Notice that these citizens are of measure zero and are therefore ignored henceforth.

[32] In addition to the two-position serious equilibria, there exist one-position serious equilibria. The set of one-position serious equilibria consists of $\{x\}$ for $x \in \left[\frac{2}{5}, \frac{3}{5}\right]$. Throughout the rest of the analysis, I shall ignore these equilibria since, as argued above, these equilibria are equivalent under every voting procedure and are the most moderate equilibria.

resp.), and every citizen ℓ with $x_\ell \in \left(\frac{5}{16}, \frac{1}{2}\right)$ $\left(x_\ell \in \left(\frac{1}{2}, \frac{11}{16}\right)\right.$, resp.) will cast one vote for the candidate at 1/2 and will cast her other vote for one of the two candidates at 1/8 (7/8, resp.). The vote total of the candidate at 1/2 is once again equal to 6/16. However, the vote total of a candidate at 1/8 and 7/8 is now equal to 13/32. Thus, a potential candidate at 1/2 does not want to enter the race since his candidacy would leave the election outcome unchanged. In other words, a candidate at 1/2 would be squeezed. The same holds true for every other potential candidate. It thus follows that $\left\{\frac{1}{8}, \frac{7}{8}\right\}$ is a two-position serious equilibrium under Dual Voting (and, for that matter, under every other Multiple Voting rule with finite q). Hence, the possibility that a Multiple Voting rule leads to policies that are more extreme as compared to Plurality Voting. □

To sum up, a Multiple Voting rule is subject to the splitting-the-vote effect if and only if there are q or more candidates standing at the same position. Under Plurality Voting, this implies that there is exactly one candidate at each position. By contrast, under a Multiple Voting rule with finite q and only completely-filled ballots permissible, there are exactly q candidates at each position. Now, these duplicate candidacies trigger a *dilution effect*, whereby the support of a centrist candidate gets diluted into a greater mass of votes. In turn, this dilution effect implies a strengthening of the squeezing effect, and thus an expansion of the two-position serious equilibrium set towards the extremes. Notice that this is in sharp contrast with the case where candidacy is exogenous. Indeed, when candidacy is exogenous the squeezing effect is weaker, not stronger, under a Multiple Voting rule compared to Plurality Voting. This follows because there is no dilution effect when candidacy is exogenous.

Negative Voting. I now restrict attention to the sole serious equilibria and examine why in this context, Negative Voting ($q = +\infty$) always yields policy moderation compared to Plurality Voting. The key to this result is that Negative Voting is never subject to the squeezing effect. To see this, observe that under Negative Voting a citizen votes for every candidate except (one of) the candidate(s) she likes the least. Now, the single-peakedness of preferences implies that a citizen's least-preferred candidate is either a leftmost candidate or a rightmost candidate. In consequence, a centrist candidate always receives unanimous vote and is therefore elected with a positive probability. In other words, a centrist candidate is never squeezed under Negative Voting.

That Negative Voting is never subject to the squeezing effect implies that every serious equilibrium is a one-position serious equilibrium. To see this, suppose a two-position serious equilibrium with candidates' positions x_L and x_R, $x_L < x_R$. If a potential candidate i with ideal policy x_i just on the right of x_L were to enter the race, he would be the only centrist candidate and would thus be elected outright. This, together with the fact that a potential candidate at x_L was willing to bear the candidacy cost in order to tie for the first place, implies that potential candidate i would be better off entering the race. Hence, this cannot be an equilibrium. Likewise, no multi-position equilibrium can be serious under Negative Voting. This is because every centrist candidate would receive unanimous vote, whereas the leftmost and rightmost candidates would receive less than unanimous vote. So, not all candidates would be tying for the first place, which implies that the equilibrium cannot be serious.

Hence the policy moderation since the one-position serious equilibria are the most moderate equilibria.

Spoiler equilibria. I now take the spoiler equilibria into account, and examine why Negative Voting need no longer yield policy moderation compared to Plurality Voting. Notice first that whereas Plurality Voting always deters spoiling candidacies, Negative Voting is unable to do so. The key to this difference is the way the votes of a defecting candidate are transferred. Under Plurality Voting the votes of a defecting candidate are transferred to his neighbor(s), thereby helping the election of the latter(s). It is then easy to see that either a leftmost candidate or a rightmost candidate (or both) would be better off defecting from the race. This rules out the existence of spoiler equilibria under Plurality Voting. By contrast, under Negative Voting the negative votes of a defecting candidate—i.e., the difference between the vote total of the defecting candidate and unanimous vote (or, in other words, the votes that the candidate does not receive)—are transferred to his neighbor(s), thereby hurting the electoral prospects of the latter(s). This may deter candidates from defecting. In consequence, spoiler equilibria can exist under Negative Voting. Moreover, these equilibria can be extreme compared to every Plurality Voting equilibrium, as shown in the example below. In other words, the inability of Negative Voting to deter spoiling candidacies renders the elimination of the squeezing effect irrelevant for policy moderation.

Example 2. Consider the community described in Example 1, and suppose that the election is held under Negative Voting. Let there be four candidates standing for election: one at 1/20; a second one at 1/10; a third one at 9/10; and a fourth one at 19/20. Then, every citizen ℓ with $x_\ell < 1/2$ votes for the candidates at 1/20, 1/10 and 9/10, and every citizen ℓ with $x_\ell > 1/2$ votes for the candidates at 1/10, 9/10 and 19/20. Thus, the candidates at 1/10 and 9/10 tie for the first place, whereas the candidates at 1/20 and 19/20 are spoilers. Neither the candidate at 1/20 nor the candidate at 1/10 would be better off defecting from the race since the candidate at 9/10 would then be the sole centrist candidate and would therefore be elected outright. By symmetry, the candidates at 9/10 and 19/20 are likewise deterred from stepping down. Finally, no other potential candidate would be better off entering the race. To see this, notice that: a second candidate at 1/20 or 19/20 would leave the election outcome unchanged; every candidate i with $x_i \in \left(\frac{1}{20}, \frac{19}{20}\right)$ would tie for the first place with the candidates at 1/10 and 9/10; and, finally, the entry of a potential candidate i with $x_i < 1/20$ ($x_i > 19/20$, resp.) would trigger a tie between the candidates at 1/20 (19/20, resp.), 1/10 and 9/10. In any case, the expected utility gain for this potential candidate would be smaller than the candidacy cost. Hence, this candidacy profile can be supported as a spoiler equilibrium under Negative Voting. Moreover, this equilibrium is extreme compared to every Plurality Voting equilibrium. (Recall that the most extreme equilibrium under Plurality Voting is a two-position serious equilibrium in which 1/6 and 5/6 are each implemented with an equal probability.) □

4.2. Truncated ballots

I now consider the case where truncated ballots are permissible. In this case, Proposition 1 establishes that a Multiple Voting rule always yields policy moderation compared to Plurality Voting if: (1) the Multiple Voting rule is Approval Voting (i.e., $q = +\infty$); (2) the voting behavior is relatively sincere; and (3) attention is restricted to the sole serious equilibria.[33] However, if any one of these

[33] Notice that whereas conditions (1) and (3) are necessary, condition (2) is only sufficient (in the sense that there might exist other restrictions on the voting behavior that would have the

three conditions is not satisfied, then a Multiple Voting rule can lead to even more extreme policies as compared to Plurality Voting. The key to this result is the greater multiplicity of voting profiles under a Multiple Voting rule as compared to Plurality Voting.[34] To make the intuition clear, I shall again proceed in several steps.

Sincere voting. I start by examining why a Multiple Voting rule that permits truncated ballots does not always yield policy moderation compared to Plurality Voting. The key to this result is that the permissibility of truncated ballots triggers a further expansion of the two-position serious equilibrium set towards the extremes.[35] To see this, consider a two-position serious equilibrium with candidates' positions x_L and x_R, $x_L < x_R$. Recall from above that if only completely-filled ballots are permissible, then there will be *exactly* q candidates standing at each position. Instead, if truncated ballots are permissible, then there will be *up to* q candidates standing at each position.[36] Indeed, the splitting-the-vote effect still puts an upper-bound equal to q on the number of candidates standing at each position. However, the permissibility of truncated ballots (together with weak undominance) eliminates the lower-bound on the number of candidates standing at each position. This is because a citizen who prefers x_L to x_R (x_R to x_L, resp.) will never vote for a candidate at x_R (x_L, resp.). In consequence, the potential candidates at x_L and x_R no longer want to stand for election in order to capture votes that would otherwise go to the candidates at the other position. Hence, there need no longer be exactly q candidates at each position. Actually, the number of candidates standing at each position is now an increasing function of the distance between x_L and x_R. This is because the utility gain from getting one's ideal policy implemented—and thus the incentive to stand for election—is bigger the more polarized x_L and x_R are. Thus, the two-position serious equilibria are now of two types. First, there are those equilibria with exactly q candidates at each position. I say that these equilibria are *saturated*. Second, there are those equilibria with less than q candidates at each position. I say that these equilibria are *non-saturated*. Needless to say that every saturated equilibrium is extreme compared to every non-saturated equilibrium.

Now, suppose that a potential candidate i with ideal policy $x_i \in (x_L, x_R)$ enters the race. Recall from above that if only completely-filled ballots are permissible, then candidate i will receive a vote from every centrist, whereas a candidate at x_L (x_R, resp.) will receive a vote from every leftist (rightist, resp.) plus a vote from a proportion $\frac{q-1}{2q}$ of the centrists. Instead, if citizens are allowed to truncate their ballots, then the voting profile can be as follows: every leftist casts a vote for the sole

same effect as the relative sincerity refinement).

[34]Interestingly, the greater multiplicity of voting profiles under Approval Voting as compared to Plurality Voting has been the object of a heated debate. On the one hand, Donald Saari argues that a greater multiplicity of voting profiles is a vice since it makes the election outcome under Approval Voting highly indeterminate (e.g., Saari and Van Newenhizen 1988, Saari 2001). On the other hand, Steven Brams argues that a greater multiplicity of voting profiles is a virtue since it makes Approval Voting responsive to voters' preferences (e.g., Brams et al. 1988, Brams and Sanver 2006).

[35]It is worth mentioning that the permissibility of truncated ballots also implies that multi-position serious equilibria can exist under a Multiple Voting rule. These multi-position serious equilibria can be extreme compared to the Plurality Voting equilibria.

[36]It is worth mentioning that the symmetry of preferences implies that there is still an equal number of candidates at each position and that the two positions must still be equidistant from the median m so that every candidate ties for the first place.

candidate(s) at x_L; every rightist casts a vote for the sole candidate(s) at x_R; and, finally, every centrist casts a vote for candidate i and casts her other votes either for the candidates at x_L or for the candidates at x_R (whichever she prefers).[37] Key to note is that candidate i's vote total is identical whether only completely-filled ballots or truncated ballots are permissible. The same holds true for the candidates at x_L and x_R if the equilibrium is saturated. If instead the equilibrium is non-saturated, then every candidate at x_L and x_R receives more votes when truncated ballots are permissible than when only completely-filled ballots are permissible. This happens because every centrist casts a vote either for *every* candidate at x_L or for *every* candidate at x_R, whereas none of the leftists and rightists reciprocates by casting a vote for candidate i. I call this the *non-reciprocity effect*. Now, this non-reciprocity effect implies that the vote share of candidate i is smaller when truncated ballots are permissible than when only completely-filled ballots are permissible. The reverse holds true for the candidates at x_L and x_R. In other words, the non-reciprocity effect triggers a further strengthening of the squeezing effect, and thus a further expansion of the two-position serious equilibrium set towards the extremes.[38]

The following example illustrates this argument.

Example 3. Consider the community described in Example 1. Suppose that the election is held under Dual Voting. Recall that when only completely-filled ballots are permissible, the set of two-position serious equilibria consists of $\{x, 1-x\}$ for $x \in \left[\frac{1}{10}, \frac{3}{10}\right)$. By contrast, when truncated ballots are permissible, the set of two-position serious equilibria consists of $\{x, 1-x\}$ for $x \in \left[0, \frac{3}{10}\right)$. To understand why the permissibility of truncated ballots triggers such an expansion of the two-position serious equilibrium set, consider the pair $\left\{\frac{1}{20}, \frac{19}{20}\right\}$.

If only completely-filled ballots are permissible, then there will be exactly two candidates at each position. However, this cannot be an equilibrium since a potential candidate at $1/2$ will then want to enter the race. Indeed, it is easy to check that his vote total will be equal to $18/40$, whereas the vote total of every candidate at $1/20$ and $19/20$ will be equal to $11/40$.

If instead truncated ballots are permissible, then there will be only one candidate at each position. To see this, suppose that a second potential candidate at $1/20$ were to enter the race. Then, every citizen ℓ with $x_\ell < 1/2$ would vote for the two candidates at $1/20$, whereas every citizen ℓ with $x_\ell > 1/2$ would vote only for the sole candidate at $19/20$. All three candidates would thus tie for the first place, and each would be elected with probability $1/3$. A second potential candidate at $1/20$ would thus be worse off entering the race since his expected utility gain would be equal to $3/20$, which is smaller than the candidacy cost. By symmetry, a potential candidate at $19/20$ is likewise deterred from entering the race.

Now, suppose that a potential candidate at $1/2$ enters the race. Let the voting profile be as follows: every citizen ℓ with $x_\ell < 11/40$ votes only for the candidate at $1/20$; every citizen ℓ with $x_\ell \in \left(\frac{11}{40}, \frac{1}{2}\right)$ votes for the candidates at $1/20$ and $1/2$;

[37] Such a voting profile could correspond, for example, to a situation where following candidate i's entry, citizens anticipate that the race is still between the candidates at x_L and x_R. (Such beliefs could result from focal manipulation by political leaders.) Given these beliefs, neither the leftists nor the rightists may thus bother casting a vote for candidate i.

