On the value and optimal allocation of water

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Abstract

We consider the problem of efficiently allocating water of a lake among different potential users. We consider two types of irreversibility: the irreversibility of an investment that creates a fixed damage to the ecosystem and the irreversibility of the right to use the resource that comes from the *Law of Water* (legislative irreversibility). First of all, we determine the value of water for users. Then, we characterize the optimal allocation of water among users. With legislative irreversibility, we show that it is sometimes optimal to reduce the amount of water allocated to the firm, even though there is no rivalry in use. Moreover, we show that it is not always optimal to prevent the damage created by the irreversible investment. We define the context, in which it is optimal to intervene to prevent the damage. Furthermore, with irreversibility, we prove that the marginal value of water at the efficient allocation is not equalized between users. Overall, we show that in the case of no rivalry in use, unused water should not be seen as a limitless resource to be used in any way whatever.

Keywords: Water allocation; Irreversibility; Value of water

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1 Introduction

Water has a strong historic presence in the economic development of the province of Quebec. From the fur industry to the hydroelectric industry passing through the forest industry, the rivers and lakes of Quebec have been and are still an important economic lever. Multiple uses of this resource have led to different conflicts of uses. The laws that rule the status of water have evolved over time trying to resolve the conflicts between the different users. The aim of the current paper is not to provide a retrospective of the status of water in Quebec, whereas one can find an interesting review of the principal highlights in Richelle & Thibaudin (2011). The most recent legislation, *Loi affirmant le caractère collectif des ressources en eau et visant à renforcer leur protection (LRQ, chapitre C-6.2)*, states, principally, that water is a collective resource (*res communis*).[1] This law gives the population of the province of Quebec the exclusive right to use the resource and names the State its unique keeper. The particularity of a *res communis* is that no one can possess the resource, and hence, no price can be assigned to it. Only fees, that are managed by the keeper of the resource, can be determined. Those fees are designed to capture only the cost of restoration, management and use of the water. In contrast with prices, fees don't reflect the value of the water. If water had a price based on the willingness to pay of potential users, only the ones that value the most the resource would use it. Consequently, some conflicts of uses might be resolved. In this context of no price for water, it is relevant to ask: how should we allocate the right to use the resource? We suggest an allocation of water based on the maximization of the welfare induced by the different uses of water. This allocation is socially optimal. The welfare of water directly comes from the value that users give to the water. The evaluation of the value of water is a challenging task. Part of the difficulties lies in the mismatch between the marginal value and the use value, as it is pointed out by Smith (1776). The value of a standard economic good is determined by the marginal value, which is based on the relative rarity. If we assume that water is a standard good, its value is derived from what it is possible to get in exchange. Is water a standard economic good? The answer to this question is really important, in the view of the fact that it has an important impact on the optimal allocation of the resource. Once the value of water is established, we can tackle the issues of water allocation.

Water is used by a variety of different users: municipalities, firms, local residents, ecosys-

[1] From now on, *Law of Water*

tems, etc. Some investments might be needed by some users before using the resource, e.g., water purification plant, road construction. Other investments are made to improve the productivity of water, e.g., repair of leaky pipes. Part of those investments leads to destruction, degradation and fragmentation of natural ecosystems. If the destruction and degradation cannot be altered, the investments create in fact irreversible damages. The size of the investment is not always a good indicator of its impact on the ecosystem, i.e., even small investments that lead to fragmentation of ecological areas can cause major environmental damages. In their analysis of the degrading effects of fragmentation of the ecosystems, Haddad *et al.* (2015) show that fragmentation is an important issue. In particular, they show that 70% of the global forest cover are affected by fragmentation. The project of a new oil port in Cacouna is another good example of investment that is irreversible and that affects the ecosystem regardless of its size. Environmentalists and biologists argue that regardless of the size of an oil port, once the habitat of beluga is affected, this species would be in danger of extinction. These investments known as irreversible developments may lead to a loss of biodiversity or to animal extinction. To take this issue into account, we incorporate in the analysis of the value and the allocation of water, an investment that irreversibly damages the ecosystem regardless of its size. The damage depends neither on the level of the net investment nor on the level of the firm production, and creates a lump sum reduction of welfare.

The model also encompasses how legislation affects the optimal allocation of water. In particular, we introduce a form of legislative irreversibility that is inspired by the *Law of Water*, which states that the current allocation of water to the firm binds its future allocation. Based on the *Law of water*, once the State gives the right to use a certain amount of water, x, to a firm, in every future period this firm will have the right to use at least x units of water. Obviously, this legislation creates an irreversibility. By choosing the allocation of water in the current period, the decision maker chooses the lower bound amount of water that some users have the right to use in the future. Legislative irreversibility should be considered by anyone who plans to study the value of water and the optimal allocation of this resource.

We develop a two-period analytical model to shed light on the optimal allocation of water of a lake among different potential users, namely: a firm, a municipality, and an ecosystem. First of all, we determine the value of water for the different users of the economy. Then, we characterize the optimal allocation of water among users in various situations.

The optimal allocation in the benchmark model without irreversibility is analyzed first. As we should expect the marginal value is equal across users. The solution of this basic case, particularly when there is no rivalry in use, highlights the fact that increasing the allocation to one particular user, doesnt necessarily increase the social benefit. In the case of no rivalry in use, unused water should not be seen as a limitless resource to be used in any way whatever.

Next, we suppose that net investment irreversibly damages the ecosystem and creates a lump sum welfare loss. We show that it is not always optimal to prevent the damage, moreover we characterize the range of welfare losses for which the decision maker intervenes to prevent the damage. To approach the reality, we introduce the irreversibility of the right to use the resource (legislative irreversibility). With legislative irreversibility, we show that it is sometimes optimal to reduce the amount of water allocated to the firm, even though there is no rivalry in use. In other words, it might be optimal to limit the access to water of the firm even if its marginal value is zero and there is unused water in the lake. Furthermore, with irreversibility, we prove that the equalization of the marginal value of water between users doesn't hold.

In Section 2, we discuss the principal characteristics of this particular good, that is water. Section 3 sets the basis of the theoretical model. The optimal value of water is analyzed in Section 4. The discussion of our principal results is provided in Section 5.

1.1 Literature review

The literature on the economic value of water and the optimal allocation of this resource is relatively modest, despite the importance of the issues. Ambec & Sprumont (2002) and Ambec & Ehlers (2008) are among the few papers that study rigorously the problem of efficiently sharing water among a group of agents. They provide an analytical framework to evaluate how to allocate the water of a river among a group of satiable agents located along the river. Gaudet *et al.* (2006) evaluate the optimal paths of production and water usage by two sectors (agricultural and non renewable resource) that share a limited amount of water. In comparison with our analysis, neither the irreversibility of investment nor the legislative irreversibility is considered. Despite their importance, it appears that optimal allocation of water and irreversibility have seldom been analyzed together.

The current paper is also related to the literature on the evaluation of the value of water. Hanemann *et al.* (2006) and Ward (2007) review the economic concepts related to water. They agree on the importance of determining the value of water, however, they do not provide any analytical model. Gibbons (1986) provides a framework for understanding water values and demand for water in various sectors in the United States. She studies different methods of estimating the value of water. Gibbons (1986) is an illustrative study of water values that results in a heterogeneous set of values which are not comparable between sectors. Furthermore, most of her estimations are based on price data. Whereas, in our model, no price can be assigned to water. Dachraoui & Harchaoui (2004) estimate the private value of water for self-supplied firms through the diminution of cost induced by the reduction of inputs caused by an extra unit of water. This method is appropriate when capital (labour) and water are substitutes, but is unsuitable when capital (labour) and water are complements. They estimate that for the majority of the 36 industries considered capital (labour) and water are substitutes and for 12 industries out of 36, capital (labour) and water are complements. Thus, for one third of the industries the method used underestimates the value of water for firms.

Our paper is also related to the literature on non market economic valuation of environmental resources. For instance, Young (2004) evaluates a monetary value on goods and services provided by water. The aim of Young (2004) is to apply non market methods for measuring benefits and costs of particular public policies relating to water. By contrast, the aim of our model is no to assign monetary measures of individuals' preferences for outcomes of policy proposals or events.

Because our model incorporates an irreversible damage created by the net investment, it is relevant to mention the vast literature of option value. The notion of option value introduced by the pioneering paper of Weisbrod (1964) has broadly evolved over time. This paper was the first of a long series: Cicchetti & FreemanIII (1971), Fisher *et al.* (1972), Arrow & Fisher (1974), Henry (1974), Hanemann (1989), Fisher (2001), Fisher & Narain (2003), Zhao (2011), and many more. More recently, option value theory has been used to study the timing of investment in the context of climate change: Fisher & Narain (2003), Fisher (2001), and Pindyck (2002). In particular Fisher & Narain (2003) show that an irreversible decision or action has to clear a higher hurdle to pass a cost-benefit analysis.

To the best of our knowledge, this paper is the first in the economic literature to develop a theoretical model that analyses the value of water and the optimal allocation of water in the presence of irreversibility.

2 Water: a particular economic good

In this section, we outline the characteristics of the water, as an economic good, that are important for our analysis.

2.1 Legal aspect

Over time, water has been used in very different usages: navigation, fur industry, forest industry and hydroelectricity. The multiplication of the uses has led to potential conflicts. Laws have evolved over time to resolve the conflicts between the different users. Richelle & Thibaudin (2011) provide a brief summary of the evolution of the importance of water for Quebec together with the evolution of the laws that rule the competing uses of water. In the present paper, we focus our attention on the current legislation, which is *Loi affirmant le caractère collectif des resources en eau et visant à renforcer leur protection (LRQ, chapter C-6.2)*.

This law has been voted in 2009. Some provisions of this law are worth our attention. First, the law confirms not only the *res communis* status of water, but goes further by affirming that water becomes now a collective resource, that is, water is now part of the heritage of Quebec society. Consequently, the appropriation of this resource is formally forbidden.[2] Furthermore, the law introduces the principles of user pays. However, it is only two years after the legislation took effect that the fees have been determined. It is worth noting that the fees are determined only to cover the cost of managing the resource, therefore there is no price associated with the resource. By asserting the exclusivity of usage by the population of Quebec, this law implies that water is neither public property nor private property. The law stipulates that the management of the resource is the responsibility of the State, in other words, the State becomes the keeper of water. The last articles of the law that ought to be mentioned are articles 33 to 38. These articles stipulate that users of water will have at their disposal during the next 10 years at least the same amount of water that they have in the current period. This piece of legislation is relatively

[2]Except under very special conditions specified in the law.

important in the study of our model since it creates irreversible situations. We refer to this type of irreversibility as legislative irreversibility.

2.2 Value of water

The *res communes* property implies that water cannot be owned by anyone, as a result, water cannot be exchanged on the market and no price can be determined. With price, conflicts could be resolved more easily based on the willingness to pay of potential users. Without any price, how could water be allocated among users? Should the allocation be organized in a hierarchy as the ecologists suggest? We don't think so. The impossibility to assign a price to water doesn't imply that it is impossible to determine its use value. The allocation based the maximization of the welfare of water is the solution that we prone. Users are those that value the most the resource. Moreover, this allocation is the optimal one. But, this alternative raises an important question: how can we determine the welfare of water and what are its components?

We first, we claim that water is an economic good. Water is indeed a good with particular characteristics, but even so it is an economic good. Let's shed light on the particularities of water. One really important characteristic of water is that in contrast to most economic goods, the use of water does not destroy it, it simply transforms it, and in turn this transformed water can be used. It is important to note that it is not the same water that is used twice, the initial water and its transformation should be considered as two different economic goods. The warming and the addition of pollution are examples of such transformation.

By way of illustration, let us consider a firm that uses as an input water pumped in a lake and rejects it one degree warmer. Water that enters and that gets out of the plant should be considered as two different economic goods. A user that needs cold water cannot utilize the water that comes out of the plant, but can utilize the water that enters the plant.

Another important property of water is its mobility. The flow of a stream carries the water to lakes or in rivers. Consequently, the use of water in a particular stream might affect the use of water in another stream in the future. Water upstream and downstream a river should not be considered as the same economic good.

The last feature that is worth mentioning is time. Time can directly transform water. For instance, time can reduce pollution of a lake through settling water. If water is subject to transformations through time, then we should consider different economic goods.

The private and the social marginal value of a private good coincide. In the following subsection, we argue that water is a private good, consequently the private value is also the social value.

2.3 Water, a private good

A public good is both non-excludable and non-rival. To understand the reason why water is not a public good, clarifications about rivalry should be made. There are two types of rivalry: rivalry in consumption and rivalry in use. There is non-rivalry in consumption, if the consumption by one consumer doesn't prevent the consumption by other consumers at a given moment. One unit of water consumed at a given moment in time, cannot be consumed by other users at the same time. Therefore, water is rival in consumption. On the other hand, water can be non-rival in use. If there is enough water in a lake, there might be several users at the same time. The simultaneous uses of the water of a lake simply reflect that there is enough water in the lake for all users, and does not reflect the non-rivalry in consumption.

Consequently, water is a private good. For that reason, we consider the private value of water for different users in the analytical model.

3 The model

Consider two discrete time periods, $t = 1, 2$. There is a lake with a fixed amount of water within a period, Θ_t. The amount of water can vary from one period to the other. Three agents want to use the water of the lake: a firm, a municipality and an ecosystem. The allocation of water among the firm, the municipality and the ecosystem is denoted by A_t^f, A_t^m, and A_t^e, respectively. Given the allocation of water, the amount of water that is actually used by each of the three agents in period t is given by: Q_t^f, Q_t^m and Q_t^e. To make it clear, an agent does not have to use the totality of its allocation of water. Therefore, the allocation of water is the upper bound of the amount of water used by each agent.

For simplicity, we assume that water can be used by any of the three agents without any transformation.

In Sections 3.1, 3.2, and 3.3, we evaluate the marginal value of water for the three agents.

3.1 The firm

A price-taking firm produces a consumption good. Every period the market for that good is assumed to be perfectly competitive.

The consumption good is produced using two intermediate inputs, y_t and z_t. The first intermediate input, y_t, is produced using capital, K_t, and labour, L_t. The second intermediate input, z_t, is produced using water, Q_t^f, factors that are perfect complements of water, X_t^c, and capital, K_t^s. The technologies F and H represent the production function of y_t and z_t, respectively:

$$y_t = F(L_t, K_t),$$
$$z_t = H(\min\{Q_t^f, f(X_t^c)\}, K_t^s).$$

The assumption of perfect complementarity of Q_t^f and X_t^c implies that optimally, the firm will combine the two inputs such that $Q_t^f = f(X_t^c)$. Consequently, z_t can be rewritten as $z_t = H(Q_t^f, K_t^s)$. There are two types of capital, K_t and K_t^s. K_t is the classical stock of capital, e.g., machinery, tools and buildings. This stock of capital has a direct positive effect on the production of the consumption good. Its depreciation rate, $0 < \delta < 1$, is assumed to be constant. The capital stock evolves as follows (where I is the investment):

$$K_2 = (1 - \delta)K_1 + I.$$

The other type of capital, K_t^s, affects the productivity of water. Its depreciation rate, $0 < \delta^s < 1$, is also assumed to be constant. We have:

$$K_2^s = (1 - \delta^s)K_1^s + I^s.$$

We make the distinction between both types of capital to distinguish between an investment I that increases the capacity of production and an investment I^s that increases the productivity of water.

The consumption good is produced using the two intermediate inputs, z_t and y_t. The quantity produced is given by:

$$G(z_t, y_t) = G(H(Q_t^f, K_t^s), F(L_t, K_t)).$$

Assumption 1 regroups the assumptions about the technologies.

Assumption 1

A1. $f(\cdot)$ is invertible with $f^{-1}(\cdot) = g(\cdot)$, $g(\cdot)$ is convex, and $\lim_{Q_t^f \to 0} g'(\cdot) \neq \infty$.

A2. The technologies G, H, and F satisfy the Inada conditions.

A3. The Hessian of G, H, and F are negative definite, as a result those functions are strictly concave in their arguments.

A4. F is homogeneous of degree 1.

A5. $G_{yz} > 0$.

