UNANIMOUS IMPLEMENTATION: A CASE FOR APPROVAL MECHANISMS*

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Abstract. We design a class of indirect mechanisms, the Approval ones, which allow the players' strategies to coincide with the subsets of the outcome space. By focusing on the single-peaked domain, we prove that: a) each of these mechanisms is characterized by a unique equilibrium outcome and b) essentially for every implementable welfare optimum (outcome of a social choice rule), including the Condorcet winner alternative, there exists an Approval mechanism that *unanimously* implements it. That is, Approval mechanisms help a society achieve every feasible welfare goal and, perhaps more importantly, they promote social coherence: the implemented outcome is approved by everyone.

Keywords. Nash Implementation, Strategy-proof, Unanimity, Indirect Mechanisms.

JEL Classification. C9, D71, D78, H41.

1. Introduction

Democratic entities, once they set their fundamental welfare goals, try to achieve them by adopting decision making procedures which allow the equal participation of all individuals. These democratic decision making procedures may be broadly split in two categories: voting and deliberation. Voting requires agents to take actions in support of certain policy alternatives and then, given the action profile and the particular voting rule, an alternative is implemented. In the literature, a voting mechanism is a (simultaneous or sequential) game with formal structure, whose unique equilibrium outcome[1] coincides with a specific welfare optimum - that is, with the outcome of a social choice rule (Maskin [1999]). Indeed, in standard decision making frameworks well-defined voting mechanisms exist and implement a variety of welfare optima/social choice rules. For example, in the context of single-peaked preferences, as recently shown by Gershkov et al. [2015], sequential quota mechanisms may implement any (generalized) median rule.[2] On the other hand, deliberation requires agents to engage in rounds of, more or less, informal discussions and negotiations until a consensual decision is reached. These procedures guarantee that the outcome reflects the interests

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[1] If, for certain individual preferences, we care to implement a certain welfare optimum, then we cannot rely on procedures that admit other equilibrium outcomes: voting games that admit multiple equilibrium outcomes are uninteresting from a Nash implementation perspective (see Maskin [1999]).
[2] The welfare optimum defined by such a rule coincides with the median of the set which consists of: a) the voters' ideal policies and b) some exogenously fixed numbers (phantom voters).

of all members (Innes and Booher [1999]) and they are employed in high-stakes decision making (for example, by the UN Security Council and by the European Council).

Both of these approaches to democracy have deep philosophical grounds and, despite their apparent differences, they complement each other. Arriving to unanimous decisions through deliberations is usually deemed superior to just voting since it leads to collective harmony. But it is also more costly. That is, a collective decision problem is commonly solved by voting when the costs of deliberation surpass the potential costs of post-decision confrontation and clash. When a collective body is composed of a small number of entities (such as the 28 countries of the EU) among which clash and confrontation has been proved to be very costly in the past (for example, by the experience of the two world wars), it is straightforward that deliberation is the optimal choice.[3] When a collective body, though, is too large for deliberation to take place in an effective manner and/or costly clash among the participating members is not likely to occur, then voting mechanisms are more likely to be adopted.

Since the advantage of deliberative democracy is the implementation of a consensual alternative and hence the minimization of post-decision confrontation and clash among participating entities (at the expense of reaching this decision after a possibly long period), while the advantage of voting is the low cost of decision making (at the expense possibly generating post-decision conflicts), would not it be desirable to implement welfare optima via voting mechanisms that generate unanimous outcomes?

In this paper, we focus on the framework of single-peaked preferences[4] and we design a class of indirect mechanisms, the *Approval* ones, which do precisely this - they bring together the described positive features of voting (low decision making costs) and deliberation (unanimous decisions). These mechanisms allow every player to support, not just a single alternative, but as many alternatives as one wants (an arbitrary interval within the unit interval). After all individuals report their sets of approved alternatives, a publicly known aggregation rule is applied and an alternative is implemented. These aggregation rules might take very simple forms. The most intuitive examples are arguably the median and the mean aggregation rule. When players submit their sets of approved alternatives a distribution of approvals is generated: the density of this distribution at $x \in [0,1]$ is identical to the number of individuals that have approved of alternative x, normalized by the total measure of approvals. The median (resp. mean) aggregation rule simply implements the median (resp. mean) of this distribution.