[38] Notice that this argument does not apply to Approval Voting since there are no two-position serious equilibria under Negative Voting (which is equivalent to Approval Voting, but with only completely-filled ballots permissible). This said, the formal proof of Proposition 1 in the Appendix establishes that the two-position serious equilibrium set under Approval Voting is an extreme superset of the two-position serious equilibrium set under every other Multiple Voting rule.

every citizen ℓ with $x_\ell \in \left(\frac{1}{2}, \frac{29}{40}\right)$ votes for the candidates at 1/2 and 19/20; and, finally, every citizen ℓ with $x_\ell > 29/40$ votes only for the candidate at 19/20. The vote total of the candidate at 1/2 is still equal to 18/40. However, the vote total of every candidate at 1/20 and 19/20 is now equal to 1/2. A potential candidate at 1/2 is thus deterred from entering the race since his candidacy would leave the election outcome unchanged. In other words, a candidate at 1/2 would be squeezed. The same holds true for every other potential candidate. Hence, $\left\{\frac{1}{20}, \frac{19}{20}\right\}$ is a two-position serious equilibrium when truncated ballots are permissible. $\square$

Interestingly, this result is in sharp contrast with what happens when voting is strategic. Indeed, in the latter case, permitting truncated ballots triggers a *contraction* of the two-position serious equilibrium set towards the center, not an expansion towards the extremes (see Proposition 1 in Dellis 2009a). This is because when voting is strategic, permitting only completely-filled ballots induces candidate entry at multiple positions, which helps support self-fulfilling prophecies that deter candidate defections and new candidate entries. This suggests that the desirability of permitting truncated ballots depends on the voting behavior. Specifically, if voters are sincere, then it may be better (in terms of policy moderation) to force them to submit completely-filled ballots. If instead voters are strategic, then it may be better to let them truncate their ballot.[39]

Relative Sincerity. One may object that the above result follows because the notion of sincere voting puts too few restrictions on the voting behavior. I now argue that even with the more restrictive notion of relative sincerity, it is still the case that a Multiple Voting rule with finite q does not always yield policy moderation compared to Plurality Voting. This is because only the most extreme two-position serious equilibria survive the relative sincerity refinement. To see this, let the election be held under a Multiple Voting rule with finite q, and suppose that truncated ballots are permissible. Consider a two-position serious equilibrium with candidates' positions x_L and x_R, $x_L < x_R$. Obviously, the relative sincerity refinement has no bite if the equilibrium is saturated; every such equilibrium thus survives the relative sincerity refinement. So, suppose instead that the equilibrium is non-saturated. Let $\overline{x} \in (x_L, x_R)$ denote the expected winning policy. Suppose that a potential candidate i with ideal policy x_i just on the right of x_L, with $x_i \leq \overline{x}$, enters the race. Given the concavity of the utility function $u(.)$, every citizen ℓ with ideal policy $x_\ell \leq x_i$ prefers candidate i to the winning lottery. Since there are less than q candidates at x_L, every such citizen must then be casting a vote for candidate i (by relative sincerity).[40] In other words, the relative sincerity refinement eliminates the non-reciprocity effect. Moreover, candidate i is the most-preferred candidate of every citizen ℓ with ideal policy $x_\ell \in [x_i, \widetilde{x}]$, where $\widetilde{x} = \frac{x_i + x_R}{2} \in (m, x_R)$ is the indifference point between x_i and x_R. Every such citizen must then be casting a vote for candidate i (by weak undominance). In consequence, candidate i receives votes from a majority of citizens. At the same time, weak undominance implies that no citizen ℓ with ideal policy $x_\ell \leq m$ ($x_\ell \geq m$, resp.) casts a vote for the candidate(s) at x_R (x_L, resp.). In consequence, the candidates at x_L and x_R

[39] Interestingly, Cox (1990) raises the question whether the voting behavior—sincere or strategic—matters for the effect of ballot truncation on policy moderation. The present paper shows it does.

[40] Implicit in this argument is the assumption that citizens still anticipate a tie between the candidates at x_L and x_R. Notice however that a similar argument would hold if citizens were to anticipate that a candidate at x_L or a candidate at x_R would be elected with probability one.

receive votes only from a minority of citizens. Thus, potential candidate i would win the election outright if he were to enter the race. This, together with the fact that a potential candidate at x_L was willing to bear the candidacy cost in order to tie for the first place, implies that potential candidate i would be better off entering the race. Hence, this cannot be an equilibrium. It follows that the set of two-position serious equilibria is now equivalent to the set of *saturated* two-position serious equilibria. Thus, the relative sincerity refinement triggers a *contraction* of the two-position serious equilibrium set *towards the extremes*. Hence, those Multiple Voting rules with finite q do not always yield policy moderation compared to Plurality Voting, even if the voting behavior is relatively sincere.

The same is not true for Approval Voting. Indeed, if one restricts attention to the sole serious equilibria, then Approval Voting always yields policy moderation compared to Plurality Voting if the voting behavior is relatively sincere. This is because in this case, Approval Voting is no longer subject to the squeezing effect. In consequence, every serious equilibrium is a one-position serious equilibrium. Indeed, no two-position serious equilibrium exists. This is because $q = +\infty$ implies that every two-position serious equilibrium is non-saturated. As argued above, neither of these equilibria survive the relative sincerity refinement. Moreover, no multi-position equilibrium can be serious. This is because at least one centrist candidate would receive votes from a majority of citizens, whereas the leftmost candidates or the rightmost candidates (or both) would receive votes only from a minority of citizens. So, it cannot be that every candidate is tying for the first place and, therefore, that the equilibrium is serious.

Spoiler equilibria. I now take the spoiler equilibria into account. In this case, Approval Voting need no longer yield policy moderation compared to Plurality Voting. The key to this result is that the presence of spoilers induces a greater multiplicity of voting profiles. This greater multiplicity of voting profiles can then help deter candidate defections and new candidate entries, and thus support spoiler equilibria. Now, these equilibria can be extreme compared to every Plurality Voting equilibrium, as shown in the example below. In other words, as for Negative Voting, the presence of spoilers renders the elimination of the squeezing effect irrelevant for policy moderation.

Example 4. Consider the community described in Example 1. Suppose that the election is held under Approval Voting. Then, the candidacy profile described in Example 2 can be supported as a spoiler equilibrium. Specifically, let there be four candidates: one at 1/20; a second one at 1/10; a third one at 9/10; and a fourth one at 19/20. Let the voting profile be as follows: every citizen ℓ with $x_\ell \leq \frac{9}{20}$ votes for the candidates at 1/20 and 1/10; every citizen ℓ with $x_\ell \in \left(\frac{9}{20}, \frac{1}{2}\right)$ votes only for the candidate at 1/10; every citizen ℓ with $x_\ell \in \left(\frac{1}{2}, \frac{11}{20}\right)$ votes only for the candidate at 9/10; and, finally, every citizen ℓ with $x_\ell \geq \frac{11}{20}$ votes for the candidates at 9/10 and 19/20. Thus, the candidates at 1/10 and 9/10 tie for the first place, whereas the candidates at 1/20 and 19/20 are spoilers. It is easy to check that neither the candidate at 1/20 nor the candidate at 1/10 is willing to defect from the race if he (correctly) anticipates that citizens will react to his defection by voting for the candidate at 9/10 and for every more-preferred candidate; the candidate at 9/10 would then receive unanimous vote and be elected outright. The candidates at 9/10 and 19/20 can be likewise deterred from defecting. Finally, no other potential candidate i is willing to enter the race if he (correctly) anticipates that citizens will likewise react to his candidacy. Hence, this candidacy profile can be supported as a spoiler equilibrium under Approval Voting. Notice that every voting profile is here relatively sincere and that this equilibrium is extreme compared to every Plurality Voting equilibrium. □

5. ORDINAL VOTING RULES

In this section, I consider elections that are held under an Ordinal Voting rule. An Ordinal Voting rule is a voting procedure in which every citizen rank-orders the candidates. Ordinal Voting rules differ in the way rankings are transformed into votes for the candidates. In this paper, I consider four commonly-studied Ordinal Voting rules, namely, the Borda Count, the Alternative Vote, Coombs Voting and Bucklin Voting. I now describe each of these in turn.

The Borda Count is an example of a weighted Ordinal Voting rule. Under this voting procedure, a citizen gives $c-1$ points to the candidate she ranks first, $c-2$ points to the candidate she ranks second, ..., and zero point to the candidate she ranks last (and to every candidate she does not rank). The candidate with the highest point total wins the election. Ties are broken randomly.

The Alternative Vote and Coombs Voting are two examples of instant-runoff procedures. Under these voting procedures, a candidate is elected if he is ranked first on a majority of ballots. If there is no such candidate, then one candidate is eliminated. The Alternative Vote and Coombs Voting differ in which candidate is eliminated. Specifically, under the Alternative Vote the candidate who is eliminated is the one with the *fewest first* place votes. By contrast, under Coombs Voting the candidate who is eliminated is the one with the *most last* place votes. In both cases, the first place votes of the eliminated candidate are transferred to whichever of the other candidates were ranked second. The process is iterated until either one candidate receives a majority of first place votes or all the ballots are exhausted. In the latter case, the election winner is the candidate who receives the most votes in the last count. As usual, ties are broken randomly.[41]

Finally, Bucklin Voting is an iterative version of Approval Voting. Specifically, under Bucklin Voting every citizen rank-orders as many of the candidates as she wishes.[42] A candidate is elected if he is ranked first on a majority of ballots. If there is no such candidate, then the second place votes are added to the first place votes. A candidate is then elected if he is ranked first or second on a majority of ballots. If there are several such candidates, then the candidate with the largest majority is elected. Ties are broken randomly. The process is iterated until either a candidate receives a majority of votes or all the ballots are exhausted. In the latter case, the election winner is the candidate with the most votes.[43]

[41] The Alternative Vote is currently used for elections to the Australian lower house, for Irish presidential elections and for local elections in San Francisco (California) and Burlington (Vermont). Its use for political elections in the USA has been advocated by some interest groups (e.g., the Center for Voting and Democracy). Coombs Voting was proposed in Coombs (1964). Although it has never been used for political elections, Coombs Voting has recently been advocated by several scholars (e.g., Grofman and Feld 2004, Nagel 2007).

[42] Nothing precludes combining Bucklin Voting with the requirement that citizens must submit complete rankings of the candidates. However, I do not consider this option here since my sole motivation for studying Bucklin Voting is as a version of Approval Voting in which citizens express differential preferences for the candidates they approve.

[43] Bucklin Voting was first proposed by James Bucklin in the early twentieth century. This voting procedure had been used for some local political elections in the USA. Notice that Bucklin Voting is sometimes referred to as *Fallback Voting* (e.g., Brams 2007).

The following proposition identifies conditions under which an Ordinal Voting rule always yields maximal policy moderation. (Policy moderation is maximal if every equilibrium is a one-position serious equilibrium.) Obviously, an Ordinal Voting rule under which the extent of policy moderation is maximal yields policy moderation compared to Plurality Voting.

PROPOSITION 2. *The extent of policy moderation is always maximal under an Ordinal Voting rule if:*

1. *the Ordinal Voting rule is Coombs Voting; or*
2. *the Ordinal Voting rule is the Borda Count, citizens are forced to submit complete rankings of the candidates, and attention is restricted to the sole serious equilibria; or*
3. *the Ordinal Voting rule is Bucklin Voting, the voting behavior is relatively sincere, and attention is once again restricted to the sole serious equilibria.* ||

Thus, Proposition 2 establishes that one Ordinal Voting rule—namely, Coombs Voting—always yields policy moderation compared to Plurality Voting. This result offers an interesting contrast with the result stated in Proposition 1. Indeed, Proposition 1 establishes that some Multiple Voting rules (i.e., Negative Voting and Approval Voting) always yield policy moderation compared to Plurality Voting, but *only if* some further restrictions on the candidacy and voting behaviors are satisfied. However, there is no a priori reason to believe that these conditions would be satisfied in actual political elections. By contrast, no such further restrictions are needed in the case of Coombs Voting.

Also, notice the similarity of the result on Bucklin Voting with the one on Approval Voting in Proposition 1. This sheds some interesting light on a criticism that is often made against Approval Voting, namely, that it does not allow voters to express differential preferences for the candidates they approve. Our results on Approval Voting and Bucklin Voting suggest that this extra information on voters' preferences might not make much (if any) difference once candidacy decisions are taken into account.

I now discuss the intuition behind Proposition 2. For this purpose, I shall consider each Ordinal Voting rule in turn.[44]

Borda Count. Proposition 2 establishes that every serious equilibrium is a one-position serious equilibrium when citizens are forced to submit complete rankings of the candidates (i.e., when only completely-filled ballots are permissible). The key to this result is that the Borda Count is then subject to an *inverse squeezing effect*, whereby a candidate *benefits* from having other candidates standing on either side of his platform. To see this, assume by way of contradiction that there is a two-position serious equilibrium with candidates' positions x_L and x_R, $x_L < x_R$. Suppose that a potential candidate i with $x_i < x_L$ enters the race. Then, a citizen will give one more point to the candidates she prefers to candidate i, and she will give the same number of points to the candidates to whom she prefers candidate i. Key to note is that no citizen prefers x_R to x_i and x_i to x_L (by single-peakedness), whereas every citizen ℓ with ideal policy $x_\ell \in \left(\frac{x_i+x_L}{2}, \frac{x_i+x_R}{2}\right)$ prefers x_L to x_i and x_i to x_R. This implies that there will be more citizens giving an extra point to the candidate(s) at x_L than citizens giving an extra point to the candidate(s) at x_R. Thus, instead of being squeezed, a candidate at x_L will actually gain from becoming

[44] In order to save space, I shall not consider Bucklin Voting as the intuition is similar to the one for Approval Voting.

a centrist candidate. Now, since the candidates at x_L and x_R were initially tying for the first place, the election winner will now be a candidate at x_L. (It is not difficult to check that candidate i will receive a smaller share of points than the candidate(s) at x_L.) This, together with the fact that a potential candidate at x_L was willing to bear the candidacy cost in order to tie for the first place, implies that potential candidate i would be better off entering the race, a contradiction. Hence, there cannot exist a two-position serious equilibrium. Multi-position serious equilibria are likewise ruled out.[45]

Interestingly, the same result holds true when the voting behavior is strategic (Dellis 2009a). However, the intuition is different. Indeed, when voting is sincere, the result follows because the Borda Count is unable to deter extremists from entering the race. By contrast, when voting is strategic, the Borda Count can always deter entry by extremists. Instead, the result follows because the Borda Count can neither deter nor accomodate duplicate candidacies.

However, this result is not robust to permitting truncated ballots. This follows because of two things. First, the Borda Count is no longer subject to the inverse squeezing effect, and can thus deter extremists from entering the race. Second, the Borda Count is now subject to the non-reciprocity effect, and can thus deter moderates from entering the race. Taken together, these two things imply that there can be two- and multi-position serious equilibria as well as spoiler equilibria. Moreover, these equilibria can be extreme compared to every Plurality Voting equilibrium, as shown in the example below. Notice that this holds true even if the voting behavior is restricted to be relatively sincere.[46]

Example 5. Consider the community described in Example 1. Suppose that the election is held under the Borda Count.

If only completely-filled ballots are permissible, then every serious equilibrium is a one-position serious equilibrium. To see this, consider the pair $\left\{\frac{1}{4}, \frac{3}{4}\right\}$. (Recall that this pair can be supported as a two-position serious equilibrium under Plurality Voting.) First, notice that there will be only one candidate standing at each position. Indeed, if a second potential candidate at 1/4 were to enter the race, then all three candidates would be tying for the first place. The probability that 1/4 is implemented would then increase from 1/2 to 2/3. The second candidate at 1/4 would thus be worse off since his expected utility gain would be equal to 1/12, which is smaller than the candidacy cost. By symmetry, no second potential candidate at 3/4 wants to stand for election.

Now, suppose that a potential candidate i with $x_i = 1/5$ were to enter the race. Then, every citizen ℓ with $x_\ell < \frac{9}{40}$ would submit a ranking $\left\{i, \frac{1}{4}, \frac{3}{4}\right\}$, where candidate i is ranked first, the candidate at 1/4 second and the candidate at 3/4 third. Also, every citizen ℓ with $x_\ell \in \left(\frac{9}{40}, \frac{19}{40}\right)$ would submit a ranking $\left\{\frac{1}{4}, i, \frac{3}{4}\right\}$, every citizen ℓ with $x_\ell \in \left(\frac{19}{40}, \frac{1}{2}\right)$ would submit a ranking $\left\{\frac{1}{4}, \frac{3}{4}, i\right\}$ and, finally, every citizen ℓ with $x_\ell > \frac{1}{2}$ would submit a ranking $\left\{\frac{3}{4}, \frac{1}{4}, i\right\}$. The vote totals of candidate i and of the candidates at 1/4 and 3/4 would be equal to 28/40, 51/40 and 41/40, respectively. The candidate at 1/4 would thus be elected outright. Anticipating this, potential candidate i will want to enter the race since his expected utility gain

[45] So far, I have been unable to construct spoiler equilibria when the election is held under the Borda Count and only completely-filled ballots are permissible. At the same time, I have also been unable to formally rule out the existence of such equilibria. In consequence, the condition in Proposition 2 that attention is restricted to the sole serious equilibria need not be necessary in actuality.

[46] Formal proofs of these results are available from the author upon request.

will be equal to 1/4, which exceeds the candidacy cost. Hence, this cannot be an equilibrium.

