Invalidity of the Bootstrap and the m Out of n Bootstrap for Interval Endpoints Defined by Moment Inequalities

Donald W. K. Andrews
Cowles Foundation for Research in Economics
Yale University

Sukjin Han
Department of Economics
Yale University

November 2005
Revised: July 2008

Abstract

This paper analyzes the finite-sample and asymptotic properties of several bootstrap and m out of n bootstrap methods for constructing confidence interval (CI) endpoints in models defined by moment inequalities. In particular, we consider using these methods directly to construct CI endpoints. By considering two very simple models, the paper shows that neither the bootstrap nor the m out of n bootstrap is valid in finite samples or in a uniform asymptotic sense in general when applied directly to construct CI endpoints.

In contrast, other results in the literature show that other ways of applying the bootstrap, m out of n bootstrap, and subsampling do lead to uniformly asymptotically valid confidence sets in moment inequality models. Thus, the uniform asymptotic validity of resampling methods in moment inequality models depends on the way in which the resampling methods are employed.

Keywords: Bootstrap, coverage probability, m out of n bootstrap, moment inequality model, partial identification, subsampling.

JEL Classification Numbers: C01.

1 Introduction

This paper considers confidence intervals (CIs) for partially-identified parameters defined by moment inequalities. The paper investigates the properties of the bootstrap and the m out of n bootstrap applied directly to CI endpoints. (Here, m is the bootstrap sample size and n is the original sample size.) By "applied directly to CI endpoints," we mean that one takes the CI upper endpoint to be the upper bound of the estimated set based on the original sample plus a (recentered and rescaled) sample quantile of the upper bounds of the estimated sets from a collection of bootstrap or m out of n bootstrap samples and analogously for the CI lower endpoint. We note that the m out of n bootstrap has been suggested in the literature as an alternative to the bootstrap in cases in which the bootstrap does not work properly.

"Backward" and "forward" bootstrap and m out of n bootstrap CIs are considered. (These are defined below.) Both finite-sample and asymptotic coverage probabilities and sizes of the CIs are obtained. In fact, one of the novelties of the paper is the determination of *exact* finite-sample coverage probabilities and sizes for some bootstrap and m out of n bootstrap procedures.

The results show that neither the bootstrap nor the m out of n bootstrap is asymptotically valid in a uniform sense in general when applied directly to CI endpoints. These results are obtained by considering the parametric bootstrap in two particular models. These two models each have normally distributed observations, a scalar parameter θ, and two moment inequalities. The two models are selected to exhibit the two common features of moment inequality models that cause difficulties for inference.

The first model exhibits a redundant, but not irrelevant, moment inequality. The model has two moment inequalities, only one of which is binding in the population, but either of which may be binding in the sample due to random fluctuations. The second model exhibits the possibility of random "reversals" of moment inequalities. The model has two moment inequalities that bound a parameter from below and above such that the identified set is a proper interval. But the length of the identified set is sufficiently short relative to the variability in the moment inequalities that there is a non-negligible probability that the estimated set is a singleton. The estimated set is a singleton when the lower bound from one moment inequality is larger than the upper bound from another moment inequality, which is referred to as a "reversal" of the moment inequalities. Redundant but not irrelevant moment inequalities and reversals of moment inequalities are common features of moment inequality models, e.g., see Andrews, Berry, and Jia (2004) and Pakes, Porter, Ishii, and Ho (2004).

The paper shows that the finite-sample and asymptotic coverage probabilities and sizes of the bootstrap and m out of n bootstrap CIs can be far from their nominal levels in these two models.[1] For example, in the first model, the nominal .95 "backward" bootstrap CI has finite-sample confidence size equal (up to simulation error) to .00 for all n. Similarly, the nominal .95 m out of n "backward" bootstrap

[1] By definition, the "size" of a CI is the infimum of the coverage probabilities of the CI over all distributions in the model. A CI has "level" $1-\alpha$ if its size is greater than or equal to $1-\alpha$.

has finite-sample confidence size equal to .00 when $m/n = .01$, .05, or .10 for all n. It has asymptotic size .00 when $m/n \to 0$ as $n \to \infty$. In the second model, the nominal .95 "forward" bootstrap CI has finite-sample confidence size equal (up to simulation error) to .51 for all n. Similarly, the nominal .95 m out of n "forward" bootstrap has finite-sample confidence size equal to .51 when $m/n = .01$, .05, or .10 for all n. It has asymptotic size .50 when $m/n \to 0$ as $n \to \infty$.

The failure of the bootstrap in these models is due to the non-differentiability of the statistics of interest as a function of the underlying sample moments. See Shao (1994) for further discussion. The failure of the m out of n bootstrap is due to the discontinuity of the asymptotic distribution of the statistics of interest as a function of the parameters. See Andrews and Guggenberger (2005a) for further discussion.

Obviously, if a method fails to deliver desirable finite-sample and asymptotic properties in the two models considered in the paper, it cannot do so in general. Hence, the results of this paper show that the bootstrap and m out of n bootstrap applied directly to CI endpoints cannot be relied upon to give valid inference in general.

As stated above, the results given here are for the parametric bootstrap and the m out of n parametric bootstrap. The asymptotic properties of the nonparametric i.i.d. bootstrap and nonparametric i.i.d. m out of n bootstrap are the same as those of the parametric bootstrap, although we do not show this explicitly in the paper. Furthermore, asymptotic results for subsampling are the same as for the nonparametric i.i.d. m out of n bootstrap provided the subsample size b equals m and $m^2/n \to 0$ as $n \to \infty$, see Politis, Romano, and Wolf (1999, p. 48). Hence, the asymptotic results of this paper for the m out of n bootstrap also should apply to subsampling methods. Such results for subsampling could be established directly using the methods in Andrews and Guggenberger (2005a). For brevity, we do not do so here.

We emphasize that there are different ways of applying the bootstrap, m out of n bootstrap, and subsampling to moment inequality models. This paper addresses one way, viz., by applying such methods to CI endpoints directly. The paper shows that this does not yield asymptotically valid CIs in a uniform sense in general. On the other hand, if one follows the approach in Chernozhukov, Hong, and Tamer (2007) and one constructs confidence sets by inverting tests based on an Anderson-Rubin-type test statistic, then subsampling and the m out of n bootstrap yield confidence sets that are asymptotically valid in a uniform sense for test statistics in a suitable class, see Andrews and Guggenberger (2008). Also see Romano and Shaikh (2008). Furthermore, confidence sets constructed by inverting tests based on an Anderson-Rubin test statistic can be coupled with a recentered bootstrap that is applied as part of a generalized moment selection (GMS) method for constructing critical values. Such bootstrapped-based confidence sets are asymptotically valid in a uniform sense, see Andrews and Soares (2007). Bugni (2007a,b) and Canay (2007) consider similar bootstrap-based confidence sets that are asymptotically valid in a uniform sense.

We now discuss additional related literature. There is a large literature on bootstrap inconsistency due to non-regularity of a model, see Efron (1979), Bickel and

Freedman (1981), Beran (1982, 1997), Babu (1984), Beran and Srivastava (1985), Athreya (1987), Romano (1988), Basawa *et al.* (1991), Putter and van Zwet (1996), Bretagnolle (1983), Deheuvels, Mason, and Shorack (1993), Dümbgen (1993), Sriram (1993), Athreya and Fukuchi (1994), Datta (1995), Bickel, Götze, and van Zwet (1997), and Andrews (2000).

When the bootstrap is not consistent, it is common in the literature to suggest using the m out of n bootstrap or subsampling as an alternative, see Bretagnolle (1983), Swanepoel (1986), Athreya (1987), Beran and Srivastava (1987), Shao and Wu (1989), Wu (1990), Eaton and Tyler (1991), Politis and Romano (1994), Shao (1994, 1996), Beran (1997), Bickel, Götze, and van Zwet (1997), Andrews (1999, 2000), Politis, Romano, and Wolf (1999), Romano and Wolf (2001), Guggenberger and Wolf (2004), Lehmann and Romano (2005), Romano and Shaikh (2005, 2008), and Chernozhukov, Hong, and Tamer (2007).

Potential problems with the m out of n bootstrap and subsampling are discussed in Dümbgen (1993), Beran (1997), Andrews (2000), Samworth (2003), Andrews and Guggenberger (2005a,b,c,d), and Mikusheva (2007).

The remainder of the paper is organized as follows. Section 2 introduces the general moment inequality model. Section 3 defines "forward" and "backward" bootstrap CIs based on bootstrapping CI endpoints. Section 4 does likewise for the m out of n bootstrap. Sections 5 and 6 treat two specific moment inequality models that are based on linear normally-distributed moment inequalities. These two sections provide finite-sample and asymptotic coverage probability and size results for bootstrap and m out of n bootstrap procedures in the two models considered.

2 Moment Inequality Model

The sample is $\{X_i : i \leq n\}$. The random variables $\{X_i : i \geq 1\}$ are assumed to be iid. We have some moment functions $m(X_i, \theta)$ $(\in R^k)$ that depend on a parameter $\theta \in \Theta \subset R^p$. The true value of the parameter is $\theta_0 \in \Theta$. The population moments satisfy

$$E_{\theta_0} m(X_i, \theta_0) \geq 0 \quad (2.1)$$

element by element. We are interested in a real-valued smooth function $g(\theta)$ $(\in R)$ of θ.

Define the *identified set* of θ values that satisfy the population moment inequalities to be

$$\Theta_0 = \{\theta \in \Theta : E_{\theta_0} m(X_i, \theta) \geq 0\}. \quad (2.2)$$

The corresponding identified set of $g(\theta)$ values is $[g_{L0}, g_{U0}]$, where

$$\begin{aligned} g_{L0} &= \inf\{g(\theta) : \theta \in \Theta_0\} \text{ and} \\ g_{U0} &= \sup\{g(\theta) : \theta \in \Theta_0\}. \end{aligned} \quad (2.3)$$

The object is to determine a random interval $[\widetilde{g}_{Ln}, \widetilde{g}_{Un}]$ that contains either the true value $g(\theta_0)$ with probability $1 - \alpha$ asymptotically or the identified interval $[g_{L0}, g_{U0}]$

with probability $1-\alpha$ asymptotically for some $\alpha \in (0,1)$. We specify a plausible bootstrap procedure for doing this and show that it does not have the correct asymptotic coverage probability in very simple normal models with linear moment functions and a scalar parameter θ. We show that using an m out of n version of the bootstrap does not solve the problem.

Define

$$\begin{aligned}
\overline{m}_n(X_i,\theta) &= n^{-1}\sum_{i=1}^{n} m(X_i,\theta),\\
Q_n(\theta) &= d(\sqrt{n}\cdot[\overline{m}_n(X_i,\theta)]_-),\\
[x]_- &= \min\{x,0\},\\
\widehat{\Theta}_n &= \{\theta\in\Theta : Q_n(\theta) = \inf_{\overline{\theta}\in\Theta} Q_n(\overline{\theta})\},\\
\widehat{g}_{Ln} &= \inf\{g(\theta) : \theta \in \widehat{\Theta}_n\}, \text{ and}\\
\widehat{g}_{Un} &= \sup\{g(\theta) : \theta \in \widehat{\Theta}_n\},
\end{aligned} \tag{2.4}$$

where $d(\cdot) : R^k \to R$ is a non-negative distance function such as $d(x) = x'x$ or $d(x) = \sum_{j=1}^{k}|x_j|$.[2] The quantities $\widehat{\Theta}_n$, $\widehat{g}_{Ln}$, and $\widehat{g}_{Un}$ are estimators of Θ_0, g_{L0}, and g_{U0}, respectively. [3]

3 Bootstrap for CI Endpoints

We now define a heuristic procedure for constructing the random interval $[\widetilde{g}_{Ln}, \widetilde{g}_{Un}]$ based on the bootstrap. (Based on the results given below, we do not recommend that this procedure be used in practice.)