Our main finding is that (under some mild restrictions) *every implementable welfare optimum may be unanimously implemented by some anonymous Approval mechanism.* An Approval

[3]Moreover, elected officials that take decisions using advice from committees of experts are much more comfortable following unanimous recommendations than suggestions which are disputed by a number of experts in the committee. Unanimous recommendations minimize the responsibility of the decision maker and make her less accountable to groups of citizens that are negatively affected by her decisions. In addition, when experts agree on a policy recommendation, it is hard for elected officials to succumb to interest groups' pressures and neglect experts' advice, and this should maximize probability of informed decision making. These are only a few extra reasons why the consensus building literature was developed (see for example Bessette [1980], Gutmann and Thompson [1996], Gutmann and Thompson [2002] and Fishkin and Laslett [2003]).

[4]The set of alternatives is $A = [0,1]$ and the set of possible preference relations consists of the single-peaked ones on A.

mechanism is understood to unanimously implement a welfare optimum/social choice rule if: a) it implements it in *every* Nash equilibrium and b) there is at least one equilibrium in which each player includes in his strategy (set of approved outcomes) the implemented outcome. The equilibrium strategies of most players take an easy "I approve every alternative at most (least) as large as the implemented alternative" form. In fact, every player with a preferred alternative to the left (right) of the implemented one approves the implemented alternative and all the alternatives to its left (right). That is, in equilibrium at most one player may not include the implemented outcome and his own ideal outcome in his strategy, and this player's ideal outcome must coincide with the implemented one. Hence, every equilibrium is substantially unanimous in the sense that, for each voter, the implemented outcome and his ideal one are either both contained in his strategy or they coincide with each other.

Notice that in the context of single-peaked voting, the implementable welfare optima essentially coincide with the outcomes of (generalized) median rules. Indeed, as proved by Moulin [1980] (generalized) median rules are the unique social choice rules that satisfy efficiency and strategy-proofness, while Berga and Moreno [2009] established strategy-proof rules that are "not too bizarre" (in the context of Sprumont [1995])[5] are the only implementable ones. This allows us to provide, beyond the existence results, a transparent characterization of the unique equilibrium outcome of each Approval mechanism. Moreover, it gives us the tools to design explicitly an Approval mechanism for each (generalized) median rule - including one for the pure median rule (also known as the Condorcet rule or, simply, majority rule). Hence, in the end of the paper we explain how to construct an Approval mechanism that implements any given (generalized) median rule and we provide the Approval mechanism that unanimously implements the ideal policy of the median voter (Condorcet winner alternative). Finally, the fact that in equilibrium players approve, not only the implemented outcome, but their ideal one as well, indicates that these rules, beyond unanimity, promote sincere revelation of preferences to a certain extent.

The Approval mechanisms can be applied to a variety of decision-making problems. Consider for example a number of judges who disagree on the quality of an athletic performance (for example, in gymnastics or in figure skating) and that have to jointly assign a score to this performance, while each of them wants the joint score to be as close as possible to her individual performance evaluation. Another potential application is the determination of LIBOR or the board members of the European Central Bank (ECB) deciding over the interest rate from a closed and convex set of interest rates (see Cai [2009], Rausser et al. [2015] and Rosar [2015] among others for recent analysis).[6] Our Approval mechanisms can be of

[5]That is, we restrict attention to anonymous rules that implement each of the alternatives for at least one preference profile (full-range).

[6]The London Interbank Offered Rate (Libor) is the interest rate at which banks can borrow from each other and plays a critical role in financial markets. Libor anchors contracts amount "to the equivalent of $45000 for every human being on the planet" (see MacKenzie [2008]). The banks are asked to submit an interest rate at which their banks could borrow money. The lowest and highest quarter of the values are discarded and the Libor corresponds to the average of the remainder. In other words, the device used to determine this index is the trimmed mean rule. Theorists have mostly focused on the pure mean rule (without trimming) and their conclusion over its properties is qualified (see Renault and Trannoy [2005] and Yamamura and Kawasaki [2013] for theoretical works on this subject and Marchese and Montefiori [2011] and Block et al. [2014] for

interest in these settings since they can improve the quality of decision making by ensuring a unanimous final decision.