If instead truncated ballots are permissible, then the pair $\{\frac{1}{8}, \frac{7}{8}\}$—and, a fortiori, the pair $\{\frac{1}{4}, \frac{3}{4}\}$—can be supported as a two-position serious equilibrium. To see this, let there be one candidate standing at each position. Clearly, neither of these candidates wants to defect from the race. Also, no other potential candidate at these positions wants to stand for election. Indeed, if a second potential candidate at 1/8 were to enter the race, then the candidate at 7/8 would be elected outright. This is because every citizen ℓ with $x_\ell < 1/2$ would then vote for the two candidates at 1/8, whereas every citizen ℓ with $x_\ell > 1/2$ would truncate her ballot and vote only for the candidate at 7/8. By symmetry, no second potential candidate at 7/8 wants to stand for election. Finally, every other potential candidate i will be deterred from entering the race if he (correctly) anticipates that citizens will react to his candidacy by keeping their ballot unchanged, adding potential candidate i on their ballot only if he is their most-preferred candidate. In this case, candidate i's entry would either leave the election outcome unchanged (which would happen if $x_i = 1/2$) or would lead to the outright election of the candidate he likes the least (which would happen if $x_i \neq 1/2$). Hence, $\{\frac{1}{8}, \frac{7}{8}\}$ is a two-position serious equilibrium. Notice that this equilibrium is extreme compared to every Plurality Voting equilibrium. □

Coombs Voting. Proposition 2 establishes that the extent of policy moderation is always maximal under Coombs Voting. This result follows because, as was noted in Coombs (1964), Coombs Voting always elects the Condorcet winner when the policy space is unidimensional and preferences are single-peaked.[47] This has two implications. First, Coombs Voting always deters spoiling candidacies, which means that every equilibrium is serious. Second, Coombs Voting is never subject to the squeezing effect, which means that every serious equilibrium is a one-position serious equilibrium. To see this, assume by way of contradiction that there is a two-position serious equilibrium with candidates' positions x_L and x_R, $x_L < x_R$. Suppose that a potential candidate i with ideal policy just on the right of x_L enters the race. Key to note is that a citizen's least-preferred candidate (and, therefore, the candidate she ranks last on her ballot) is either a leftmost candidate or a rightmost candidate (by single-peakedness). This means that the first candidate to be eliminated is either a candidate at x_L or a candidate at x_R. Without loss of generality, suppose the former. Now, if there are several candidates at x_L, then the last place votes of the eliminated candidate will be transferred to the other candidates at x_L, and the second candidate to be eliminated will then be another candidate at x_L. This will be repeated until all the candidates at x_L are eliminated. At that time, the first place votes of the candidates at x_L will be transferred to candidate i. At the next vote count, candidate i will then receive a majority of first place votes and will therefore be elected outright. This, together with the fact that a potential candidate at x_L was willing to bear the candidacy cost in order to tie for the first place, implies that potential candidate i would be better off entering the race, a contradiction. Hence, there cannot be a two-position serious equilibrium. Multi-position serious equilibria and spoiler equilibria are likewise ruled out.

[47] Nanson's method is another Ordinal Voting rule that always elects the Condorcet winner and, therefore, always yields maximum policy moderation. Under this voting procedure, at each vote count every candidate with a Borda score at or below the average Borda score is eliminated. The process is repeated until all but one candidates are eliminated or until all the remaining candidates are tying for the first place. Thus, Nanson's method is an iterative version of the Borda Count.

This result offers an interesting contrast between Negative Voting and Coombs Voting. Neither of these two voting procedures is subject to the squeezing effect. However, whereas Coombs Voting always deters spoiling candidacies, Negative Voting does not. This is important since, as argued above, the presence of spoilers renders the elimination of the squeezing effect irrelevant. Now, the key to this difference is that Coombs Voting is an iterative version of Negative Voting. This has an important implication. Under Negative Voting every centrist candidate receives unanimous vote and is therefore elected with a positive probability. (Notice that the Condorcet winner is a centrist candidate and is thus always elected with a positive probability under Negative Voting.) However, this probability is smaller than one if there are several centrist candidates. This leaves room for a spoiling motivation. By contrast, the iterative nature of Coombs Voting implies that it always elects the Condorcet winner with probability one. This rules out any spoiling motivation since the identity of the Condorcet winner is independent of whether or not there are spoilers in the race.

Alternative Vote. It remains to understand why the extent of policy moderation is not maximal under the Alternative Vote. This follows because the Alternative Vote is subject to the squeezing effect. In consequence, the Alternative Vote can support two-position serious equilibria, as shown in the following example.

EXAMPLE 6. Consider the community described in Example 1. Suppose that the election is held under the Alternative Vote. Then, $\left\{\frac{1}{4}, \frac{3}{4}\right\}$ is a two-position serious equilibrium. To see this, let there be two candidates standing for election, one at 1/4 and another one at 3/4. Clearly, neither of these two candidates wants to defect from the race. Moreover, no other potential candidate at these positions wants to stand for election.[48] This is because the Alternative Vote is subject to the splitting-the-vote effect. Indeed, if a second potential candidate at, say, 1/4 were to enter the race, then he would be splitting the first place votes with the other candidate at 1/4. One of these two candidates would thus be the first candidate to be eliminated. At the second vote count, the remaining candidate at 1/4 would be tying for the first place with the candidate at 3/4, in which case the policy outcome would be left unchanged. Hence, no second potential candidate at 1/4 wants to stand for election.

Likewise, no other potential candidate wants to enter the race. This is because he would be receiving the smallest number of first place votes and would be the first candidate to be eliminated. This is easy to see in the case of a potential candidate i with $x_i \notin \left[\frac{1}{4}, \frac{3}{4}\right]$. To see that it is also the case for a potential candidate i with $x_i \in \left(\frac{1}{4}, \frac{3}{4}\right)$, suppose that such a potential candidate were to enter the race. Then, every citizen ℓ with $x_\ell < \frac{x_i+1/4}{2}$ would submit a ranking $\left\{\frac{1}{4}, i, \frac{3}{4}\right\}$, every citizen ℓ with $x_\ell \in \left(\frac{x_i+1/4}{2}, \frac{1}{2}\right)$ would submit a ranking $\left\{i, \frac{1}{4}, \frac{3}{4}\right\}$, every citizen ℓ with $x_\ell \in \left(\frac{1}{2}, \frac{x_i+3/4}{2}\right)$ would submit a ranking $\left\{i, \frac{3}{4}, \frac{1}{4}\right\}$, and, finally, every citizen ℓ with $x_\ell > \frac{x_i+3/4}{2}$ would submit a ranking $\left\{\frac{3}{4}, i, \frac{1}{4}\right\}$. Thus, candidate i would receive one quarter of the first place votes, whereas each of the candidates at 1/4 and 3/4 would receive strictly more than one quarter of the first place votes. In other words, candidate i would be squeezed. So, in any case candidate i would

[48] Notice that this result is not specific to the present example. Indeed, in every Alternative Vote equilibrium no two candidates are standing at the same position. A formal proof of this result is available from the author.

be eliminated, and at the second vote count the candidates at 1/4 and 3/4 would once again be tying for the first place. The election outcome would thus be left unchanged. Anticipating this, no other potential candidate wants to enter the race. Hence, $\left\{\frac{1}{4}, \frac{3}{4}\right\}$ is a two-position serious equilibrium under the Alternative Vote. $\square$

Recall that the Alternative Vote and Coombs Voting are two examples of runoff voting procedures. They differ only in the algorithm that determines the sequence in which candidates are eliminated. However, whereas Coombs Voting always yields maximum policy moderation, the Alternative Vote does not. This is because the Alternative Vote is subject to the squeezing effect, whereas Coombs Voting is not. The key to this difference is that under Coombs Voting, the elimination sequence starts at the extremes and moves monotonically towards the median. By contrast, under the Alternative Vote, the elimination sequence can start anywhere and need not be monotonic. This underlines the importance for runoff voting procedures of the algorithm that determines the sequence in which candidates are eliminated.[49]

6. CONCLUSION

In this paper, I have investigated the claim that giving every voter multiple votes to cast for the different candidates would eliminate the squeezing effect and lead to policies that are moderate compared to the policies that are implemented under Plurality Voting. I have shown that this claim needs qualifications when candidacy decisions are taken into account. Specifically, I have identified a set of conditions under which this claim holds true and shown that if any of these conditions is not satisfied, then giving every voter multiple votes to cast for the different candidates can then lead to policies that are actually even more extreme than under Plurality Voting. I have argued that this happens because of two features: (1) the inability to deter spoiling and duplicate candidacies; and (2) a greater multiplicity of voting profiles as compared to Plurality Voting, which can help deter candidate defections and new candidate entries.

As any study, the present analysis has several limitations. First, the policy-making process is here modelled following the citizen-candidate approach to electoral competition. One of the key features of this approach is the existence of multiple equilibria. Because of this, the notion of policy moderation had to be defined over sets of equilibria. This means that the present analysis is not necessarily the best-suited to address the issue of electoral *reform*. Indeed, replacing Plurality Voting with another voting procedure might not bring any change. This can happen if the equilibrium set under the new voting procedure is a superset of the equilibrium set under Plurality Voting. Indeed, the equilibrium that was reached under Plurality Voting might then become focal under the new voting procedure.[50] In such a case, addressing the question of electoral reform requires addressing the

[49] It is worth mentioning that Plurality Runoff is another runoff voting procedure which is subject to the squeezing effect and, therefore, does not yield maximum policy moderation. Plurality Runoff is the voting procedure in which every citizen votes for one candidate. A candidate is elected if he receives a majority of votes. If there is no such candidate, then a runoff is held with the two front-runners. The election winner is then the candidate who receives a majority of votes in the runoff. Plurality Runoff is currently used in many countries and in various southern states in the USA (e.g., Louisiana).

[50] Notice however that this is unlikely to be an issue in the case of Coombs Voting. This is because: (1) any equilibrium under Coombs Voting is a one-position serious equilibrium; and (2) empirical evidence (and, for that matter, experimental evidence – see Cadigan 2005) tend to support the two-position serious equilibria under Plurality Voting.

questions of equilibrium selection and path-dependence. Experimental methods may help shed light on these questions.

A second limitation of the present analysis lies in the assumption that candidates are purely policy-motivated. This assumption was made to offer a complete contrast with the assumption of pure office-motivation that was made in the previous literature. Allowing for a mix of policy- and office-motivation would be of interest. Notice that this would actually strengthen some of the key results. For example, Coombs Voting would then lead to even more policy moderation.

A third limitation lies in the assumptions that were made in order to isolate the effect of endogenizing candidacy by making the same assumptions as in the previous literature. Specifically, the policy space was here assumed to be unidimensional. Allowing for a multidimensional policy space would be of interest. Also, information was assumed to be complete and perfect. Future research should allow for information imperfections. Finally, I considered one-shot elections. Allowing for repeated elections represents an important avenue for future research. This might allow for candidacy motivations other than winning or spoiling the election.

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APPENDIX

I first introduce some extra notation. Given a candidacy profile e and a voting profile $\alpha(.)$, let $\pi^i(e,\alpha)$ denote the vote total of candidate $i \in \mathcal{C}(e)$. Also, define the winning set, $W(e,\alpha)$, as the set of candidates with the most votes. Formally,

$$W(e,\alpha) \equiv \left\{ i \in \mathcal{C}(e) : \pi^i(e,\alpha) \geq \pi^j(e,\alpha) \text{ for every } j \in \mathcal{C}(e) \right\}.$$

To simplify notation, I shall write a citizen ℓ's voting strategy $\alpha_\ell(e)$ as $\alpha_\ell(e) = \{i; A; j, k\}$ if citizen ℓ ranks candidate i first with probability one, ranks every candidate in A at positions $2, ..., (1 + \#A)$ with an equal probability, ranks candidates j and k at positions $(2 + \#A)$ and $(3 + \#A)$ with an equal probability, and does not rank any other candidate (or has exhausted all her votes).

I start by characterizing the equilibrium set under Plurality Voting. I do so via a sequence of lemmas. The first lemma establishes that in every Plurality Voting equilibrium, no two candidates are standing at the same position.

Lemma 1. *Let (e,α) be an equilibrium under Plurality Voting. Then, $x_i \neq x_j$ for every $i, j \in \mathcal{C}(e)$, $i \neq j$.*‖

Proof. Suppose that the election is held under Plurality Voting. Let (e,α) be an equilibrium. Assume by way of contradiction that $x_i = x_j$ for some $i, j \in \mathcal{C}(e)$, $i \neq j$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_i = 0$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq i$. Pick an arbitrary citizen $\ell \in \mathcal{N}$. Let $p^h_\ell \in [0,1]$ denote the probability that citizen ℓ casts a vote for a candidate $h \in \mathcal{C}(e)$. Either $p^i_\ell = 0$, in which case $\widetilde{p}^k_\ell = p^k_\ell$ for every $k \in \mathcal{C}(\widetilde{e})$. Or $p^i_\ell > 0$, in which case $\widetilde{p}^j_\ell > p^j_\ell$ and $\widetilde{p}^k_\ell = p^k_\ell$ for every $j, k \in \mathcal{C}(\widetilde{e})$ with $x_j = x_i$ and $x_h \neq x_i$.[51] It thus follows that $\pi^j(\widetilde{e},\alpha) > \pi^j(e,\alpha)$ and $\pi^k(\widetilde{e},\alpha) = \pi^k(e,\alpha)$. This, together with the fact that candidacy is costly, implies $U_i(\widetilde{e},\alpha) > U_i(e,\alpha)$, which contradicts that (e,α) is an equilibrium. ∎

The second lemma establishes that every Plurality Voting equilibrium is a one- or two-position serious equilibrium.

Lemma 2. *Let (e,α) be an equilibrium under Plurality Voting. Then, $W(e,\alpha) = \mathcal{C}(e)$ and $\#W(e,\alpha) \leq 2$.*‖

Proof. Suppose that the election is held under Plurality Voting. Let (e,α) be an equilibrium. We know from Lemma 1 that there is only one candidate at each position. I first establish that the winning set $W(e,\alpha)$ contains both the leftmost and rightmost candidates. Pick $h \in \mathcal{C}(e)$ with $x_h \leq x_k$ for every $k \in \mathcal{C}(e)$. Assume by way of contradiction that $h \notin W(e,\alpha)$. Let $L \in \mathcal{C}(e)$ be the leftmost candidate in $W(e,\alpha)$. Obviously, $x_L \leq \overline{x}$, where $\overline{x} \equiv \frac{1}{\#W(e,\alpha)} \sum_{k \in W(e,\alpha)} x_k$ is the expected winning policy. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 0$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. Let i be the leftmost candidate in $\mathcal{C}(\widetilde{e})$. Thus, $\pi^i(\widetilde{e},\alpha) = \pi^i(e,\alpha) + \pi^h(e,\alpha)$ and $\pi^k(\widetilde{e},\alpha) = \pi^k(e,\alpha)$ for every $k \in \mathcal{C}(\widetilde{e})$, $k \neq i$. This, together with the concavity of $u(.)$ and the fact that candidacy is costly, implies $U_h(\widetilde{e},\alpha) > U_h(e,\alpha)$, which contradicts that (e,α) is an equilibrium. Likewise for $h \in \mathcal{C}(e)$ with $x_h \geq x_k$ for every $k \in \mathcal{C}(e)$.

I now establish that $\#W(e,\alpha) \leq 2$. Assume by way of contradiction that $\#W(e,\alpha) \geq 3$. Let $h, j \in \mathcal{C}(e)$ be the leftmost and rightmost candidates, respectively. We know from above that $h, j \in W(e,\alpha)$. Moreover, $\#W(e,\alpha) \geq 3$ and

[51] It can be that $\widetilde{p}^k_\ell > p^k_\ell$ if $u_\ell(x_k) = u_\ell(x_i)$. However, I shall ignore this possibility since such citizens ℓ are of measure zero.