The bootstrap procedure is as follows. (i) Generate B independent bootstrap samples $\{X^*_{ir} : i \le n\}$ for $r = 1, ..., B$ using some method of bootstrapping—as discussed further below. (ii) Compute $\widehat{g}^*_{Lnr}$ and $\widehat{g}^*_{Unr}$ using the definitions of $\widehat{g}_{Ln}$ and $\widehat{g}_{Un}$ in (2.4) with $\{X^*_{ir} : i \le n\}$ in place of $\{X_i : i \le n\}$ for $r = 1, ..., B$. (iii) Compute the $\alpha/2$ sample quantile of $\{g^*_{Lnr} : r = 1, ..., B\}$, call it $c^{**}_{LnB}(\alpha/2)$. (iv) Compute the $1-\alpha/2$ sample quantile of $\{g^*_{Unr} : r = 1, ..., B\}$, call it $c^{**}_{UnB}(1-\alpha/2)$. (v) Take the random interval $[\widetilde{g}_{Ln}, \widetilde{g}_{Un}]$ to be

$$[\widetilde{g}_{Ln}, \widetilde{g}_{Un}] = [c^{**}_{LnB}(\alpha/2), c^{**}_{UnB}(1-\alpha/2)]. \tag{3.1}$$

The intuitive idea behind this interval is that the bootstrap quantities g^*_{Lnr} and g^*_{Unr} for $r = 1, ..., B$ behave like B iid realizations of $\widehat{g}_{Ln}$ and $\widehat{g}_{Un}$. Hence, the interval from the $\alpha/2$ sample quantile of g^*_{Lnr} to the $1-\alpha/2$ quantile of g^*_{Unr} should include $[g_{L0}, g_{U0}]$ with probability $1-\alpha$. This intuition is not completely correct because it

[2] One also could consider data-dependent weight functions $d(\cdot)$ of $[\overline{m}_n(X_i,\theta)]_-$, but for simplicity we do not do so here.

[3] If $E_{\theta_0} m(X_i,\theta)$ is not well-behaved, then it is possible for these estimators to be inconsistent, e.g., see Manski and Tamer (2002). But this is not the issue that is of concern here. We consider cases in which $E_{\theta_0} m(X_i,\theta)$ is sufficiently well-behaved that $\widehat{\Theta}_n$, $\widehat{g}_{Ln}$, and $\widehat{g}_{Un}$ are consistent.

ignores the issues of (i) proper centering of the bootstrap quantities and (ii) the non-differentiability in the mapping between the sample moments and the estimators $\widehat{g}_{Ln}$ and $\widehat{g}_{Un}$. In "regular" cases the first issue does not cause problems and the second issue does not arise. In the present case with moment inequalities, these issues cause problems.

In practice, one often is interested in a two-sided CI such as the one in (3.1). However, for simplicity, we focus on a one-sided interval

$$(-\infty, \widetilde{g}_{Un}] = (-\infty, c^{**}_{UnB}(1-\alpha)], \tag{3.2}$$

where $\widetilde{g}_{Un}$ and $c^{**}_{UnB}(1-\alpha)$ are defined as above.

The bootstrap that is employed to generate the bootstrap samples can be the usual iid nonparametric bootstrap (in which $\{X^*_{ir} : i \leq n\}$ are iid draws from the empirical distribution of $\{X_i : i \leq n\}$), a parametric bootstrap (if the distribution of the data is specified up to an unknown parameter), or an "asymptotic normal" bootstrap (in which $\{\overline{m}_n(X^*_{ir}, \theta) : \theta \in \Theta\}$ for $r = 1, ..., B$ are iid draws from a k-variate Gaussian process with mean $\overline{m}_n(X_i, \theta)$ and covariance function $C_n(\theta_1, \theta_2) = n^{-1}\sum_{i=1}^n (m(X_i, \theta_1) - \overline{m}_n(X_i, \theta_1))(m(X_i, \theta_2) - \overline{m}_n(X_i, \theta_2))')$.

It is standard in the bootstrap literature to analyze the properties of the bootstrap when $B = \infty$ because the simulation error due to the use of B bootstrap repetitions can be made arbitrarily small by taking B large. We do this here. When $B = \infty$, $c^{**}_{UnB}(1-\alpha)$ is the population $1-\alpha$ quantile of the distribution of g^*_{Unr} given the original sample $\{X_i : i \leq n\}$. For notational simplicity, when $B = \infty$, we let $c^{**}_{Un}(1-\alpha)$ denote $c^{**}_{UnB}(1-\alpha)$.

Let $c^*_{UnB}(\alpha)$ denote the α quantile of $n^{1/2}(\widehat{g}^*_{UnB} - \widehat{g}_{Un})$ conditional on the original sample $\{X_i : i \leq n\}$. Notice that

$$c^{**}_{UnB}(\alpha) = \widehat{g}_{Un} + n^{-1/2} c^*_{UnB}(\alpha). \tag{3.3}$$

In consequence, the interval in (3.2) can be written equivalently as

$$(-\infty, \widetilde{g}_{Un}] = (-\infty, \widehat{g}_{Un} + n^{-1/2} c^*_{UnB}(1-\alpha)]. \tag{3.4}$$

This way of writing the CI makes it clear that the CI is based on an estimator of the upper bound plus a bootstrap adjustment that takes sampling error into account. For reasons given below, the CI of (3.2) and (3.4) will be referred to as a "backward" bootstrap CI.

Suppose one is interested in a CI for the true value $g(\theta_0)$. Let AS_{tv} denote the "asymptotic size of the CI for the true value." By definition and simple algebra,

$$\begin{aligned} AS_{tv} &= \liminf_{n\to\infty} \inf_{(\theta_0,\gamma_0)\in\Theta\times\Gamma} P_{\theta_0,\gamma_0}(g(\theta_0) \leq \widehat{g}_{Un} + n^{-1/2} c^*_{Un}(1-\alpha)) \\ &= \liminf_{n\to\infty} \inf_{(\theta_0,\gamma_0)\in\Theta\times\Gamma} P_{\theta_0,\gamma_0}(n^{1/2}(\widehat{g}_{Un} - g(\theta_0)) \geq -c^*_{Un}(1-\alpha)), \end{aligned} \tag{3.5}$$

where $c^*_{Un}(1-\alpha)$ denotes $c^*_{UnB}(1-\alpha)$ when $B = \infty$ and γ_0 is a nuisance parameter with parameter space Γ. The nuisance parameter γ_0 arises because θ_0 does not completely determine the distribution of the sample $\{X_i : i \leq n\}$ in the moment inequality context. In later sections when we consider simple examples, the nuisance parameter γ_0 is specified explicitly.

Taking the infimum over $(\theta_0, \gamma_0) \in \Theta \times \Gamma$ in (3.5) is standard in the definition of the size of a CI. In particular, by definition, without the $\liminf_{n\to\infty}$ the expression in (3.5) for AS_{tv} is the confidence size of the random interval. Taking the infimum over $(\theta_0, \gamma_0) \in \Theta \times \Gamma$ guarantees that no matter what is the (unknown) true value of the parameter of interest θ_0 or the (unknown) nuisance parameter γ_0, the asymptotic coverage probability is at least AS_{tv}.

Next, suppose one is interested in a CI for the identified interval $(-\infty, g_{U0}]$. For this case, let AS_{int} be defined analogously to AS_{tv}, but with g_{U0} in place of the true value $g(\theta_0)$. Then,

$$AS_{int} = \liminf_{n\to\infty} \inf_{(\theta_0,\gamma_0)\in\Theta\times\Gamma} P_{\theta_0,\gamma_0}(n^{1/2}(\widehat{g}_{Un} - g_{U0}) \geq -c^*_{Un}(1-\alpha)). \tag{3.6}$$

One would like the bootstrap intervals to be such that

$$\begin{aligned} &AS_{tv} = 1-\alpha \text{ and } AS_{int} = 1-\alpha \text{ or, at least,} \\ &AS_{tv} \geq 1-\alpha \text{ and } AS_{int} \geq 1-\alpha. \end{aligned} \tag{3.7}$$

If the CI satisfies $AS_{tv} > 1-\alpha$ (or $AS_{int} > 1-\alpha$), then it has asymptotic level $1-\alpha$, but may be longer than desirable. We show later that the bootstrap interval $(-\infty, \widetilde{g}_{Un}]$ in (3.2) (or equivalently in (3.4)) does not necessarily satisfy $AS_{tv} \geq 1-\alpha$ or $AS_{int} \geq 1-\alpha$ even if the infimum over $(\theta_0, \gamma_0) \in \Theta \times \Gamma$ in (3.5) is deleted.

Given the definition of AS_{tv} in (3.5), the bootstrap interval in (3.4) has the desired (first-order) asymptotic property in (3.7) if the difference between the α quantile of $n^{1/2}(\widehat{g}_{Un} - g(\theta_0))$ and -1 times the $1-\alpha$ quantile of $n^{1/2}(\widehat{g}^*_{UnB} - \widehat{g}_{Un})$ (given the sample $\{X_i : i \leq n\}$) converges in probability to zero uniformly over $(\theta_0, \gamma_0) \in \Theta \times \Gamma$. This seems "backwards" because in scenarios in which the bootstrap works properly, the distribution of the normalized bootstrap estimator $n^{1/2}(\widehat{g}^*_{UnB} - \widehat{g}_{Un})$ is close to that of the normalized estimator $n^{1/2}(\widehat{g}_{Un} - g(\theta_0))$ when n is large. Hence, in such cases it makes sense to have $c^*_{Un}(\alpha)$ appear in place of $-c^*_{Un}(1-\alpha)$ in the right-hand side expression for AS_{tv} in (3.5). Indeed, Hall (1992, pp. 12, 36) refers to a bootstrap interval of the type in (3.4) as the "other percentile" or "backward percentile" bootstrap CI. If $n^{1/2}(\widehat{g}_{Un} - g(\theta_0))$ and $n^{1/2}(\widehat{g}^*_{UnB} - \widehat{g}_{Un})$ are both asymptotically normal, then the "backward percentile" bootstrap CI typically has the desired (first-order) asymptotic properties because $c^*_{Un}(\alpha)$ and $-c^*_{Un}(1-\alpha)$ both converge in probability to z_α using the symmetry of the asymptotic normal distribution. In the present case, however, neither $n^{1/2}(\widehat{g}_{Un} - g(\theta_0))$ nor $n^{1/2}(\widehat{g}^*_{UnB} - \widehat{g}_{Un})$ are asymptotically normal.