In what follows we discuss the relevant literature (section 2), we describe the model (section 3) and present an example (section 4) and the necessary and sufficient conditions for the described unanimous implementation through approval mechanisms (section 5).

2. Relevant literature

The objective of the current work is to show the usefulness of indirect mechanisms to encourage unanimous agreements. To do so, we focus on the single-peaked domain and prove the following result: using approval mechanisms, one can unanimously implement any anonymous, efficient and strategy-proof social choice function. The remainder of this section reviews the related literature and underlines the interest of our contribution with regards to the implementation theory.

Recall that, as proved by Moulin [1980], any peak-only social choice function is efficient, anonymous and strategy-proof if and only if it is a generalized median rule (GMR) with $n-1$ phantoms.[7] It is among the few general positive results in social choice theory. Its interpretation is not very intuitive since the meaning of the phantoms or fixed ballots is at first sight far from clear. In order to clarify ideas, let us now briefly explain how these rules work. We assume throughout that the outcome space A is the interval [0,1]. A GMR f is characterized by the phantom vector $(p_1,\ldots,p_{n-1})$: given the peaks $(t_1,\ldots,t_n)$ of the voters, it selects $f(t_1,t_2,\ldots,t_n)$ as an outcome with

$$f(t_1,t_2,\ldots,t_n) = med(t_1,t_2,\ldots,t_n,p_1,p_2,\ldots,p_{n-1}),$$

such that each phantom p_i is in the interval $[0,1]$.[8] Note that each phantom is not required to be in the interior of A and that plays a key role. Indeed, if $p_i = 0$ for any $i = 1,\ldots,n-1$, then

$$f(t_1,t_2,\ldots,t_n) = min(t_1,t_2,\ldots,t_n),$$

whereas if $p_i = 1$ for any $i = 1,\ldots,n-1$, then

$$f(t_1,t_2,\ldots,t_n) = max(t_1,t_2,\ldots,t_n).$$

More interestingly, when n is odd, letting $p_i = 0$ for any $i \leq \frac{n-1}{2}$ and $p_i = 1$ for any $i \geq \frac{n-1}{2}+1$ and when n is even setting $p_i = 0$ for any $i \leq \frac{n}{2}$ and $p_i = 1$ for any $i \geq \frac{n}{2}+1$, leads to

$$f(t_1,t_2,\ldots,t_n) = med(t_1,t_2,\ldots,t_n),$$

experimental ones). In environments with a large number of voters, this rule seems to be a natural candidate as it is the unique one satisfying a weakening of strategy-proofness (see Ehlers et al. [2004]). While the latter feature is quite desirable, the former violates the usual desiderata of voting theory: a voter drops his most preferred policy to announce an extremist policy that maximizes his impact on the final outcome. This extreme polarization of the voters' positions seems to posit a fundamental problem with the average method.

[7]One may wonder how restrictive is the peak-only assumption. Arguably, a lot of information is neglected by restricting attention from the preference relations to just their peaks. However, this intuition turns out to be false: when preferences are single-peaked, it turns out that every strategyproof rule whose range is an interval must be peak-only. See Ching [1997] and Sprumont [1995] for a direct proof and Barberà et al. [1993] for an extension in a multidimensional discrete model.

[8]Moulin's original work assumes that the outcome space is the set of real numbers. Our results can be extended to such a framework.

and hence coincides with the Condorcet Winner alternative. A similar reasoning shows that, carefully tuning the phantoms, leads to the implementation of the k^th ranked type.

While these rules are obviously anonymous, it is less evident that they are also efficient and strategy-proof. Efficiency arises from having not more than $n-1$ phantoms and hence ensuring that the final outcome lies in the interval defined by the lowest and the highest type. Strategy-proofness holds since no agent strictly prefers to misreport his type independently of the announcements of the rest of the players. Indeed, if a voter's peak is to the left of the chosen alternative, any announcement different than his peak has two possible consequences: either it makes the final decision greater than the chosen alternative or it does not affect the decision (see Border and Jordan [1983] regarding the notion of uncopromisingness). This naturally implies that the game generated by each GMR has a very appealing (Nash) equilibrium in which every voter sincerely reveals his true peak.