Lemma 1 imply there exists $i \in W(e,\alpha)$ with $x_i \in (x_h, x_j)$. Without loss of generality suppose $x_i \leq \overline{x}$. Proceeding as above, one can show $U_h(\widetilde{e},\alpha) > U_h(e,\alpha)$, which contradicts that (e,α) is an equilibrium. Hence, $\#W(e,\alpha) \leq 2$. This rules out the existence of multi-position serious equilibria.

It remains to rule out the existence of spoiler equilibria. Assume by way of contradiction that $W(e,\alpha) \neq \mathcal{C}(e)$. Given that no two candidates are standing at the same position and that the winning set $W(e,\alpha)$ must contain both the leftmost and rightmost candidates, it cannot be that $W(e,\alpha) \neq \mathcal{C}(e)$ and $\#W(e,\alpha) = 1$. It must then be that $W(e,\alpha) = \{h,i\}$ with h and i the leftmost and rightmost candidates in $\mathcal{C}(e)$, respectively. As $W(e,\alpha) \neq \mathcal{C}(e)$, there must exist $j \in \mathcal{C}(e)$ with $x_j \in (x_h, x_i)$. Without loss of generality suppose $x_j \leq \overline{x}$. Proceeding as above, one can show $U_h(\widetilde{e},\alpha) > U_h(e,\alpha)$, which contradicts that (e,α) is an equilibrium. Hence $W(e,\alpha) = \mathcal{C}(e)$. ∎

The next lemma characterizes the set of one-position serious equilibria.

LEMMA 3. *Let the election be held under any voting procedure. Then, there exists an equilibrium (e,α) in which $i \in \mathcal{P}$ runs unopposed if and only if*
1. $-u_i(x_0) > \delta$; *and*
2. *for every $j \in \mathcal{P}$, $j \neq i$, we have that:*
   (a) $u_m(x_i) > u_m(x_j)$; *or*
   (b) $u_m(x_i) = u_m(x_j)$ *and* $-u_j(x_i) \leq 2\delta$; *or*
   (c) $u_m(x_i) < u_m(x_j)$ *and* $-u_j(x_i) \leq \delta$.$\|$

Proof. Trivial. ∎

Finally, the characterization of a two-position serious equilibrium set under Plurality Voting is obtained by letting $q = 1$ in Claim 1.2 below. It is easy to check that every one-position serious equilibrium is moderate compared to every two-position serious equilibrium.

PROOF OF PROPOSITION 1. Let the election be held under a Multiple Voting rule with number of votes q. I shall distinguish two cases.

Case 1. Only completely-filled ballots are permissible. I first introduce some extra notation. Given a candidacy profile e with two candidates' positions, x_L and x_R, let $\mathcal{C}_L(e)$ be the set of candidates at x_L and define $c_L \equiv \#\mathcal{C}_L(e)$ as the number of candidates at this position. Define $\mathcal{C}_R(e)$ and c_R likewise.

I proceed via a sequence of lemmas. The first lemma establishes that only Negative Voting ($q = +\infty$) can always yield policy moderation compared to Plurality Voting. Given that the one-position serious equilibria are equivalent under every voting procedure and are moderate compared to the two-position serious equilibria and that the most extreme equilibrium under Plurality Voting is a two-position serious equilibrium, it is sufficient to establish that the two-position serious equilibrium set under Plurality Voting is a moderate subset of the two-position serious equilibrium set under any Multiple Voting rule with finite q.

LEMMA 4. *Suppose that the election is held under a Multiple Voting rule with finite q and only completely-filled ballots permissible. Then, the two-position serious equilibrium set is an extreme superset of the two-position serious equilibrium set under Plurality Voting.*$\|$

Proof. I establish the result via a sequence of two claims.

Claim 1.1. Suppose that the election is held under a Multiple Voting rule with finite q and only completely-filled ballots permissible. Let (e, α) be a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$. Then, $u_m(x_L) = u_m(x_R)$ and $c_L = c_R = q$.‖

Proof of Claim 1.1. Consider the situation described in the statement. Notice that $x_\ell \lesseqgtr \frac{x_L + x_R}{2}$ implies $u_\ell(x_L) \gtreqless u_\ell(x_R)$. Define $\gamma_L \equiv F\left(\frac{x_L+x_R}{2}\right)$ and $\gamma_R \equiv 1 - F\left(\frac{x_L+x_R}{2}\right)$.

I start by establishing that $m = \frac{x_L+x_R}{2}$. Assume by way of contradiction that $m < \frac{x_L+x_R}{2}$. Thus, $\gamma_L > \gamma_R$. To establish a contradiction, I now proceed in three steps.

Step 1: $c_L \leq q$ and $c_R \leq q$. Assume by way of contradiction that $c_L > q$. Candidates' vote totals are thus given by

$$\begin{cases} \pi^i(e,\alpha) = \frac{\gamma_L q + \gamma_R \max\{0, q - c_R\}}{c_L} & \text{for every } i \in \mathcal{C}_L(e) \\ \pi^j(e,\alpha) = \frac{\gamma_R \min\{c_R, q\}}{c_R} & \text{for every } j \in \mathcal{C}_R(e). \end{cases}$$

As (e, α) is a serious equilibrium, it must then be that $\pi^i(e,\alpha) = \pi^j(e,\alpha)$.

Pick $h \in \mathcal{C}_L(e)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 0$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. Key to note is that there are still q or more candidates at x_L. Candidates' vote totals are now given by

$$\begin{cases} \pi^i(\widetilde{e},\alpha) = \frac{\gamma_L q + \gamma_R \max\{0, q - c_R\}}{c_L - 1} & \text{for every } i \in \mathcal{C}_L(\widetilde{e}) \\ \pi^j(\widetilde{e},\alpha) = \frac{\gamma_R \min\{c_R, q\}}{c_R} & \text{for every } j \in \mathcal{C}_R(\widetilde{e}). \end{cases}$$

Clearly, $\pi^i(\widetilde{e},\alpha) > \pi^i(e,\alpha)$ and $\pi^j(\widetilde{e},\alpha) = \pi^j(e,\alpha)$. This, together with $\pi^i(e,\alpha) = \pi^j(e,\alpha)$, implies $W(\widetilde{e},\alpha) = \mathcal{C}_L(\widetilde{e})$. Thus, $U_h(\widetilde{e},\alpha) > U_h(e,\alpha)$, which contradicts that (e,α) is an equilibrium. Hence $c_L \leq q$. One can likewise establish that $c_R \leq q$.

Step 2: $c_L < c_R$. Given Step 1, we know that candidates' vote totals are given by

$$\begin{cases} \pi^i(e,\alpha) = \gamma_L + \gamma_R \frac{\min\{c-1, q\} - c_R}{c_L} & \text{for every } i \in \mathcal{C}_L(e) \\ \pi^j(e,\alpha) = \gamma_L \frac{\min\{c-1, q\} - c_L}{c_R} + \gamma_R & \text{for every } j \in \mathcal{C}_R(e). \end{cases}$$

Since (e,α) is a serious equilibrium, it must be that $\pi^i(e,\alpha) = \pi^j(e,\alpha)$, and thus $c_L \gamma_L = c_R \gamma_R$. This, together with $\gamma_L > \gamma_R$, implies $c_L < c_R$.

Step 3. It is easy to check that a candidate at x_L does not want to defect if and only if $\delta < -\frac{c_L}{c} u(|x_L - x_R|)$.

Pick $h \in \mathcal{P} \backslash \mathcal{C}(e)$ with $x_h = x_L$. Notice that $U_h(e,\alpha) = \frac{c_R}{c} u(|x_L - x_R|)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 1$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. Given that $c_L < c_R \leq q$, candidates' vote totals are now given by

$$\begin{cases} \pi^i(\widetilde{e},\alpha) = \gamma_L + \gamma_R \frac{\min\{c, q\} - c_R}{c_L + 1} & \text{for every } i \in \mathcal{C}_L(\widetilde{e}) \\ \pi^j(\widetilde{e},\alpha) = \gamma_L \frac{\min\{c, q\} - (c_L + 1)}{c_R} + \gamma_R & \text{for every } j \in \mathcal{C}_R(\widetilde{e}). \end{cases}$$

It is easy to check that $\pi^i(\widetilde{e},\alpha) > \pi^j(\widetilde{e},\alpha)$, which implies $W(\widetilde{e},\alpha) = \mathcal{C}_L(\widetilde{e})$. It thus follows that $U_h(\widetilde{e},\alpha) = -\delta$. This, together with $c_L < c_R$ and the above condition on δ, implies $U_h(\widetilde{e},\alpha) > U_h(e,\alpha)$, which contradicts that (e,α) is an equilibrium. Hence, it must be that $m \geq \frac{x_L+x_R}{2}$. Proceeding likewise, one can establish that it must also be that $m \leq \frac{x_L + x_R}{2}$. Taken together, these two inequalities imply $m = \frac{x_L + x_R}{2}$ and, therefore, $u_m(x_L) = u_m(x_R)$.

It remains to establish that $c_L = c_R = q$. Note first that $m = \frac{x_L + x_R}{2}$ implies $\gamma_L = \gamma_R = 1/2$. Moreover, proceeding as in Step 2 above one can show that (e, α) is a serious equilibrium only if $c_L \gamma_L = c_R \gamma_R$ which, together with $\gamma_L = \gamma_R$, implies $c_L = c_R$. Given Step 1, we also know that $c_L = c_R \leq q$. Proceeding as in Step 3 above, one can establish that $c_L = c_R < q$ yields a contradiction. Hence $c_L = c_R = q$. $\square$

Claim 1.2. Suppose that the election is held under a Multiple Voting rule with finite q and only completely-filled ballots permissible. There exists a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$, $x_L < x_R$, if and only if there exist $i, j \in \mathcal{P}$ with $x_i = x_L$ and $x_j = x_R$, and

1. $m = \frac{x_L + x_R}{2}$;
2. $-\frac{u(|x_L - x_R|)}{2} > \delta$;
3. for every $h \in \mathcal{P}$ with $x_h \in (x_L, m)$, we have:
   (a) $\left[F\left(\frac{x_R + x_h}{2}\right) - \frac{q}{q+1} F\left(\frac{x_L + x_h}{2}\right)\right] < \frac{1}{2}$; or
   (b) $\left[F\left(\frac{x_R + x_h}{2}\right) - \frac{q}{q+1} F\left(\frac{x_L + x_h}{2}\right)\right] = \frac{1}{2}$ and $\delta \geq \frac{q-1}{q+1} \frac{u_h(x_R)}{2} - \frac{u_h(x_L)}{2}$; or
   (c) $\left[F\left(\frac{x_R + x_h}{2}\right) - \frac{q}{q+1} F\left(\frac{x_L + x_h}{2}\right)\right] > \frac{1}{2}$ and $\delta \geq -\frac{u_h(x_L) + u_h(x_R)}{2}$.

   Likewise for every $h \in \mathcal{P}$ with $x_h \in (m, x_R)$; and
4. for every $h \in \mathcal{P}$ with $x_h = m$, we have:
   (a) $F\left(\frac{x_R + x_h}{2}\right) < \frac{1}{2} \frac{3q+1}{2q+1}$; or
   (b) $F\left(\frac{x_R + x_h}{2}\right) = \frac{1}{2} \frac{3q+1}{2q+1}$ and $\delta \geq -\frac{u_m(x_L)}{2q+1}$; or
   (c) $F\left(\frac{x_R + x_h}{2}\right) > \frac{1}{2} \frac{3q+1}{2q+1}$ and $\delta \geq -u_m(x_L)$.$\|$

Proof of Claim 1.2. **(Sufficiency)** The proof is by construction. Suppose that all the conditions in the statement are satisfied. Let the candidacy profile e be such that exactly $2q$ candidates are standing for election, with q candidates at x_L and q other candidates at x_R. Then, the sincere voting profile $\alpha(e)$ is such that

$$\alpha_\ell(e) = \begin{cases} \{\mathcal{C}_L(e)\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < m \\ \{\mathcal{C}_R(e)\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > m. \end{cases}$$

This, together with condition (1), implies $\pi^i(e, \alpha) = 1/2$ for every $i \in \mathcal{C}(e)$. Hence $W(e, \alpha) = \mathcal{C}(e)$.

Pick $h \in \mathcal{C}_L(e)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 0$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. Then, the sincere voting profile $\alpha(\widetilde{e})$ is such that

$$\alpha_\ell(\widetilde{e}) = \begin{cases} \{\mathcal{C}_L(\widetilde{e}); \mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < m \\ \{\mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > m. \end{cases}$$

It is easy to check that $\pi^j(\widetilde{e}, \alpha) = \frac{1}{2} \frac{q+1}{q} > \frac{1}{2} = \pi^i(\widetilde{e}, \alpha)$ for every $i \in \mathcal{C}_L(\widetilde{e})$ and $j \in \mathcal{C}_R(\widetilde{e})$. Thus, $W(\widetilde{e}, \alpha) = \mathcal{C}_R(\widetilde{e})$ and $U_h(\widetilde{e}, \alpha) = u(|x_L - x_R|)$. As $U_h(e, \alpha) = \frac{u(|x_L - x_R|)}{2} - \delta$, condition (2) thus implies $U_h(e, \alpha) > U_h(\widetilde{e}, \alpha)$. Hence, a candidate $h \in \mathcal{C}_L(e)$ would be worse off defecting. Likewise for a candidate $h \in \mathcal{C}_R(e)$.

Pick $h \in \mathcal{P} \backslash \mathcal{C}(e)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 1$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. There are three cases to consider.

Case 1: $x_h \in \{x_L, x_R\}$. Then, the sincere voting profile $\alpha(\widetilde{e})$ is such that

$$\alpha_\ell(\widetilde{e}) = \begin{cases} \{\mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < m \\ \{\mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > m. \end{cases}$$

If $x_h = x_L$, then $\pi^j(\widetilde{e}, \alpha) = \frac{1}{2} > \frac{1}{2}\frac{q}{q+1} = \pi^i(\widetilde{e}, \alpha)$ for every $i \in \mathcal{C}_L(\widetilde{e})$ and $j \in \mathcal{C}_R(\widetilde{e})$. It thus follows that $W(\widetilde{e}, \alpha) = \mathcal{C}_R(\widetilde{e})$, which implies $U_h(e, \alpha) > U_h(\widetilde{e}, \alpha)$. Likewise if $x_h = x_R$.

Case 2: $x_h \notin [x_L, x_R]$. If $x_h < x_L$, then the sincere voting profile $\alpha(\widetilde{e})$ is such that

$$\alpha_\ell(\widetilde{e}) = \begin{cases} \{h; \mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < \frac{x_h + x_L}{2} \\ \{\mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell \in \left(\frac{x_h + x_L}{2}, m\right) \\ \{\mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > m. \end{cases}$$

Again, $W(\widetilde{e}, \alpha) = \mathcal{C}_R(\widetilde{e})$ and $U_h(e, \alpha) > U_h(\widetilde{e}, \alpha)$. Likewise if $x_h > x_R$.