An alternative "forward" bootstrap CI is given by taking $(-\infty, \widetilde{g}_{Un}]$ to be

$$(-\infty, \widetilde{g}_{Un}] = (-\infty, \widehat{g}_{Un} - n^{-1/2}c^*_{UnB}(\alpha)]. \tag{3.8}$$

This "forward" bootstrap CI has coverage probabilities for covering for $g(\theta_0)$ and $(-\infty, g_{U0}]$ given by

$$\begin{aligned} AS_{tv}^{for} &= \liminf_{n\to\infty} \inf_{(\theta_0,\gamma_0)\in\Theta\times\Gamma} P_{\theta_0,\gamma_0}(n^{1/2}(\widehat{g}_{Un} - g(\theta_0)) \geq c_{Un}^*(\alpha)) \text{ and} \\ AS_{int}^{for} &= \liminf_{n\to\infty} \inf_{(\theta_0,\gamma_0)\in\Theta\times\Gamma} P_{\theta_0,\gamma_0}(n^{1/2}(\widehat{g}_{Un} - g_{U0}) \geq c_{Un}^*(\alpha)), \end{aligned} \quad (3.9)$$

respectively.

We show below that the "forward" bootstrap interval $(-\infty, \widetilde{g}_{Un}]$ in (3.8) does not necessarily satisfy $AS_{tv}^{for} \geq 1-\alpha$ or $AS_{int}^{for} \geq 1-\alpha$ even if the infimum over $(\theta_0, \gamma_0) \in \Theta \times \Gamma$ in (3.9) is deleted. Hence, neither the "backward" nor the "forward" bootstrap CI has the desired asymptotic properties in general.

Notice that the coverage probabilities of the bootstrap CIs given in (3.5), (3.6), and (3.9) depend on the asymptotic distributions of $n^{1/2}(\widehat{g}_{Un} - g(\theta_0))$, $n^{1/2}(\widehat{g}_{Un} - g_{U0})$, and $n^{1/2}(\widehat{g}_{UnB}^* - \widehat{g}_{Un})$. Hence, in subsequent sections, we determine what these normalized distributions are.

4 m Out of n Bootstrap for CI Endpoints

We now consider the m out of n bootstrap for CI endpoints. (Based on the results given below, we do not recommend that this procedure be used in practice either.) As in the previous section, we simplify the arguments by focusing on a one-sided CI.

The m out of n "backward" bootstrap procedure is defined as in (3.4) but with the bootstrap sample size n replaced by m $(< n)$. One computes it as follows: (i) Generate B independent bootstrap samples $\{X_{ir}^* : i \leq m\}$ of size m $(< n)$ for $r = 1, ..., B$ using some method of bootstrapping. (ii) Compute $\widehat{g}_{Umr}^*$ using the definitions of $\widehat{g}_{Un}$ in (2.4) with $\{X_{ir}^* : i \leq m\}$ in place of $\{X_i : i \leq n\}$ for $r = 1, ..., B$. (iii) Compute the $1-\alpha$ sample quantile of $\{m^{1/2}(\widehat{g}_{Umr}^* - \widehat{g}_{Un}) : r = 1, ..., B\}$, call it $c_{UmB}^*(1-\alpha)$.[4] (iv) Take the random interval $(-\infty, \widetilde{g}_{Un}]$ to be

$$(-\infty, \widetilde{g}_{Un}] = (-\infty, \widehat{g}_{Un} + n^{-1/2} c_{UmB}^*(1-\alpha)]. \quad (4.1)$$

The bootstrap that is employed can be any of those discussed in the previous section.

For a CI for the true value $g(\theta_0)$, AS_{tv} is defined as in (3.5) but with $c_{Um}^*(1-\alpha)$ in place of $c_{Un}^*(1-\alpha)$. Analogously, for a CI for the identified interval $(-\infty, g_{U0}]$, AS_{int} is defined (3.6) but with $c_{Um}^*(1-\alpha)$ in place of $c_{Un}^*(1-\alpha)$. As above, we would like AS_{tv} and AS_{int} to satisfy (3.7). We show below that the m out of n bootstrap interval $(-\infty, \widetilde{g}_{Un}]$ in (4.1) does not necessarily satisfy $AS_{tv} \geq 1-\alpha$ or $AS_{int} \geq 1-\alpha$ for any value of m/n including $m/n = 0$ (which gives the asymptotic size when $m/n \to 0$ as $n \to \infty$ as is usually assumed for the m out of n bootstrap). In fact, this is true even if the infima in (3.5) and (3.6) are deleted.

[4] The critical value $c_{UmB}^*(1-\alpha)$ depends on n as well as m because the bootstrap distribution depends on the original sample which is of size n. However, for notational simplicity, we suppress the dependence on n.

As discussed above, the m out of n bootstrap of (4.1) is "backward" in a certain sense. The m out of n "forward" bootstrap CI is defined by

$$(-\infty, \widetilde{g}_{Un}] = (-\infty, \widehat{g}_{Un} - n^{-1/2} c^*_{UmB}(\alpha)]. \tag{4.2}$$

This m out of n "forward" bootstrap CI has coverage probabilities for covering for $g(\theta_0)$ and $(-\infty, g_{U0}]$ given by the first and second expressions in (3.9), respectively, with $c^*_{Um}(\alpha)$ in place of $c^*_{Un}(\alpha)$. We show in the next sections that the m out of n "forward" bootstrap interval $(-\infty, \widetilde{g}_{Un}]$ in (4.2) does not necessarily satisfy $AS_{tv} \geq 1-\alpha$ or $AS_{int} \geq 1-\alpha$ for any value of m/n including $m/n = 0$.

5 Linear Inequalities I

5.1 Model and Estimators

Next we consider a special case of the moment inequality model discussed above. The model considered is one in which a moment inequality is potentially redundant but not irrelevant. We choose this particular model for the reasons given in the Introduction and for its analytic tractability—we can derive the finite-sample distribution of the bootstrap and m out of n bootstrap statistics that are considered. If neither the bootstrap nor m out of n bootstrap work in this simple model, then they are not procedures that work in general.

Let

$$\begin{aligned} X_i &= \begin{pmatrix} X_{1i} \\ X_{2i} \end{pmatrix} \sim N(\mu, \Sigma), \text{ where} \\ \mu &= \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \text{ for some } \rho \in (-1,1). \end{aligned} \tag{5.1}$$

For simplicity, we assume that ρ is known.

The moment functions, sample moments, and population moment inequalities are

$$\begin{aligned} m(X_i, \theta) &= \begin{pmatrix} X_{1i} - \theta \\ X_{2i} - \theta \end{pmatrix}, \ \overline{m}_n(X_i, \theta) = \begin{pmatrix} \overline{X}_{1n} - \theta \\ \overline{X}_{2n} - \theta \end{pmatrix}, \text{ and} \\ E_{\theta_0} m(X_i, \theta_0) &= \begin{pmatrix} \mu_1 - \theta_0 \\ \mu_2 - \theta_0 \end{pmatrix} \geq 0, \text{ where } \overline{X}_{sn} = n^{-1} \sum_{i=1}^n X_{si} \text{ for } s = 1, 2. \end{aligned} \tag{5.2}$$

In consequence, $\theta_0 \leq \min\{\mu_1, \mu_2\}$ and the identified set Θ_0 equals $(-\infty, \min\{\mu_1, \mu_2\}]$.

The function $g(\theta)$ of interest is just the identity function $g(\theta) = \theta$. Hence, $(g_{L0}, g_{U0}] = (-\infty, \min\{\mu_1, \mu_2\}]$. In the present case, $\widehat{\Theta}_n$, $\widehat{g}_{Ln}$, and $\widehat{g}_{Un}$ can be determined analytically. It is easy to see that for the distance functions $d(x) = x'x$ and $d(x) = |x_1| + |x_2|$ (and many other distance functions symmetric in x_1 and x_2), we have

$$\widehat{g}_{Ln} = -\infty, \ \widehat{g}_{Un} = \min\{\overline{X}_{1n}, \overline{X}_{2n}\}, \text{ and } \widehat{\Theta}_n = (-\infty, \min\{\overline{X}_{1n}, \overline{X}_{2n}\}]. \tag{5.3}$$

Without loss of generality, we assume that $\mu_1 \leq \mu_2$ (although this is not known to the investigator using the moment inequalities). Hence, $\min\{\mu_1, \mu_2\} = \mu_1$.

Notice that $\widehat{g}_{Un}$ is a non-differentiable function of the sample mean vector $(\overline{X}_{1n}, \overline{X}_{2n})$. In consequence, the asymptotic distribution of $\widehat{g}_{Un}$ turns out to be a discontinuous function of the parameters (μ_1, μ_2). In particular, the asymptotic distribution is different between the cases where $\mu_1 < \mu_2$ and $\mu_1 = \mu_2$. Furthermore, the asymptotic distribution is different again if the true difference $\mu_2 - \mu_1 = h_D/n^{1/2}$ for some positive constant h_D. Because of this, the bootstrap does not perform as desired.

For $s \in R$, define

$$U(s) = \min\{Z_1, Z_2 + s\}, \text{ where } Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N(0, \Sigma). \tag{5.4}$$

Combining (5.3) and (5.4) gives

$$\begin{aligned} n^{1/2}(\widehat{g}_{Un} - g_{U0}) &= \min\{n^{1/2}(\overline{X}_{1n} - \mu_1), n^{1/2}(\overline{X}_{2n} - \mu_1)\} \\ &=_d U(n^{1/2}(\mu_2 - \mu_1)) \text{ and} \\ n^{1/2}(\widehat{g}_{Un} - \theta_0) &=_d U(n^{1/2}(\mu_2 - \mu_1)) + n^{1/2}(\mu_1 - \theta_0), \end{aligned} \tag{5.5}$$

where "$=_d$ " denotes equality in distribution.

5.2 Bootstrap and m Out of n Bootstrap

We now introduce the m out of n bootstrap for the linear moment inequality model of (5.1)-(5.2). The (standard) bootstrap is obtained as a special case by taking $m = n$. We consider the parametric bootstrap for which a bootstrap sample $\{X_i^* : i \leq m\}$ consists of iid $N(\overline{X}_n, \Sigma)$ random variables. For specificity, we take

$$X_i^* = Z_i^{**} + \overline{X}_n, \text{ where } Z_i^{**} \sim N(0, \Sigma) \text{ for } i \leq m \tag{5.6}$$

and $\{Z_i^{**} : i \leq m\}$ are iid and independent of $\{X_i : i \leq n\}$. In the present model, the parametric bootstrap is the same as the "asymptotic normal" bootstrap referred to above. The issues that arise below with the parametric bootstrap are the same as those that arise with the nonparametric bootstrap. The parametric bootstrap, however, has the advantage of making these issues as clear as possible. We write $\overline{X}^*_{sm} = \overline{Z}^{**}_{sm} + \overline{X}_{sn}$ for $s = 1, 2$, where $\overline{Z}^{**}_{sm} = m^{-1}\sum_{i=1}^m Z_{si}^{**}$ and $Z_i^{**} = (Z_{1i}^{**}, Z_{2i}^{**})'$.