A6. $H_{K^sQ}H_{K^s} - H_{K^sK^s}H_Q > 0$.[3]

To alleviate the notation, we use the following simplification. For a function $f(x_1, x_2)$, f_{x_1}, f_{x_2}, $f_{x_1x_1}$, $f_{x_2x_2}$, and $f_{x_1x_2}$ stand for $\partial f(x_1, x_2)/\partial x_1$, $\partial f(x_1, x_2)/\partial x_2$, $\partial^2 f(x_1, x_2)/\partial x_1^2$, $\partial^2 f(x_1, x_2)/\partial x_2^2$, and $\partial^2 f(x_1, x_2)/\partial x_1 \partial x_2$, respectively.

The *res communis* status of water implies that there is no price associated with the resource.[4] The prices of the consumption good, labour, the complement investment, the substitute investment, and the complement inputs are represented respectively by p_t, w_t, r, r^s and r_t^c. The decision maker gives to the firm the right to use at most A_t^f units of water in period t. In other words, in period t, the firm has at its disposal A_t^f units of water. The Lagrange multiplier of its profit maximization problem associated with this constraint represents the marginal value of water for the firm. In other words, it is the

[3]The explanation of this assumption is provided in the Appendix.

[4]There are fees that cover the cost of managing the resource. Because those fees are negligible, we do not include them in the analysis.

marginal profit of relaxing the constraint, $Q_t^f \leq A_t^f$. The profit maximization problem of the firm is given by:

$$\max_{\{\{Q_t^f, L_t, X_t^c\}_{t=1,2}, I, K_2, I^s, K_2^s\}} \sum_{t=1,2} \left(p_t G(H(Q_t^f, K_t^s), F(L_t, K_t)) - w_t L_t - r_t^c X_t^c \right) - rI - r^s I^s \tag{1}$$

$$\begin{aligned} \text{s.t.} \quad & X_t^c = g(Q_t^f) \\ & K_2 = (1-\delta)K_1 + I, \quad K_1 > 0 \quad \text{given} \\ & K_2^s = (1-\delta^s)K_1^s + I^s, \quad K_1^s > 0 \quad \text{given} \\ & Q_t - A_t^f \leq 0 \quad (\lambda_t^f), \quad t = 1, 2. \end{aligned}$$

The maximization problem (1) can be rewritten by:[5]

$$\begin{aligned} \max_{\{\{Q_t^f, L_t\}_{t=1,2}, K_2, K_2^s\}} \sum_{t=1,2} & \left(p_t G(H(Q_t^f, K_t^s), F(L_t, K_t)) - w_t L_t - r_t^c g(Q_t^f) \right) \\ & - r\left(K_2 - (1-\delta)K_1\right) - r_1^s\left(K_2^s - (1-\delta^s)K_1^s\right) \\ & + \lambda_1^f (A_1^f - Q_1^f) + \lambda_2^f (A_2^f - Q_2^f). \end{aligned} \tag{2}$$

The first order conditions of (2) are given by:

$$(Q_t^f): \quad p_t G_z H_Q - r_t^c g_Q - \lambda_t^f = 0, \quad t = 1, 2, \tag{3}$$

$$(L_t): \quad p_t G_y F_L - w_t = 0, \quad t = 1, 2, \tag{4}$$

$$(K_2): \quad p_2 G_y F_K - r = 0, \tag{5}$$

$$(K_2^s): \quad p_2 G_z H_{K^s} - r^s = 0, \tag{6}$$

and the Kuhn-Tucker conditions of (2) are given by:

$$\lambda_t^f (Q_t^f - A_t^f) = 0 \quad \lambda_t^f \geq 0 \quad A_t^f - Q_t \geq 0, \quad t = 1, 2.$$

We denote by $\bar{Q}_t^f$, the amount of water that maximizes the profit of the firm without any constraint on the amount of water that can be used. The Inada conditions of the technology G ensure that there exists such $\bar{Q}_t^f$. When $A_t^f \geq \bar{Q}_t^f$, the solution of the profit maximisation problem is interior, i.e., $Q_t^f = \bar{Q}_t^f$ and the marginal value of water for the

[5]The assumption that G, F and H satisfy the Inada conditions implies that every input is essential to the production. Combining the Inada conditions with the assumption that the limit of the first derivative of $g(\cdot)$ isn't positive infinity as the amount of water used by the firm approaches 0, implies that all the inputs are strictly positive at the optimum.

firm, λ_t^f, is zero. Otherwise, the optimum is achieved at a corner solution and the optimum amount of water used by the firm, Q_t^f, equals A_t^f. In that case, the marginal value of water for the firm is given by the expression of λ_t^f obtained by (3) evaluated at the solution of (2): [6]

$$(7) \quad \lambda_t^f = p_t G_z H_Q - r_t^c g_Q > 0.$$

Consequently, for a given amount of water allocated to the firm, A_t^f, the amount of water actually used by the firm is the minimum between A_t^f and $\bar{Q}_t^f$:

$$Q_t^f = \min\{A_t^f, \bar{Q}_t^f\}.$$

For an amount of water used by the firm, x, the marginal value of water for the firm is defined by:

$$\lambda_t^f(x) = \begin{cases} 0 & \text{if} \quad x \geq \bar{Q}_t^f \\ p_t G_z(H(x, K_t^s), F(L_t, K_t)) H_Q(x, K_t^s) - r_t^c g_Q(x) & \text{if} \quad x < \bar{Q}_t^f \end{cases} .$$

Consequently, in period t, given an allocation of water, A_t^f, the use value of the water used by the firm, Q_t^f, is given by:

$$(8) \quad \Pi_t = \int_0^{Q_t^f} \lambda_t^f(x) dx = \int_0^{Q_t^f} \left[p_t G_z(H(x, K_t^s), F(L_t, K_t)) H_Q(x, K_t^s) - r_t^c g_Q(x) \right] dx,$$

with $Q_t^f = \min\{A_t^f, \bar{Q}_t^f\}$. We define the use value of water for the firm by the sum of Π_1 and Π_2:

$$(9) \quad \Pi = \sum_{t=1,2} \Pi_t.$$

It is worth noting that the use value of water for the firm equals its profit. It raises the important point that even though the marginal value of water is zero, the use value of water isn't zero.

[6] In other words, Q_1^f, Q_2^f. L_1, L_2, K_2, and K_2^s are the optimal quantity chosen by the firm. Henceforth, when we refer to (7), it is implied that we consider (3) evaluated at the solution of (2).

3.1.1 Comparative statics

This section provides a series of propositions about the comparative statics. Throughout this section, we assume that $A_t^f \leq \bar{Q}_t^f$, otherwise a variation of the firm allocation of water has no effect on the optimal variables. All the proofs are in the Appendix. Proposition 1 establishes the impact of the allocation of water on the optimal L_2, K_2, and K_2^s.

Proposition 1 *The firm allocation of water in period two has a positive effect on L_2 and K_2, whereas its effect on K_2^s is ambiguous:*

$$\frac{dK_2}{dA_2^f} > 0, \quad \frac{dL_2}{dA_2^f} > 0, \quad \textit{and} \quad \frac{dK_2^s}{dA_2^f} \lesseqqgtr 0.$$

An increase of the allocation of water to the firm, A_2^f, has two opposite effects on the capital that affects the productivity of water, K_2^s. On one hand, an increase of A_2^f leads to a substitution effect. The firm reduces the capital K_2^s relatively to its use of water. On the other hand, an increase of A_2^f has a level effect, that affects positively K_2^s. Which of those two effects dominates determines the overall effect of an increase of A_2^f on K_2^s. Even though, the global effect of A_2^f on K_2^s is unknown, we prove that A_2^f affects positively the intermediate input, z_2. Which in turn, induces the firm to choose a larger amount of labour and capital, L_2 and K_2.

We turn next to the impact of prices on labour and on the capital stocks.

Proposition 2 *The factor prices, r, r^s, and w_2 have a negative impact on K_2^s, K_2 and L_2, whereas the price p_2 has a positive impact.*

$$\frac{dK_2^s}{dr} < 0, \quad \frac{dK_2^s}{dw_2} < 0, \quad \frac{dK_2^s}{dr^s} < 0, \quad \textit{and} \quad \frac{dK_2^s}{dp_2} > 0;$$
$$\frac{dK_2}{dr} < 0, \quad \frac{dK_2}{dw_2} \lesseqqgtr 0, \quad \frac{dK_2}{dr^s} < 0, \quad \textit{and} \quad \frac{dK_2}{dp_2} > 0;$$
$$\frac{dL_2}{dr} \lesseqqgtr 0, \quad \frac{dL_2}{dw_2} < 0, \quad \frac{dL_2}{dr^s} < 0, \quad \textit{and} \quad \frac{dL^s}{dp_2} > 0.$$

When r_s increases, the firm cannot substitute water for K_2^s, because the amount of water used by the firm is bounded by the amount of water allocated to the firm. Therefore, an increase of r^s affects negatively the intermediate good z_2. This reduction of z_2 induces the firm to reduce the labour and the capital, L_2 and K_2. As expected, an increase of r leads to a reduction of the classical stock of capital, K_2, whereas the effect on labour is ambiguous. The change of r has two opposite effect on L_2: a substitution effect and a level effect. Which of these effects dominates determines the global effect of a variation of r on L_2. Even though the effect of a change in r on labour is ambiguous, the overall effect on the intermediate input, y_2, is negative. Consequently, r affects negatively K_2^s. An increase of w_2 leads to a reduction of labour, L_2, whereas the effect on the classic stock of capital, K_2, is ambiguous. The change of w_2 has two opposite effect on K_2: a substitution effect and a level effect. Which of these effects dominates determines the global effect of a variation of w_2 on K_2. Even though the effect of a change in w_2 on labour is ambiguous, the overall effect on the intermediate input, y_2, is negative. Consequently, w_2 affects negatively K_2^s. An increase in the value of the consumption good, p_2, induces the firm to produce more, and therefore, it brings the firm to increase its inputs, L_2, K_2, and K_2^s.

So far, we have analyzed the impact of different parameters on the labour and the different types of capital. The following step is to evaluate the impact of those parameters on the marginal value of water for the firm in period two, λ_2^f.

In Proposition 3, we proceed with the analysis of the effects of the factor prices on λ_2^f.

Proposition 3 *All factor prices have a negative impact on λ_2^f, whereas p_2 affects positively the marginal value of water for the firm in period two:*

$$\frac{d\lambda_2^f}{dr} < 0, \quad \frac{d\lambda_2^f}{dr^s} < 0, \quad \frac{d\lambda_2^f}{dw_2} < 0, \quad \frac{d\lambda_2^f}{dr_2^c} < 0, \quad \textit{and} \quad \frac{d\lambda_2^f}{dp_2} > 0.$$

From Proposition 2, an increase of either one of r, r^s, and w_2 leads to a reduction of both intermediate goods. Consequently, the production of the consumption good is reduced. In other words, the firm produces less with the same amount of water, therefore the marginal value of water falls. The effect of r_2^c on the marginal value of water is direct. Because we restrict our attention on cases where $A_2^f \leq \bar{Q}_2^f$, even though r_2^c increases the firm continues to use the same amount of water. However, the cost of each unit of water used is larger, hence, the value of water falls. An increase of p_2 leads to an increase of both intermediate goods. As a result, the production of the consumption good increases. The firm produces more with the same amount of water. Consequently, the marginal value of water raises.

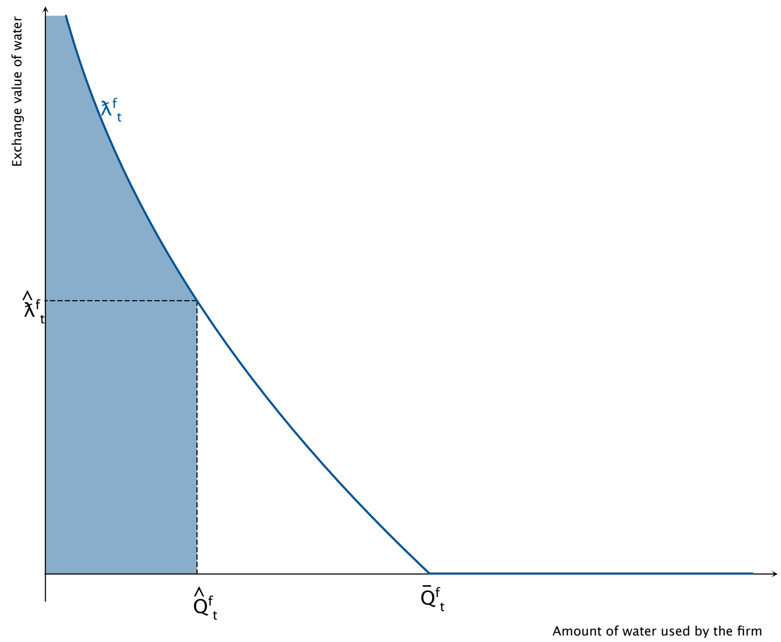


Figure 1: The value of water for the firm

Finally, Proposition 4 establishes the impact of the firm allocation of water on the marginal value of water.

Proposition 4 *The marginal value of water for the firm is decreasing in its amount of water used.*

Proposition 4 highlights an important feature of the value of water for the firm. As it is illustrated in Figure 1, the marginal value of water for the firm decreases as its use of water increases. Figure 1 illustrates also the difference between the marginal value and the use value of water. If the amount of water allocated to the firm is $\hat{A}_t^f$, the amount of water used by the firm is $\hat{Q}_t^f$, the marginal value of water is given by $\hat{\lambda}_t^f$ and the use value, (8), is represented by the blue area.[7]

[7] No assumption is made on the third derivative of the production functions, thus, the marginal value of water can take various shapes. Figure 1 illustrates a convex marginal value of water for the firm.

3.2 The municipality

Every period, the municipality is allocated various types of resources, among which is water. We denote by Q_t^m the amount of water used by the municipality in period $t = 1, 2$. Apart from water, all the resources are gathered together and are represented by M_t. The municipality has to choose the optimal allocation of its resources, $\{Q_t^m, M_t\}_{t=1,2}$, among its citizens. We define the value function of the municipality by $v(Q_t^m, M_t)$. We assume that the value function, $v(\cdot,\cdot)$, is concave in its arguments. The decision maker gives to the municipality the right to use at most A_t^m units of water in period t, i.e., $Q_t^m \leq A_t^m$. The marginal value of water for the municipality, λ_t^m, is defined as the marginal benefit of relaxing the constraint $Q_t^m \leq A_t^m$. In addition, we make the assumption that for a given M_t, there exists a level of water that maximizes the value function of the municipality without any constraint on the amount of water that can be used, which we denote by $\bar{Q}_t^m$. When the allocation of water to the municipality is larger than $\bar{Q}_t^m$, the municipality uses $\bar{Q}_t^m$ and the marginal value of water is zero, i.e., $\lambda_t^m = 0$. Thus, as for the firm, there exists an amount of water above which the marginal value of water is zero. If the allocation of water to the municipality is smaller than $\bar{Q}_t^m$, then the municipality uses A_t^m. In that case, the marginal value of water for the municipality is given by:

$$\lambda_t^m = v_Q(A_t^m, M_t).$$

Consequently, for a given amount of water allocated to the municipality, A_t^m, the amount of water actually used by the municipality is the minimum between A_t^m and $\bar{Q}_t^m$:

$$Q_t^m = \min\{A_t^m, \bar{Q}_t^m\}.$$

For an amount of water used by the municipality, x, the marginal value of water for the municipality is defined by:

$$\lambda_t^m(x) = \begin{cases} 0 & \text{if} \quad x \geq \bar{Q}_t^m \\ v_Q(x, M_t) & \text{if} \quad x < \bar{Q}_t^m \end{cases}.$$

The concavity assumption of $v(\cdot,\cdot)$ implies that the marginal value of water for the municipality is decreasing in x.[8] In period t, given an allocation of water A_t^m, the use value

[8]The curvature of the marginal value of water for the municipality in period t depends on the third derivative of $v(\cdot,\cdot)$ with respect to the amount of water used. Since no assumption is made on the third derivative, the marginal value of water for the municipality can take various shapes.

of the water used by the municipality, Q_t^m, is given by:

$$(10) \quad Y_t = \int_0^{Q_t^m} \lambda_t^m(x)dx = \int_0^{Q_t^m} v_Q(x, M_t)dx,$$

with $Q_t^m = \min\{A_t^m, \bar{Q}_t^m\}$. The use value of water for the municipality for both periods is simply defined by the sum of Y_1 and Y_2:

$$(11) \quad Y = \sum_{t=1,2} Y_t.$$

The use value Y is the monetary equivalent of the welfare of the municipality.