Case 3: $x_h \in (x_L, x_R)$. Then, the sincere voting profile $\alpha(\widetilde{e})$ is such that

$$\alpha_\ell(\widetilde{e}) = \begin{cases} \{\mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < \frac{x_h + x_L}{2} \\ \{h; \mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell \in \left(\frac{x_h + x_L}{2}, m\right) \\ \{h; \mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell \in \left(m, \frac{x_h + x_R}{2}\right) \\ \{\mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > \frac{x_h + x_R}{2}. \end{cases}$$

Notice that $x_h \lesseqqgtr m$ implies $F\left(\frac{x_h + x_L}{2}\right) \lesseqqgtr 1 - F\left(\frac{x_h + x_R}{2}\right)$. Suppose first that $x_h \leq m$. Notice that it is then the case that $\pi^j(\widetilde{e}, \alpha) \geq \pi^i(\widetilde{e}, \alpha)$ for every $i \in \mathcal{C}_L(\widetilde{e})$ and $j \in \mathcal{C}_R(\widetilde{e})$ (with a strict inequality if $x_h < m$). We can thus restrict attention to the vote totals of candidate h and of the candidate(s) $j \in \mathcal{C}_R(\widetilde{e})$:

$$\begin{cases} \pi^h(\widetilde{e}, \alpha) = F\left(\frac{x_h + x_R}{2}\right) - F\left(\frac{x_h + x_L}{2}\right) \\ \pi^j(\widetilde{e}, \alpha) = \frac{1}{2} - \frac{1}{q}\left[F\left(\frac{x_h + x_R}{2}\right) - \frac{1}{2}\right]. \end{cases}$$

One of the following must then be true:

1. $\left[F\left(\frac{x_R + x_h}{2}\right) - \frac{q}{q+1}F\left(\frac{x_L + x_h}{2}\right)\right] < \frac{1}{2}$, in which case

$$W(\widetilde{e}, \alpha) = \begin{cases} \mathcal{C}_R(\widetilde{e}) & \text{if } x_h < m \\ W(e, \alpha) & \text{if } x_h = m. \end{cases}$$

   It follows that $U_h(\widetilde{e}, \alpha) = u_h(x_R) - \delta$ anyway.

2. $\left[F\left(\frac{x_R + x_h}{2}\right) - \frac{q}{q+1}F\left(\frac{x_L + x_h}{2}\right)\right] = \frac{1}{2}$, in which case

$$W(\widetilde{e}, \alpha) = \begin{cases} \mathcal{C}_R(\widetilde{e}) \cup \{h\} & \text{if } x_h < m \\ \mathcal{C}(\widetilde{e}) & \text{if } x_h = m. \end{cases}$$

   It follows that $U_h(\widetilde{e}, \alpha) = \beta u_h(x_R) - \delta$ where $\beta = \frac{q}{q+1}$ if $x_h < m$ and $\beta = \frac{2q}{2q+1}$ if $x_h = m$.

3. $\left[F\left(\frac{x_R + x_h}{2}\right) - \frac{q}{q+1}F\left(\frac{x_L + x_h}{2}\right)\right] = \frac{1}{2}$, in which case $W(\widetilde{e}, \alpha) = \{h\}$ and $U_h(\widetilde{e}, \alpha) = -\delta$.

In all cases, conditions (3) and (4) imply $U_h(e, \alpha) \geq U_h(\widetilde{e}, \alpha)$. Likewise if $x_h > m$.

Finally, for every other candidacy profile $\widehat{e}$, let $\alpha(\widehat{e})$ be a sincere voting profile. Hence, (e, α) is a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$.

(Necessity) Let (e, α) be a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$. The necessity of condition (1) has already been proven in Claim 1.1. To see that condition (2) is necessary as well, pick $h \in \mathcal{C}_L(e)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 0$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. Proceeding as in the sufficiency part, one can show that $W(\widetilde{e}, \alpha) = \mathcal{C}_R(\widetilde{e})$. Thus, the violation of condition (2) implies $U_h(\widetilde{e}, \alpha) \geq U_h(e, \alpha)$, and candidate h would be better off defecting. This contradicts that (e, α) is an equilibrium. The necessity of conditions (3) and (4) is likewise established. $\square$

Let V be a Multiple Voting rule with finite q. One can easily infer from Claim 1.2 that every two-position serious equilibrium under Plurality Voting has an equivalent two-position serious equilibrium under V. At the same time, Example 1 shows that there can exist two-position serious equilibria under V for which there is no equivalent two-position serious equilibrium under Plurality Voting. Thus, the two-position serious equilibrium set under V is a superset of the two-position serious equilibrium set under Plurality Voting.

It remains to prove that the superset is extreme. Let (e, α) be a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$ when the election is held under V. Let $(\widehat{e}, \widehat{\alpha})$ be a two-position serious equilibrium with policy outcome $\{\widehat{x}_L, \widehat{x}_R\}$ when the election is held under Plurality Voting. Suppose that $[x_L, x_R] \subsetneq [\widehat{x}_L, \widehat{x}_R]$. To prove extremism, it is sufficient to show that there exists under Plurality Voting a two-position serious equilibrium $(\widetilde{e}, \widetilde{\alpha})$ equivalent to (e, α). It is easily done by proceeding as in Claim 1.2. (The proof is omitted here, but is available from the author upon request.). ▮

The next lemma establishes that every serious equilibrium under Negative Voting is a one-position serious equilibrium. As the one-position serious equilibria are equivalent under every voting procedure and are moderate compared to the two-position serious equilibria, together Lemmas 2 and 5 imply that the serious equilibrium set under Negative Voting is a moderate subset of the serious equilibrium set under Plurality Voting.

LEMMA 5. *Suppose that the election is held under Negative Voting. Then, every serious equilibrium is a one-position serious equilibrium.*$\|$

Proof. I establish the result via a sequence of two claims.

CLAIM 1.3. Suppose that the election is held under a Negative Voting. Then, no two-position serious equilibrium exists.$\|$

PROOF OF CLAIM 1.3. Assume by way of contradiction that (e, α) is a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$. Without loss of generality, suppose that $c_R \geq c_L$. Then, the sincere voting profile $\alpha(e)$ is such that

$$\alpha_\ell(e) = \begin{cases} \{\mathcal{C}_L(e); \mathcal{C}_R(e)\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < \frac{x_L + x_R}{2} \\ \{\mathcal{C}_R(e); \mathcal{C}_L(e)\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > \frac{x_L + x_R}{2}. \end{cases}$$

Candidates' vote totals are thus given by

$$\begin{cases} \pi^i(e, \alpha) = 1 - \frac{1 - \gamma_L}{c_L} & \text{for every } i \in \mathcal{C}_L(e) \\ \pi^j(e, \alpha) = 1 - \frac{\gamma_L}{c_R} & \text{for every } i \in \mathcal{C}_R(e). \end{cases}$$

Since (e,α) is a serious equilibrium, it must be that $\pi^i(e,\alpha)=\pi^j(e,\alpha)$.

Pick $h\in\mathcal{C}_L(e)$. Notice that $U_h(e,\alpha)=\frac{c_R}{c}u(|x_L-x_R|)-\delta$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h=0$ and $\widetilde{e}_k=e_k$ for every $k\in\mathcal{P}$, $k\neq h$. Either $c_L=1$, in which case $W(\widetilde{e},\alpha)=\mathcal{C}_R(\widetilde{e})$. Or $c_L>1$, in which case candidates' vote totals are given by

$$\begin{cases}\pi^i(\widetilde{e},\alpha)=1-\frac{1-\gamma_L}{c_L-1} & \text{for every } i\in\mathcal{C}_L(\widetilde{e})\\ \pi^j(\widetilde{e},\alpha)=1-\frac{\gamma_L}{c_R} & \text{for every } i\in\mathcal{C}_R(\widetilde{e}).\end{cases}$$

Clearly, $\pi^i(\widetilde{e},\alpha)<\pi^i(e,\alpha)$ and $\pi^j(\widetilde{e},\alpha)=\pi^j(e,\alpha)$. This, together with $\pi^i(e,\alpha)=\pi^j(e,\alpha)$, implies $W(\widetilde{e},\alpha)=\mathcal{C}_R(\widetilde{e})$. Thus, in any case, $U_h(\widetilde{e},\alpha)=u(|x_L-x_R|)$. Since (e,α) is an equilibrium, it must be that $U_h(e,\alpha)>U_h(\widetilde{e},\alpha)$, which implies $\delta<-\frac{c_L}{c}u(|x_L-x_R|)$.

Pick $h\in\mathcal{P}$ with $x_h\in(x_L,x_L+\varepsilon)$. Notice that $U_h(e,\alpha)=\frac{c_L u(|x_h-x_L|)+c_R u(|x_h-x_R|)}{c}$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h=1$ and $\widetilde{e}_k=e_k$ for every $k\in\mathcal{P}$, $k\neq h$. It is easy to check that candidates' vote totals are now such that

$$\begin{cases}\pi^i(\widetilde{e},\alpha)=\pi^i(e,\alpha) & \text{for every } i\in\mathcal{C}_L(\widetilde{e})\\ \pi^h(\widetilde{e},\alpha)=1 & \\ \pi^j(\widetilde{e},\alpha)=\pi^j(e,\alpha) & \text{for every } j\in\mathcal{C}_R(\widetilde{e}).\end{cases}$$

Thus, $W(\widetilde{e},\alpha)=\{h\}$ and $U_h(\widetilde{e},\alpha)=-\delta$. Since (e,α) is an equilibrium, it must be that $U_h(e,\alpha)\geq U_h(\widetilde{e},\alpha)$. Considering the limit case where ε goes to zero, one then gets $\delta\geq-\frac{c_R}{c}u(|x_L-x_R|)$. This, together with $c_R\geq c_L$ and $u(|x_L-x_R|)<0$, contradicts the above condition on δ. Hence, (e,α) cannot be an equilibrium. □

Claim 1.4. Suppose that the election is held under a Negative Voting. Then, no multi-position serious equilibrium exists.‖

Proof of Claim 1.4. Assume by way of contradiction that (e,α) is a multi-position serious equilibrium. Define $x_L\equiv\min_{k\in\mathcal{C}(e)}x_k$ to be the leftmost candidate position. Likewise, define $x_R\equiv\max_{k\in\mathcal{C}(e)}x_k$ to be the rightmost candidate position. Notice that for every $\ell\in\mathcal{N}$ with $x_\ell<\frac{x_L+x_R}{2}$, $i\notin b_\ell(\mathcal{C}(e))$ if and only if $x_i=x_L$. Likewise, for every $\ell\in\mathcal{N}$ with $x_\ell>\frac{x_L+x_R}{2}$, $i\notin b_\ell(\mathcal{C}(e))$ if and only if $x_i=x_R$. In consequence, candidates' vote totals are such that

$$\begin{cases}\pi^i(e,\alpha)<1 & \text{for every } i\in\mathcal{C}(e) \text{ with } x_i\in\{x_L,x_R\}\\ \pi^j(e,\alpha)=1 & \text{for every } j\in\mathcal{C}(e) \text{ with } x_j\in(x_L,x_R).\end{cases}$$

This contradicts that (e,α) is a serious equilibrium. □ ∎

Finally, Example 2 shows that there can exist spoiler equilibria under Negative Voting, and that these equilibria can be extreme compared to every Plurality Voting equilibrium.

Case 2. Truncated ballots are permissible. I proceed via a sequence of two lemmas. The first lemma establishes that the two-position serious equilibrium set under any Multiple Voting rule that permits truncated ballots is an extreme superset of the two-position serious equilibrium set under Plurality Voting.

Lemma 6. *Suppose that the election is held under a Multiple Voting rule with truncated ballots permissible. Then, the two-position serious equilibrium set is an extreme superset of the two-position serious equilibrium set under Plurality Voting.*‖

Proof. I establish the result via a sequence of two claims. The first claim parallels Claim 1.1 above.

Claim 1.5. Suppose that the election is held under a Multiple Voting rule with number of votes q and truncated ballots permissible. Let (e, α) be a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$. Then, $u_m(x_L) = u_m(x_R)$ and $c_L = c_R \leq q$. Moreover,

$$-\frac{1}{c-1}\frac{u(|x_L - x_R|)}{2} > \delta \geq -\frac{1}{c+1}\frac{u(|x_L - x_R|)}{2}$$

with the latter inequality only if $c_L = c_R < q$.‖

Proof of Claim 1.5. Consider the situation described in the statement. Weak undominance implies that the voting profile $\alpha(e)$ is given by

$$\alpha_\ell(e) = \begin{cases} \{\mathcal{C}_L(e)\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < \frac{x_L + x_R}{2} \\ \{\mathcal{C}_R(e)\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > \frac{x_L + x_R}{2}. \end{cases}$$

Candidates' vote totals are thus given by

$$\begin{cases} \pi^i(e, \alpha) = \frac{\min\{q, c_L\}}{c_L}\gamma_L & \text{for every } i \in \mathcal{C}_L(e) \\ \pi^j(e, \alpha) = \frac{\min\{q, c_R\}}{c_R}\gamma_R & \text{for every } j \in \mathcal{C}_R(e). \end{cases}$$

As (e, α) is a serious equilibrium, it must be that $\pi^i(e, \alpha) = \pi^j(e, \alpha)$.

I start by establishing that $c_L \leq q$ and $c_R \leq q$. Assume by way of contradiction that $c_L > q$. Pick $h \in \mathcal{C}_L(e)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 0$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. Candidates' vote totals are such that $\pi^i(\widetilde{e}, \alpha) = \frac{q}{c_L - 1}\gamma_L > \frac{q}{c_L}\gamma_L = \pi^i(e, \alpha)$ and $\pi^j(\widetilde{e}, \alpha) = \pi^j(e, \alpha)$. This, together with $\pi^i(e, \alpha) = \pi^j(e, \alpha)$, implies $W(\widetilde{e}, \alpha) = \mathcal{C}_L(\widetilde{e})$. Thus, $U_h(\widetilde{e}, \alpha) > U_h(e, \alpha)$, which contradicts that (e, α) is an equilibrium. Hence $c_L \leq q$. Likewise, $c_R \leq q$.

Now, $\pi^i(e, \alpha) = \pi^j(e, \alpha)$ implies $\gamma_L = \gamma_R$ and, therefore, $m = \frac{x_L + x_R}{2}$. Hence, $u_m(x_L) = u_m(x_R)$.

One can easily check that $\delta < -\frac{1}{c-1}\frac{u(|x_L - x_R|)}{2}$ is necessary for every candidate to be worse off defecting. Likewise, $\delta \geq -\frac{1}{c+1}\frac{u(|x_L - x_R|)}{2}$ is necessary for every other potential candidate at x_L and x_R to be worse off entering the race if either $c_L < q$ or $c_R < q$ (or both).

Finally, $c_L = c_R$ follows from the symmetry of $u(.)$. □

The next claim characterizes the set of two-position serious equilibria under a Multiple Voting rule that permits truncated ballots. It parallels Claim 1.2.