Using (5.3) and (5.6), the bootstrap estimator $\widehat{g}^*_{Um}$ is defined by

$$\widehat{g}^*_{Um} = \min\{\overline{Z}^{**}_{1m} + \overline{X}_{1n}, \overline{Z}^{**}_{2m} + \overline{X}_{2n}\}. \tag{5.7}$$

For $s \in R$, define

$$\begin{aligned} U^*(m/n, s) &= \min\{Z_1^* + (m/n)^{1/2}\, Z_1,\ Z_2^* + (m/n)^{1/2}\,(Z_2 + s)\}, \text{ where} \\ Z^* &= \begin{pmatrix} Z_1^* \\ Z_2^* \end{pmatrix} \sim N(0, \Sigma), \end{aligned} \tag{5.8}$$

Z is as defined in (5.4), and Z^* and Z are independent. Using (5.7) and (5.8), we have

$$\begin{aligned} & m^{1/2}(\widehat{g}^*_{Um} - g_{U0}) \\ = & \min\{m^{1/2}\overline{Z}^{**}_{1m} + (m/n)^{1/2} n^{1/2}(\overline{X}_{1n} - \mu_1), \\ & m^{1/2}\overline{Z}^{**}_{2m} + (m/n)^{1/2} [n^{1/2}(\overline{X}_{2n} - \mu_2) + n^{1/2}(\mu_2 - \mu_1])\} \\ =_d & \ U^*(m/n, n^{1/2}(\mu_2 - \mu_1)). \end{aligned} \tag{5.9}$$

Combining (5.5) and (5.9) gives

$$\begin{aligned} m^{1/2}(\widehat{g}^*_{Um} - \widehat{g}_{Un}) &= m^{1/2}(\widehat{g}^*_{Um} - g_{U0}) - m^{1/2}(\widehat{g}_{Un} - g_{U0}) \\ &=_d U^*(m/n, n^{1/2}(\mu_2 - \mu_1)) - (m/n)^{1/2} U(n^{1/2}(\mu_2 - \mu_1)). \end{aligned} \tag{5.10}$$

5.3 Coverage Probabilities and Size

We now use the results of the previous subsection to provide expressions for the coverage probabilities of the m out of n "backward" and "forward" bootstrap CIs. As above, results for the standard bootstrap are obtained by taking $m = n$. For notational convenience, let

$$h_1 = n^{1/2}(\mu_1 - \theta_0),\ h_D = n^{1/2}(\mu_2 - \mu_1),\ \text{and } h = (h_1, h_D)'. \tag{5.11}$$

Note that $h_1, h_D \geq 0$ and h_D denotes the scaled difference between μ_1 and μ_2. Let $c^*_U(m/n, s, \alpha)$ be the conditional α quantile of $U^*(m/n, s) - (m/n)^{1/2} U(s)$ given Z. Using (4.1), (5.5), and (5.10), the probability that the m out of n "backward" bootstrap CI covers the true value θ_0, denoted $CP_{tv}(m/n, h)$, is

$$\begin{aligned} CP_{tv}(m/n, h) &= P_{\theta_0,\mu}(\theta_0 \leq \widehat{g}_{Un} + n^{-1/2} c^*_{Um}(1-\alpha)) \\ &= P_{\theta_0,\mu}(n^{1/2}(\widehat{g}_{Un} - \theta_0) \geq -c^*_{Um}(1-\alpha)) \\ &= P(U(h_D) + h_1 \geq -c^*_U(m/n, h_D, 1-\alpha)), \end{aligned} \tag{5.12}$$

where $c^*_{Um}(1-\alpha)$ is the $1-\alpha$ quantile of $n^{1/2}(\widehat{g}^*_{Um} - \widehat{g}_{Un})$ conditional on $\{X_i : i \leq n\}$. Note that $CP_{tv}(m/n, h)$ only depends on $h \in R^2_+$, ρ (the correlation coefficient between X_{1i} and X_{2i}), and m/n.

The finite-sample coverage probability $CP_{tv}(m/n, h)$ is exactly the same as the asymptotic coverage probability that arises when (i) m/n is fixed for all n, (ii) $n^{1/2}(\mu_1 - \theta_0) \to h_1$ and (iii) $n^{1/2}(\mu_2 - \mu_1) \to h_D$ as $n \to \infty$ for some fixed $h_1, h_D \in [0, \infty]$. Hence, the results given here are both exact and asymptotic.

If the true value θ_0 is on the edge of the identified interval (i.e., $\theta_0 = g_{U0}$ and $h_1 = 0$) and the difference between μ_1 and μ_2 is "arbitrarily large" (i.e., $h_D = \infty$), then $U(h_D) = Z_1$, $U^*(m/n, h_D) = Z^*_1 + (m/n)^{1/2} Z_1$, $U^*(m/n, h_D) - (m/n)^{1/2} U(h_D) = Z^*_1$, $c^*_U(m/n, h_D, 1-\alpha) = z_{1-\alpha}$, and $CP_{tv}(m/n, h) = P(Z_1 \geq -z_{1-\alpha}) = 1-\alpha$, as desired. However, if the difference between μ_1 and μ_2 is not "arbitrarily large," then this desired result does not hold, as shown in the next subsection.

The finite-sample size of an m out of n "backward" bootstrap CI for the true value θ_0 is

$$\begin{aligned} Size_{tv}(m/n) &= \inf_{h \in R_+^2} CP_{tv}(m/n, h) \\ &= \inf_{h_1, h_D \in R_+} P(U(h_D) + h_1 \geq -c_U^*(m/n, h_D, 1-\alpha)) \\ &= \inf_{h_D \in R_+} P(U(h_D) \geq -c_U^*(m/n, h_D, 1-\alpha)), \end{aligned} \tag{5.13}$$

which depends on m and n only through m/n. Provided $r = \lim_{n\to\infty} m/n$ $(\in [0,1])$ exists, the asymptotic size AS_{tv} of the CI is given by (5.13) with r in place of m/n. Hence, the size results given here also are both exact and asymptotic. Note that for the bootstrap we have $r = 1$, and for the usual choices of m for the m out of n bootstrap we have $r = 0$.

For the "forward" bootstrap CI, the coverage probability and size are the same as in (5.12) and (5.13), respectively, but with $c_{Um}^*(1-\alpha)$ and $-c_U^*(m/n, h_D, 1-\alpha)$ replaced by $-c_{Um}^*(\alpha)$ and $c_U^*(m/n, h_D, \alpha)$, respectively.

Next, suppose one wants a CI for the identified interval $(-\infty, g_{U0}]$, rather than the true value. Then, using (5.5) and (5.10), the probability that the CI covers the identified interval $(-\infty, g_{U0}]$, denoted $CP_{int}(m/n, h)$, is

$$\begin{aligned} CP_{int}(m/n, h) &= P_{\theta_0,\mu}(g_{U0} \leq \widehat{g}_{Un} + n^{-1/2} c_{Um}^*(1-\alpha)) \\ &= P(U(h_D) \geq -c_U^*(m/n, h_D, 1-\alpha)). \end{aligned} \tag{5.14}$$

If the difference between μ_1 and μ_2 is "arbitrarily large" (i.e., $h_D = \infty$), then $U(h_D) = Z_1$, $U^*(m/n, h_D) = Z_1^* + (m/n)^{1/2} Z_1$, $U^*(m/n, h_D) - (m/n)^{1/2} U(h_D) = Z_1^*$, $c_U^*(m/n, h_D, 1-\alpha) = z_{1-\alpha}$, and $CP_{int}(m/n, h) = P(Z_1 \geq -z_{1-\alpha}) = 1-\alpha$, as desired. If the difference between μ_1 and μ_2 is not "arbitrarily large," then the desired result for $CP_{int}(m/n, h)$ does not hold. The size of the m out of n "backward" bootstrap for the identified interval is given by

$$Size_{int}(m/n) = \inf_{h_D \in R_+} P(U(h_D) \geq -c_U^*(m/n, h_D, 1-\alpha)), \tag{5.15}$$

which is the same as that for the corresponding CI for the true value.

For the "forward" bootstrap CI for the identified interval $(-\infty, g_{U0}]$, the coverage probability and size are the same as in (5.14) and (5.15), respectively, but with $c_{Um}^*(1-\alpha)$ and $-c_U^*(m/n, h_D, 1-\alpha)$ replaced by $-c_{Um}^*(\alpha)$ and $c_U^*(m/n, h_D, \alpha)$, respectively.

5.4 Coverage Probability Simulations for Bootstrap CIs

Table 1 provides values of $CP_{tv}(1, h)$ and $CP_{int}(1, h)$ based on the formulae in (5.12) and (5.14), respectively, computed by simulation for the "backward" and "forward" bootstrap CIs (for which $m = n$). Table 1 provides results for $1-\alpha = .95$ and for a variety of values of $h \in R_+^2$ and $\rho \in [-1, 1]$. In particular, we consider the cases where $h_1 = 0$, $h_D = 0, .125, .25, .5, 1.0, 2.0$, and $\rho = -1.0, -.99, -.95, -.5, 0, .5, .95, .99, 1.0$.

Given that $h_1 = 0$, we have $CP_{tv}(1, h) = CP_{int}(1, h)$. Forty thousand bootstrap repetitions are used here (and in all Tables below) to compute the bootstrap critical value for each simulation repetition. Forty thousand simulation repetitions are used here (and in all Tables below) to compute each coverage probability.

Table 1(a) shows that the coverage probabilities for the "backward" bootstrap are much less than the nominal level .95 when $\rho \leq .5$ and $h_D \leq .5$. For a given value of ρ, the exact (and asymptotic) confidence size of the bootstrap CI is less than or equal to the minimum value in each column. For example, for $\rho = -1.0$, the confidence size is .00 rather than .95. When $\rho = -.99$, the confidence size is .13 rather than .95. Clearly, the "backward" bootstrap fails dramatically to deliver a CI with asymptotic size equal to its nominal size.

Table 1(b) shows that the coverage probabilities for the "forward" bootstrap are less than the nominal level when $\rho \leq .5$ and $h_D \leq .5$. But, the differences are much smaller than with the "backward" bootstrap. The results of the Table indicate that the finite-sample (and asymptotic) confidence size (over all $\rho \in [-1.0, 1.0]$) of the "forward" bootstrap is .90 or less rather than .95.

The pointwise asymptotic coverage probabilities (ACPs) of the "backward" and "forward" bootstrap CIs when $\theta_0 = \mu_1 = \mu_2$ are given by the values in the first rows of Table 1(a) and 1(b), respectively (which correspond to $h_D = 0$). Table 1 shows that the pointwise ACPs of the bootstrap CIs are less than the nominal .95 coverage probability for many values of ρ. In some cases, they are much below .95. Hence, these bootstrap CIs do not yield correct pointwise ACP.

To conclude, Table 1 illustrates that neither the "backward" nor the "forward" bootstrap yields CIs with finite-sample or asymptotic confidence size equal to the nominal level. In particular, the bootstrap CIs are not asymptotically valid in a pointwise or uniform sense.

5.5 Coverage Probability Simulations for m out of n Bootstrap CIs

Table 2 provides values of $CP_{tv}(m/n, h)$ and $CP_{int}(m/n, h)$ computed by simulation for the m out of n "backward" and "forward" bootstrap CIs for $m/n = 0, .01, .05, .1, .5$ and for the same confidence level and parameters as in Table 1. Given that $h_1 = 0$, we have $CP_{tv}(m/n, h) = CP_{int}(m/n, h)$.