3.3 The ecosystem

The ecosystem is a particular user of water, indeed, neither preference system nor profit function can be assigned to the ecosystem. At first sight, it is not an easy task to attribute a value of water to this user. We base our analysis on the *Millennium Ecosystem Assessment (MA)* called for by the United Nations Secretary-General in 2000.[9] The *MA* assesses the consequences of ecosystem change for human well-being. They conceptualize the ecosystem as an input/output model. The relation between the input (water, sun,...) and output (ecosystem services) is exclusively technical. For that reason, the marginal value of water assigned to the ecosystem can be derived using a similar method as the one used for the municipality.

The ecosystem produces services that are denoted by S. The ecosystem services are produced using water, Q_t^e, and other inputs that are gathered together and are represented by E_t. The inputs are allocated in the ecosystem, in order to maximize the ecosystem services. We make the assumption that the utility function associated to the ecosystem services is increasing in S. As a result, the allocation of inputs that maximizes the ecosystem services, maximizes also the utility function. We define the value function associated to the ecosystem services by $\phi(Q_t^e, E_t)$. We make the assumption that the value function, $\phi(\cdot, \cdot)$, is concave in its arguments. In period t, the decision maker allocates A_t^e units of water to the ecosystem, i.e, $Q_t^e \leq A_t^e$. The marginal value of water for the ecosystem, λ_t^e, is defined as the marginal benefit of relaxing the constraint, $Q_t^e \leq A_t^e$. As for the firm and

[9]www.maweb.org

the municipality, we assume that for a given E_t, there exists a level of water that maximizes the ecosystem services without any constraint, which we denote by $\bar{Q}^e_t$. When the allocation of water to the ecosystem is larger than $\bar{Q}^e_t$, then the ecosystem uses $\bar{Q}^e_t$ and the marginal value of water is zero. If the allocation of water to the municipality is smaller than $\bar{Q}^e_t$, then the ecosystem uses A^e_t. In that case, the marginal value of water for the ecosystem is given by:

$$(12) \quad \lambda^e_t = \phi_Q(A^e_t, E_t), \quad t = 1, 2.$$

Therefore, for a given amount of water allocated to the ecosystem, A^e_t, the amount of water actually used by the municipality is the minimum between A^e_t and $\bar{Q}^e_t$, that is:

$$Q^e_t = \min\{A^e_t, \bar{Q}^e_t\}.$$

Given the amount of water used by the ecosystem, x, the marginal value of water for the ecosystem is defined by:

$$\lambda^e_t(x) = \begin{cases} 0 & \text{if} \quad x \geq \bar{Q}^e_t \\ \phi_Q(x, M_t) & \text{if} \quad x < \bar{Q}^e_t \end{cases}.$$

The concavity of $\phi(\cdot,\cdot)$ implies that the marginal value of water for the ecosystem services is decreasing in the amount of water used by the ecosystem.[10] The use value of water for the ecosystem in period t is given by:

$$(13) \quad \Phi_t = \int_0^{Q^e_t} \lambda^e_t(x)dx = \int_0^{Q^e_t} \phi_Q(x, E_t)dx,$$

with $Q^e_t = \min\{A^e_t, \bar{Q}^e_t\}$. The use value of water for the ecosystem for both periods is simply defined by the sum of Φ_1 and Φ_2:

$$(14) \quad \Phi = \sum_{t=1,2} \Phi_t.$$

The use value Φ is the monetary equivalent of the welfare from the ecosystem services.

[10]The curvature of the marginal value of water for the ecosystem services in period t depends on the third derivative of $\phi(\cdot,\cdot)$ with respect to the amount of water.

4 The optimal value of water

According to environmentalists, there exist investments and economic developments that, regardless of their sizes, lead to irreversible degradation and destruction of the natural ecosystem. While several examples of this type of investment are provided in Section 1, we now provide a formal structure to analyze the issue. More specifically, we build a model where the firms net investments create an irreversible damage to the ecosystem. In this setting, we characterize the optimal allocation of water among the different users and determine the optimal intervention of the decision maker given the damage irreversibility.

Following the discussion in Section 2, the model also encompasses how legislation affects the optimal allocation of water. In particular, we introduce a form of legislative irreversibility that is inspired by the Law of Water, which states that the current allocation of water to the firm binds its future allocation.

Our analysis proceeds in several steps. In Section 4.1, we study the simplest case, which is the case without any form of irreversibility. Then, in Sections 4.2 and 4.3, we analyze legislation irreversibility and damage irreversibility, respectively. At last, in Section 4.4, we analyze the combination of the two types of irreversibility.

Given an allocation of water among the different users, $\{A_t^f, A_t^m, A_t^e\}$, we define the welfare of water in period t by the sum of the value in use of water for each agent:

$$\mathcal{W}_t = \Pi_t + \Upsilon_t + \Phi_t, \quad t = 1, 2.$$

Combining the preceding definition with (8), (10), and (13) implies that:

$$\text{(15)} \quad \mathcal{W}_t = \int_0^{Q_t^f} \lambda_t^f(x)dx + \int_0^{Q_t^m} \lambda_t^m(x)dx + \int_0^{Q_t^e} \lambda_t^e(x)dx, \quad t = 1, 2,$$

with $Q_t^i = \min\{A_t^i, \bar{Q}_t^i\}$, $i = f, m, e$. The welfare of water for both periods is defined by the sum of $\mathcal{W}_1$ and $\mathcal{W}_2$:

$$\mathcal{W} = \sum_{t=1,2} \mathcal{W}_t.$$

Prior to this analysis, one remark has to be discussed. It concerns the composition of the welfare of water. The decision maker chooses the allocation of water among the different users by maximizing the welfare of water. Υ and Φ represent the use value associated with the use of water by the municipality and the ecosystem, respectively, whereas Π represents the producer use value of water, which doesn't take in consideration the use value of water for the consumers. Before going further into the analysis, let us provide some clarifications. In the consumption good market, the use value of water is the addition of the use value for the consumers and for the firm. Assuming that the consumers use value of water is independent of the production of the firm, the decision maker can consider only Π in its analysis of the allocation of water. This assumption can be satisfied in various contexts, e.g., when the consumption good is partly imported and the local firm is small relatively to the rest of the world. In that context, the production of the local firm has no impact on the total consumption and the price. As a result, the allocation of water among the different users has no impact on the consumer use value. Therefore the decision maker can consider only Π. Moreover, we make the assumption that there is no distortion in the labour market.

4.1 No irreversibility

We evaluate the marginal value of water and the optimal allocation of water among the firm, the municipality and the ecosystem without irreversibility. To do so, we maximize the welfare of water:

$$\begin{aligned}
(16) \qquad & \max_{\{A_t^f, A_t^m, A_t^e\}} \sum_{t=1,2} \left[\int_0^{Q_t^f} \lambda_t^f(x)dx + \int_0^{Q_t^m} \lambda_t^m(x)dx + \int_0^{Q_t^e} \lambda_t^e(x)dx \right] \\
& \text{s.t.} \quad Q_t^f + Q_t^m + Q_t^e \leq \Theta_t \quad t = 1,2 \\
& \qquad\quad Q_t^i = \min\{A_t^i, \bar{Q}_t^i\} \quad t = 1,2 \quad i = f, m, e.
\end{aligned}$$

It is worth noting that without irreversibility, at optimum, $A_t^i \leq \bar{Q}_t^i$ for $t = 1,2$ and $i = f, m, e$. Hence, (16) can be rewritten as:

$$\begin{aligned}
& \max_{\{A_t^f, A_t^m, A_t^e\}} \sum_{t=1,2} \left[\int_0^{A_t^f} \lambda_t^f(x)dx + \int_0^{A_t^m} \lambda_t^m(x)dx + \int_0^{A_t^e} \lambda_t^e(x)dx \right] \\
& \text{s.t.} \quad A_t^f + A_t^m + A_t^e \leq \Theta_t \quad t = 1,2.
\end{aligned}$$

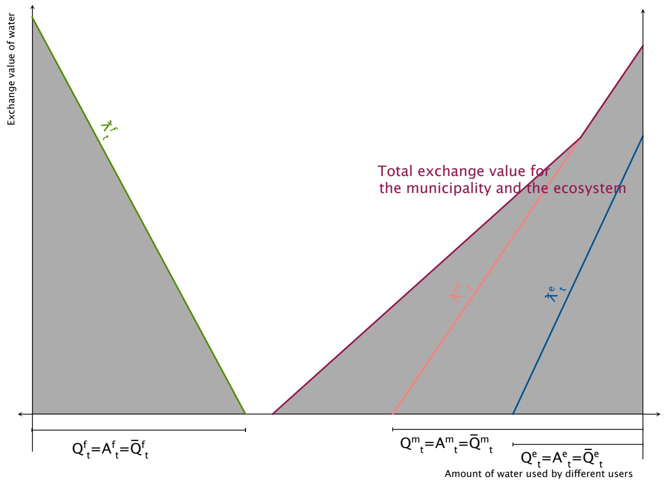


Figure 2: Optimal allocation of water without rivalry in use

The first order conditions and the Kuhn-tucker conditions are given by:

$$(A_t^i): \quad \lambda_t^i(A_t^i) - \mu_t = 0, \quad i = f, m, e, \quad t = 1, 2$$
$$\mu_t(\Theta_t - A_t^f - A_t^m - A_t^e) = 0, \quad \mu_t \geq 0, \quad \Theta_t - A_t^f - A_t^m - A_t^e \geq 0.$$

If in period t, $\bar{Q}_t^f + \bar{Q}_t^m + \bar{Q}_t^e \leq \Theta_t$, there is no rivalry in use. Hence, every user can use the amount of water he wants, consequently, the optimal allocation is characterized by $\{A_t^f, A_t^m, A_t^e\} = \{\bar{Q}_t^f, \bar{Q}_t^m, \bar{Q}_t^e\}$. Therefore, the marginal value of water at the optimal allocation is zero. It is important to note that it does not mean that water is worth nothing for the agents. The welfare of water is the area under the curve λ_t^f, λ_t^m, and λ_t^e. Graphic 2 illustrates this case with simple linear values of water.[11] The welfare of water is represented by the grey area of Figure 2.

If $\bar{Q}_t^f + \bar{Q}_t^m + \bar{Q}_t^e > \Theta_t$, there is rivalry in use, the optimal allocation, $\{A_t^f, A_t^m, A_t^e\}$, is characterized by the equalization of the marginal value of water for each agent:

$$\lambda_t^f(A_t^f) = \lambda_t^m(A_t^m) = \lambda_t^e(A_t^e) > 0.$$

[11] In Figures 2 and 3, the marginal value of water for the firm is larger than the marginal value of water for the municipality, which in turn is larger than the marginal value of water for the ecosystem. This figure illustrates one possible relation between the values of water, whereas in our analytical analysis, no assumption is made on this relationship.

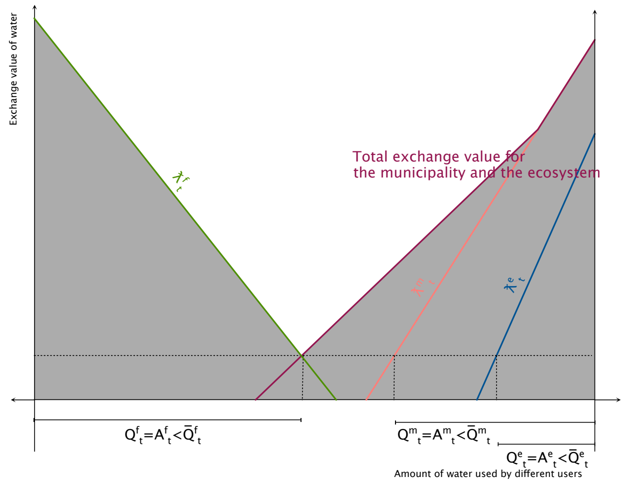


Figure 3: Optimal allocation of water with rivalry in use

As a result, the marginal value of water is strictly positive. Graphic 3 illustrates this situation with linear marginal values of water.[11]The total social value of water is represented by the grey area of Figure 3.

We denote by $\{A_t^{f,(16)}, A_t^{m,(16)}, A_t^{e,(16)}\}_{t=1,2}$, the optimal allocation without irreversibility and the amount of water used by the different users induced by this optimal allocation of water is denoted by $\{Q_t^{f,(16)}, Q_t^{m,(16)}, Q_t^{e,(16)}\}_{t=1,2}$.

The solution of this basic case, particularly when there is no rivalry in use, highlights the fact that increasing the allocation to one particular user, doesn't necessarily increase the social benefit. In the case of no rivalry in use, unused water shouldn't be seen as a limitless resource to be used in any way whatever.

One remark has to be made upon this case. This basic situation shows evidence that the optimal allocation among different users is based on the marginal value of water, not on the use value. And most importantly, even though the marginal value of water is null, agents value water.

Turning next to the legislative irreversibility, we show that even though the marginal value of water is zero in the first period, at the optimum the decision maker might want to limit the access to water.

4.2 Legislative irreversibility

The current amount of water allocated to the firm binds its future allocation. The *Law of Water* states that once the keeper of the water gives the right to a firm to use a given amount of water, the firm should have the right to use at least the same amount in the future.[12] In other words, the current legislation creates an irreversibility that should be taken into account at the optimal allocation. Once a water-taking permit is issued, the decision maker cannot change the authorization for a predetermined period of time.

The legislative irreversibility adds to the maximization of the welfare of water (16) the constraint $A_1^f \leq A_2^f$. Thus, the optimal allocation of water is given by the solution of the following maximization:

$$\begin{aligned}
(17) \quad & \max_{\{A_t^f, A_t^m, A_t^e\}_{t=1,2}} \sum_{t=1,2} \left[\int_0^{Q_t^f} \lambda_t^f(x)dx + \int_0^{Q_t^m} \lambda_t^m(x)dx + \int_0^{Q_t^e} \lambda_t^e(x)dx \right] \\
& \text{s.t.} \quad Q_t^f + Q_t^m + Q_t^e \leq \Theta_t, \quad t = 1,2 \\
& \qquad A_1^f \leq A_2^f \\
& \qquad Q_t^i = \min\{A_t^i, \bar{Q}_t^i\} \quad t = 1,2 \quad i = f, m, e.
\end{aligned}$$

The solution of (17) and the amount of water used by the different users induced by the solution of (17) are denoted respectively by:

$$\{A_t^{f(17)}, A_t^{m(17)}, A_t^{e(17)}\}_{t=1,2} \quad \text{and} \quad \{Q_t^{f(17)}, Q_t^{m(17)}, Q_t^{e(17)}\}_{t=1,2}.$$

It is worth noting that the amount of water allocated to the firm in period two is linked to its allocation in the first period. For that reason, it might be optimal to reduce the allocation of water to the firm in the first period in order to restrain its rights to use water in the second period. The decision maker reduces the amount of water allocated to the firm in period one as long as the welfare loss in that period is compensated by the welfare gain in period two. On the other hand, if the welfare loss in period one is relatively large compared to the gain in period two, it might be optimal to allocate to the firm more than $\bar{Q}_2^f$ units of water in period two, in order to let it uses more water in period one. The amount of water allocated to the firm in period two is the only one that might be larger than $\bar{Q}_2^f$ at the optimum. There is no benefit in allowing the municipality, the ecosystem

[12] The *Law of Water* states that the water-taking permits are valid for a period of 10 years.

and the firm in period one, to use more than their respective $\bar{Q}$. Therefore, $Q_t^i = A_t^i \leq \bar{Q}_t^i$, $t = 1, 2$ and $i = m, e$, and $Q_1^f = A_1^f \leq \bar{Q}_1^f$.

Maximization problem (17) is the constrained version of (16). If the solution of the unconstrained maximization problem (16) satisfies the inequality constraint $A_1^{f(16)} \leq A_2^{f(16)}$, then, $A_t^{i(17)} = A_t^{i(16)}$ for $i = f, m, e$ and $t = 1, 2$. We concentrate our attention to the other case, i.e., $A_1^{f(16)} > A_2^{f(16)}$.