Claim 1.6. Suppose that the election is held under a Multiple Voting rule with number of voters q and truncated ballots permissible. There exists a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$, $x_L < x_R$, if and only if there exist $i, j \in \mathcal{P}$ with $x_i = x_L$ and $x_j = x_R$, and

1. $m = \frac{x_L + x_R}{2}$;
2. $c_L = c_R \leq q$ and $-\frac{1}{c-1}\frac{u(|x_L - x_R|)}{2} > \delta \geq -\frac{1}{c+1}\frac{u(|x_L - x_R|)}{2}$, with the latter inequality only if $c_L = c_R < q$;
3. for every $h \in \mathcal{P}$ with $x_h \in (x_L, m)$, we have:
   (a) $\left[F\left(\frac{x_R + x_h}{2}\right) - \beta F\left(\frac{x_L + x_h}{2}\right)\right] < \frac{1}{2}$; or

(b) $\left[F\left(\frac{x_R+x_h}{2}\right)-\beta F\left(\frac{x_L+x_h}{2}\right)\right]=\frac{1}{2}$ and $\delta \geq \frac{c-2}{c+2}\frac{u_h(x_R)}{2}-\frac{u_h(x_L)}{2}$; or

(c) $\left[F\left(\frac{x_R+x_h}{2}\right)-\beta F\left(\frac{x_L+x_h}{2}\right)\right]>\frac{1}{2}$ and $\delta \geq -\frac{u_h(x_L)+u_h(x_R)}{2}$,

where $\beta = 1$ if $q > \frac{1}{2}-\frac{u(|x_L-x_R|)}{4\delta}$ and $\beta=\frac{q}{q+1}$ otherwise. Likewise for every $h \in \mathcal{P}$ with $x_h \in (m, x_R)$; and

4. for every $h \in \mathcal{P}$ with $x_h = m$, we have:
   (a) $F\left(\frac{x_R+x_h}{2}\right)<\frac{\beta}{2}$; or
   (b) $F\left(\frac{x_R+x_h}{2}\right)=\frac{\beta}{2}$ and $\delta \geq -\frac{u_m(x_L)}{c+1}$; or
   (c) $F\left(\frac{x_R+x_h}{2}\right)>\frac{\beta}{2}$ and $\delta \geq -u_m(x_L)$,

   where $\beta=\frac{3}{2}$ if $q > \frac{1}{2}-\frac{u(|x_L-x_R|)}{4\delta}$ and $\beta = \frac{3q+1}{2q+1}$ otherwise.$\|$

Proof of Claim 1.6. **(Sufficiency)** The proof is by construction. Suppose that all the conditions in the statement are satisfied. Let the candidacy profile e be such that c_L and c_R satisfy condition (2). Given weak undominance, the sincere voting profile $\alpha(e)$ is given by

$$\alpha_\ell(e)=\begin{cases}\{\mathcal{C}_L(e)\} & \text{for every } \ell\in\mathcal{N} \text{ with } x_\ell<m\\ \{\mathcal{C}_R(e)\} & \text{for every } \ell\in\mathcal{N} \text{ with } x_\ell>m.\end{cases}$$

This, together with condition (1), implies $\pi^i(e,\alpha)=1/2$ for every $i\in\mathcal{C}(e)$. Hence $W(e,\alpha)=\mathcal{C}(e)$.

Pick $h\in\mathcal{C}_L(e)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h=0$ and $\widetilde{e}_k=e_k$ for every $k\in\mathcal{P}$, $k\neq h$. Either $c_L=1$, in which case $W(\widetilde{e},\alpha)=\mathcal{C}_R(\widetilde{e})$. Or $c_L>1$, in which case the sincere voting profile $\alpha(\widetilde{e})$ is given by

$$\alpha_\ell(\widetilde{e})=\begin{cases}\{\mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell\in\mathcal{N} \text{ with } x_\ell<m\\ \{\mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell\in\mathcal{N} \text{ with } x_\ell>m.\end{cases}$$

Thus, $W(\widetilde{e},\alpha)=\mathcal{C}(\widetilde{e})$. In either case, $U_h(\widetilde{e},\alpha)=\frac{c}{c-1}\frac{u(|x_L-x_R|)}{2}$. Condition (2) implies $U_h(e,\alpha)>U_h(\widetilde{e},\alpha)$. Hence, a candidate $h\in\mathcal{C}_L(e)$ would be worse off defecting. Likewise for a candidate $h\in\mathcal{C}_R(e)$.

Pick $h\in\mathcal{P}\backslash\mathcal{C}(e)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h=1$ and $\widetilde{e}_k=e_k$ for every $k\in\mathcal{P}$, $k\neq h$. There are four cases to consider.

Case 1: $x_h\in\{x_L,x_R\}$. Suppose that $x_h=x_L$. Candidates' vote totals are then given by

$$\begin{cases}\pi^i(\widetilde{e},\alpha)=\frac{1}{2}\frac{\min\{c_L+1,q\}}{c_L+1} & \text{for every } i\in\mathcal{C}_L(\widetilde{e})\\ \pi^j(\widetilde{e},\alpha)=\frac{1}{2} & \text{for every } j\in\mathcal{C}_R(\widetilde{e}).\end{cases}$$

Either $c_L=q$, in which case $W(\widetilde{e},\alpha)=\mathcal{C}_R(\widetilde{e})$ and $U_h(\widetilde{e},\alpha)=u(|x_L-x_R|)-\delta$. Clearly, $U_h(e,\alpha)>U_h(\widetilde{e},\alpha)$. Or $c_L<q$, in which case $W(\widetilde{e},\alpha)=\mathcal{C}(\widetilde{e})$ and $U_h(\widetilde{e},\alpha)=\frac{c}{c+1}\frac{u(|x_L-x_R|)}{2}-\delta$. It follows from condition (2) that $U_h(e,\alpha)\geq U_h(\widetilde{e},\alpha)$. Thus, in either case potential candidate h would be worse off entering the race. Likewise if $x_h=x_R$.

Case 2: $x_h\notin[x_L,x_R]$. Suppose that $x_h<x_L$. Then, construct the voting profile $\alpha(\widetilde{e})$ as follows:

$$\alpha_\ell(\widetilde{e})=\begin{cases}\{h\} & \text{for every } \ell\in\mathcal{N} \text{ with } x_\ell<\frac{x_h+x_L}{2}\\ \{\mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell\in\mathcal{N} \text{ with } x_\ell\in\left(\frac{x_h+x_L}{2},m\right)\\ \{\mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell\in\mathcal{N} \text{ with } x_\ell>m.\end{cases}$$

Clearly, $W(\widetilde{e},\alpha)=\mathcal{C}_R(\widetilde{e})$ and $U_h(e,\alpha)>U_h(\widetilde{e},\alpha)$. Proceed likewise for $x_h>x_R$.

Case 3: $x_h \in (x_L, x_R)$, $x_h \neq m$. Suppose that $x_h \in (x_L, m)$. Construct the voting profile $\alpha(\widetilde{e})$ as follows:

$$\alpha_\ell(\widetilde{e}) = \begin{cases} \{\mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < \frac{x_h + x_L}{2} \\ \{h\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell \in \left(\frac{x_h + x_L}{2}, m\right) \\ \{h; \mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell \in \left(m, \frac{x_h + x_R}{2}\right) \\ \{\mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > \frac{x_h + x_R}{2}. \end{cases}$$

Notice that $x_h < m$ implies $F\left(\frac{x_h + x_L}{2}\right) < 1 - F\left(\frac{x_h + x_R}{2}\right)$. It follows that $\pi^i(\widetilde{e}, \alpha) < \pi^j(\widetilde{e}, \alpha)$ for every $i \in \mathcal{C}_L(\widetilde{e})$ and $j \in \mathcal{C}_R(\widetilde{e})$. We can thus restrict attention to the vote totals of candidate h and of the candidate(s) $j \in \mathcal{C}_R(\widetilde{e})$:

$$\begin{cases} \pi^h(\widetilde{e}, \alpha) = F\left(\frac{x_h + x_R}{2}\right) - F\left(\frac{x_h + x_L}{2}\right) \\ \pi^j(\widetilde{e}, \alpha) = \begin{cases} \frac{1}{2} & \text{if } c_R < q \\ \frac{1}{2} - \frac{1}{q}\left[F\left(\frac{x_h + x_R}{2}\right) - \frac{1}{2}\right] & \text{if } c_R = q. \end{cases} \end{cases}$$

Let $\beta \equiv \frac{q}{q+1}$ if $c_R = q$ and $\beta \equiv 1$ otherwise. One of the following must be true.

1. $\left[F\left(\frac{x_R + x_h}{2}\right) - \beta F\left(\frac{x_L + x_h}{2}\right)\right] < \frac{1}{2}$, in which case $W(\widetilde{e}, \alpha) = \mathcal{C}_R(\widetilde{e})$ and $U_h(\widetilde{e}, \alpha) = u_h(x_R) - \delta$; or
2. $\left[F\left(\frac{x_R + x_h}{2}\right) - \beta F\left(\frac{x_L + x_h}{2}\right)\right] = \frac{1}{2}$, in which case $W(\widetilde{e}, \alpha) = \mathcal{C}_R(\widetilde{e}) \cup \{h\}$ and $U_h(\widetilde{e}, \alpha) = \frac{c}{c+2} u_h(x_R) - \delta$; or
3. $\left[F\left(\frac{x_R + x_h}{2}\right) - \beta F\left(\frac{x_L + x_h}{2}\right)\right] > \frac{1}{2}$, in which case $W(\widetilde{e}, \alpha) = \{h\}$ and $U_h(\widetilde{e}, \alpha) = -\delta$.

In all three cases, condition (3) implies $U_h(e, \alpha) \geq U_h(\widetilde{e}, \alpha)$. Proceed likewise if $x_h \in (m, x_R)$.

Case 4: $x_h = m$. Construct the voting profile $\alpha(\widetilde{e})$ as follows:

$$\alpha_\ell(\widetilde{e}) = \begin{cases} \{\mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < \frac{x_L + m}{2} \\ \{h; \mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell \in \left(\frac{x_L + m}{2}, m\right) \\ \{h; \mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell \in \left(m, \frac{x_R + m}{2}\right) \\ \{\mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > \frac{x_R + m}{2}. \end{cases}$$

Notice that $F\left(\frac{x_L + m}{2}\right) = 1 - F\left(\frac{x_R + m}{2}\right)$ implies $\pi^i(\widetilde{e}, \alpha) = \pi^j(\widetilde{e}, \alpha)$ for every $i \in \mathcal{C}_L(\widetilde{e})$ and $j \in \mathcal{C}_R(\widetilde{e})$. Let $\beta = \frac{3q+1}{2q+1}$ if $c_L = c_R = q$ and $\beta = 3/2$ if $c_L = c_R < q$. One of the following must be true:

1. $F\left(\frac{x_R + x_h}{2}\right) < \frac{\beta}{2}$, in which case $W(\widetilde{e}, \alpha) = \mathcal{C}(e)$ and $U_h(\widetilde{e}, \alpha) = u_m(x_L) - \delta$; or
2. $F\left(\frac{x_R + x_h}{2}\right) = \frac{\beta}{2}$, in which case $W(\widetilde{e}, \alpha) = \mathcal{C}(\widetilde{e})$ and $U_h(\widetilde{e}, \alpha) = \frac{c}{c+1} u_m(x_L) - \delta$; or
3. $F\left(\frac{x_R + x_h}{2}\right) > \frac{\beta}{2}$, in which case $W(\widetilde{e}, \alpha) = \{h\}$ and $U_h(\widetilde{e}, \alpha) = -\delta$.

In all three cases, condition (4) implies $U_h(e, \alpha) \geq U_h(\widetilde{e}, \alpha)$.

Finally, for every other candidacy profile $\widehat{e}$, let $\alpha(\widehat{e})$ be any sincere voting profile. Hence, (e, α) is a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$.

(Necessity) Let (e, α) be a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$. The necessity of conditions (1) and (2) has already been proven in Claim 1.5. The necessity of conditions (3) and (4) is proven by noting that the voting profiles in cases 1-4 of the sufficiency part were constructed so to minimize the expected utility of a potential candidate $h \in \mathcal{P} \backslash \mathcal{C}(e)$ if he were to enter the race. In consequence, if either condition (3) or condition (4) is violated, then one such potential candidate will clearly be better off entering the race. This would contradict that (e, α) is an equilibrium. $\square$

The rest of the proof of Lemma 6 follows the same lines as the proof of Lemma 4. ▮

The next lemma considers the case where the voting behavior is relatively sincere.

LEMMA 7. *Suppose that the voting behavior is relatively sincere. Consider the set of serious equilibria. Then, a Multiple Voting rule with truncated ballots permissible always yields policy moderation compared to Plurality Voting if and only if it is Approval Voting.*$\|$

Proof. I establish the result via a sequence of three claims.

CLAIM 1.7. Suppose that the election is held under a Multiple Voting rule with truncated ballots permissible. Then, the set of relatively sincere two-position serious equilibria is equivalent to the set of saturated two-position serious equilibria.$\|$

PROOF OF CLAIM 1.7. **(Sufficiency)** Let $(\widehat{e}, \widehat{\alpha})$ be a saturated two-position serious equilibrium with policy outcome $\{x_L, x_R\}$, $x_L < x_R$. I now construct a relatively sincere two-position serious equilibrium (e, α) which is equivalent to $(\widehat{e}, \widehat{\alpha})$. Specifically, let $e = \widehat{e}$. Also, let $\alpha(e) = \widehat{\alpha}(e)$. Thus, $W(e, \alpha) = \mathcal{C}(e)$. Moreover, $c_L = c_R = q$ implies that $\alpha(e)$ is relatively sincere.

Pick $h \in \mathcal{C}(e)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 0$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. Let $\alpha(\widetilde{e}) = \widehat{\alpha}(\widetilde{e})$. Since there are only two candidate positions and voting is weakly undominated, the voting profile is necessarily relatively sincere. Moreover, since $(\widehat{e}, \widehat{\alpha})$ is an equilibrium, it must be that $U_h(e, \alpha) > U_h(\widetilde{e}, \alpha)$.

Pick $h \in \mathcal{P} \backslash \mathcal{C}(e)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 1$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. There are three cases to consider.

<u>Case 1: $x_h \in \{x_L, x_R\}$.</u> Let $\alpha(\widetilde{e}) = \widehat{\alpha}(\widetilde{e})$. Again, it is easy to see that the voting profile is relatively sincere.

<u>Case 2: $x_h \notin [x_L, x_R]$.</u> Construct $\alpha(\widetilde{e})$ as in the corresponding case in the sufficiency proof of Claim 1.6, letting $\alpha_\ell(\widetilde{e}) = \{h; \mathcal{C}_L(\widetilde{e})\}$ instead of $\{h\}$ for every $\ell \in \mathcal{N}$ with $x_\ell < \frac{x_h + x_L}{2}$. Thus, $W(\widetilde{e}, \alpha) = \mathcal{C}_R(\widetilde{e})$ and $\alpha(\widetilde{e})$ is relatively sincere. Proceed likewise for $x_h > x_R$.

<u>Case 3: $x_h \in (x_L, x_R)$.</u> Let $\alpha(\widetilde{e})$ be such that

$$\alpha_\ell(\widetilde{e}) = \begin{cases} \{\mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < \frac{x_L + x_h}{2} \\ \{h; \mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell \in \left(\frac{x_L + x_h}{2}, m\right) \\ \{h; \mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell \in \left(m, \frac{x_R + x_h}{2}\right) \\ \{\mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > \frac{x_R + x_h}{2}. \end{cases}$$

Again, $c_L = c_R = q$ implies that $\alpha(\widetilde{e})$ is relatively sincere.

As $(\widehat{e}, \widehat{\alpha})$ is a two-position serious equilibrium, it satisfies conditions $(2) - (4)$ in Claim 1.6. This implies that in any of the three cases above, $U_h(\widetilde{e}, \alpha) \geq U_h(e, \alpha)$. Thus, potential candidate h would be worse off entering the race.

Finally, for every other candidacy profile $\overline{e}$, let $\alpha(\overline{e})$ be any relatively sincere voting profile. Hence, (e, α) is a relatively sincere two-position serious equilibrium.

(Necessity) Let (e, α) be a relatively sincere two-position serious equilibrium. Since a relatively sincere two-position serious equilibrium is a two-position serious equilibrium, condition (2) of Claim 1.6 implies $c_L = c_R \leq q$ and $\delta < -\frac{u(|x_L - x_R|)}{2}$.