For each value of m/n, Table 2(a) (i.e., Table 2(i)(a) through 2(v)(a)) shows that the coverage probabilities for the m out of n "backward" bootstrap are much lower than the nominal level .95 when $\rho \leq .5$ for all values of h_D. For a given value of ρ, the exact (and asymptotic) confidence size of the m out of n bootstrap CI is less than or equal to the minimum value in each column and each Table. Hence, when $\rho = -1.0$ and for all values of m/n, the confidence size is .00 rather than .95. Clearly, the m out of n "backward" bootstrap fails dramatically to deliver a CI with asymptotic size equal to its nominal size.

For all $m/n \in [.01, .5]$, for certain values of h_D, and for all $\rho \leq .5$, Table 2(b) (i.e., Table 2(i)(b) through 2(v)(b)) shows that the m out of n "forward" bootstrap has coverage probability that is less than the nominal level .95. But, the differences are much smaller than with the m out of n "backward" bootstrap. More specifically, the finite-sample confidence size of the nominal .95 m out of n "forward" bootstrap is less than or equal to .93 for $m/n = .01$, .91 for $m/n = .05, .10$, and .90 for $m/n = .5$. Table 2(i)(b) shows that the m out of n "forward" bootstrap has correct asymptotic size when $m/n \to 0$ as $n \to \infty$ (i.e., $AS_{tv} = AS_{int} = .95$).

The pointwise ACPs of the m out of n "backward" and "forward" bootstrap CIs when $\theta_0 = \mu_1 = \mu_2$ are given by the values in the first rows of Table 2(i)(a) and 2(i)(b), respectively (which correspond to $h_D = 0$ and $m/n = 0$). Table 2(i)(a) shows that the pointwise ACPs of the m out of n "backward" bootstrap CIs are less than the nominal .95 coverage probability for many values of ρ. This is because of the asymmetry of the distribution of $m^{1/2}(\widehat{g}^*_{Um} - \widehat{g}_{Un})$ $(= U^*(0,0)$, here) in (5.10). In some cases, the ACPs are much below .95. Hence, the m out of n "backward" bootstrap CIs do not yield correct pointwise ACP. Table 2(i)(b) shows that the pointwise ACP probabilities of the m out of n "forward" bootstrap are greater than or equal to .95, as desired.

To conclude, Table 2 illustrates that in model I the m out of n "backward" bootstrap yields CIs with finite-sample and asymptotic confidence size substantially less than its nominal level. In particular, the m out of n "backward" bootstrap CI is not asymptotically valid in a pointwise or uniform sense. On the other hand, the m out of n "forward" bootstrap is asymptotically valid in pointwise and uniform senses when $\lim_{n\to\infty} m/n = 0$. In finite samples, the m out of n "forward" bootstrap yields CIs with confidence sizes that are somewhat lower than their nominal level.

6 Linear Inequalities II

6.1 Model and Estimators

In this section, we consider a model with X_i defined as in (5.1), but with different moment inequalities. The main purpose of this section is to see the quantitative difference between the finite-sample/asymptotic sizes and the nominal sizes of the bootstrap CIs in a model scenario in which "reversals" of moment inequalities may occur. In particular, we are interested in whether the m out of n "forward" bootstrap yields CIs whose confidence size is close to the nominal size (because these CIs work fairly well in model I).

The moment functions, sample moments, and population moment inequalities are

$$m(X_i, \theta) = \begin{pmatrix} \theta_0 - X_{1i} \\ X_{2i} - \theta_0 \end{pmatrix}, \ \overline{m}_n(X_i, \theta) = \begin{pmatrix} \theta_0 - \overline{X}_{1n} \\ \overline{X}_{2n} - \theta_0 \end{pmatrix}, \text{ and}$$

$$E_{\theta_0} m(X_i, \theta) = \begin{pmatrix} \theta_0 - \mu_1 \\ \mu_2 - \theta_0 \end{pmatrix} \geq 0, \text{ where } \overline{X}_{sn} = n^{-1} \sum_{i=1}^{n} X_{si} \text{ for } s = 1, 2. \quad (6.1)$$

In consequence, $\mu_1 \leq \theta_0 \leq \mu_2$ and the identified set Θ_0 equals $[\mu_1, \mu_2]$.

Note that the model considered here differs from the "interval-censored variable" model considered in Imbens and Manski (2004) because the latter assumes that $X_{1i} \leq X_{2i}$ almost surely. In contrast, the model defined by (5.1) and (6.1) allows for sample "reversals" of the moment conditions which lead to no solution to the sample moment inequalities $\overline{m}_n(X_i, \theta) \geq 0$ even though the population inequalities $E_{\theta_0} m(X_i, \theta) \geq 0$

hold. This is a common feature of more complicated moment inequality models. In the model considered here, a "reversal" occurs whenever $\overline{X}_{1n} \geq \overline{X}_{2n}$.

The function $g(\theta)$ of interest is just the identity function $g(\theta) = \theta$. Hence, $[g_{L0}, g_{U0}] = [\mu_1, \mu_2]$. In the present case, $\widehat{\Theta}_n$, $\widehat{g}_{Ln}$, and $\widehat{g}_{Un}$ can be determined analytically. It is easy to see that if $\overline{X}_{1n} \leq \overline{X}_{2n}$, then

$$\widehat{\Theta}_n = [\overline{X}_{1n}, \overline{X}_{2n}],\ \widehat{g}_{Ln} = \overline{X}_{1n},\ \text{and } \widehat{g}_{Un} = \overline{X}_{2n}. \tag{6.2}$$

On the other hand, provided $d(x)$ is symmetric in its $k = 2$ components and nondecreasing in each component (as is true if $d(x) = x'x$ or $d(x) = |x_1| + |x_2|$), it is easy to see that if $\overline{X}_{1n} \geq \overline{X}_{2n}$, then

$$\widehat{\Theta}_n = \{(\overline{X}_{1n} + \overline{X}_{2n})/2\},\ \widehat{g}_{Ln} = (\overline{X}_{1n} + \overline{X}_{2n})/2,\ \text{and } \widehat{g}_{Un} = (\overline{X}_{1n} + \overline{X}_{2n})/2. \tag{6.3}$$

Notice that $\widehat{g}_{Ln}$ and $\widehat{g}_{Un}$ are non-differentiable functions of the sample mean vector $(\overline{X}_{1n}, \overline{X}_{2n})$. In consequence, the asymptotic distributions of $\widehat{g}_{Ln}$ and $\widehat{g}_{Un}$ turn out to be discontinuous functions of the parameters (μ_1, μ_2). In particular, the asymptotic distributions are different between the cases where $\mu_1 < \mu_2$ and $\mu_1 = \mu_2$. Furthermore, the asymptotic distribution is different again if the true difference $\mu_2 - \mu_1 = h_D/n^{1/2}$ for some positive constant h_D. Because of this, it is shown below that the bootstrap and the m out of n bootstrap do not perform as desired.

Because we only consider one-sided CIs here, we focus on $\widehat{g}_{Un}$ from now on. Combining (6.2) and (6.3) gives

$$\widehat{g}_{Un} = \max\{\overline{X}_{2n}, (\overline{X}_{1n} + \overline{X}_{2n})/2\}. \tag{6.4}$$

For $s \in R$, define

$$U(s) = \max\{Z_2, (Z_1 + Z_2 - s)/2\},\ \text{where } Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N(0, \Sigma). \tag{6.5}$$

Combining (6.4) and (6.5) gives

$$\begin{aligned} n^{1/2}(\widehat{g}_{Un} - g_{U0}) &= \max\{n^{1/2}(\overline{X}_{2n} - \mu_2), n^{1/2}(\overline{X}_{1n} + \overline{X}_{2n} - 2\mu_2)/2\} \\ &=_d U(n^{1/2}(\mu_2 - \mu_1)). \end{aligned} \tag{6.6}$$

In turn, (6.6) yields

$$n^{1/2}(\widehat{g}_{Un} - \theta_0) =_d U(n^{1/2}(\mu_2 - \mu_1)) + n^{1/2}(\mu_2 - \theta_0). \tag{6.7}$$

6.2 Bootstrap and m Out of n Bootstrap

We now consider the m out of n bootstrap for the linear moment inequality model of (5.1)-(6.1). As above, the (standard) bootstrap is obtained by taking $m = n$. The bootstrap sample $\{X_i^* : i \leq m\}$ is defined exactly as in (5.6).

Using (5.6) and (6.4), the bootstrap estimator $\widehat{g}_{Um}^*$ satisfies

$$\widehat{g}_{Um}^* = \max\{\overline{Z}_{2m}^{**} + \overline{X}_{2n}, (\overline{Z}_{1m}^{**} + \overline{Z}_{2m}^{**} + \overline{X}_{1n} + \overline{X}_{2n})/2\}. \tag{6.8}$$

For $s \in R$, define

$$U^*(m/n,s) = \max\{Z_2^* + (m/n)^{1/2} Z_2, (Z_1^* + Z_2^* + (m/n)^{1/2}(Z_1 + Z_2 - s))/2\}, \text{ where}$$
$$Z^* = \begin{pmatrix} Z_1^* \\ Z_2^* \end{pmatrix} \sim N(0,\Sigma), \tag{6.9}$$

Z is as defined in (6.5), and Z^* and Z are independent. Using (6.8) and (6.9), we have

$$\begin{aligned} & m^{1/2}(\widehat{g}_{Um}^* - g_{U0}) \\ &= \max\Big\{m^{1/2}\overline{Z}_{2m}^{**} + (m/n)^{1/2} n^{1/2}(\overline{X}_{2n} - \mu_2), \\ &\quad \Big(m^{1/2}\overline{Z}_{1m}^{**} + m^{1/2}\overline{Z}_{2m}^{**} + (m/n)^{1/2} n^{1/2}(\overline{X}_{1n} + \overline{X}_{2n} - 2\mu_2)\Big)/2\Big\} \\ &=_d U^*(m/n, n^{1/2}(\mu_2 - \mu_1)). \end{aligned} \tag{6.10}$$

Combining (6.6) and (6.10) gives

$$\begin{aligned} m^{1/2}(\widehat{g}_{Um}^* - \widehat{g}_{Un}) &= m^{1/2}(\widehat{g}_{Um}^* - g_{U0}) - m^{1/2}(\widehat{g}_{Un} - g_{U0}) \\ &=_d U^*(m/n, n^{1/2}(\mu_2 - \mu_1)) \\ &\quad - (m/n)^{1/2} U(n^{1/2}(\mu_2 - \mu_1)). \end{aligned} \tag{6.11}$$

6.3 Coverage Probabilities and Size

Next, we use the results of the previous subsection to provide expressions for the coverage probabilities of the m out of n bootstrap CIs considered above. For notational convenience, let

$$\begin{aligned} h_1 &= n^{1/2}(\theta_0 - \mu_1),\ h_2 = n^{1/2}(\mu_2 - \theta_0), \\ h &= (h_1, h_2)',\ \text{and } h_D = h_1 + h_2 = n^{1/2}(\mu_2 - \mu_1). \end{aligned} \tag{6.12}$$

Note that $h_1, h_2, h_D \geq 0$ and h_D denotes the scaled difference between μ_1 and μ_2. Let $c_U^*(m/n, s, \alpha)$ be the conditional α quantile of $U^*(m/n,s) - (m/n)^{1/2} U(s)$ given Z.