First, we provide an example to highlight some important features generalized in Proposition 5. We consider an example with the marginal value of water in period one equals to zero, i.e., $Q_1^{i(16)} = \bar{Q}_1^i$ for $i = f, m, e$. If for any reason $A_1^{f(16)} > A_2^{f(16)}$, the legislative irreversibility implies that the allocation of water, $\{A_t^{f,(16)}, A_m^{f,(16)}, A_t^{e,(16)}\}_{t=1,2}$, is no longer achievable.[13] Indeed, with legislative irreversibility if the firm has the right to use $A_1^{f(16)}$ in period one, its allocation of water in period two cannot be limited to $A_2^{f(16)}$. With legislative irreversibility, the allocation of water to the firm is the same in both periods and lies between $A_2^{f(16)}$ and $A_1^{f(16)}$. As a result, even though the marginal value of water is zero, it might be optimal to restrict its usage. In this example, the marginal value of water for the firm in period one (two) is larger (smaller) with legislative irreversibility than without. If the decision maker believes that the marginal value of water would be larger in the future, the current marginal value of water becomes larger. Consequently, the marginal value of water for the firm in period one is not zero any more. Moreover, the marginal value of water is no longer equal among users within a period. Proposition 5 generalizes this example.

Proposition 5 *With legislative irreversibility, if $A_1^{f(16)} > A_2^{f(16)}$:*

- *at the optimum, the marginal value of water for the firm is different from the marginal value of water for the municipality and the ecosystem:*

$$\lambda_1^{f(17)} \neq \lambda_1^{m(17)} = \lambda_1^{e(17)} \quad \textit{and} \quad \lambda_2^{f(17)} \neq \lambda_2^{m(17)} = \lambda_2^{e(17)};$$

- *the marginal value of water for the firm is larger (smaller) in period one (two) and the marginal value of water for the municipality and for the ecosystem is smaller (larger) in*

[13]There are several reasons why $A_1^{f(16)}$ might be larger than $A_2^{f(16)}$, among them, there is the possibility that the marginal value of water in first period is larger than in period two *ceteris paribus*, or the possibility that $\Theta_1 > \Theta_2$ *ceteris paribus*

period one (two):

$$\lambda_1^{f(17)} \geq \lambda_1^{f(16)}, \quad \lambda_1^{m(17)} \leq \lambda_1^{m(16)}, \quad \textit{and} \quad \lambda_1^{e(17)} \leq \lambda_1^{e(16)},$$
$$\lambda_2^{f(17)} \leq \lambda_2^{f(16)}, \quad \lambda_2^{m(17)} \geq \lambda_2^{m(16)}, \quad \textit{and} \quad \lambda_2^{e(17)} \geq \lambda_2^{e(16)}.$$

Proof of Proposition 5

To prove Proposition 5, we proceed with the characterization of the optimal allocation of water, $\{A_t^{f(17)}, A_t^{m(17)}, A_t^{e(17)}\}_{t=1,2}$. Four different allocations characterize the solution of (17): (18), (19), (20), and (21). To refer to one specific allocation we use the notation $\{A_t^{fj}, A_t^{mj}, A_t^{ej}\}_{t=1,2}$ for $j = (18), (19), (20), (21)$. To refer to the amount of water used by the different users induced by this specific optimal allocation, we use the notation $\{Q_t^{fj}, Q_t^{mj}, Q_t^{ej}\}_{t=1,2}$ for $j = (18), (19), (20), (21)$.

If $A_2^{f(16)} = \bar{Q}_2^f < A_1^{f(16)}$, there is no benefit of lowering the amount of water allocated to the firm in period one. If the decision maker reduces the allocation of water in period one, then this reduction creates a loss of welfare in period one and doesn't create a gain in period two. Consequently, the optimal allocation of water is given by:

$$(18) \quad A_1^f = A_2^f = A_1^{f(16)} \quad \text{and} \quad A_t^i = A_t^{i(16)}, \quad t = 1,2 \quad i = m, e.$$

Given the optimal allocation of water, the amount of water used by the different users is given by:

$$Q_1^f = A_1^f, \quad Q_2^f = \bar{Q}_2^f, \quad \text{and} \quad Q_t^i = A_t^i, \quad t = 1,2 \quad i = m, e.$$

Even though, the firm has the right to use $A_1^f > \bar{Q}_2^f$ units of water in period two, it uses only $\bar{Q}_2^f$. In period two, the difference between the amount of water allocated to the firm and the amount of water used by the firm, $A_1^f - \bar{Q}_2^f$, is useless to the firm. As a consequence, the decision maker can allocate to the municipality and the ecosystem $\bar{Q}_2^m$ and $\bar{Q}_2^e$ unit of water, respectively.[14] It is worth noting that in that case, the legislative irreversibility affects neither the marginal value of water nor the welfare at the optimal allocation. One example of the context in which the conditions for this solution to be optimal hold is when in period one the marginal value of water for the firm is larger than in period two and when there is no rivalry in use in period two. Figure 4.a illustrates this example.[15]

[14]Note that even if $\sum_{i=f,m,e} A_2^i$ is larger than Θ_2, an allocation of water is achievable if $\sum_{i=f,m,e} Q_2^i \leq \Theta_2$.

[15]Without loss of generality, we represent the marginal value of water for the municipality and the ecosystem as a single curve in Figure 4.

If $A_2^{f(16)} < A_1^{f(16)} < \bar{Q}_2^f$, the solution is characterized by:

$$\begin{aligned}
(19)\quad & A_t^i < \bar{Q}_t^i, \quad t = 1,2 \quad \text{and} \quad i = f,m,e, \quad \text{with} \quad A_1^f = A_2^f, \\
& A_t^f + A_t^m + A_t^e = \Theta_t, \quad t = 1,2, \\
& \lambda_1^f(A_1^f) - \lambda_1^m(A_1^m) + \lambda_2^f(A_1^f) = \lambda_2^m(A_2^m) = \lambda_2^e(A_2^e), \\
& \lambda_1^f(A_1^f) + \lambda_2^f(A_1^f) - \lambda_2^m(A_2^m) = \lambda_1^m(A_1^m) = \lambda_1^e(A_1^e).
\end{aligned}$$

Therefore,

$$\lambda_1^f(A_1^f) + \lambda_2^f(A_1^f) = \lambda_1^m(A_1^m) + \lambda_2^m(A_2^m) = \lambda_1^e(A_1^e) + \lambda_2^e(A_2^e).$$

Given the optimal allocation of water, every user uses exactly the amount of water allocated to them, i.e., $Q_t^i = A_t^i$ for $t = 1,2$ and $i = f,m,e$. The first important remark about this solution is that the one period equality of the marginal value of water among the different users doesn't hold anymore. Moreover, at the allocation (19), the marginal value of water for the firm in period one (two) is larger (smaller) in comparison of the optimal allocation without legislative irreversibility. The inverse relationship holds for the municipality and the ecosystem. If the marginal value of water for the municipality or for the ecosystem is larger in period two than in period one and there is rivalry in use in both periods, the conditions for this cas to be the solution hold. Figure 4.b illustrates this example.

If $A_2^{f(16)} < \bar{Q}_2^f \leq A_1^{f(16)}$, it is not always optimal to reduce the allocation of water to the firm in period one. The welfare loss in period one caused by the limitation of the allocation of water to the firm might not be compensated by the welfare gain in period two. Before determining a condition under which it is optimal to limit the allocation of water to the firm in period one, we characterize the optimal allocation in both situations.

If the decision maker wants to reduce the allocation of water to the firm in period one, the optimal allocation is characterized by:

$$\begin{aligned}
(20)\quad & A_1^f \in [A_2^{f(16)}, \bar{Q}_2^f], \\
& A_2^f = A_1^f, \\
& A_t^m, A_t^e \quad \text{s.t.} \quad \lambda_t^m(A_t^m) = \lambda_t^e(A_t^e) \quad \text{and} \quad A_t^f + A_t^m + A_t^e = \Theta_t \quad t = 1,2.
\end{aligned}$$

(a) Optimal allocation characterized by (18) (b) Optimal allocation characterized by (19)

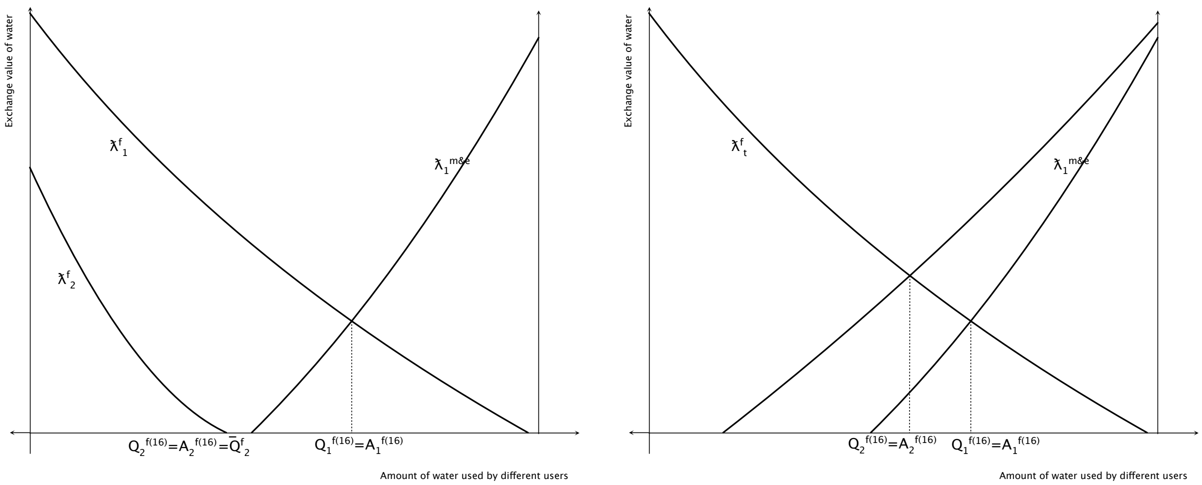


Figure 4: Legislative irreversibility

We refer to the welfare associated with the allocation (20) as $\mathcal{W}^{(20)}$:

$$\mathcal{W}^{(20)} = \int_0^{A_1^{f(20)}} \lambda_1^f(x)dx + \int_0^{A_1^{m(20)}} \lambda_1^m(x)dx + \int_0^{A_1^{e(20)}} \lambda_1^e(x)dx + \int_0^{A_1^{f(20)}} \lambda_2^f(x)dx + \int_0^{A_2^{m(20)}} \lambda_2^m(x)dx + \int_0^{A_2^{e(20)}} \lambda_2^e(x)dx.$$

If the decision maker doesn't limit the allocation of water to the firm in period one, the optimal allocation is characterized by:

$$\begin{aligned} (21) \quad & A_1^i = A_1^{i(16)}, \quad i = f, m, e \\ & A_2^f = A_1^f \\ & A_2^m, A_2^e \quad \text{s.t.} \quad \lambda_2^m(A_2^m) = \lambda_2^e(A_2^e) \quad \text{and} \quad \bar{Q}_2^f + A_2^m + A_2^e = \Theta_2. \end{aligned}$$

We refer to the welfare associated with the allocation (21) as $\mathcal{W}^{(21)}$:

$$\begin{aligned} \mathcal{W}^{(21)} &= \sum_{t=1,2} \sum_{i=f,m,e} \int_0^{A_2^{i(21)}} \lambda_t^i(x)dx \\ &= \int_0^{A_1^{f(16)}} \lambda_1^f(x)dx + \int_0^{A_1^{m(16)}} \lambda_1^m(x)dx + \int_0^{A_1^{e(16)}} \lambda_1^e(x)dx \\ &\quad + \int_0^{\bar{Q}_2^f} \lambda_2^f(x)dx + \int_0^{A_2^{m(21)}} \lambda_2^m(x)dx + \int_0^{A_2^{e(21)}} \lambda_2^e(x)dx. \end{aligned}$$

(a) Optimal allocation characterized by (20) (b) Optimal allocation characterized by (21)

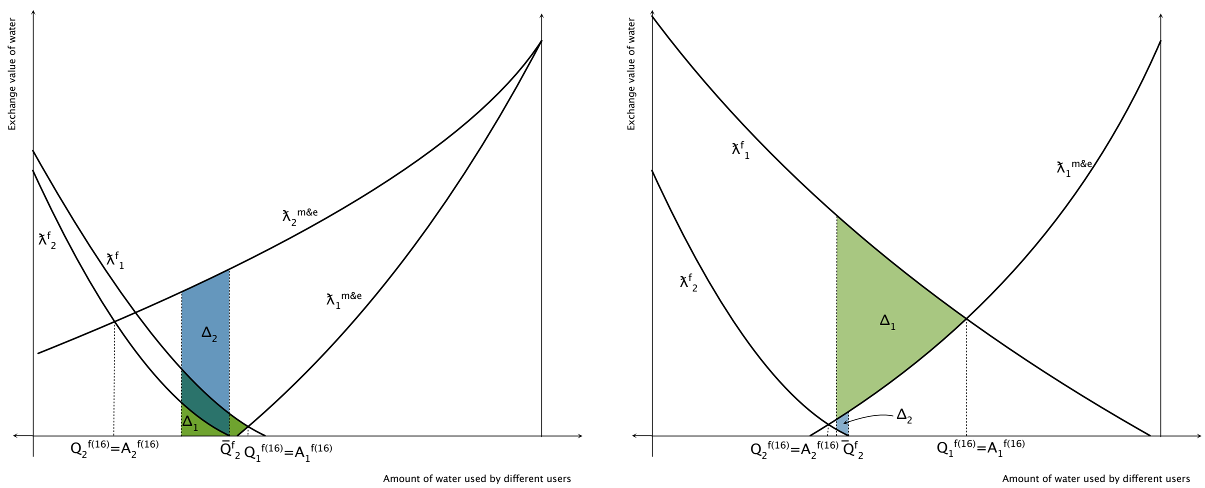


Figure 5: Legislative irreversibility

To determine whether it is optimal to limit the amount of water allocated to the firm in period one, we compare $\mathcal{W}^{(20)}$ to $\mathcal{W}^{(21)}$. It is optimal to reduce the allocation of water to the firm in period one if $\mathcal{W}^{(20)} \geq \mathcal{W}^{(21)}$, that is:

$$\mathcal{W}^{(20)} - \mathcal{W}^{(21)} = \underbrace{-\int_{A_1^{f(20)}}^{A_1^{f(16)}} \lambda_1^f(x)dx + \int_{A_1^{m(16)}}^{A_1^{m(20)}} \lambda_1^m(x)dx + \int_{A_1^{e(16)}}^{A_1^{e(20)}} \lambda_1^e(x)dx}_{\Delta_1<0}$$
$$\underbrace{-\int_{A_1^{f(20)}}^{\bar{Q}_2^f} \lambda_2^f(x)dx + \int_{A_2^{m(16)}}^{A_2^{m(20)}} \lambda_2^m(x)dx + \int_{A_2^{e(16))}}^{A_2^{e(20)}} \lambda_2^e(x)dx}_{\Delta_2>0} \geq 0.$$

The reduction of the allocation of water to the firm leads to a welfare loss in period one, which we denote by Δ_1, whereas, it also leads to a welfare gain in period two defined by Δ_2.

If there exists $A_1^f \in [A_2^{f(16)}, \bar{Q}_2^f]$, such that the welfare loss in period one is totally compensated by the gain in period two, the decision maker reduces the amount of water allocated to the firm in period one compared to the optimal allocation of the unconstraint maximisation problem (16). The optimal allocation is characterized by (20). The equality of the marginal value of water among the different users doesn't hold anymore. Moreover, at the allocation (20), the marginal value of water for the firm in period one (two) is larger (smaller) compared to the optimal allocation without legislative irreversibility

(16). The inverse relationship holds for the municipality and the ecosystem. Figure 5.a illustrates this case.[16] The welfare loss is represented by the blue and the blue turquoise area, whereas the gain is represented by the green and the blue turquoise area. This case might happen if the marginal value of water for the municipality and the firm is larger in period two than in period one and if the marginal value of water for the firm is slightly smaller in period two than in period one.