To prove the result, it is sufficient to establish that $c_L = c_R = q$. I do so by contradiction. Assume $c_L = c_R < q$. Pick $h \in \mathcal{P}$ with $x_h \in (x_R - \varepsilon, x_R)$. Consider the candidacy profile $\widetilde{e}$ where $\widetilde{e}_h = 1$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. Suppose that $x_h \geq \overline{x}$, where $\overline{x} \equiv \frac{1}{\#W(\widetilde{e}, \alpha)} \sum_{k \in W(\widetilde{e}, \alpha)} x_k$. (As shown below, there is no loss of generality in making this assumption.) This, together with the concavity of $u(.)$, implies $u_\ell (x_h) \geq U_\ell (\widetilde{e}, \alpha)$ for every $\ell \in \mathcal{N}$ with $x_\ell \geq x_h$. As $c_R < q$, relative sincerity implies $h \in b_\ell (\mathcal{C}(\widetilde{e}))$ for every such citizen ℓ. Notice also that candidate h is a most-preferred candidate for every $\ell \in \mathcal{N}$ with $x_\ell \in \left(\frac{x_L + x_h}{2}, x_h\right)$. In consequence, $h \in b_\ell (\mathcal{C}(\widetilde{e}))$ for every such citizen ℓ. It follows that $\pi^h (\widetilde{e}, \alpha) \geq 1 - F\left(\frac{x_L + x_h}{2}\right)$. Now, $m = \frac{x_L + x_R}{2}$ (by condition (1) in Claim 1.6) implies $\frac{x_h + x_L}{2} < m$ and, therefore, $\pi^h (\widetilde{e}, \alpha) > 1/2$.

At the same time, for every $i \in \mathcal{C}_L (\widetilde{e})$, we have $i \in b_\ell (\mathcal{C}(\widetilde{e}))$ implies $x_\ell \leq m$, and thus $\pi^i (\widetilde{e}, \alpha) \leq 1/2$. Likewise, for every $j \in \mathcal{C}_R (\widetilde{e})$, we have $j \in b_\ell (\mathcal{C}(\widetilde{e}))$ implies $x_\ell \geq m$, and thus $\pi^j (\widetilde{e}, \alpha) \leq 1/2$.

It follows that $W(\widetilde{e}, \alpha) = \{h\}$ and $U_h (\widetilde{e}, \alpha) = -\delta$. Since (e, α) is an equilibrium, it must be that $U_h (e, \alpha) \geq U_h (\widetilde{e}, \alpha)$. Considering the limit case where ε goes to zero, one then gets $\delta \geq -\frac{u(|x_L - x_R|)}{2}$. The latter inequality contradicts the above condition on δ. It must then be that $c_L = c_R = q$. $\square$

Recall from Lemma 6 that the two-position serious equilibrium set under any Multiple Voting rule with truncated ballots permissible is an extreme superset of the two-position serious equilibrium set under Plurality Voting. Moreover, one can easily infer from Claim 1.6 that every saturated two-position serious equilibrium is extreme compared to every non-saturated two-position serious equilibrium. It follows that a Multiple Voting rule with truncated ballots permissible can always yield policy moderation compared to Plurality Voting only if no two-position serious equilibrium is ever saturated. The following claim establishes that the latter condition is satisfied if and only if the Multiple Voting rule is Approval Voting.

Claim 1.8. Suppose that the election is held under a Multiple Voting rule with truncated ballots permissible. Let the voting behavior be relatively sincere. Then, the two-position serious equilibrium set is always empty if and only if the Multiple Voting rule is Approval Voting.$\|$

Proof of Claim 1.8. Define $S \equiv \left\{\{x_L, x_R\} \in X^2 : x_L < x_R \text{ and } m = \frac{x_L + x_R}{2}\right\}$ to be the set of possible two-position serious equilibrium outcomes. Let $\{\overline{x}_L, \overline{x}_R\} \equiv \sup_{\{x_L, x_R\} \in S} |x_L - x_R|$. Finally, define $\overline{q} \in \mathbb{N}$ such that

$$\frac{1}{2}\left(-1 - \frac{u(|\overline{x}_L - \overline{x}_R|)}{2\delta}\right) < \overline{q} \leq \frac{1}{2}\left(1 - \frac{u(|\overline{x}_L - \overline{x}_R|)}{2\delta}\right).$$

Thus, $\overline{q}$ is the maximum number of candidates standing at each position in a two-position serious equilibrium. Clearly, $q > \overline{q}$ then implies that no two-position serious equilibrium is saturated and, therefore, that the set of relatively sincere two-position serious equilibria is empty. This condition is always satisfied under Approval Voting ($q = +\infty$). However, for every other Multiple Voting rule, q is finite. One can then construct communities with sufficiently small δ in which $q < \overline{q}$ and saturated two-position serious equilibria exist. $\square$

Together, Claims 1.7 and 1.8 prove the necessity part in Lemma 7. The sufficiency part is now proven by establishing that every relatively sincere serious equilibrium under Approval Voting is a one-position serious equilibrium.

CLAIM 1.9. Suppose that the election is held under Approval Voting and that the voting behavior is relatively sincere. Then, every serious equilibrium is a one-position serious equilibrium.$\|$

PROOF OF CLAIM 1.9. Let (e, α) be a serious equilibrium. We already know from Claim 1.8 that (e, α) cannot be a two-position serious equilibrium. Assume by way of contradiction that (e, α) is a multi-position serious equilibrium. For every $\ell \in \mathcal{N}$, define $G_\ell(e) \equiv \left\{ i \in \mathcal{C}(e) : i \in \underset{k \in \mathcal{C}(e)}{\arg\max} u_\ell(x_k) \right\}$ as the set of candidates citizen ℓ likes the most. Likewise, define $L_\ell(e) \equiv \left\{ i \in \mathcal{C}(e) : i \in \underset{k \in \mathcal{C}(e)}{\arg\min} u_\ell(x_k) \right\}$ as the set of candidates citizen ℓ likes the least.

Pick $i \in L_m(e)$. Given the single-peakedness of preferences, candidate i is either a leftmost or a rightmost candidate. Without loss of generality, suppose the former. Then, by weak undominance, $i \in b_\ell(\mathcal{C}(e))$ implies $x_\ell < m$. Hence $\pi^i(e, \alpha) \leq 1/2$.

Pick $j \in G_m(e)$. Without loss of generality, suppose that $x_j \geq \overline{x}$, where $\overline{x}$ is the expected winning policy. By relative sincerity, $x_\ell \geq x_j$ implies $j \in b_\ell(\mathcal{C}(e))$. Now, there are two cases to consider.

First, suppose that $x_h = x_j$ for every $h, j \in G_m(e)$. Either $x_j < m$, in which case $\pi^j(e, \alpha) > 1/2$. Or $x_j \geq m$, in which case there exists $\underline{x} < m$ such that $x_\ell \in (\underline{x}, x_j)$ implies $j \in G_\ell(e)$ and, thus, $j \in b_\ell(\mathcal{C}(e))$. Again, $\pi^j(e, \alpha) > 1/2$. This, together with $\pi^i(e, \alpha) \leq 1/2$, contradicts that (e, α) is a serious equilibrium.

Second, suppose that $x_h < x_j$ for some $h, j \in G_m(e)$. Thus, $x_h < m < x_j$. Moreover, it must be that $x_h < \overline{x} < x_j$; otherwise, relative sincerity would imply either $\pi^h(e, \alpha) > 1/2$ or $\pi^j(e, \alpha) > 1/2$ (or both), which would again contradict that (e, α) is a serious equilibrium. By the same argument as in the first case above, one can establish that $x_\ell \geq m$ implies $j \in b_\ell(\mathcal{C}(e))$ and, thus, $\pi^j(e, \alpha) \geq 1/2$. Since (e, α) is a serious equilibrium, it must then be that $\pi^i(e, \alpha) = \pi^j(e, \alpha) = 1/2$, which implies $i \in b_\ell(\mathcal{C}(e))$ if and only if $x_\ell < m$. Now, $u_m(x_j) > u_m(x_i)$ and $x_i < m < x_j$ imply that there exists $\underline{x} < m$ such that $u_\ell(x_j) > u_\ell(x_i)$ for every $\ell \in \mathcal{N}$ with $x_\ell > \underline{x}$. As voting is sincere, $i \in b_\ell(\mathcal{C}(e))$ implies $j \in b_\ell(\mathcal{C}(e))$ for every $\ell \in \mathcal{N}$ with $x_\ell \in (\underline{x}, m)$. But then, $\pi^j(e, \alpha) > 1/2$, which contradicts that (e, α) is a serious equilibrium. $\square$

Finally, Example 4 shows that there can exist relatively sincere spoiler equilibria under Approval Voting, and that these equilibria can be extreme compared to every Plurality Voting equilibrium. **Q.E.D.**

PROOF OF PROPOSITION 2. I prove the result via a sequence of three lemmas.

LEMMA 8. *Suppose that the election is held under the Borda Count with only completely-filled ballots permissible. Then, every serious equilibrium is a one-position serious equilibrium.*$\|$

Proof. For a given number of candidates c, let $s_k(c) = c - k$ be the number of points a citizen gives to a candidate by ranking him at the k^{th} position on her ballot.

I proceed via a sequence of three claims.

Claim 2.1. Suppose that the election is held under the Borda Count with only completely-filled ballots permissible. Let (e, α) be a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$, $x_L < x_R$. Then, $u_m(x_L) = u_m(x_R)$ and $c_L = c_R$. Moreover,

$$-\frac{1}{c-1}\frac{u(|x_L - x_R|)}{2} > \delta \geq -\frac{1}{c+1}\frac{u(|x_L - x_R|)}{2}.\|$$

Proof of Claim 2.1. Consider the situation described in the statement. The voting profile $\alpha(e)$ is then such that

$$\alpha_\ell(e) = \begin{cases} \{\mathcal{C}_L(e); \mathcal{C}_R(e)\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < \frac{x_L + x_R}{2} \\ \{\mathcal{C}_R(e); \mathcal{C}_L(e)\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > \frac{x_L + x_R}{2}. \end{cases}$$

The vote total of a candidate $i \in \mathcal{C}_L(e)$ is thus given by

$$\pi^i(e, \alpha) = \frac{1}{c_L}\left\{F\left(\frac{x_L + x_R}{2}\right)\sum_{k=1}^{c_L} s_k(c) + \left[1 - F\left(\frac{x_L + x_R}{2}\right)\right]\sum_{k=c_R+1}^{c} s_k(c)\right\}$$

and the vote total of a candidate $j \in \mathcal{C}_R(e)$ by

$$\pi^j(e, \alpha) = \frac{1}{c_R}\left\{F\left(\frac{x_L + x_R}{2}\right)\sum_{k=c_L+1}^{c} s_k(c) + \left[1 - F\left(\frac{x_L + x_R}{2}\right)\right]\sum_{k=1}^{c_R} s_k(c)\right\}.$$

As (e, α) is a serious equilibrium, it must be that $\pi^i(e, \alpha) = \pi^j(e, \alpha)$, which implies $F\left(\frac{x_L + x_R}{2}\right) = \frac{1}{2}$. Hence, $m = \frac{x_L + x_R}{2}$, which implies $u_m(x_L) = u_m(x_R)$. Moreover, the symmetry of $u(.)$ implies $c_L = c_R$.

Pick $h \in \mathcal{C}_L(e)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 0$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. Either $c_L = 1$, in which case $W(\widetilde{e}, \alpha) = \mathcal{C}_R(\widetilde{e})$. Or $c_L > 1$, in which case $\alpha(\widetilde{e})$ is such that

$$\alpha_\ell(\widetilde{e}) = \begin{cases} \{\mathcal{C}_L(\widetilde{e}); \mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < m \\ \{\mathcal{C}_R(\widetilde{e}); \mathcal{C}_L(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > m. \end{cases}$$

It is easy to check that $W(\widetilde{e}, \alpha) = \mathcal{C}(\widetilde{e})$. It follows that $U_h(e, \alpha) > U_h(\widetilde{e}, \alpha)$ if and only if $\delta < -\frac{1}{c-1}\frac{u(|x_L - x_R|)}{2}$.

Pick $h \in \mathcal{P} \backslash \mathcal{C}(e)$ with $x_h = x_L$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 1$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. Then, the voting profile $\alpha(\widetilde{e})$ is as defined above and $W(\widetilde{e}, \alpha) = \mathcal{C}(\widetilde{e})$. It follows that $U_h(e, \alpha) \geq U_h(\widetilde{e}, \alpha)$ if and only if $\delta \geq -\frac{1}{c+1}\frac{u(|x_L - x_R|)}{2}$. □

Claim 2.2. Suppose that the election is held under the Borda Count with only completely-filled ballots permissible. Then, no two-position serious equilibrium exists.‖

Proof of Claim 2.2. Assume by way of contradiction that (e, α) is a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$, $x_L < x_R$. Pick $h \in \mathcal{P}$

with $x_h < x_L$.[52] Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 1$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. The voting profile $\alpha(\widetilde{e})$ is such that

$$\alpha_\ell(\widetilde{e}) = \begin{cases} \{h; \mathcal{C}_L(\widetilde{e}); \mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < \frac{x_h + x_L}{2} \\ \{\mathcal{C}_L(\widetilde{e}); h; \mathcal{C}_R(\widetilde{e})\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell \in \left(\frac{x_h+x_L}{2}, \frac{x_h+x_R}{2}\right) \\ \{\mathcal{C}_L(\widetilde{e}); \mathcal{C}_R(\widetilde{e}); h\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell \in \left(\frac{x_h+x_R}{2}, m\right) \\ \{\mathcal{C}_R(\widetilde{e}); \mathcal{C}_L(\widetilde{e}); h\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > m. \end{cases}$$

Candidates' vote totals are thus given by

$$\begin{cases} \pi^h(\widetilde{e}, \alpha) = F\left(\frac{x_h+x_L}{2}\right) + \left[F\left(\frac{x_h+x_R}{2}\right) - F\left(\frac{x_h+x_L}{2}\right)\right] s_{c_L+1}(c+1) \\ \pi^i(\widetilde{e}, \alpha) = \frac{1}{c_L} \left\{ F\left(\frac{x_h+x_L}{2}\right) \sum_{k=2}^{c_L+1} s_k(c+1) + \left[\frac{1}{2} - F\left(\frac{x_h+x_L}{2}\right)\right] \sum_{k=1}^{c_L} s_k(c+1) + \frac{1}{2} \sum_{k=c_R+1}^{c} s_k(c+1) \right\} \\ \pi^j(\widetilde{e}, \alpha) = \frac{1}{c_R} \left\{ \begin{array}{c} F\left(\frac{x_h+x_R}{2}\right) \sum_{k=c_L+2}^{c+1} s_k(c+1) + \left[\frac{1}{2} - F\left(\frac{x_h+x_R}{2}\right)\right] \sum_{k=c_L+1}^{c} s_k(c+1) \\ +\frac{1}{2} \sum_{k=1}^{c_R} s_k(c+1) \end{array} \right\} \end{cases}$$

for every $i \in \mathcal{C}_L(\widetilde{e})$ and $j \in \mathcal{C}_R(\widetilde{e})$. One can show that $F\left(\frac{x_h+x_L}{2}\right) < F\left(\frac{x_h+x_R}{2}\right) < \frac{1}{2}$ implies $W(\widetilde{e}, \alpha) = \mathcal{C}_L(\widetilde{e})$. It follows that $U_h(e, \alpha) \geq U_h(\widetilde{e}, \alpha)$ if and only if $\delta \geq \frac{u_h(x_L) - u_h(x_R)}{2}$. This inequality, together with the condition from Claim 2.1 that $\delta < -\frac{1}{c-1} \frac{u(|x_L - x_R|)}{2}$, implies $-u(|x_L - x_R|) > u_h(x_L) - u_h(x_R)$.