However, one should note that the game generated by each GMR need not lead to the generalized median of the true peaks. In this respect, the GMRs share a common flaw with other strategy-proof mechanisms: they admit a large multiplicity of Nash equilibria, some of which produce the wrong outcome. For instance, the game triggered by the pure median rule exhibits a large set of equilibria: as long as every player announces the same alternative x, this constitutes an equilibrium that implements x since no unilateral deviation affects the aggregate outcome.[9] This leads to the following conclusion: the direct game associated to a GMR does not Nash implement the GMR (see Repullo [1985]) for similar results).

When presented with the previous observation, two main questions appear: (i) why would one care about implementing GMRs? and (ii) how should we implement them?

As far as the first question is concerned, a careful examination of the literature shows that these rules are essentially the only Nash implementable ones in this environment. To see why, consider the following line of reasoning. First, Maskin [1999] proves that any Nash implementable social choice function must be Maskin monotonic (for any domain of preferences). Second, Berga and Moreno [2009] prove that with single peaked preferences, a rule is Maskin monotonic if and only if it is strategy proof and (weakly) non-bossy[10]. Moreover, they prove that non-bossiness is equivalent to convex range in this preference domain (note that the GMRs have convex range). If f is implementable but f is not a GMR, then it is strategy proof rule but without a convex range. If a rule has a non-convex range, then it fails unanimity in the sense that an alternative need not be implemented even if it is the most preferred one of all voters. This constitutes a strong argument against the use of rules without a convex range. As Sprumont [1995] puts it, the GMRs are the only rules which are not "too bizarre" in this environment. In other words, if the social planner's

[9]Experimental evidence shows that strategy-proof mechanisms need not lead a large share of the agents to reveal their true type: see Attiyeh et al. [2000] , Kawagoe and Mori [2001], Kagel and Levin [1993] and Cason et al. [2006] among others.

[10]The relation between strategy-proofness and Maskin monotonicity has produced a rich literature. Muller and Satterthwaite [1977] show that Maskin monotonicity and strategy-proofness are equivalent when preferences are the unrestricted. Dasgupta et al. [1979] obtains strategy-proofness as necessary condition for Maskin monotonicity under some restricted preferences. For recent contributions, see also Takamiya [2007] and Klaus and Bochet [2013].

objective is to implement an efficient, unanimous and anonymous social choice function, he must opt to implement a GMR.

In order to answer the second question, the natural answer would be the use of the integer game. In this game, the players send messages to the social planner; to ensure efficiency, the players name integers, and when their messages contradict each other, the one announcing the largest integer is rewarded. Yet, the integer games were built to be applicable in very general settings rather than for their plausibility. For this reason, these mechanisms are often quite complex and this has stimulated researchers to investigate the implementation problem using different approaches as argued by Jackson [2001].

The literature on designing appealing indirect mechanisms is rather huge. Yet, it often lacks general results but succeeds in tailoring interesting mechanisms for particular situations. In this literature, the most closely related contributions are, on the one hand, the ones by Yamamura and Kawasaki [2013] and Gershkov et al. [2015] and on the other hand the one by Saijo et al. [2007].

Yamamura and Kawasaki [2013] proves how to implement the GMRs through a class of simple direct mechanisms: the average rules. As they show, the agents tend to adopt an extremist behavior (either 0 or 1) in equilibrium. Moreover, the equilibrium outcome coincides with the GMR of the true peaks with an important restriction: all phantoms must be interior (i.e. different from 0 or 1). Hence, one cannot implement the Condorcet winner using the average rules. Gershkov et al. [2015] show how to implement the GMRs through sequential quota mechanisms. More precisely, their sequential mechanisms are obtained by modifying a sequential voting scheme suggested by Bowen [1943]. Our approach is orthogonal to theirs since our Approval mechanisms are simultaneous. In short, the implementation results that these papers obtain are related to ours: yet, our main contribution is to show that Approval mechanisms give incentives for reaching unanimous agreements.