If for all $A_1^f \in [A_2^{f(16)}, \bar{Q}_2^f]$, the welfare loss in period one is larger than the gain in period two, it is not optimal to reduce the amount of water allocated to the firm in period one compared to the optimal allocation of the unconstrained maximisation problem (16). The optimal allocation is characterized by (21). The equality of the marginal value of water among the different users in period two doesn't hold any more, however, in period one the marginal value of water doesn't change. At the allocation (21), the marginal value of water for the firm in period two is smaller compared to the optimal allocation without legislative irreversibility. The inverse relationship holds for the municipality and the ecosystem. Figure 5.b represents this case. There does not exist $A_1^f \in [A_2^{f(16)}, \bar{Q}_2^f]$ such that the welfare loss in period one, represented by the green area, is lower than welfare gain in period two represented by the blue area. This case might happen if the marginal value of water for firm in period one is really larger than in period two. *Q.E.D.*

In the following sections, we analyze the impact of an irreversible damage to the ecosystem created by the net investment of the firm on the optimal allocation of water . In Section 4.3, we assume that there is no legislative irreversibility. This assumption is relaxed in Section 4.4.

4.3 Investment

We suppose that a positive net investment, $K_2 - (1-\delta)K_1 > 0$, creates an irreversible damage to the ecosystem. The damage is fixed, in other words, it depends neither on the level of the net investment nor on the level of production. We suppose that the damage is independent of the level of the net investment to represent investments that affect the ecosystem regardless of their size as discussed in Section 1. Some types of investment

[16] Without loss of generality, we represent the marginal value of water for the municipality and the ecosystem as a single curve in Figure 5.

affect in an irreversible way the ecosystem, as the project of new oil port in Cacouna. Regardless of the size of an oil port, once the habitat of beluga is affected, this species would be in danger. Damage that depends solely on the size of the investment would not model properly this reality.

As long as the investment is larger than the capital depreciation, δK_1, the ecosystem suffers the damage. The damage creates a lump sum reduction of welfare denoted by D. If $I > \delta K_1$, then regardless of the level of investment there is a damage. On the other hand, if $I \leq \delta K_1$, there is no damage. We define the damage function by:

$$\mathcal{D} = \begin{cases} D & \text{if} \quad I > \delta K_1 \\ 0 & \text{if} \quad I \leq \delta K_1 \end{cases} .$$

It is important to note that from the *Law of Water*, the manager of the water has no taxing power. To prevent the damage, his only way is to implement a change in the investment of the firm through a modification of its allocation of water. In that section, we characterize the range welfare losses created by the damage that induce a modification of the optimal allocation of water.

The decision maker chooses the allocation of water among the firm, the municipality and the ecosystem by maximizing the welfare:

$$(22) \quad \max_{\{A_t^f, A_t^m, A_t^e\}} \sum_{t=1,2} \left[\int_0^{Q_t^f} \lambda_t^f(x)dx + \int_0^{Q_t^m} \lambda_t^m(x)dx + \int_0^{Q_t^e} \lambda_t^e(x)dx \right] - \mathcal{D}$$

$$\text{s.t.} \quad Q_t^f + Q_t^m + Q_t^e \leq \Theta_t, \quad t = 1, 2$$

$$Q_t^i = \min \{A_t^i, \bar{Q}_t^i\} \quad t = 1, 2 \quad i = f, m, e.$$

The solution of (22) and the amount of water used by the different users induced by the solution of (22) are denoted respectively by:

$$\{A_t^{f(22)}, A_t^{m(22)}, A_t^{e(22)}\}_{t=1,2} \quad \text{and} \quad \{Q_t^{f(22)}, Q_t^{m(22)}, Q_t^{e(22)}\}_{t=1,2}.$$

The decision maker has to decide whether he intervenes or not. From (37), $dK_2/dA_2^f > 0$, hence, to prevent the damage the decision maker has to limit the allocation of water to the firm. The decision maker intervenes only if the welfare loss induced by the damage is larger than the welfare loss induced by his intervention.

If the decision maker doesn't intervene, the optimal allocation of water among users is characterized by $\{A_t^{f,(16)}, A_t^{m,(16)}, A_t^{e,(16)}\}_{t=1,2}$.

It is worth mentioning that if the investment induced by the allocation of water $A_2^{f(16)}$ is smaller than the depreciation of the capital, then there is no damage. In other words, if the net investment induced by the optimal allocation $A_2^{f(16)}$ is negative, the decision maker doesn't have to intervene to prevent to damage, because there is no damage. We restrict our attention to cases where the optimal allocation induces a positive level of net investment.

The allocation of water that leads to the prevention of the damage is noted by $\tilde{A}_2^f$ and is defined as the level of allocation of water that induces $I = \delta K_1$. Since the amount of water allocated to the firm has a positive impact on K_2:

$$\tilde{A}_2^f \leq A_2^{f(16)}.$$

To prevent the damage, the decision maker allocates $\tilde{A}_2^f$ units of water to the firm.

In order to characterize a set of welfare losses created by the net investment of the firm that induce the decision maker to intervene by reducing the allocation of water to the firm in period two, we determine the smallest of these losses. We define $\tilde{D}$ as welfare loss induced by the damage such that the decision maker is indifferent between intervening to prevent the damage and not. In order to evaluate $\tilde{D}$ in that situation, we should first analyze the optimal allocation of water if the decision maker intervenes to prevent the damage.

Proposition 6 *If the decision maker intervenes to prevent the damage,*

- *in period 1, the optimal allocation of water is characterized by* $\{A_1^{f(16)}, A_1^{m(16)}, A_1^{e(16)}\}$*;*
- *in period 2, the optimal allocation of water is characterized by:*

$$A_2^f = \tilde{A}_2^f, \quad A_2^m + A_2^e \leq \Theta_2 - \tilde{A}_2^f, \quad \textit{and} \quad \lambda_2^m(A_2^m) = \lambda_2^e(A_2^e).$$

Proof of Proposition 6

The prevention of the damage has no impact on the first period, therefore the solution of (16) is the optimal allocation.

In period 2, to prevent the damage the net investment should be equal to the depreciation of the capital. In order to induce this level of investment the allocation of water to the firm is set to $\tilde{A}_2^f$.

If $\tilde{A}_2^f + \bar{Q}_2^m + \bar{Q}_2^e \leq \Theta_t$, then $A_2^i = \bar{Q}_2^i$, $i = m, e$. Consequently, $A_2^m + A_2^e \leq \Theta_2 - \tilde{A}_2^f$.

If $\tilde{A}_2^f + \bar{Q}_2^m + \bar{Q}_2^e > \Theta_t$, then $A_2^i < \bar{Q}_2^i$, $i = m, e$ and $\lambda_2^m(A_2^m) = \lambda_2^e(A_2^e)$. Consequently, $A_2^m + A_2^e = \Theta_2 - \tilde{A}_2^f$. *Q.E.D.*

To avoid any confusion, we denote respectively by $\tilde{A}_2^m$ and $\tilde{A}_2^e$ the allocation of water to the municipality and to the ecosystem when the decision maker intervenes to prevent the damage.

Proposition 7 *If the decision maker intervenes to prevent the damage,*

- *in period 2, the marginal value of water for the firm is different from the marginal value of water for the municipality and the ecosystem;*
- *in period 2, the intervention leads to an increase of the marginal value of water for the firm and to a decrease of the marginal value of water for the municipality and the ecosystem.*

Proof of Proposition 7

The proof of Proposition 7 comes directly from the proof of Proposition 6.

If the decision maker doesn't intervene to prevent the damage, from the solution of (16), the marginal value of water among users are equal.

If the decision maker intervenes to prevent the damage, from Proposition 6, he reduces the amount of water allocated to the firm and increases the amount of water allocated to the municipality and the ecosystem. As a result, the marginal value of water for the firm increases and the marginal value of water for the municipality and the ecosystem falls. As a result, the marginal value of water among users are not equal anymore. *Q.E.D.*

In the following step, we evaluate the welfare when the decision maker doesn't intervene and when he intervenes to prevent the damage. If he doesn't intervene, the welfare is given by:

$$(23) \quad \mathcal{W} = \sum_{t=1,2} \sum_{i=f,m,e} \int_0^{Q_t^{i(16)}} \lambda_t^i(x)dx - D.$$

We refer to this welfare as $\mathcal{W}^{(23)}$. If the decision maker intervenes, the welfare is given by:

$$\text{(24)} \quad \mathcal{W} = \sum_{i=f,m,e} \int_0^{Q_1^{i(16)}} \lambda_1^i(x)dx + \sum_{i=f,m,e} \int_0^{\tilde{A}_2^i} \lambda_2^i(x)dx.$$

We refer to this welfare as $\mathcal{W}^{(24)}$. In that situation, $\tilde{D}$ is the level of damage such that $\mathcal{W}^{(23)} = \mathcal{W}^{(24)}$, that is:

$$\begin{aligned}
\tilde{D} &= \sum_{t=1,2} \sum_{i=f,m,e} \int_0^{Q_t^{i(16)}} \lambda_t^i(x)dx - \sum_{i=f,m,e} \int_0^{Q_1^{i(16)}} \lambda_1^i(x)dx - \sum_{i=f,m,e} \int_0^{\tilde{A}_2^i} \lambda_2^i(x)dx \\
&= \sum_{i=f,m,e} \int_0^{Q_2^{i(16)}} \lambda_2^i(x)dx - \sum_{i=f,m,e} \int_0^{\tilde{A}_2^i} \lambda_2^i(x)dx \\
&= \int_{\tilde{A}_2^f}^{Q_2^{f(16)}} \lambda_2^i(x)dx - \int_{Q_2^{m(16)}}^{\tilde{A}_2^m} \lambda_2^i(x)dx - \int_{Q_2^{e(16)}}^{\tilde{A}_2^e} \lambda_2^i(x)dx. \qquad (25)
\end{aligned}$$

If $D < \tilde{D}$, the decision maker doesn't intervene to prevent the damage, otherwise he intervenes to prevent the damage.

Parameters that affect the intervention

Every parameter that affects the marginal value of water for the firm, for the municipality or for the ecosystem in period two affects also $\tilde{D}$.

An increase of every parameter that has a positive impact on the marginal value of water for the municipality or the marginal value of water for the ecosystem reduces $\tilde{D}$. Consequently, the decision maker intervenes to prevent damages that create smaller welfare loss. Figure 6 illustrates an increase in the marginal value of water for the municipality.[17] The addition of the dark-blue and the pale-green areas represents $\tilde{D}$ prior to the increase of the marginal value of water for the municipality. The dark-blue area represents $\tilde{D}$ after the increase in the marginal value of water for the municipality.

The impact of a variation of the input prices on the marginal value of water for the firm has been calculated in Section 3.1.1. The variation of the marginal value of water for the firm that is induced by a variation of r, w_2, r^s, r_2^c, and p_2, is represented respectively

[17] An increase in the marginal value of water for the ecosystem can similarly be illustrated.

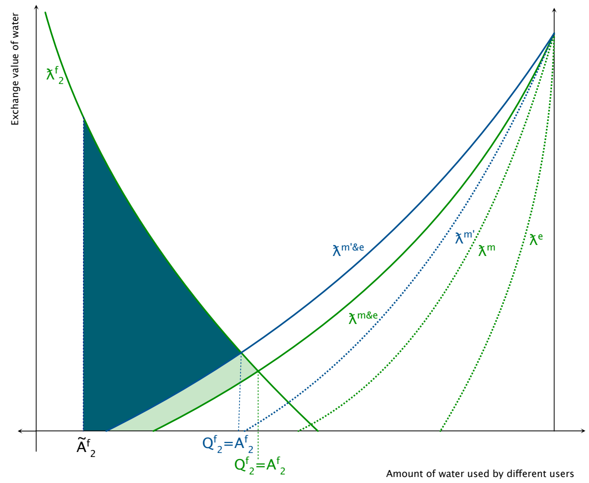


Figure 6: Impact of an increase of the value of water for the municipality

by (63), (64), (65), (66), and (67). The marginal value of water for the firm is negatively affected by r, w_2, r^s, and r_2^c, whereas p_2 has a positive impact on λ_2^f.

The level of capital that leads to the prevention of the damage, δK_1, is not affected by the variation of input prices. Does it imply that the allocation of water that induces the firm to choose the level of capital that leads the to prevention of the damage is not affected? The answer to that question is no. Proposition 8 establishes the impact on the intervention of the decision maker of a variation of r, w_2, r^s, r_2^c, and p_2.

Proposition 8 *The input prices affect negatively $\tilde{D}$, whereas p_2 has a positive impact on $\tilde{D}$:*

$$\frac{d\tilde{D}}{dr_2^c} < 0, \quad \frac{d\tilde{D}}{dr} < 0, \frac{d\tilde{D}}{dr^s} < 0, \frac{d\tilde{D}}{dw_2} < 0, \quad \textit{and} \quad \frac{d\tilde{D}}{dp_2} < 0.$$

Proof of Proposition 8

The price of the inputs that are perfect complements of water has no impact on K_2, L_2, and K_2^s. Consequently, the choice of A_2^f is not affected by the variation of r_2^c. However, the value of water for the firm is negatively affected. Figure 7.a illustrates those effects. If r_2^c increases, the range of welfare losses induced by the damage for which the decision maker intervenes increases as well.

From (46), (48), and (50), we know that r, w_2 and r^s affect negatively K_2. Therefore, an increase of r, w_2 or r^s induces the decision maker to choose a larger $\tilde{A}_2^f$. Moreover, from

(63), (64), and (65), the impact of these input prices on the value of water for the firm in period two is negative. Consequently, the range of welfare losses induced by the damage for which the decision maker intervenes increases. The decision maker intervenes for smaller losses of welfare induced by the damage. Figure 7.b illustrates these effects.

In Figures 7.a and 7.b, the addition of the dark-blue and the pale-green areas represents the minimum damage for which the decision maker intervenes prior to the increase of one of the input prices. The dark-blue area represents the minimum damage for which the decision maker intervenes to prevent the damage after the increase of one of the input prices.

From (60), the impact of p_2 on the capital, K_2, is positive. Therefore, an increase of p_2 induces the decision maker to reduce the amount of water allocated to the firm to prevent the damage, $\tilde{A}_2^f$. From (67), the marginal value of water for the firm is also affected positively by a change of p_2. Figure 7.c illustrates these effects. The range of losses of welfare induced by the damage for which the decision maker intervenes is reduced.

In Figure 7.c, the green area represents the minimum damage for which the decision maker intervenes prior to the increase of p_2. The dark-blue area represents the minimum damage for which the decision maker intervenes to prevent the damage after the increase of p_2. *Q.E.D.*

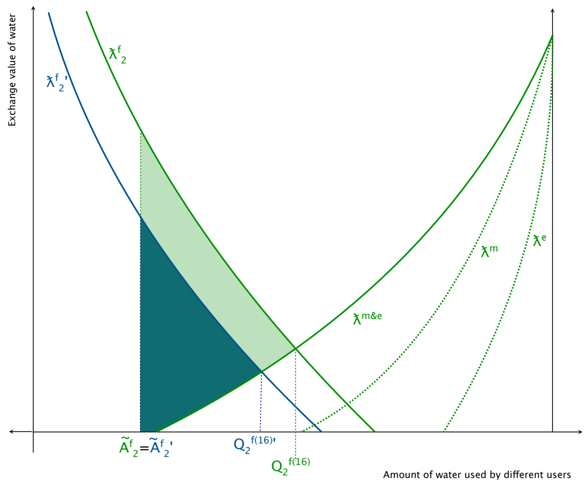
Impact of I^s

We must question ourselves on the impact of the investment that affects the productivity of water, I^s, on the decision maker's intervention. In order to prevent the damage, does the decision maker allocate more or less water to the firm when the firm can invest in K^s? Does the investment in K^s induce the decision maker to intervene more or less often to prevent the damage? To address these questions, we analyze the behaviour of the firm when this investment is so expensive that the firm doesn't invest in capital K^s, therefore $K_2^s = (1 - \delta^s)K_1^s$. Proposition 9 addresses these questions.