Using (6.7) and (6.11), the coverage probability of the m out of n "backward" bootstrap CI for the true value, denoted $CP_{tv}(m/n, h)$, is

$$\begin{aligned} CP_{tv}(m/n,h) &= P_{\theta_0,\mu}(g(\theta_0) \leq \widehat{g}_{Un} + n^{-1/2}c_{Um}^*(1-\alpha)) \\ &= P_{\theta_0,\mu}(n^{1/2}(\widehat{g}_{Un} - g(\theta_0)) \geq -c_{Um}^*(1-\alpha)) \\ &= P(U(h_D) + h_2 \geq -c_U^*(m/n, h_D, 1-\alpha)), \end{aligned} \tag{6.13}$$

where $c_{Um}^*(1-\alpha)$ denotes the $1-\alpha$ quantile of $m^{1/2}(\widehat{g}_{Um}^* - \widehat{g}_{Un})$ conditional on $\{X_i : i \leq n\}$. This finite-sample probability is exactly the same as the asymptotic probability that arises when m/n is fixed for all n, $n^{1/2}(\theta_0 - \mu_1) \to h_1$, and $n^{1/2}(\mu_2 - \theta_0) \to h_2$ as $n \to \infty$ for some fixed $h_1, h_2 \in [0, \infty]$. Hence, the results given here are both exact finite-sample and asymptotic. The probability in (6.13) depends only on $h_D, h_2 \in R_+$, ρ, and m/n.

If θ_0 is on the right edge of the identified interval (i.e., $\theta_0 = g_{U0}$ and $h_2 = 0$) and the interval is "arbitrarily wide" (i.e., $h_1 = \infty$), then $U(h_D) = Z_2$, $U^*(m/n, h_D) = Z_2^* + (m/n)^{1/2} Z_2$, $U^*(m/n, h_D) - (m/n)^{1/2} U(h_D) = Z_2^*$, $c_U^*(m/n, h_D, 1-\alpha) = z_{1-\alpha}$, and $CP_{tv}(m/n, h) = P(Z_2 \geq -z_{1-\alpha}) = 1 - \alpha$. However, if the identified interval is not "arbitrarily wide," then this desired result does not hold.

The finite-sample size of an m out of n "backward" bootstrap CI for the true value θ_0 is

$$\begin{aligned} Size_{tv}(m/n) &= \inf_{h \in R_+^2} CP_{tv}(m/n, h) \\ &= \inf_{h_2, h_D \in R_+} P(U(h_D) + h_2 \geq -c_U^*(m/n, h_D, 1-\alpha)) \\ &= \inf_{h_D \in R_+} P(U(h_D) \geq -c_U^*(m/n, h_D, 1-\alpha)). \end{aligned} \tag{6.14}$$

As above, provided $r = \lim_{n\to\infty} m/n$ ($\in [0,1]$) exists, the asymptotic size AS_{tv} of the CI is given by (6.14) with r in place of m/n. Hence, the size results given here also are both exact and asymptotic.

For the "forward" bootstrap CI, the coverage probability and size are the same as in (6.13) and (6.14), respectively, but with $c_{Um}^*(1-\alpha)$ and $-c_U^*(m/n, h_D, 1-\alpha)$ replaced by $-c_{Um}^*(\alpha)$ and $c_U^*(m/n, h_D, \alpha)$, respectively.

Using (6.6) and (6.11), the probability that the m out of n "backward" bootstrap CI covers the identified interval $(-\infty, g_{U0}]$, denoted $CP_{int}(m/n, h)$, is

$$\begin{aligned} CP_{int}(m/n, h) &= P_{\theta_0,\mu}(g_{U0} \leq \widehat{g}_{Un} + n^{-1/2} c_{Um}^*(1-\alpha)) \\ &= P(U(h_D) \geq -c_U^*(m/n, h_D, 1-\alpha)). \end{aligned} \tag{6.15}$$

If the identified interval is "arbitrarily wide" (i.e., $h_D = \infty$), then $U(h_D) = Z_2$, $U^*(m/n, h_D) = Z_2^* + (m/n)^{1/2} Z_2$, $U^*(m/n, h_D) - (m/n)^{1/2} U(h_D) = Z_2^*$, $c_U^*(m/n, h_D, 1-\alpha) = z_{1-\alpha}$, and $CP_{int}(m/n, h) = P(Z_2 \geq -z_{1-\alpha}) = 1 - \alpha$, as desired. If the identified interval is not "arbitrarily wide," then the desired result for $CP_{int}(m/n, h)$ does not hold.

The size of the m out of n "backward" bootstrap for the identified interval is given by the same expression as in (5.15) (but with $U(h_D)$ and $c_U^*(m/n, h_D, 1-\alpha)$ defined as in this section). For the "forward" bootstrap CI for the identified interval $(-\infty, g_{U0}]$, the coverage probability and size are the same as in (6.15) and (5.15), respectively, but with $c_{Um}^*(1-\alpha)$ and $-c_U^*(m/n, h_D, 1-\alpha)$ replaced by $-c_{Um}^*(\alpha)$ and $c_U^*(m/n, h_D, \alpha)$, respectively.

6.4 Coverage Probability Simulations of Bootstrap CIs

Table 3 provides values of $CP_{tv}(1, h)$ and $CP_{int}(1, h)$ for "backward" and "forward" bootstrap CIs for the moment inequality model of (5.1)-(6.1). These results are analogous to those in Table 1, but apply to the second linear inequality model rather than the first. The parameters considered are $h_2 = 0$ and $h_1 =$

$h_D = 0, .125, .25, .5, 1.0, 2.0, 4.0, 6.0, 8.0$. Given that $h_2 = 0$, we have $CP_{tv}(1, h) = CP_{int}(1, h)$.

Table 3(a) indicates that the "backward" bootstrap has exact and asymptotic size equal to the nominal level .95, as desired. (That is, the coverage probabilities are .95 or greater with equality for some parameter values.) The coverage probability exceeds .95 in a variety of cases, however, so the CI is not asymptotically similar. In consequence, the CI may be longer than necessary. This does not occur in model scenarios in which the bootstrap performs properly.

Table 3(b) shows that the "forward" bootstrap has asymptotic size substantially less than its nominal level when $\rho \leq -.5$.[5] For example, when ρ is fixed at -1.0, the exact and asymptotic size is less than or equal to .51. When $\rho = -.99$, the exact and asymptotic size is less than or equal to .57. This demonstrates that the "forward" bootstrap CI can behave quite poorly depending upon the moment inequalities and the parameter values considered.

6.5 Coverage Probability Simulations of m out of n Bootstrap CIs

Table 4 provides values of $CP_{tv}(m/n, h)$ and $CP_{int}(m/n, h)$ for m out of n "backward" and "forward" bootstrap CIs for the moment inequality model of (5.1)-(6.1). These results are analogous to those in Table 2 and use the same values of m/n as in Table 2, but apply to the second linear inequality model rather than the first. The nominal confidence level .95 and parameter values are the same as in Table 3. Given that $h_2 = 0$, we have $CP_{tv}(m/n, h) = CP_{int}(m/n, h)$.

Table 4(a) indicates that the m out of n "backward" bootstrap has exact and asymptotic size equal to the nominal level .95, as desired. (That is, the coverage probabilities are .95 or greater with equality for some parameter values.) The coverage probabilities for the m out of n "backward" bootstrap are not very sensitive to the value of m/n. The coverage probabilities exceed .95 in a variety of cases, however, so the CI is not asymptotically similar. In consequence, the CI may be longer than necessary.

Table 4(b) shows that the m out of n "forward" bootstrap has asymptotic size substantially less than its nominal level when $\rho \leq -.5$. For example, when $\rho = -1.0$ and $m/n = 0$, the exact and asymptotic size is less than or equal to .50. When $\rho = -.99$ and $m/n = 0$, it is less than or equal to .58. This demonstrates that the m out of n "forward" bootstrap CI can behave quite poorly depending upon the moment inequalities and the parameter values considered.

[5] Table 3(b) and Tables 4(i)(b)-4(v)(b) show a discontinuity in the coverage probability at $(h_D, \rho) = (0.0, -1.0)$. To see why this discontinuity occurs, consider the case where $m/n = 0$. In this case, (i) $U(h_D) = \max\{Z_2, (Z_1 + Z_2 - h_D)/2\} = \max\{Z_2, -h_D/2\}$, where the last equality holds because $Z_1^* = -Z_2^*$ when $\rho = -1$, (ii) $U^*(m/n, h_D) = U^*(0, h_D) = \max\{Z_2^*, (Z_1^* + Z_2^*)/2\} = \max\{Z_2^*, 0\}$, (iii) the $1 - \alpha$ quantile of $U^*(0, h_D)$ is $c_U^*(0, h_D, 1 - \alpha) = 0.0$ for $\alpha \leq 1/2$, (iv) $CP_{tv}(m/n, h) = CP_{tv}(0, (h_D, 0)) = P(U(h_D) \geq -c_U^*(0, h_D, 1 - \alpha)) = P(\max\{Z_2, -h_D/2\} \geq 0)$, and (v) $P(\max\{Z_2, -h_D/2\} \geq 0) = 1$ when $h_D = 0$ and $P(\max\{Z_2, -h_D/2\} \geq 0) = 1/2$ when $h_D > 0$.

ACKNOWLEDGMENTSCKNOWLEDGEMENTS

Andrews gratefully acknowledges the research support of the National Science Foundation via grant numbers SES-0417911 and SES-0751517.

References

Andrews, D. W. K. (1999). Estimation when a parameter is on a boundary. *Econometrica 67,* 1341–1383.

Andrews, D. W. K. (2000). Inconsistency of the bootstrap when a parameter is on the boundary of the parameter space. *Econometrica 68,* 399–405.

Andrews, D. W. K., S. Berry, and P. Jia (2004). Confidence regions for parameters in discrete games with multiple equilibria, with an application to discount chain store location. Working Paper, Cowles Foundation, Yale University.

Andrews, D. W. K. and P. Guggenberger (2005a). The limit of finite sample size and a problem with subsampling and with the m out of n bootstrap. Working paper, Cowles Foundation, Yale University.

Andrews, D. W. K. and P. Guggenberger (2005b). Hybrid and size-corrected subsample methods. Cowles Foundation Discussion Paper No. 1606, Yale University. Available at http://cowles.econ.yale.edu.

Andrews, D. W. K. and P. Guggenberger (2005c). Applications of subsampling, hybrid, and size-correction methods. Cowles Foundation Discussion Paper No. 1608, Yale University.

Andrews, D. W. K. and P. Guggenberger (2005d). Invalidity of subsampling inference based on post-consistent model selection estimators. Working paper, Cowles Foundation, Yale University.

Andrews, D. W. K. and P. Guggenberger (2008). Validity of subsampling and "plug-in asymptotic" inference for parameters defined by moment inequalities. *Econometric Theory,* forthcoming.

Andrews, D. W. K. and G. Soares (2007). Inference for parameters defined by moment inequalities using generalized moment selection. Cowles Foundation Discussion Paper No. 1631, Yale University.