Once we have commented on these closely related works, we state some final remarks on two literatures to which this paper is connected.

The first one is the one focusing on strategic voting and more precisely on the unanimity rule (see Feddersen and Pesendorfer [1996, 1997, 1998] for classical references in the area and Koriyama and Szentes [2009] and Bouton et al. [2014] for recent contributions). The comparison between the current results and the ones in such a literature seems to be far from pertinent. Indeed, in broad terms, these works often evaluate the consequences of honest and strategic behavior of voters when confronted with the unanimity rule. They tend to perform a welfare analysis of this rule in several settings: private/common values, complete/incomplete information, optimal size of the jury, etc.. Their main message is that the unanimity rule tends to be inefficient whenever strategic voting is present: that is voters do not reveal their true information if the collective decision is to be made by unanimity. Our objective is different: we posit incentives to endogenously achieve unanimity assuming from the outset that voters are strategic. Having said so, our paper is also related to the literature on Approval Voting (see Brams and Fishburn [1983] and Laslier and Sanver [2010] for a recent account), to which Approval mechanisms borrow both its name and its flexibility.

Laslier et al. [2015] design a bargaining device over lotteries based on Approval voting and derive conditions for consensus reaching in equilibrium with just two agents.

Finally, this work is of course related to implementation theory (see Maskin [1999] and Jackson [2001] for a review). Our notion of implementation refines the one of Nash implementation in the sense that it requires that all equilibria of the game form implement the desired social choice rule.[11] It is stronger since it requires the existence of unanimous equilibrium in which all voters agree on the implemented policy. Our objective is hence twofold: to reduce the multiplicity of equilibria associated to strategy-proof mechanisms while ensuring a unanimous agreement. The former requirement is hence closely related to the nice contribution by Saijo et al. [2007] which proposes a novel concept of implementation named secure implementation. This implementation notion[12] aims to get rid of the multiplicity of equilibria inherent to the direct mechanisms associated to strategy-proof rules previously described by coarsening the notion of implementation. Their proposal manages to derive securely implementable functions in some situations (such as quasi-linear preferences) but fails to do so in our framework. Our contribution is hence hinting at a possible manner of overcoming this theoretical objection: in order to implement a strategy-proof social choice function, rather than using its associated direct mechanism one could make use of indirect (approval) mechanisms that foster unanimity while ensuring the uniqueness of the equilibrium outcome.

3. The setting

Let $A := [0,1]$ denote the set of alternatives and $N := \{1,\ldots,n\}$ with $n \geq 2$ stand for the finite set of players. Each player is endowed with preferences over A. The utility for player i when $x \in A$ is the implemented policy equals $u_i(x)$ with $u_i : A \to \mathbb{R}$ where each $u_i \in U$, the set of single-peaked preferences. Note that each player i has a unique peak denoted t_i so that $u_i(x') < u_i(x'')$ when $x' < x'' \leq t_i$ and when $t_i \leq x'' < x'$.[13] We let $(t_1,\ldots,t_n)$ stand for a distribution of the players' peaks and $u = (u_1,\ldots,u_n) \in U := \prod_{j=1}^n U_i$.

A social choice function is a function $f : U \to A$ that associates to every $u \in U$, a unique alternative $f(u)$ in A. A mechanism is a function $\theta : S \to A$ that assigns to every $s \in S$, a unique element $\theta(s)$ in A, where $S := \prod_{i=1}^n S_i$ and S_i is the strategy space of agent i. The mechanism θ is the direct revelation mechanism associated to a SCF f if $S_i = U_i$ for all $i \in N$ and $\theta(u) = f(u)$ for every $u \in U$. A *SCF* f is strategy-proof if for all $i \in N$, all $u_i, \tilde{u}_i \in U_i$, and all $u_{-i} \in U_{-i}$, $u_i(f(u_i,u_{-i})) \geq u_i(f(\tilde{u}_i,u_{-i}))$. As shown by Moulin [1980], these rules admit a simple characterization: they implement as an outcome the median of the peaks

[11] There is a large literature on implementation and different notions have been proposed. While Nash implementation is arguably the most well-known, scholars have focused on other concepts such that Bayesian implementation (Jackson [1991]), virtual implementation (Abreu and Sen [1991],**?**), implementation in mixed strategies (Mezzetti and Renou [2012]) and implementation with partial honesty (Dutta and Sen [2012]) among others.