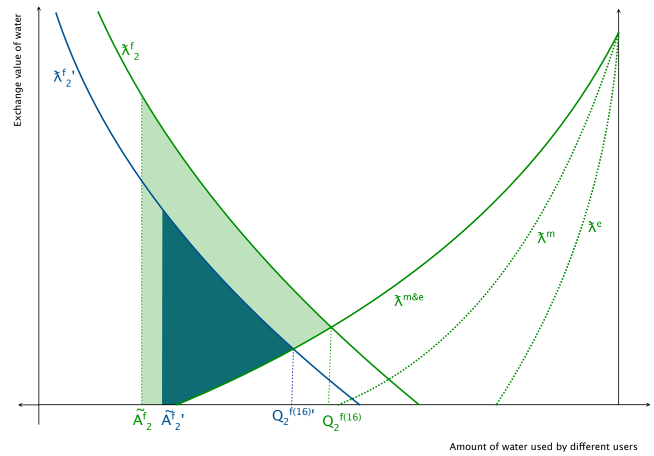
Proposition 9 *The investment in K^s affects positively $\tilde{D}$.*

Proof of Proposition 9

(a) Impact of r_2^c on $\tilde{D}$



(b) Impact of w_2, r, or r^s on $\tilde{D}$



(c) Impact of p_2 on $\tilde{D}$

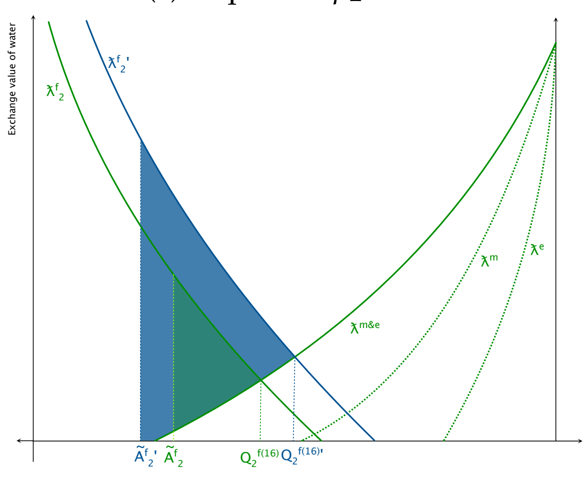


Figure 7: Impact r_2^c, r^s, w_2, r, and p_2 on $\tilde{D}$

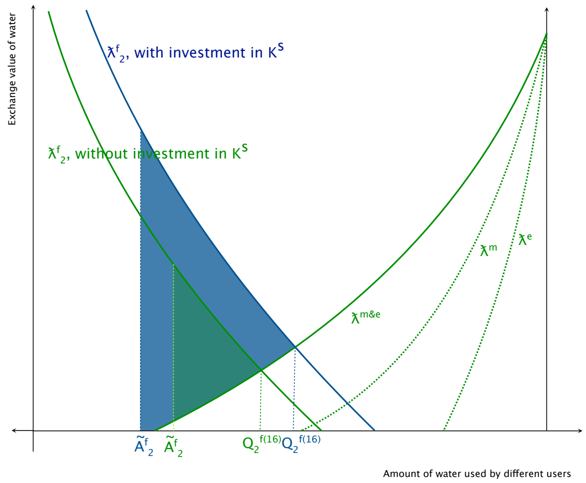


Figure 8: Impact of the presence of the investment in K^s on $\tilde{D}$

It is worth mentioning that the level of investment needed to prevent the damage, δK^1, is independent of the investment in capital K^s. There exists a unique z_2 such that $I = \delta K^1$. Because two inputs compose z_2, Q_2 and K_2^s, the investment in K^s induces the decision maker to allocate a smaller amount of water to the firm.

If the firm doesn't invest in the capital that affects the productivity of water, then each unit of water is less productive. Consequently, the marginal value of water is negatively affected. From (62), the impact of K^s on λ_2^f is positive, therefore if the there is no investment in K^s, then the marginal value of water is smaller. Consequently, $\tilde{D}$ is positively affected by the presence of K^s. *Q.E.D.*

In Figure 8, the green (blue) area represents $\tilde{D}$ without (with) the investment in K^s. The presence of the investment in K^s induces the decision maker to intervene only for larger welfare losses created by the damage. In other words, the range of losses of welfare induced by the damage for which the decision maker intervenes is reduced.

4.4 Investment and legislative irreversibility

How does the presence of the legislative irreversibility introduced in Section 4.2 affect the intervention of the decision maker to prevent a fixed irreversible damage caused by the net investment of the firm that creates a lump sum reduction of welfare as in Section 4.3? To address this question, we determine the impact of the legislative irreversibility on the level of loss created by the damage that induces the decision maker to be indifferent between intervening or not, which is $\tilde{D}$.

First, we specify the maximization of the welfare taking into account the fixed damage function, $\mathcal{D}$, and the condition that the allocation of water to the firm in period 1 is the lower bound of by its allocation in period 2.

The decision maker chooses the allocation of water among the firm, the municipality, and the ecosystem by maximizing the welfare:

$$(26) \quad \max_{\{A_t^f, A_t^m, A_t^e\}} \sum_{t=1,2} \left[\int_0^{Q_t^f} \lambda_t^f(x)dx + \int_0^{Q_t^m} \lambda_t^m(x)dx + \int_0^{Q_t^e} \lambda_t^e(x)dx \right] - \mathcal{D}$$
$$\text{s.t.} \quad A_1^f < A_2^f$$
$$Q_t^f + Q_t^m + Q_t^e \leq \Theta_t$$
$$Q_t^i = \min\{A_t^i, \bar{Q}_t^i\} \quad t = 1,2 \quad i = f, m, e.$$

The solution of (26) and the amount of water used by the different users induced by the solution of (26) are denoted respectively by:

$$\{A_t^{f(26)}, A_t^{m(26)}, A_t^{e(26)}\}_{t=1,2} \quad \text{and} \quad \{Q_t^{f(26)}, Q_t^{m(26)}, Q_t^{e(26)}\}_{t=1,2}.$$

As in Section 4.3, the decision maker has to decide whether he intervenes to prevent the damage or not. He intervenes if the welfare loss induced by the damage is larger than the welfare loss induced by his intervention. Moreover, as in Section 4.2, the decision maker has to decide whether he limits the allocation of water in period one or not. By contrast with Section 4.3, with the legislative irreversibility, the intervention of the decision maker might affect the optimal allocation of water in the first period.

Before determining the impact of the legislative irreversibility on $\tilde{D}$, we characterize the solution of (26).

If the decision maker doesn't intervene, the allocation of water among users is characterized by the optimal allocation of (17), i.e., $\{A_t^{f(17)}, A_t^{m(17)}, A_t^{e(17)}\}_{t=1,2}$.

The presence of the legal irreversibility doesn't affect the amount of water the decision maker has to allocate to the firm to prevent the damage. In Section 4.3, this amount of water has been defined as $\tilde{A}_2$, such that $I = \delta K_1$. As in Section 4.3 we restrict our attention to cases where the optimal allocation induces a positive level of net investment, with legislative irreversibility the condition becomes $A_t^{f(17)} > \tilde{A}_2$. Proposition 10 specifies the optimal allocation of water among users if the damage is prevented.

Proposition 10 *If the decision maker intervenes to prevent the damage, the solution of (26) is characterized by:*

- *if $A_1^{f(16)} \leq \tilde{A}_2^f$, the optimal allocation is characterized by the solution (22);*
- *if $A_1^{f(16)} > \tilde{A}_2^f$, the optimal allocation is characterized by:*

$$(27) \quad A_1^f = A_2^f = \tilde{A}_2^f,$$
$$A_t^m, A_t^e \quad s.t. \quad \lambda_t^m(A_t^m) = \lambda_t^e(A_t^e) \quad and \quad \tilde{A}_2^f + A_t^m + A_t^e = \Theta_t \quad t = 1, 2.$$

Proof of Proposition 10

If $A_1^{f(16)} \leq \tilde{A}_2^f$, the optimal allocation with and without legal irreversibility are identical. If the decision maker wants to prevent the damage, he doesn't have to restrict the amount of water allocated to the firm in period 1. In that case, the solution of (26) is given by the solution of (22).

If $A_1^{f(16)} > \tilde{A}_2^f$, to prevent the damage, the net investment should be equal to the depreciation of the capital. In order to induce this level of investment the allocation of water to the firm in period 2 is set to $\tilde{A}_2^f$. Due to the legislative irreversibility, the amount of water allocated to the firm in period 1 should be limited to $\tilde{A}_2^f$ as well.

If $\tilde{A}_t^f + \bar{Q}_t^m + \bar{Q}_t^e \leq \Theta_t$, then $A_t^i = \bar{Q}_t^i$, $i = m, e$ and $t = 1, 2$. Consequently, $A_t^m + A_t^e \leq \Theta_2 - \tilde{A}_2^f$.

If $\tilde{A}_t^f + \bar{Q}_t^m + \bar{Q}_t^e < \Theta_t$, then $A_t^i < \bar{Q}_t^i$, $i = m, e$ and $t = 1, 2$, and $\lambda_t^m(A_t^m) = \lambda_t^e(A_t^e)$. Consequently, $A_t^m + A_t^e = \Theta_2 - \tilde{A}_2^f$. To avoid any confusion, we denote by $\tilde{A}_1^m$ and $\tilde{A}_1^e$ the allocation of water to the municipality and the ecosystem when the decision maker intervenes to prevent the damage. *Q.E.D.*

Proposition 7 establishes that without legislative irreversibility, if the decision maker intervenes to prevent the damage, the marginal value of water among users are not equalized in period 2. The intervention leads to an increase (decrease) in the marginal value of water for the firm (the municipality and the ecosystem). In period 1, the intervention has no effect on the optimal allocation and hence, on the marginal value of water. Proposition 11 specifies that the legislative irreversibility has no impact on the effect of the intervention on the marginal value of water in period 2, by contrast with the first period.

Proposition 11 *If the decision maker intervenes to prevent the damage,*

- *the legislative irreversibility has no impact on the effect of the intervention on the marginal value of water in period 2;*
- *in period 1, if $A_1^{f(16)} > \tilde{A}_2^f$, the intervention leads to an increase in the marginal value of water for the firm and to a decrease of the marginal value of water for the municipality and the ecosystem. Consequently, the marginal value of water for the firm is different from the marginal value of water for the municipality and the ecosystem;*
- *in period 1, if $A_1^{f(16)} \leq \tilde{A}_2^f$, the intervention has no impact on the marginal value of water.*

Proof of Proposition 11

Suppose that the decision maker decides to intervene the prevent the damage.

Period 2

The allocation of water among users is the same regardless the presence of legislative irreversibility. Therefore, the presence of legislative irreversibility has no impact on the effect of the intervention on the value of water.

Period 1

If $A_1^{f(16)} \leq \tilde{A}_2^f$, the allocation of water is characterized by the solution of (22). From Proposition 7, the intervention has no impact.

If $A_1^{f(16)} > \tilde{A}_2^f$, from Proposition 10, the amount of water allocated to the firm (the municipality and the ecosystem) with the legislative irreversibility is lower (larger) than without. Therefore, the marginal value of water for the firm (the municipality and the ecosystem) increases (decreases). Since, without legislative the marginal value of water is equal among users, with legislative irreversibility, the equality does hold. *Q.E.D.*

The following proposition establishes the effect of legislative irreversibility on the welfare loss created by the damage for which the decision maker is indifferent between intervening and not intervening.

Proposition 12 *The effect on $\tilde{D}$ of legislative irreversibility is:*

- *null if $A_1^{f(16)} \leq \tilde{A}_2^f$;*
- *positive if $A_2^{f(16)} = \bar{Q}_2^f < A_1^{f(16)}$, or if $\tilde{A}_2^f < A_1^{f(16)} \leq A_2^{f(16)}$;*
- *ambiguous if $A_2^{f(16)} < \bar{Q}_2^f \leq A_1^{f(16)}$.*

Proof of Proposition 12

If the decision maker intervenes to prevent the damage, the legislative irreversibility is an issue and the welfare is given by:

$$(28) \quad \mathcal{W} = \sum_{t=1,2} \sum_{i=f,m,e} \int_0^{\tilde{A}_t^i} \lambda_t^i(x)dx.$$

We refer to this welfare as $\mathcal{W}^{(28)}$. Since $\tilde{A}_1^i > Q_1^{i(16)}$ $i = m, e$ and $\tilde{A}_2^f < Q_1^{f(16)}$, $\mathcal{W}^{(28)}$ is smaller than $\mathcal{W}^{(24)}$, which is the welfare when the decision maker intervenes to prevent the damage without legislative irreversibility. In other words, when the decision maker intervenes to prevent the damage, the welfare is always smaller with legislative irreversibility than without.

Case $A_1^{f(16)} \leq \tilde{A}_2^f$

From Proposition 10, the optimal allocation of (26) is characterized by the optimal allocation of (22). Therefore, legislative irreversibility is not an issue and it has no impact on $\tilde{D}$.

Case $\tilde{A}_2^f < A_1^{f(16)} \leq A_2^{f(16)}$

If the decision maker doesn't intervene to prevent the damage, the legislative irreversibility is not an issue and the welfare is given by $\mathcal{W}^{(23)}$.

The combination of the preceding remark and the fact that $\mathcal{W}^{(28)}$ is smaller than $\mathcal{W}^{(24)}$, implies that legislative irreversibility has a positive effect on $\tilde{D}$.

Case $A_2^{f(16)} = \bar{Q}_2^f < A_1^{f(16)}$

If the decision maker intervenes to prevent the damage, the welfare is given by $\mathcal{W}^{(28)}$ which is smaller than $\mathcal{W}^{(24)}$.

If the decision maker doesn't intervene to prevent the damage, the allocation of water among users is the same regardless the presence of legislative irreversibility and the welfare is given by $\mathcal{W}^{(23)}$.

Consequently, legislative irreversibility leads to an increase of $\tilde{D}$.

Case $A_2^{f(16)} < \bar{Q}_2^f \leq A_1^{f(16)}$

If the decision maker doesn't intervene to prevent the damage, the optimal allocation is characterized by either (19), (20), or (21) and the welfare is characterized by:

$$(29) \quad W = \sum_{t=1,2} \sum_{i=f,m,e} \int_0^{A_t^{ij}} \lambda_t^f(x)dx - D, \quad j = (19), (20), (21).$$

We refer to this welfare as $\mathcal{W}^{(29)}$. We evaluate the level of welfare loss that induces the decision maker to be independent between intervening and not, that is $\tilde{D}$. In that situation, $\tilde{D}$ is the welfare loss created by the damage that induces the equality between $\mathcal{W}^{(29)}$ and $\mathcal{W}^{(28)}$, that is:

$$(30) \quad \tilde{D} = \sum_{i=f,m,e} \int_{\tilde{A}_1^i}^{A_1^{ij}} \lambda_1^f(x)dx + \sum_{i=f,m,e} \int_{\tilde{A}_2^i}^{A_2^{ij}} \lambda_t^f(x)dx, \quad j = (19), (20), (21).$$

The first argument of (30) represents the welfare loss in period 1 caused by the intervention of the decision maker to prevent the damage in the presence of legislative irreversibility. It is worth noting that without legislative irreversibility, there is no such loss in the first period. The second argument of (30) summarizes two opposite effects of the intervention on the welfare in period 2. First, the reduction of the allocation of water to the firm from $\tilde{A}_2^f$ to $A_2^{f(16)}$ has a positive impact on the welfare, whereas, the reduction of the allocation of water to the firm from $A_2^{f(16)}$ to A^{fj}, j=(19), (20), (21), affects negatively the welfare.

In the following, we prove that the effect of legislative irreversibility on $\tilde{D}$ is ambiguous. In order to do so, we compare (25) to (30). The difference between (25) and (30) is defined as $\Delta_{\tilde{D}}$:

$$\Delta_{\tilde{D}} = -\sum_{i=f,m,e} \int_{\tilde{A}_1^i}^{A_1^{ij}} \lambda_1^f(x)dx - \sum_{i=f,m,e} \int_{A_2^{i(16)}}^{A_2^{ij}} \lambda_t^f(x)dx, \quad j = (19), (20), (21).$$

Since for j=(19), (20), (21),

$$A_1^{f(j)} \leq A^{f(16)}, \quad A_1^{m(j)} \geq A_1^{m(16)}, \quad A_1^{e(j)} \geq A_1^{e(16)},$$
$$A_2^{f(j)} > A_2^{f(16)}, \quad A_2^{m(j)} < A_2^{m(16)}, \quad \text{and} \quad A_2^{e(j)} < A_2^{e(16)},$$

the sign of $\Delta_{\tilde{D}}$ is undetermined. If $\Delta_{\tilde{D}} > 0$, the legislative irreversibility has a positive effect on $\tilde{D}$, and negative otherwise. *Q.E.D.*

We have proved that the presence of legislative irreversibility has an ambiguous effect on the range of loss created by the damage for which the decision maker intervene.