Athreya, K. B. (1987). Bootstrap of the mean in the infinite variance case. *Annals of Statistics 15,* 724–731.

Athreya, K. B. and J. Fukuchi (1994). Bootstrapping extremes of i.i.d. random variables. *Proceedings of the Conference on Extreme Value Theory.* NIST.

Babu, J. (1984). Bootstrapping statistics with linear combinations of chi-squares as weak limit. *Sankhya 46,* 86-93.

Basawa, I. V., A. K. Mallik, W. P. McCormick, J. H. Reeves, and R. L. Taylor (1991). Bootstrapping unstable first-order autoregressive processes. *Annals of Statistics 19,* 1098–1101.

Beran, R. (1982). Estimated sampling distributions: the bootstrap and competitors. *Annals of Statistics 10,* 212–225.

Beran, R. (1997). Diagnosing bootstrap success. *Annals of the Institute of Statistical Mathematics 49,* 1–24.

Beran, R. and M. S. Srivastava (1985). Bootstrap tests and confidence regions for functions of a covariance matrix. *Annals of Statistics 13,* 95–115.

Bickel, P. J. and D. Freedman (1981). Some asymptotic theory for the bootstrap. *Annals of Statistics 9,* 1196-1217.

Bickel, P. J., F. Götze, and W. R. van Zwet (1997). Resampling fewer than n observations: gains, losses, and remedies for losses. *Statistica Sinica 7,* 1–31.

Bretagnolle, J. (1983). Lois limites du bootstrap de certaines fonctionnelles. *Annales de L'institut Henri Poincaré Sec. B, 19,* 281-296.

Bugni, F. (2007a). Bootstrap inference in partially identified models. Working paper, Department of Economics, Northwestern University.

Bugni, F. (2007b). Bootstrap inference in partially identified models: pointwise construction. Working paper, Department of Economics, Northwestern University.

Canay, I. A. (2007). EL inference for partially identified models: large deviations optimality and bootstrap validity. Working paper, Department of Economics, University of Wisconsin.

Chernozhukov, V., H. Hong, and E. Tamer (2007). Estimation and confidence regions for parameter sets in econometric models. *Econometrica 75,* 1243-1284.

Datta, S. (1995). On a modified bootstrap for certain asymptotically non-normal statistics. *Statistics and Probability Letters 24,* 91–98.

Deheuvels, P., D. Mason, and G. Shorack (1993). Some results on the influence of extremes on the bootstrap. *Annales de L'institut Henri Poincaré 29,* 83–103.

Dümbgen, L. (1993). On nondifferentiable functions and the bootstrap. *Probability Theory and Related Fields 95,* 125–140.

Eaton, M. L. and D. E. Tyler (1991). On Wieland's inequality and its applications to the asymptotic distribution of the eigenvalues of a random symmetric matrix. *Annals of Statistics 19,* 260–271.

Efron, B. (1979). Bootstrap methods: another look at the jackknife. *Annals of Statistics 7,* 1-26.

Guggenberger, P. and M. Wolf (2004). Subsampling tests of parameter hypotheses and overidentifying restrictions with possible failure of identification. Working paper, Department of Economics, UCLA.

Hall, P. (1992). *The Bootstrap and Edgeworth Expansion.* New York: Springer.

Imbens, G. and C. F. Manski (2004). Confidence intervals for partially identified parameters. *Econometrica 72,* 1845–1857.

Lehmann, E. L. and J. P. Romano (2005). *Testing Statistical Hypotheses* (3rd ed.). New York: Wiley.

Manski, C. F. and E. Tamer (2002). Inference on regressions with interval data on a regressor or outcome. *Econometrica 70,* 519-546.

Mikusheva, A. (2007). Uniform inferences in autoregressive models. *Econometrica 75,* 1411-1452.

Pakes, A., J. Porter, K. Ho, and J. Ishii (2004). Applications of moment inequalities. Working paper, Department of Economics, Harvard University.

Politis, D. N. and J. P. Romano (1994). Large sample confidence regions based on subsamples under minimal assumptions. *Annals of Statistics 22,* 2031–2050.

Politis, D. N., J. P. Romano, and M. Wolf (1999). *Subsampling.* New York: Springer.

Putter, H. and W. R. van Zwet (1996). Resampling: consistency of substitution estimators. *Annals of Statistics 24,* 2297–2318.

Romano, J. (1988). Bootstrapping the mode. *Annals of the Institute of Statistical Mathematics 40,* 565-586.

Romano, J. P. and A. M. Shaikh (2005). Inference for the identified set in partially identified econometric models. Working paper, Department of Economics, University of Chicago.

Romano, J. P. and A. M. Shaikh (2008). Inference for identifiable parameters in partially identified econometric models. *Journal of Statistical Inference and Planning (Special Issue in Honor of T. W. Anderson),* forthcoming.

Romano, J. P. and M. Wolf (2001). Subsampling intervals in autoregressive models with linear time trend. *Econometrica 69,* 1283–1314.

Samworth, R. (2003). A note on methods of restoring consistency to the bootstrap. *Biometrika 90,* 985-990.

Shao, J. (1994). Bootstrap sample size in nonregular cases. *Proceedings of the American Mathematical Society 112,* 1251–1262.

Shao, J. (1996). Bootstrap model selection. *Journal of the American Statistical Association 91,* 655–665.

Shao, J. and C. J. F. Wu (1989). A general theory for jackknife variance estimation. *Annals of Statistics 15,* 1563–1579.

Sriram, T. N. (1993). Invalidity of the bootstrap for critical branching processes with immigration. *Annals of Statistics 22,* 1013–1023.

Swanepoel, J. W. H. (1986). A note on proving that the (modified) bootstrap works. *Communications in Statistics–Theory and Methods Section 15,* 3193–3203.

Wu, C. F. J. (1990). On the asymptotic properties of the jackknife histogram. *Annals of Statistics 18,* 1438–1452.

Table 1. Linear moment inequalities I: bootstrap coverage probabilities of nominal 95% confidence intervals when $h_1 = 0$.

(a) "Backward" bootstrap confidence interval.

$m/n = 1$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	.00	.13	.36	.70	.80	.87	.93	.94	.95
	.125	.03	.36	.50	.74	.82	.88	.94	.95	.95
h_D	.250	.76	.63	.63	.77	.84	.89	.94	.95	.95
	.500	.90	.88	.82	.83	.87	.91	.95	.95	.95
	1.00	.94	.94	.93	.90	.91	.93	.95	.95	.95
	2.00	.95	.95	.95	.94	.94	.95	.95	.95	.95

(b) "Forward" bootstrap confidence interval.

$m/n = 1$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	.90	.90	.90	.90	.91	.92	.94	.95	.95
	.125	.91	.91	.91	.91	.92	.93	.95	.95	.95
h_D	.250	.92	.92	.92	.92	.93	.94	.95	.95	.95
	.500	.94	.94	.94	.93	.94	.94	.95	.95	.95
	1.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	2.00	.95	.95	.95	.95	.95	.95	.95	.95	.95

Table 2. Linear moment inequalities I: m out of n bootstrap coverage probabilities of nominal 95% confidence intervals when $h_1 = 0$.

(i) $m/n = 0$.

(a) "Backward" bootstrap confidence interval.

$m/n = 0$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
h_D	0.00	.00	.01	.05	.37	.60	.77	.92	.94	.95
	.125	.00	.03	.08	.40	.63	.79	.93	.94	.95
	.250	.05	.06	.11	.43	.65	.81	.93	.94	.95
	.500	.14	.15	.19	.49	.69	.83	.93	.94	.95
	1.00	.30	.31	.34	.58	.74	.86	.94	.94	.95
	2.00	.45	.45	.47	.65	.77	.86	.94	.94	.95

(b) "Forward" bootstrap confidence interval.

$m/n = 0$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
h_D	0.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	.125	.96	.96	.96	.96	.96	.96	.96	.96	.95
	.250	.96	.96	.96	.96	.96	.96	.96	.96	.95
	.500	.97	.97	.97	.97	.97	.97	.96	.96	.95
	1.00	.97	.97	.97	.97	.98	.97	.96	.96	.95
	2.00	.98	.98	.98	.97	.98	.97	.96	.96	.95

Table 2 (cont.).

(ii) $m/n = 0.01$.

(a) "Backward" bootstrap confidence interval.

$m/n = 0.01$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	.00	.01	.06	.40	.63	.79	.92	.94	.95
	.125	.00	.03	.09	.43	.66	.81	.93	.94	.95
h_D	.250	.06	.07	.12	.47	.68	.83	.93	.94	.95
	.500	.16	.17	.21	.53	.72	.85	.94	.95	.95
	1.00	.33	.34	.37	.62	.77	.87	.94	.95	.95
	2.00	.50	.50	.52	.70	.81	.89	.94	.95	.95

(b) "Forward" bootstrap confidence interval.

$m/n = 0.01$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	.93	.93	.93	.93	.94	.94	.95	.95	.95
	.125	.94	.94	.94	.94	.95	.95	.95	.95	.95
h_D	.250	.95	.95	.95	.95	.95	.95	.96	.95	.95
	.500	.96	.96	.95	.96	.96	.96	.96	.95	.95
	1.00	.96	.96	.96	.96	.97	.97	.96	.95	.95
	2.00	.96	.96	.96	.96	.97	.96	.95	.95	.95

Table 2 (cont.).

(iii) $m/n = 0.05$.

(a) "Backward" bootstrap confidence interval.

$m/n = 0.05$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	.00	.01	.06	.43	.66	.80	.92	.94	.95
	.125	.00	.03	.10	.47	.69	.82	.93	.94	.95
h_D	.250	.07	.08	.15	.51	.71	.84	.94	.95	.95
	.500	.19	.19	.25	.57	.75	.86	.94	.95	.95
	1.00	.38	.39	.43	.67	.80	.89	.95	.95	.95
	2.00	.57	.57	.60	.76	.85	.91	.95	.95	.95

(b) "Forward" bootstrap confidence interval.

$m/n = 0.05$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	.91	.91	.91	.92	.93	.94	.95	.95	.95
	.125	.92	.92	.92	.93	.94	.94	.95	.95	.95
h_D	.250	.93	.93	.93	.94	.94	.95	.96	.95	.95
	.500	.94	.94	.94	.95	.95	.96	.96	.95	.95
	1.00	.95	.95	.95	.95	.96	.96	.95	.95	.95
	2.00	.95	.95	.95	.95	.96	.96	.95	.95	.95

Table 2 (cont.).

(iv) $m/n = 0.1$.

(a) "Backward" bootstrap confidence interval.

$m/n = 0.1$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	.00	.01	.08	.47	.68	.81	.92	.94	.95
	.125	.00	.04	.11	.51	.71	.83	.93	.95	.95
h_D	.250	.07	.09	.16	.54	.73	.85	.94	.95	.95
	.500	.21	.21	.28	.61	.77	.87	.94	.95	.95
	1.00	.43	.43	.47	.71	.83	.90	.95	.95	.95
	2.00	.63	.64	.66	.80	.88	.92	.95	.95	.95

(b) "Forward" bootstrap confidence interval.