The concavity of $u(.)$, together with $x_h < x_L < x_R$, implies

$$\begin{cases} \frac{u_h(x_L)}{x_L - x_h} \geq \frac{u_h(x_R)}{x_R - x_h} \geq \frac{u_h(x_R) - u_h(x_L)}{x_R - x_h} \\ \frac{u_L(x_R)}{x_R - x_L} \geq \frac{u_h(x_R)}{x_R - x_h} \geq \frac{u_h(x_R) - u_L(x_L)}{x_L - x_h}. \end{cases}$$

One can infer from this set of inequalities that $u_h(x_L) - u_h(x_R) \geq -u(|x_L - x_R|)$, which contradicts the above condition. Hence, (e, α) cannot be an equilibrium. $\square$

Claim 2.3. Suppose that the election is held under the Borda Count with only completely-filled ballots permissible. Then, no multi-position serious equilibrium exists.‖

Proof of Claim 2.3. Assume by way of contradiction that (e, α) is a multi-position serious equilibrium. Define $x_L \equiv \min_{k \in \mathcal{C}(e)} x_k$ and $y \equiv \min_{\substack{k \in \mathcal{C}(e) \\ x_k \neq x_L}} x_k$ as the two leftmost candidate positions. Likewise, define $x_R \equiv \max_{k \in \mathcal{C}(e)} x_k$ and $z \equiv \max_{\substack{k \in \mathcal{C}(e) \\ x_k \neq x_R}} x_k$ as the two rightmost candidate positions. Thus, $x_L < y \leq z < x_R$. Finally, denote the number of candidates at x_L (y, z, x_R, resp.) by c_L (c_y, c_z, c_R, resp.).

Pick $h \in \mathcal{C}_L(e)$ and $i \in \mathcal{C}_y(e)$. Define $\Delta_{h,i} \equiv \left[\pi^h(e, \alpha) - \pi^i(e, \alpha)\right]$ to be candidate h's plurality over candidate i. On the interval $\left(-\infty, \frac{x_L+y}{2}\right)$, it is given

[52] In case $x_k \in [x_L, x_R]$ for every $k \in \mathcal{P}$, then the result is proven by showing that a potential candidate h with x_h close to m would win the election outright if he were to enter the race. This, together with $x_0 \in X$ and $-u_m(x_0) > \delta$, would imply that such a potential candidate would be better off entering the race. This would contradict that (e, α) is an equilibrium.

by

$$\begin{aligned}\Delta_{h,i}\left(-\infty,\frac{x_L+y}{2}\right) &= F\left(\frac{x_L+y}{2}\right)\left[\frac{1}{c_L}\sum_{k=1}^{c_L}s_k(c)-\frac{1}{c_y}\sum_{k=c_L+1}^{c_L+c_y}s_k(c)\right]\\ &= F\left(\frac{x_L+y}{2}\right)\frac{c_L+c_y}{2}.\end{aligned}$$

On the interval $\left(\frac{x_L+y}{2},\infty\right)$, it is such that

$$\begin{aligned}\Delta_{h,i}\left(\frac{x_L+y}{2},\infty\right) &\leq \left[1-F\left(\frac{x_L+y}{2}\right)\right]\left[\frac{1}{c_L}\sum_{k=x+c_y+1}^{x+c_y+c_L}s_k(c)-\frac{1}{c_y}\sum_{k=x+1}^{x+c_y}s_k(c)\right]\\ &= \left[F\left(\frac{x_L+y}{2}\right)-1\right]\frac{c_L+c_y}{2}\end{aligned}$$

for some $x\in\{0,1,...,c-c_L-c_y\}$. As the citizens with ideal policy $\frac{x_L+y}{2}$ are of measure zero, they can be ignored. Thus, $\Delta_{h,i}=\Delta_{h,i}\left(-\infty,\frac{x_L+y}{2}\right)+\Delta_{h,i}\left(\frac{x_L+y}{2},\infty\right)$. Since (e,α) is a serious equilibrium, it must be that $\Delta_{h,i}=0$, which implies $F\left(\frac{x_L+y}{2}\right)\geq\frac{1}{2}$ and, therefore, $\frac{x_L+y}{2}\geq m$.

Pick $j\in\mathcal{C}_z(e)$ and $k\in\mathcal{C}_R(e)$. Proceeding as above, one gets $\frac{z+x_R}{2}\leq m$.

Taken together, the two inequalities above imply $\frac{z+x_R}{2}\leq\frac{x_L+y}{2}$, which yields a contradiction. $\square$ ∎

Lemma 9. *Suppose that the election is held under Coombs Voting. Then, every equilibrium is a one-position serious equilibrium.*‖

Before proving this result, I must introduce some extra notation. For a given candidacy profile e and voting profile $\alpha(e)$, let σ be an elimination sequence, i.e., a sequence in which candidates are eliminated. A candidate is said to be active at elimination round $t\in\{1,2,...,c\}$ if he has not yet been eliminated by that round. Let $\mathcal{C}_t(e,\sigma)\subseteq\mathcal{C}(e)$ be the set of candidates who are active at elimination round t.[53] Let $\gamma_t^i(e,\sigma)$ denote the mass of voters who, at elimination round t, are ranking candidate i first. I call $\gamma_t^i(e,\sigma)$ candidate i's support at elimination round t.

Proof. I start by establishing that in every equilibrium, every candidate in the winning set is a Condorcet winner.

Claim 2.4. Suppose that the election is held under Coombs Voting. Then, $i\in W(e,\alpha)$ implies $i\in G_m(e)$.‖

Proof of Claim 2.4. Pick $j\in G_m(e)$. Without loss of generality, suppose that $x_j\leq m$. Assume by way of contradiction that $i\in W(e,\alpha)$, but $i\notin G_m(e)$. Suppose that $x_i<x_j$. (A similar argument holds for $x_i>x_j$.) Since $i\in W(e,\alpha)$, there must be an elimination sequence σ in which candidate i is the first to receive majority support. Formally, $i\in\mathcal{C}_\tau(e,\sigma)$ for some $\tau\in\{1,2,...,c\}$ and

$$\begin{cases}\gamma_t^h(e,\sigma)\leq\frac{1}{2}\text{ for every }h\in\mathcal{C}_t(e,\sigma)\text{ and }t<\tau\\ \gamma_t^j(e,\sigma)\leq\frac{1}{2}<\gamma_\tau^i(e,\sigma)\text{ for every }j\in\mathcal{C}_\tau(e,\sigma),\ j\neq i.\end{cases}$$

[53] To simplify notation, I have omitted the voting profile $\alpha(.)$ from the arguments of $\mathcal{C}_t(.)$. I shall do so whenever there is no risk of confusion.

As $x_i < x_j$, we have $u_m(x_i) < u_m(x_j)$ for every $\ell \in \mathcal{N}$ with $x_\ell \geq m$. These citizens must then be ranking candidate j above candidate i. It follows that candidate i cannot receive majority support before every candidate h with $x_h \geq x_j$ is eliminated. Let t $(< \tau)$ be the elimination round at which the last active candidate at x_j is eliminated. (If need be, relabel the candidates such that this candidate is candidate j.) Since only a leftmost or a rightmost candidate is eliminated, it must then be that at elimination round t candidate j is the (unique) rightmost active candidate. Pick an active candidate h with ideal policy $x_h < x_j$ and who is the closest active neighbor of candidate j. Notice that $u_\ell(x_j) > u_\ell(x_h)$ for every $\ell \in \mathcal{N}$ with $x_\ell > \frac{x_h + x_j}{2}$. Since candidate j is the rightmost active candidate at elimination round t, all these citizens must be ranking candidate j first. It follows that $\gamma_t^j(e, \sigma) \geq 1 - F\left(\frac{x_h + x_j}{2}\right)$. Notice that $x_h < x_j \leq m$ implies $F\left(\frac{x_h + x_j}{2}\right) < \frac{1}{2}$. It follows that $\gamma_t^j(e, \sigma) > 1/2$ and, therefore, that candidate j receives majority support before candidate i. Hence the contradiction. $\square$

Let (e, α) be an equilibrium. We know from Claim 2.4 that $u_m(x_i) = u_m(x_j) > u_m(x_k)$ for every $i, j \in W(e, \alpha)$ and every $k \in \mathcal{C}(e) \backslash W(e, \alpha)$. There are two cases to consider.

First, suppose that $x_i = x_j$ for every $i, j \in W(e, \alpha)$. In this case, (e, α) is a one-position serious equilibrium. Indeed, every $h \in \mathcal{C}(e)$ with $x_h \neq x_i$ would be a spoiler candidate. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 0$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. It is easy to see that $W(\widetilde{e}, \alpha) = W(e, \alpha)$, in which case $U_h(\widetilde{e}, \alpha) > U_h(e, \alpha)$. This would contradict that (e, α) is an equilibrium.

Second, suppose that $x_i < x_j$ for some $i, j \in W(e, \alpha)$. In this case, Claim 2.4 implies $x_i < m < x_j$ and $\{h \in \mathcal{C}(e) : x_h \in (x_i, x_j)\} = \emptyset$. Letting p denote the probability with which x_i is implemented (and $1 - p$ the probability with which x_j is implemented), we have $U_i(e, \alpha) = (1 - p)\, u(|x_i - x_j|) - \delta$. Thus, $U_i(e, \alpha) > U_i(\widetilde{e}, \alpha)$ if and only if $-pu(|x_i - x_j|) > \delta$. Likewise, $U_j(e, \alpha) > U_j(\widetilde{e}, \alpha)$ if and only if $-(1 - p)\, u(|x_i - x_j|) > \delta$. Taken together, these two inequalities imply $-\frac{u(|x_i - x_j|)}{2} > \delta$.

Without loss of generality, suppose that $p \leq 1/2$. Pick $h \in \mathcal{P}$ with $x_h \in (x_i, x_i + \varepsilon)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 1$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq h$. Notice that $U_h(e, \alpha) = pu(|x_i - x_h|) + (1 - p)\, u(|x_j - x_h|)$. It follows from Claim 2.4 that $W(\widetilde{e}, \alpha) = \{h\}$ and $U_h(\widetilde{e}, \alpha) = -\delta$. Since (e, α) is an equilibrium, it must be that $U_h(e, \alpha) \geq U_h(\widetilde{e}, \alpha)$. Considering the limit case where ε goes to zero, one then gets $\delta \geq -\frac{u(|x_i - x_j|)}{2}$. This contradicts the above condition on δ. Hence, (e, α) cannot be an equilibrium. ∎

Lemma 10. *Suppose that the election is held under Bucklin Voting. Then, every relatively sincere serious equilibrium is a one-position serious equilibrium.*‖

Proof. I establish the result via a sequence of two claims.

Claim 2.5. Suppose that the election is held under Bucklin Voting. Let the voting behavior be relatively sincere. Then, no two-position serious equilibrium exists.‖

Proof of Claim 2.5. Assume by way of contradiction that (e, α) is a two-position serious equilibrium with policy outcome $\{x_L, x_R\}$, $x_L < x_R$. I start by establishing that $m = \frac{x_L + x_R}{2}$. Assume by way of contradiction that $m < \frac{x_L + x_R}{2}$.

(A similar argument holds for $m > \frac{x_L + x_R}{2}$.) The voting profile $\alpha(e)$ is thus such that

$$\alpha_\ell(e) = \begin{cases} \{\mathcal{C}_L(e)\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell < \frac{x_L+x_R}{2} \\ \{\mathcal{C}_R(e)\} & \text{for every } \ell \in \mathcal{N} \text{ with } x_\ell > \frac{x_L+x_R}{2}. \end{cases}$$

Let $\pi_t^k(e, \alpha)$ denote candidate k's vote total at vote count $t = 1, 2, ..., c$. Notice that $\pi_t^k(e, \alpha) \leq \pi_{t+1}^k(e, \alpha)$ for every $k \in \mathcal{C}(e)$ and every $t = 1, 2, ..., c-1$.

Given $\alpha(e)$, candidates' vote totals at the last vote count c are thus given by

$$\begin{cases} \pi_c^i(e, \alpha) = F\left(\frac{x_L+x_R}{2}\right) & \text{for every } i \in \mathcal{C}_L(e) \\ \pi_c^j(e, \alpha) = 1 - F\left(\frac{x_L+x_R}{2}\right) & \text{for every } j \in \mathcal{C}_R(e). \end{cases}$$

Now, $m < \frac{x_L+x_R}{2}$ implies $\pi_c^i(e, \alpha) > 1/2 > \pi_c^j(e, \alpha)$. This, together with $\pi_t^k(e, \alpha) \leq \pi_{t+1}^k(e, \alpha)$, implies $W(e, \alpha) = \mathcal{C}_L(e)$, which contradicts that (e, α) is an equilibrium.

Pick $i \in \mathcal{C}_L(e)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_i = 0$ and $\widetilde{e}_k = e_k$ for every $k \in \mathcal{P}$, $k \neq i$. As there are only two candidate positions and $m = \frac{x_L+x_R}{2}$, we have $W(\widetilde{e}, \alpha) = \mathcal{C}(\widetilde{e})$ and $U_i(\widetilde{e}, \alpha) = \frac{c_R}{c-1} u(|x_L - x_R|)$. It follows that $U_i(e, \alpha) > U_i(\widetilde{e}, \alpha)$ if and only if $\delta < -\frac{c_R}{c(c-1)} u(|x_L - x_R|)$.

Pick $h \in \mathcal{P}$ with $x_h \in (x_L, x_L + \varepsilon)$. Consider the candidacy profile $\widetilde{e}$ in which $\widetilde{e}_h = 1$ and $\widetilde{e}_k = e_k$, $k \neq h$. Suppose that $x_h \leq \overline{x}$, where $\overline{x} \equiv \frac{1}{\#W(\widetilde{e},\alpha)} \sum_{k \in W(\widetilde{e},\alpha)} x_k$. (As we shall see below, there is no loss of generality in making this assumption.) Proceeding as in the necessity part of Claim 1.7, one can establish that

$$\pi_c^h(\widetilde{e}, \alpha) > \frac{1}{2} \geq \pi_c^k(\widetilde{e}, \alpha)$$

for every $k \in \mathcal{C}(e)$. Thus, $W(\widetilde{e}, \alpha) = \{h\}$. Since (e, α) is an equilibrium, it must be that $U_h(e, \alpha) \geq U_h(\widetilde{e}, \alpha)$. Considering the limit case where ε goes to zero, one then gets $\delta \geq -\frac{c_R}{c} u(|x_L - x_R|)$. This contradicts the above condition on δ. Hence, (e, α) cannot be an equilibrium. □

CLAIM 2.6. Suppose that the election is held under Bucklin Voting. Let the voting behavior be relatively sincere. Then, no multi-position serious equilibrium exists.‖

PROOF OF CLAIM 2.6. Assume by way of contradiction that (e, α) is a multi-position serious equilibrium. Proceeding as in the necessity part of Claim 1.7, one can establish that $\pi_c^h(\widetilde{e}, \alpha) > 1/2 \geq \pi_c^i(\widetilde{e}, \alpha)$ for some $h \in G_m(e)$ and $i \in L_m(e)$. It thus follows that $i \notin W(e, \alpha)$, which contradicts that (e, α) is a serious equilibrium. □ ∎

Example 6 shows that two-position serious equilibria can exist under the Alternative Vote and, therefore, that the extent of policy moderation is not always maximal under the Alternative Vote. Also, one can prove by construction that there can be (non-relatively-sincere) two-position serious equilibria under Bucklin Voting. Finally, one can construct relatively sincere spoiler equilibria under Bucklin Voting that are extreme compared to every Plurality Voting. (Examples of such equilibria are available from the author upon request.) **Q.E.D.**