5 Conclusion

We develop a two-period analytical model to analyze the value and the optimal allocation of water in the context of sharing water from a lake among different users in the presence of irreversibility.

The optimal allocation in the benchmark model without irreversibility is analyzed first. If there is neither legislative irreversibility nor irreversible damage, then the marginal value is equal across users and might be zero.

The introduction of legislative irreversibility to the benchmark model implies that it is sometimes optimal to reduce the amount of water allocated to the firm, even though there is unused water in the lake. The amount of water allocated to the firm in the first period falls, whereas in period 2 the allocation of water to the firm raises. The inverse relationship holds for the municipality and the ecosystem. At the optimal allocation of water, the marginal value is not equal across users.

Our analysis establishes that it is not always optimal to prevent an irreversible damage created by an investment that generates a lump sum welfare loss. We first analyze the optimal allocation of water when investment creates a damage without legislative irreversibility. To prevent the damage, in period 2, the allocation of water to the firm falls.

Consequently, the equalization of marginal value among users doesn't hold anymore. The damage has no impact on the value and the optimal allocation of water in period 1.

At last, to approach the reality, we introduce legislative irreversibility in the analysis of the optimal allocation of water when an investment generates a damage to the ecosystem. Our work suggests that the legislative irreversibility has an ambiguous impact on the intervention to prevent the damage. Contrary to the case without legislative irreversibility, when the damage is prevented, the amount of water allocated to the firm falls in both periods. Consequently, the equalization of the marginal value between users doesn't hold.

Finally, we see several extensions of the model that would further our understanding of the optimal allocation of water among different users. First, it would be worthwhile to extend the model to include uncertainty about the total amount of water in the lake. This uncertainty would add realism into the model. In the current analysis, we consider investments that affect the ecosystem regardless of their size, to illustrate that the magnitude of investments is not always a good indicator of their impact on the ecosystem. Some projects create irreversible damages to the ecosystem that are independent of their size. Nevertheless, it would be interesting to consider the possibility that investments might also create variable damages. In order to do so, we would need to generalize the damage function, by adding to the lump sum welfare loss created by the damage a variable component. It would be interesting to address those possibilities in future research.

Appendix

Proof of Proposition 1

To determine the impact of A_2^f on L_2, K_2, and K_2^s, we first study the impact of z_2 on the optimal choice of K_2 and L_2. In order to do so, we totally differentiate the first order conditions (4) et (5) with respect to K_2, L_2, and z_2:

$$(31)\quad \underbrace{\begin{pmatrix} p_2 G_{yy}(F_K)^2 + p_2 G_y F_{KK} & p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} \\ p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & p_2 G_{yy}(F_L)^2 + p_2 G_y F_{LL} \end{pmatrix}}_{\Delta_{KL}} \begin{pmatrix} dK \\ dL \end{pmatrix} = \begin{pmatrix} -\, p_2 G_{yz} F_K dz \\ -\, p_2 G_{yz} F_L dz \end{pmatrix}.$$

The concavity assumption of G and F ensures that $|\Delta_{KL}|$ is strictly positive:

$$\begin{aligned} |\Delta_{KL}| =& p_2^2 \Big[\left(G_{yy}\right)^2 \left((F_L)^2 (F_K)^2 - (F_L F_K)^2 \right) \\ & + G_{yy} G_y \left((F_L)^2 F_{KK} + (F_K)^2 F_{LL} - 2 F_L F_K F_{LK} \right) \\ & + \left(G_y\right)^2 \left(F_{LL} F_{KK} - (F_{LK})^2 \right) \Big] > 0. \end{aligned}$$

It is worth mentioning that the concavity of F also implies that Σ_F is negative:

$$\Sigma_F = (F_L)^2 F_{KK} + (F_K)^2 F_{LL} - 2 F_L F_K F_{LK} = \begin{pmatrix} -F_L & F_K \end{pmatrix} \begin{pmatrix} F_{KK} & F_{LK} \\ F_{LK} & F_{LL} \end{pmatrix} \begin{pmatrix} -F_L \\ F_K \end{pmatrix} < 0.$$

From the Cramer rule:

$$(32)\quad \frac{dK_2}{dz_2} = \frac{-1}{|\Delta_{KL}|} \begin{vmatrix} p_2 G_{yz} F_K & p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} \\ p_2 G_{yz} F_L & p_2 G_{yy}(F_L)^2 + p_2 G_y F_{LL} \end{vmatrix} = \frac{-p_2^2 G_{yz} G_y \left(F_K F_{LL} - F_L F_{LK} \right)}{|\Delta_{KL}|}.$$

The assumption of homogeneity of degree 1 of F implies that F_L is homogeneous of degree 0 and F_{LK} is strictly positive:

$$\begin{aligned} & F_L L + F_K K = F \quad \Rightarrow F_K = \frac{F}{K} - F_L \frac{L}{K}, \\ & F_{LL} L + F_{LK} K = 0 \quad \Rightarrow F_{LK} K = -F_{LL} L > 0. \end{aligned}$$

Therefore,

$$\begin{aligned} F_K F_{LL} - F_L F_{LK} &= (F/K) F_{LL} - F_L (F_{LL} L / K + F_{LK}) \\ &= (F/K) F_{LL} - \frac{F_L}{K} (F_{LL} L + F_{LK} K) = \frac{F F_{LL}}{K} < 0. \end{aligned}$$

Consequently, under the assumption of homogeneity of degree 1 and the assumption that the cross derivative of G is positive, $G_{yz} > 0$, an increase of z_2 leads to an increase of K_2, i.e., $dK_2/dz_2 > 0$. Correspondingly,

$$\frac{dL_2}{dz_2} = \frac{-1}{|\Delta_{LK}|}\begin{vmatrix} p_2G_{yy}(F_K)^2 + p_2G_yF_{KK} & p_2G_{yz}F_K \\ p_2G_{yy}F_KF_L + p_2G_yF_{LK} & p_2G_{yz}F_L \end{vmatrix}$$

$$= \frac{-p_2^2G_{yz}G_y\left(F_LF_{KK} - F_KF_{LK}\right)}{|\Delta_{LK}|} > 0. \tag{33}$$

We can combine these two results and determine the impact of a variation of z on the optimal level of y that is chosen by the firm:

$$\frac{dy_2}{dz_2} = F_K\frac{dK_2}{dz_2} + F_L\frac{dL_2}{dz_2} = -\frac{p_2^2G_yG_{yz}\Sigma_F}{|\Delta_{LK}|} > 0.$$

Let us proceed to the analysis of the impact of a variation of y_2 on K_2^s. The total derivative of (6) implies that:

$$\frac{dK_2^s}{dy_2} = -\frac{G_{yz}H_{K^s}}{G_{zz}\left(H_{K^s}\right)^2 + G_zH_{K^sK^s}}.$$

The concavity of G and H, and the assumption that $G_{yz} > 0$ ensure that $dK_2^s/dy_2 > 0$.

Before proceeding with the evaluation of the impact of a variation of the amount of water allocated to the firm, A_2^f, on the optimal choice of K_2, L_2 et K_2^s, we define by Ω_2 the Hessian matrix associated with the first order conditions (4), (5), and (6):

$$\Omega_2 = \begin{bmatrix} p_2G_{yy}\left(F_K\right)^2 + p_2G_yF_{KK} & p_2G_{yy}F_LF_K + p_2G_yF_{LK} & p_2G_{yz}F_KH_{K^s} \\ p_2G_{yy}F_LF_K + p_2G_yF_{LK} & p_2G_{yy}\left(F_L\right)^2 + p_2G_yF_{LL} & p_2G_{yz}F_LH_{K^s} \\ p_2G_{yz}F_KH_{K^s} & p_2G_{yz}F_LH_{K^s} & p_2G_{zz}\left(H_{K^s}\right)^2 + p_2G_zH_{K^sK^s} \end{bmatrix}.$$

The determinant of Ω_2 is strictly negative:

$$|\Omega_2| = p_2G_zH_{K^sK^s}|\Delta_{KL}| + p_2^3\left(G_y\right)^2G_{zz}\left(H_{K^s}\right)^2\left(F_{LL}F_{KK} - \left(F_{LK}\right)^2\right)$$
$$+ P_2^3G_y\left(H_{K^s}\right)^2\left(G_{zz}G_{yy} - \left(G_{yz}\right)^2\right)\Sigma_F < 0.$$

The impact of a variation of the amount of water allocated to the firm on K_2^s is determined using the Cramer rule:

$$\frac{dK_2^s}{dA_2^f} = -\frac{|\Omega_{A_2^fK_2^s}|}{|\Omega_2|}, \tag{34}$$

with,

$$\Omega_{A_2^f K_2^s} = \begin{bmatrix} p_2 G_{yy} (F_K)^2 + p_2 G_y F_{KK} & p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & p_2 G_{yz} F_K H_Q \\ p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & p_2 G_{yy} (F_L)^2 + p_2 G_y F_{LL} & p_2 G_{yz} F_L H_Q \\ p_2 G_{yz} F_K H_{K^s} & p_2 G_{yz} F_L H_{K^s} & p_2 G_{zz} H_{K^s} H_Q + p_2 G_z H_{QK^s} \end{bmatrix}.$$

The sign of the determinant of $\Omega_{A_2^f K_2^s}$ is ambiguous:

$$\begin{aligned} |\Omega_{A_2^f K_2^s}| =& p_2 G_z |\Delta_{LK}| H_{QK^s} + p_2^3 G_{zz} H_Q \left(F_{LL} F_{KK} - (F_{LK})^2 \right) \\ & + p_2^3 G_y H_Q H_{K^s} \left(G_{zz} G_{yy} - (G_y z)^2 \right) \Sigma_F. \end{aligned}$$

Therefore, the impact of the amount of water allocated to the firm on the optimal level of K_2^s is ambiguous. For the upcoming analysis, it is useful to rewrite the determinant of $\Omega_{A_2^f K^s}$ by:

$$|\Omega_{A_2^f K_2^s}| = |\Omega| \frac{H_Q}{H_{K^s}} + p_2 G_z |\Delta_{LK}| \left(\frac{H_{QK^s}}{H_Q} - \frac{H_{K^s K^s}}{H_{K^s}} \right) H_Q.$$

Therefore,

$$\frac{dK_2^s}{dA_2^f} = -\frac{H_Q}{H_{K^s}} - \frac{p_2 G_z |\Delta_{LK}| (H_{QK^s} H_{K^s} - H_{K^s K^s} H_Q)}{H_{K^s} |\Omega_2|}. \tag{35}$$

Using (35), we can evaluate the variation of z that follows the variation of water allocated to the firm:

$$\begin{aligned} \frac{dz_2}{dA_2^f} =& H_Q \frac{dQ_2^f}{dA_2^f} + H_{K^s} \frac{dK_2^s}{dA_2^f} \\ =& - \frac{p_2 G_z |\Delta_{LK}| (H_{QK^s} H_{K^s} - H_{K^s K^s} H_Q)}{|\Omega_2|} > 0. \end{aligned} \tag{36}$$

The Assumption A6 ensures that an increase of water used by the firm induces an increase of services produced by that water.

The impact of a variation of the amount of water allocated to the firms on K_2 and L_2 is determined by the combination of (32), (33), and (36):

$$\begin{aligned} \frac{dK_2}{dA_2^f} &= \frac{dK_2}{dz_2} \frac{dz_2}{dA_2^f} = \frac{p_2^3 G_y G_z G_{yz} (F_{LL} F_K - F_{LK} F_L) \left(H_{QK^s} K_{K^s} - H_{K^s K^s} H_Q \right)}{|\Omega_2|}, \\ \frac{dL_2}{dA_2^f} &= \frac{dK_2}{dz_2} \frac{dz_2}{dA_2^f} = \frac{p_2^3 G_y G_z G_{yz} (F_L F_{KK} - F_{LK} F_K) \left(H_{QK^s} K_{K^s} - H_{K^s K^s} H_Q \right)}{|\Omega_2|}, \end{aligned} \tag{37}$$

which is positive under the assumptions taken throughout the paper. *Q.E.D.*

Proof of Proposition 2

First, we determine the impact of r on K_2^s using the Cramer rule:

$$(38)\quad \frac{dK_2^s}{dr} = \frac{|\Omega_{rK_2^s}|}{|\Omega_2|},$$

with,

$$\Omega_{rK_2^s} = \begin{bmatrix} p_2 G_{yy} (F_K)^2 + p_2 G_y F_{KK} & p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & 1 \\ p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & p_2 G_{yy} (F_L)^2 + p_2 G_y F_{LL} & 0 \\ p_2 G_{yz} F_K H_{K^s} & p_2 G_{yz} F_L H_{K^s} & 0 \end{bmatrix}.$$

Therefore,

$$(39)\quad \frac{dK_2^s}{dr} = \frac{p_2^2 G_y G_{yz} H_{K^s} (F_L F_{LK} - F_K F_{LL})}{|\Omega_2|} < 0.$$

Next, we determine the impact of w_2 on K^s:

$$(40)\quad \frac{dK_2^s}{dw_2} = \frac{|\Omega_{w_2K_2^s}|}{|\Omega_2|},$$

with,

$$\Omega_{w_2K_2^s} = \begin{bmatrix} p_2 G_{yy} (F_K)^2 + p_2 G_y F_{KK} & p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & 0 \\ p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & p_2 G_{yy} (F_L)^2 + p_2 G_y F_{LL} & 1 \\ p_2 G_{yz} F_K H_{K^s} & p_2 G_{yz} F_L H_{K^s} & 0 \end{bmatrix}.$$

Therefore,

$$(41)\quad \frac{dK_2^s}{dw_2} = \frac{p_2^2 G_y G_{yz} H_{K^s} (F_K F_{LK} - F_L F_{KK})}{|\Omega_2|} < 0.$$

We analyse the impact of r^s on K_2^s:

$$\frac{dK_2^s}{dr^s} = \frac{|\Omega_{r^sK_2^s}|}{|\Omega_2|},$$

with,

$$\Omega_{r^s K_2^s} = \begin{bmatrix} p_2 G_{yy} (F_K)^2 + p_2 G_y F_{KK} & p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & 0 \\ p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & p_2 G_{yy} (F_L)^2 + p_2 G_y F_{LL} & 0 \\ p_2 G_{yz} F_K H_{K^s} & p_2 G_{yz} F_L H_{K^s} & 1 \end{bmatrix}.$$

Therefore,

$$\frac{dK_2^s}{dr^s} = \frac{|\Delta_{LK}|}{|\Omega_2|} < 0. \tag{42}$$

We prove that p_2 affects positively K_2^s. Using the Cramer rule, the impact of p_2 on K_2^s is defined by:

$$\frac{dK_2^s}{dp_2} = \frac{|\Omega_{p_2 K_2^s}|}{|\Omega_2|}, \tag{43}$$

with,

$$\Omega_{p_2 K_2^s} = \begin{bmatrix} p_2 G_{yy} (F_K)^2 + p_2 G_y F_{KK} & p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & -G_y F_K \\ p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & p_2 G_{yy} (F_L)^2 + p_2 G_y F_{LL} & -G_y F_L \\ p_2 G_{yz} F_K H_{K^s} & p_2 G_{yz} F_L H_{K^s} & -G_z H_{K^s} \end{bmatrix}.$$

Therefore,

$$\frac{dK_2^s}{dp_2} = \frac{(G_y)^2 G_{yz} H_{K^s} \Sigma_F - G_z H_{K^s} |\Delta_{LK}|}{|\Omega_2|} > 0. \tag{44}$$

Then, we evaluate the impact of factor prices on K_2 using the Cramer rule. We first determine dK_2/dr:

$$\frac{dK_2}{dr} = \frac{|\Omega_{rK_2}|}{|\Omega_2|}, \tag{45}$$

with:

$$\Omega_{rK_2} = \begin{bmatrix} 1 & p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & p_2 G_{yz} F_K H_{K^s} \\ 0 & p_2 G_{yy} (F_L)^2 + p_2 G_y F_{LL} & p_2 G_{yz} F_L H_{K^s} \\ 0 & p_2 G_{yz} F_L H_{K^s} & p_2 G_{zz} (H_{K^s})^2 + p_2 G_z H_{K^s K^s} \end{bmatrix}.$$