$m/n = 0.1$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	.91	.91	.91	.91	.92	.93	.95	.95	.95
	.125	.92	.92	.92	.92	.93	.94	.95	.95	.95
h_D	.250	.93	.93	.93	.93	.94	.95	.95	.95	.95
	.500	.94	.94	.94	.94	.95	.95	.95	.95	.95
	1.00	.95	.95	.95	.95	.95	.96	.95	.95	.95
	2.00	.95	.95	.95	.95	.95	.95	.95	.95	.95

Table 2 (cont.).

(v) $m/n = 0.5$.

(a) "Backward" bootstrap confidence interval.

$m/n = 0.5$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
h_D	0.00	.00	.02	.15	.61	.76	.85	.93	.94	.95
	.125	.00	.09	.23	.65	.78	.87	.94	.95	.95
	.250	.17	.20	.32	.69	.80	.88	.94	.95	.95
	.500	.45	.46	.52	.75	.84	.90	.95	.95	.95
	1.00	.77	.77	.77	.84	.89	.93	.95	.95	.95
	2.00	.93	.93	.92	.92	.93	.95	.95	.95	.95

(b) "Forward" bootstrap confidence interval.

$m/n = 0.5$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
h_D	0.00	.90	.90	.90	.90	.91	.92	.95	.95	.95
	.125	.91	.91	.91	.91	.92	.93	.95	.95	.95
	.250	.92	.92	.92	.92	.93	.94	.95	.95	.95
	.500	.94	.94	.94	.93	.94	.95	.95	.95	.95
	1.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	2.00	.95	.95	.95	.95	.95	.95	.95	.95	.95

Table 3. Linear moment inequalities II: bootstrap coverage probabilities of nominal 95% confidence intervals when $h_2 = 0$.

(a) "Backward" bootstrap confidence interval.

$m/n = 1$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	1.00	1.00	1.00	.99	.98	.98	.96	.95	.95
	.125	.95	.99	.99	.99	.98	.97	.96	.95	.95
	.250	.95	.98	.99	.99	.98	.97	.95	.95	.95
	.500	.95	.95	.98	.98	.98	.97	.95	.95	.95
h_D	1.00	.95	.95	.95	.97	.97	.96	.95	.95	.95
	2.00	.95	.95	.95	.96	.96	.95	.95	.95	.95
	4.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	6.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	8.00	.95	.95	.95	.95	.95	.95	.95	.95	.95

(b) "Forward" bootstrap confidence interval.

$m/n = 1$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	1.00	.96	.97	.96	.96	.96	.96	.95	.95
	.125	.51	.86	.93	.96	.96	.96	.95	.95	.95
	.250	.53	.71	.89	.95	.95	.95	.95	.95	.95
	.500	.55	.57	.77	.93	.94	.95	.95	.95	.95
h_D	1.00	.60	.61	.65	.89	.92	.94	.95	.95	.95
	2.00	.69	.71	.72	.85	.91	.94	.95	.95	.95
	4.00	.84	.85	.86	.91	.94	.95	.95	.95	.95
	6.00	.93	.93	.93	.94	.95	.95	.95	.95	.95
	8.00	.95	.95	.95	.95	.95	.95	.95	.95	.95

Table 4. Linear moment inequalities II: m out of n bootstrap coverage probabilities of nominal 95% confidence intervals when $h_2 = 0$.

(i) $m/n = 0$.

(a) "Backward" bootstrap confidence interval.

$m/n = 0$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	1.00	1.00	1.00	1.00	.99	.98	.96	.96	.95
	.125	1.00	1.00	1.00	1.00	.99	.98	.96	.96	.95
	.250	1.00	1.00	1.00	1.00	.99	.98	.96	.95	.95
	.500	1.00	1.00	1.00	1.00	.99	.98	.96	.95	.95
h_D	1.00	1.00	1.00	1.00	1.00	.98	.97	.96	.95	.95
	2.00	1.00	1.00	1.00	.98	.96	.96	.96	.95	.95
	4.00	.95	.95	.95	.95	.95	.96	.96	.95	.95
	6.00	.95	.95	.95	.95	.95	.96	.96	.95	.95
	8.00	.95	.95	.95	.95	.95	.96	.96	.95	.95

(b) "Forward" bootstrap confidence interval.

$m/n = 0$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	1.00	.95	.95	.95	.95	.95	.95	.95	.95
	.125	.50	.83	.90	.94	.95	.95	.95	.95	.95
	.250	.50	.67	.85	.93	.94	.94	.95	.95	.95
	.500	.50	.54	.72	.90	.92	.93	.95	.95	.95
h_D	1.00	.50	.54	.59	.85	.90	.92	.95	.95	.95
	2.00	.50	.54	.58	.77	.86	.91	.95	.95	.95
	4.00	.50	.54	.58	.75	.85	.90	.95	.95	.95
	6.00	.50	.54	.58	.75	.85	.90	.95	.95	.95
	8.00	.50	.54	.58	.75	.85	.90	.95	.95	.95

Table 4 (cont.).

(ii) $m/n = 0.01$.

(a) "Backward" bootstrap confidence interval.

$m/n = 0.01$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	1.00	1.00	1.00	1.00	.99	.98	.96	.96	.95
	.125	1.00	1.00	1.00	1.00	.99	.98	.96	.95	.95
	.250	1.00	1.00	1.00	1.00	.99	.98	.96	.95	.95
	.500	1.00	1.00	1.00	1.00	.99	.97	.95	.95	.95
h_D	1.00	1.00	1.00	1.00	1.00	.98	.97	.95	.95	.95
	2.00	1.00	1.00	1.00	.98	.96	.96	.95	.95	.95
	4.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	6.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	8.00	.95	.95	.95	.95	.95	.95	.95	.95	.95

(b) "Forward" bootstrap confidence interval.

$m/n = 0.01$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	1.00	.95	.95	.95	.95	.95	.95	.95	.95
	.125	.50	.84	.91	.94	.95	.95	.95	.95	.95
	.250	.51	.68	.86	.93	.94	.94	.95	.95	.95
	.500	.51	.55	.73	.91	.93	.94	.95	.95	.95
h_D	1.00	.52	.55	.60	.86	.90	.92	.95	.95	.95
	2.00	.54	.57	.61	.79	.87	.91	.95	.95	.95
	4.00	.57	.60	.64	.79	.87	.92	.95	.95	.95
	6.00	.61	.63	.67	.81	.88	.93	.95	.95	.95
	8.00	.64	.67	.70	.83	.89	.93	.95	.95	.95

Table 4 (cont.).

(iii) $m/n = 0.05$.

(a) "Backward" bootstrap confidence interval.

$m/n = 0.05$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	1.00	1.00	1.00	1.00	.99	.98	.96	.96	.95
	.125	1.00	1.00	1.00	1.00	.99	.98	.96	.95	.95
	.250	1.00	1.00	1.00	1.00	.99	.98	.96	.95	.95
	.500	1.00	1.00	1.00	1.00	.99	.97	.95	.95	.95
h_D	1.00	1.00	1.00	1.00	1.00	.98	.97	.95	.95	.95
	2.00	1.00	1.00	1.00	.98	.96	.96	.95	.95	.95
	4.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	6.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	8.00	.95	.95	.95	.95	.95	.95	.95	.95	.95

(b) "Forward" bootstrap confidence interval.

$m/n = 0.05$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	1.00	.95	.96	.95	.95	.95	.95	.95	.95
	.125	.51	.84	.92	.94	.95	.95	.95	.95	.95
	.250	.51	.68	.87	.93	.94	.95	.95	.95	.95
	.500	.52	.55	.74	.91	.93	.94	.95	.95	.95
h_D	1.00	.54	.56	.61	.86	.90	.93	.95	.95	.95
	2.00	.57	.60	.63	.80	.88	.92	.95	.95	.95
	4.00	.65	.67	.70	.82	.89	.93	.95	.95	.95
	6.00	.71	.73	.75	.85	.91	.94	.95	.95	.95
	8.00	.77	.79	.80	.88	.93	.95	.95	.95	.95

Table 4 (cont.).

(iv) $m/n = 0.1$.

(a) "Backward" bootstrap confidence interval.

$m/n = 0.1$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	1.00	1.00	1.00	1.00	.99	.98	.96	.96	.95
	.125	1.00	1.00	1.00	1.00	.99	.98	.96	.95	.95
	.250	1.00	1.00	1.00	1.00	.99	.98	.96	.95	.95
	.500	1.00	1.00	1.00	1.00	.98	.97	.95	.95	.95
h_D	1.00	1.00	1.00	1.00	.99	.98	.97	.95	.95	.95
	2.00	1.00	1.00	1.00	.97	.96	.96	.95	.95	.95
	4.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	6.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	8.00	.95	.95	.95	.95	.95	.95	.95	.95	.95

(b) "Forward" bootstrap confidence interval.

$m/n = 0.1$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
	0.00	1.00	.96	.96	.96	.96	.96	.95	.95	.95
	.125	.51	.85	.92	.95	.95	.95	.95	.95	.95
	.250	.51	.69	.87	.94	.94	.95	.95	.95	.95
	.500	.52	.56	.75	.91	.93	.94	.95	.95	.95
h_D	1.00	.55	.57	.62	.87	.91	.93	.95	.95	.95
	2.00	.60	.62	.65	.81	.88	.92	.95	.95	.95
	4.00	.69	.70	.73	.84	.90	.94	.95	.95	.95
	6.00	.77	.78	.80	.88	.93	.95	.95	.95	.95
	8.00	.83	.84	.85	.91	.94	.95	.95	.95	.95

Table 4 (cont.).

(v) $m/n = 0.5$.

(a) "Backward" bootstrap confidence interval.

$m/n = 0.5$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
h_D	0.00	1.00	1.00	1.00	.99	.99	.98	.96	.96	.95
	.125	.99	1.00	1.00	.99	.99	.98	.96	.95	.95
	.250	.99	1.00	1.00	.99	.98	.97	.96	.95	.95
	.500	.99	.99	.99	.99	.98	.97	.95	.95	.95
	1.00	.98	.98	.98	.98	.97	.96	.95	.95	.95
	2.00	.97	.97	.97	.96	.96	.95	.95	.95	.95
	4.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	6.00	.95	.95	.95	.95	.95	.95	.95	.95	.95
	8.00	.95	.95	.95	.95	.95	.95	.95	.95	.95

(b) "Forward" bootstrap confidence interval.

$m/n = 0.5$		ρ								
		-1.00	-0.99	-0.95	-0.50	0.00	0.50	0.95	0.99	1.00
h_D	0.00	1.00	.96	.96	.96	.96	.96	.96	.95	.95
	.125	.51	.86	.93	.95	.96	.96	.95	.95	.95
	.250	.52	.70	.88	.94	.95	.95	.95	.95	.95
	.500	.54	.57	.75	.92	.94	.95	.95	.95	.95
	1.00	.58	.60	.64	.88	.92	.94	.95	.95	.95
	2.00	.66	.68	.70	.83	.90	.94	.95	.95	.95
	4.00	.80	.81	.82	.89	.93	.95	.95	.95	.95
	6.00	.89	.89	.90	.93	.95	.95	.95	.95	.95
	8.00	.95	.95	.94	.94	.95	.95	.95	.95	.95