[12] More specifically, they focus on a dual notion of implementation that requires dominant strategy implementation and Nash implementation simultaneously. The notion of secure implementation is equivalent to the one of robust implementation (see Bergemann and Morris [2009] among others) in any private values setting, as ours, as shown by Adachi [2014].

[13] For simplicity, we assume that $t_i \neq t_j$ for any $i,j \in N$. Our results are not affected when relaxing this constraint.

of the players plus $(n-1)$ exogenous parameters (phantoms). More formally, for any finite collection of points $x_1,\dots,x_m$ in $[0,1]$, we let $m(x_1,\dots,x_m)$ denote their median, that is the smallest number $m(x_1,\dots,x_m) \in x_1,\dots,x_m$, which satisfies: $\frac{1}{m}\#\{x_i \mid x_i \le m(x_1,\dots,x_m)\} \ge \frac{1}{2}$ and $\frac{1}{m}\#\{x_i \mid x_i \ge m(x_1,\dots,x_m)\} \ge \frac{1}{2}$. In the domain U and assuming that each agent's message is one element of A, a SCF f is anonymous, efficient and strategy-proof if and only if there exist $(n-1)$ real numbers, $\kappa_1,\dots,\kappa_{n-1}$ such that $f(t_1,\dots,t_n) = m(t_1,\dots,t_n,\kappa_1,\dots,\kappa_{n-1})$.

We let $\mathcal{B}$ denote the collection of closed intervals of A and define an approval mechanism as a mechanism such that $S_i = \mathcal{B}$ for every $i \in N$.[14] We write $\underline{b_i} = \min b_i$ and $\overline{b_i} = \max b_i$ for each $b_i \in \mathcal{B}$. Note that the strategy set $\mathcal{B}$ allows elements of different dimensions: singletons and positive length intervals. To accommodate this fact, we let λ_d denote the Lebesgue measure on $\mathbb{R}^d$ with $d = 0,1$. Since each b_i is a convex set, its dimension is well-defined so that for each approval profile $b = (b_i, b_{-i})$, we let $\dim(b) = \max_{i\in N} \dim(b_i)$.

Given a mechanism $\theta : S \to A$, the strategy profile $s \in S$ is a Nash equilibrium of θ at $u \in U$, if $u_i(\theta(s_i,s_{-i})) \ge u_i(\theta(s_i',s_{-i})$ for all $i \in N$ and any $s_i' \in S_i$. Let $N^\theta(u)$ be the set of Nash equilibria of θ at u. The mechanism θ implements the SCF f in Nash equilibria if for each $u \in U$, (i) there exists $s \in N^\theta(u)$ such that $\theta(s) = f(u)$ and (ii) for any $s \in N^\theta(u)$, $\theta(s) = f(u)$. The SCF f is implementable if there exists a mechanism that implements f in Nash equilibria. An Approval Mechanism θ unanimously implements the SCF f if (i) θ implements f in Nash equilibria and (ii) there exists $s \in N^\theta(u)$ such that $\cap_{i=1}^n s_i \neq \emptyset$ with $\theta(s) = \cap_{i=1}^n s_i$. Our focus is on the unanimous implementation of strategy-proof rules.

4. An Example: the Median Approval Mechanism

In order to clarify the main ideas behind unanimous implementation, this section presents an example that illustrates how an approval mechanism works. We are concerned here with the median approval mechanism that associates, to any distribution of approvals, its *median*. Therefore, we assume that the median approval mechanism associates to every strategy profile b (i.e. any announcement of intervals), the median $\theta(b)$ of these intervals with

$$\theta(b) := \min\{x \in [0,1] \mid \int_0^x f_b(t)dt = \frac{1}{2}\},$$

$$\text{with } f_b(t) = \frac{\#\{i \in N \mid t \in b_i\}}{\sum_{i\in N} \lambda_{\dim(b)}(b_i)} \text{ for any } t \in [0,1].$$

In order to understand the definition of θ, it suffices to understand that $f_b(t)$ stands for the "score" of alternative t normalized by the size of the intervals in b and therefore $\int_0^x f_b(t)dt$ counts the share of "approvals" located between 0 and x. It is hence a cumulative distribution in the usual sense[15] and therefore θ implements the median $\theta(b)$ as the value in which the share of approvals located below and above it is equal to 1/2, the lowest value being chosen in case of ties.