Therefore,

$$\frac{dK_2}{dr} = \frac{p_2^2}{|\Omega_2|} \Big[\left(G_{yy} G_{zz} - (G_{yz})^2 \right) (F_L H_{K^s})^2 + G_z G_{yy} (F_L)^2 H_{K^s K^s} + G_y G_{zz} F_{LL} (H_{K^s})^2 + G_y G_z F_{LL} H_{K^s K^s} \Big] < 0. \tag{46}$$

We determine dK_2/dw_2:

$$\text{(47)} \quad \frac{dK}{dw_2} = \frac{|\Omega_{w_2K_2}|}{|\Omega_2|},$$

with,

$$\Omega_{w_2K_2} = \begin{bmatrix} 0 & p_2G_{yy}F_LF_K + p_2G_yF_{LK} & p_2G_{yz}F_KH_{K^s} \\ 1 & p_2G_{yy}\left(F_L\right)^2 + p_2G_yF_{LL} & p_2G_{yz}F_LH_{K^s} \\ 0 & p_2G_{yz}F_LH_{K^s} & p_2G_{zz}\left(H_{K^s}\right)^2 + p_2G_zH_{K^sK^s} \end{bmatrix}.$$

Therefore,

$$\text{(48)} \quad \frac{dK_2}{dw_2} = \frac{p_2^2}{|\Omega_2|}\Big[\left(G_{yz}\right)^2 F_LF_K\left(H_{K^s}\right)^2 - G_{yy}G_{zz}F_LF_K\left(H_{K^s}\right)^2 - G_zG_{yy}F_LF_KH_{K^sK^s} - G_yG_{zz}F_{LK}\left(H_{K^s}\right)^2 - G_yG_zF_{LK}H_{K^sK^s}\Big].$$

We calculate the impact of r^s on K_2:

$$\text{(49)} \quad \frac{dK_2}{dr^s} = \frac{|\Omega_{r^sK_2}|}{|\Omega_2|},$$

with:

$$\Omega_{r^sK_2} = \begin{bmatrix} 0 & p_2G_{yy}F_LF_K + p_2G_yF_{LK} & p_2G_{yz}F_KH_{K^s} \\ 0 & p_2G_{yy}\left(F_L\right)^2 + p_2G_yF_{LL} & p_2G_{yz}F_LH_{K^s} \\ 1 & p_2G_{yz}F_LH_{K^s} & p_2G_{zz}\left(H_{K^s}\right)^2 + p_2G_zH_{K^sK^s} \end{bmatrix}.$$

As a result,

$$\text{(50)} \quad \frac{dK_2}{dr^s} = \frac{p_2^2G_yG_{yz}H_{K^s}\left(F_LF_{LK} - F_KF_{LL}\right)}{|\Omega_2|} < 0.$$

We also evaluate the impact of p_2 on K_2:

$$\text{(51)} \quad \frac{dK_2}{dp_2} = \frac{|\Omega_{p_2K_2}|}{|\Omega_2|},$$

with:

$$\Omega_{p_2K_2} = \begin{bmatrix} -G_yF_K & p_2G_{yy}F_LF_K + p_2G_yF_{LK} & p_2G_{yz}F_KH_{K^s} \\ -G_yF_L & p_2G_{yy}\left(F_L\right)^2 + p_2G_yF_{LL} & p_2G_{yz}F_LH_{K^s} \\ -G_zH_{K^s} & p_2G_{yz}F_LH_{K^s} & p_2G_{zz}\left(H_{K^s}\right)^2 + p_2G_zH_{K^sK^s} \end{bmatrix}.$$

Consequently,

$$\frac{dK_2}{dp_2} = \frac{p_2^2 G_y \left(F_K F_{LL} - F_L F_{LK}\right) \left(\left(G_z G_{yz} - G_y G_{zz}\right) \left(H_{K^s}\right)^2 - G_y G_z H_{K^s K^s} \right)}{|\Omega_2|} > 0. \tag{52}$$

At last, we determine the impact of factor prices on L_2 using the Cramer rule. We begin with the impact of r:

$$\frac{dL_2}{dr} = \frac{|\Omega_{rL_2}|}{|\Omega|}, \tag{53}$$

with:

$$\Omega_{rL_2} = \begin{bmatrix} p_2 G_{yy} \left(F_K\right)^2 + p_2 G_y F_{KK} & 1 & p_2 G_{yz} F_K H_{K^s} \\ p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & 0 & p_2 G_{yz} F_L H_{K^s} \\ p_2 G_{yz} F_K H_{K^s} & 0 & p_2 G_{zz} \left(H_{K^s}\right)^2 + p_2 G_z H_{K^s K^s} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \frac{dL_2}{dr} =& \frac{p_2^2}{|\Omega_2|} \Big[- \left(G_{yy} G_{zz} - \left(G_{yz}\right)^2 \right) F_L F_K \left(H_{K^s}\right)^2 - G_z G_{yy} F_L F_K H_{K^s K^s} \\ & - G_y G_{zz} F_{LK} \left(H_{K^s}\right)^2 - G_y G_z F_{LK} H_{K^s K^s} \Big]. \end{aligned} \tag{54}$$

We determine the impact of w_2 on L_2:

$$\frac{dL_2}{dw_2} = \frac{|\Omega_{w_2 L_2}|}{|\Omega_2|}, \tag{55}$$

with,

$$\Omega_{w_2 L_2} = \begin{bmatrix} p_2 G_{yy} \left(F_K\right)^2 + p_2 G_y F_{KK} & 0 & p_2 G_{yz} F_K H_{K^s} \\ p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & 1 & p_2 G_{yz} F_L H_{K^s} \\ p_2 G_{yz} F_K H_{K^s} & 0 & p_2 G_{zz} \left(H_{K^s}\right)^2 + p_2 G_z H_{K^s K^s} \end{bmatrix}.$$

As a result,

$$\begin{aligned} \frac{dL_2}{dw_2} =& \frac{p_2^2}{|\Omega_2|} \Big[\left(G_{yy} G_{zz} - \left(G_{yz}\right)^2 \right) \left(F_K H_{K^s}\right)^2 + G_z G_{yy} \left(F_K\right)^2 H_{K^s K^s} \\ & + G_y G_{zz} F_{KK} \left(H_{K^s}\right)^2 + G_y G_z F_{KK} H_{K^s K^s} \Big] < 0. \end{aligned} \tag{56}$$

Then, we determine the impact of r^s on L_2:

$$\text{(57)} \quad \frac{dL_2}{dr^s} = \frac{|\Omega_{r^s L_2}|}{|\Omega_2|},$$

with,

$$\Omega_{r^s L_2} = \begin{bmatrix} p_2 G_{yy} (F_K)^2 + p_2 G_y F_{KK} & 0 & p_2 G_{yz} F_K H_{K^s} \\ p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & 0 & p_2 G_{yz} F_L H_{K^s} \\ p_2 G_{yz} F_K H_{K^s} & 1 & p_2 G_{zz} (H_{K^s})^2 + p_2 G_z H_{K^s K^s} \end{bmatrix}.$$

Then,

$$\text{(58)} \quad \frac{dL_2}{dr^s} = \frac{p_2^2 G_y G_{yz} H_{K^s} (F_K F_{LK} - F_{KK} F_L)}{|\Omega_2|} < 0.$$

At last, we evaluate the impact of p_2 on L_2:

$$\text{(59)} \quad \frac{dL_2}{dp_2} = \frac{|\Omega_{p_2 L_2}|}{|\Omega_2|},$$

with:

$$\Omega_{p_2 L_2} = \begin{bmatrix} p_2 G_{yy} (F_K)^2 + p_2 G_y F_{KK} & -G_y F_K & p_2 G_{yz} F_K H_{K^s} \\ p_2 G_{yy} F_L F_K + p_2 G_y F_{LK} & -G_y F_L & p_2 G_{yz} F_L H_{K^s} \\ p_2 G_{yz} F_K H_{K^s} & -G_z H_{K^s} & p_2 G_{zz} (H_{K^s})^2 + p_2 G_z H_{K^s K^s} \end{bmatrix}.$$

Therefore,

$$\text{(60)} \quad \frac{dL_2}{dp_2} = \frac{p_2^2 G_y (F_L F_{KK} - F_K F_{LK}) \left((G_z G_{yz} - G_y G_{zz}) (H_{K^s})^2 - G_y G_z H_{K^s K^s} \right)}{|\Omega_2|} > 0.$$

Q.E.D.

Proof of Proposition 3

First, we determine the impact of K_2^s on λ_2^f. To do so we combine (3) and (6):

$$\text{(61)} \quad \lambda_2^f = r^2 \frac{H_Q}{H_{K^s}} - r_2^c g_Q.$$

The positive impact of K_2^s on the value of water for the firm is shown by derivating equation (61) with respect of K_2^s:

$$\frac{d\lambda_2^f}{dK_2^s} = \frac{r^s \left(H_{QK^s} H_{K^s} - H_{K^s K^s} H_Q\right)}{\left(H_{K^s}\right)^2} > 0. \tag{62}$$

The impact of r on λ_2^f can be written as the combination of (39) and (62):

$$\begin{aligned}\frac{d\lambda_2^f}{dr} &= \frac{d\lambda_2^f}{dK_2^s}\frac{dK_2^s}{dr}\\ &= \frac{p_2^2 r^s G_{yz} G_y \left(F_L F_{LK} - F_K F_{LL}\right)\left(H_{QK^s} H_{K^s} - H_Q H_{K^s K^s}\right)}{H_{K^s}|\Omega_2|}\\ &= \frac{p_2^3 G_y G_z G_{yz} \left(F_L F_{LK} - F_K F_{LL}\right)\left(H_{QK^s} H_{K^s} - H_Q H_{K^s K^s}\right)}{|\Omega_2|} < 0. \end{aligned} \tag{63}$$

Next, the impact of w_2 on λ_2^f can be written as the combination of (41) and (62):

$$\begin{aligned}\frac{d\lambda_2^f}{dw_2} &= \frac{d\lambda_2^f}{dK_2^s}\frac{dK_2^s}{dw_2}\\ &= \frac{p_2^2 r^s G_y G_{yz} \left(F_K F_{LK} - F_L F_{KK}\right)\left(H_{QK^s} H_{K^s} - H_Q H_{K^s K^s}\right)}{H_{K^s}|\Omega_2|}\\ &= \frac{p_2^3 G_y G_z G_{yz} \left(F_K F_{LK} - F_L F_{KK}\right)\left(H_{QK^s} H_{K^s} - H_Q H_{K^s K^s}\right)}{|\Omega_2|} < 0. \end{aligned} \tag{64}$$

The effect of r^s on λ_2^f can be written as the combination of (42) and (62):

$$\begin{aligned}\frac{d\lambda_2^f}{dr^s} &= \frac{d\lambda_2^f}{dK_2^s}\frac{dK_2^s}{dr^s}\\ &= \frac{r^s \left(H_{QK^s} H_{K^s} - H_Q H_{K^s K^s}\right)}{\left(H_{K^s}\right)^2}\frac{|\Delta_{LK}|}{|\Omega_2|} < 0. \end{aligned} \tag{65}$$

The impact of r_2^c on the marginal value of water in period 2 can be directly calculate from (3), that is:

$$\frac{d\lambda_2^f}{dr_2^c} = -g_Q < 0. \tag{66}$$

The impact of p_2 on λ_2^f can be written as the combination of (44) and (62):

$$\begin{aligned}\frac{d\lambda_2^f}{dp_2} =& \frac{d\lambda_2^f}{dK_2^s}\frac{dK_2^s}{dp_2} \\ =& \frac{r^s\left(H_{QK^s}H_{K^s} - H_Q H_{K^sK^s}\right)\left(\left(G_y\right)^2 G_{yz}H_{K^s}\Sigma_F - G_z H_{K^s}|\Delta_{LK}|\right)}{\left(H_{K^s}\right)^2|\Omega_2|} > 0. \qquad (67)\end{aligned}$$

Q.E.D.

Proof of Proposition 4

We first prove that the impact of A_1^f on λ_1^f is negative. The combination of (3) and (4) implies that:

$$\begin{aligned}\frac{d\lambda_1^f}{\partial A_1^f} =& p_2 G_{zz}(H_Q)^2 + p_2 G_z H_{QQ} - r_1^c g_{QQ} - \frac{\left(p_2 G_{yz} F_L H_Q\right)^2}{p_2 G_{yy}(F_L)^2 + p_2 G_y F_{LL}} \\ =& \frac{p_2}{G_{yy}(F_L)^2 + G_y F_{LL}}\Big[G_{yy}G_{zz}(F_L H_Q)^2 + G_y G_{zz}(H_Q)^2 F_{LL} \\ &+ G_z G_{yy}(F_L)^2 H_{QQ} + G_y G_z F_{LL} H_{QQ} - \left(G_{yz}F_L H_Q\right)^2\Big] - r_1^c g_{QQ} \\ =& \frac{p_2}{G_{yy}(F_L)^2 + G_y F_{LL}}\Big[(G_{yy}G_{zz} - G_{yz})(F_L H_Q)^2 + G_y G_{zz}(H_Q)^2 F_{LL} \\ &+ G_z G_{yy}(F_L)^2 H_{QQ} + G_y G_z F_{LL} H_{QQ}\Big] - r_1^c g_{QQ} < 0.\end{aligned}$$

Before proceeding with the analysis of the impact of A_2^f on λ_2^f, it is worth mentioning that the concavity of H implies that:

$$\Sigma_H = H_{QQ}(H_{K^s})^2 + H_{K^sK^s}(H_Q)^2 - 2H_Q H_{K^s} H_{QK^s} < 0.$$

To determine $d\lambda_2^f/dA_2^f$, we calculate the total derivative of (61):

$$
\begin{aligned}
\frac{d\lambda_2^f}{dA_2^f} =& r_s \frac{H_{QQ}H_{K^s} - H_{QK^s}H_Q}{(H_{K^s})^2} - r_2^c g_{QQ} + \frac{d\lambda_2^f}{dK_2^s}\frac{dK_2^s}{dA_2^f} \\
=& \frac{r^s}{(H_{K^s})^2}\left(\frac{H_{QQ}(H_{K^s})^2 + H_{K^sK^s}(H_Q)^2 - 2H_Q H_{K^s} H_{QK^s}}{H_{K^s}}\right. \\
& \left. - \frac{p_2 G_z |\Delta_{LK}| \left(H_{QK^s}H_{K^s} - H_Q H_{K^sK^s}\right)^2}{H_{K^s}|\Omega_2|}\right) - r_2^c g_{QQ} \\
=& \frac{r^s}{(H_{K^s})^3 |\Omega_2|}\left[p_2 G_z |\Delta_{LK}| \left(\Sigma_H H_{K^sK^s} - \left(H_{QK^s}H_{K^s} - H_Q H_{K^sK^s}\right)^2\right)\right. \\
& + \Sigma_H p_2^3 \left(G_{zz}(H_{K^s})^2 (G_y)^2 \left(F_{LL}F_{KK} - (F_{LK})^2\right)\right. \\
& \left.\left. + G_y (H_{K^s})^2 \left(G_{zz}G_{yy} - (G_{yz})^2\right)\Sigma_F\right)\right] - r_2^c g_{QQ} \\
=& \frac{r^s}{(H_{K^s})^3 |\Omega_2|}\left[p_2 G_z |\Delta_{LK}| (H_{K^s})^2 (H_{QQ}H_{K^sK^s} - (H_{QK^s})^2)\right. \\
& + \Sigma_H p_2^3 \left((G_y)^2 G_{zz}(H_{K^s})^2 \left(F_{LL}F_{KK} - (F_{LK})^2\right)\right. \\
& \left.\left. + G_y (H_{K^s})^2 \left(G_{zz}G_{yy} - (G_{yz})^2\right)\Sigma_F\right)\right] - r_2^c g_{QQ} < 0.
\end{aligned}
$$

Consequently, the marginal value of water for firm in period t is decreasing in A_t^f. *Q.E.D.*

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