[14]This assumption can be relaxed by allowing as a pure strategy any finite union of closed and convex subsets of A. Relaxing it however would imply more cumbersome notation and proofs since then two strategies that differ by a zero-measure set can have equivalent consequences. Moreover, it will not affect much the result so that we prefer to stick to the simpler definition of strategy to keep the main message as simple as possible.

[15]A formal proof of this statement is provided by Lemma 1.

Summarizing, the median approval mechanism works as follows:

(1) Every player simultaneously and independently announces a closed interval b_i in A and
(2) The mechanism implements $\theta(b)$ with $b = (b_1, \ldots, b_n)$.

As will be shown, with n players, the median approval mechanism unanimously implements the generalized median rule $m(t_1, t_2, \ldots, t_n, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n})$. In the particular situation in which $N = \{1,2,3\}$ and (t_1, t_2, t_3) with $t_1 < t_2 < \frac{1}{3} < \frac{2}{3} < t_3$,[16] it follows that the unique equilibrium outcome should equal $\frac{1}{3}$. This outcome can be thought as a compromise between the extreme types of the players. Moreover, such an outcome is supported by a unique equilibrium $b^* = (b_1^*, b_2^*, b_3^*)$ such that

$$b_1^* = b_2^* = \Big[0, \frac{1}{3}\Big] \quad \text{and} \quad b_3^* = \Big[\frac{1}{3}, 1\Big].$$

It follows that $\sum_{i \in N} \lambda_{\dim(b^*)}(b_i^*) = \frac{4}{3}$ and hence that $f_{b^*}(t) = \frac{3}{2}$ whenever $0 \leq t \leq \frac{1}{3}$ and $f_{b^*}(t) = \frac{3}{4}$ otherwise. One can hence easily check that $\theta(b^*) = \frac{1}{3}$. Figure 1 depicts the distribution of approvals generated by b^*. The alternatives lower than $\frac{1}{3}$ are selected by two players whereas the rest of them just by one. Hence it is graphically simple to understand that $\theta(b^*) = \frac{1}{3}$ since it splits the area below the curve in two exact halfs.

To explain why b^* is an equilibrium, we now describe the consequences of a possible deviation of player 1. Asume that 1 deviates to $b_1' = [0, x]$. The size of $b' = (b_1', b_2^*, b_3^*)$ is equal to $1 + x$ so that $f_{b'}(t) = \frac{2}{1+x}$ whenever $0 \leq t \leq x$ and $f_{b^*}(t) = \frac{1}{1+x}$ otherwise. When x is larger than $\frac{1}{3}$, the total size of b' is higher than the one of b^*; hence the median must be located to the right of $\frac{1}{3}$ since it is the value that divides the area below f_b in two exact halfs. On the contrary, when x is lower than $\frac{1}{3}$, the total size of b' is smaller than the one of b^*. However, the area located to the left of $\frac{1}{3}$ now equals $\frac{2x}{1+x} + \frac{\frac{1}{3}-x}{1+x}$ and hence represents less than half of the approvals. This leads again to a median larger than $\frac{1}{3}$. [17]

A similar reasoning proves the claim for the different deviation of this player and the different players.

If the distribution becomes less polarized and (t_1, t_2, t_3) with $t_1 < \frac{1}{3} < t_2 < \frac{2}{3} < t_3$, it follows that the unique equilibrium outcome should equal t_2. By similar reasonings as before, one can show that this outcome can be supported by an equilibrium b^+ with, for some pair $0 < \delta_1, \delta_2 < \min\{t_2, 1 - t_2\}$

$$b_1^+ = \Big[0, t_2\Big], \quad b_2^+ = \Big[t_2 - \delta_1, t_2 + \delta_2\Big] \quad \text{and} \quad b_3^+ = \Big[t_2, 1\Big]$$

[16] A similar example is analyzed in Austen-Smith and Banks [2005], chapter 6, p.233. In their model, the three players also reach a consensus over an interior policy in the interval [0,1]. The reasons for consensus depend on the discount factors, which is to define the no-delay equilibrium. See also Banks and Duggan [2000] for a bargaining model of collective choice.

[17] More generally, assume by contradiction that player 1 has a best response b_1' such that $b_1' \cap [\frac{1}{3}, 1]$ has positive Lebesgue measure. Then, take the strategy $b_1'' := b_1' \setminus \{b_1' \cap [\frac{1}{3}, 1]\}$. It is simple to see that $\theta(b_1'', b_2^*, b_3^*) < \theta(b_1', b_2^*, b_3^*)$ so that b_1'' leads to a median closer to player 1' s ideal policy than b_1', a contradiction. Similarly, assume again by contradiction that there is some best response b_1' with $b_1^* \setminus b_1'$ having positive Lebesgue measure. Then, $\theta(b_1^*, b_2^*, b_3^*) < \theta(b_1', b_2^*, b_3^*)$ since all these points are located to the left of $\theta(b^*)$.

Two features of the equilibria b^* and b^+ deserve to be highlighted. The first one is that all players include the implemented alternative in their interval. The second one is that the voters' strategies are divided in tree blocks: the ones with a peak to the left of the implemented alternative, the ones to the right and the one in which it coincides. Both characteristics are present in any equilibrium of any approval mechanism discussed in this work.

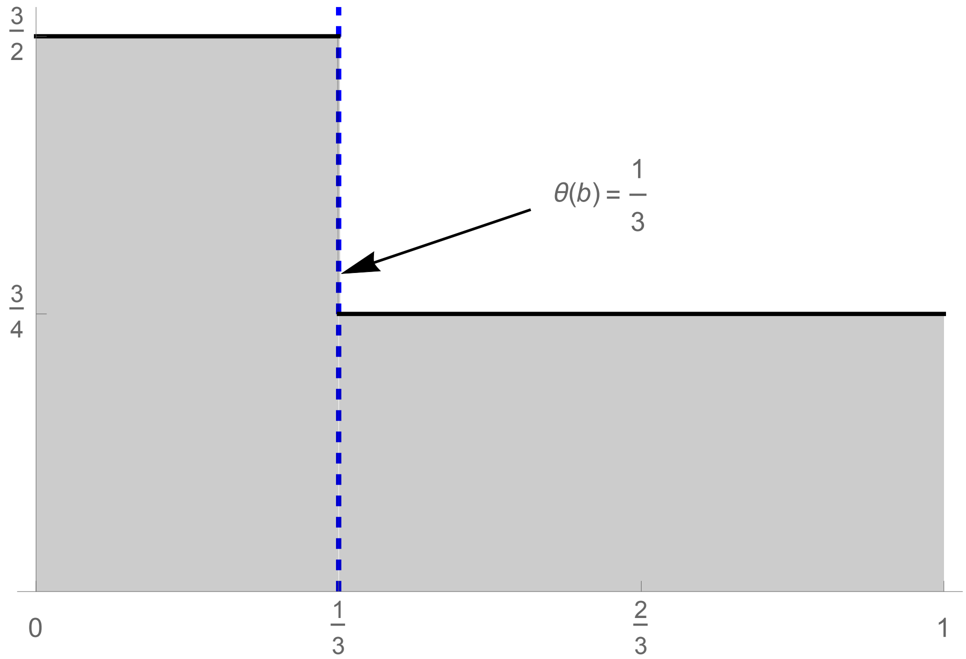

FIGURE 1. Distribution of approvals associated to b^*.

5. CONDITIONS FOR UNANIMOUS IMPLEMENTATION

This section presents the main results of this work. After describing some axioms for the Approval mechanisms to satisfy, it proves that these axioms are enough to characterize unanimous implementation under Approval mechanisms.

5.1. **Axioms on Approval Mechanisms .** We restrict ourselves to anonymous Approval mechanisms[18] such that for each $x \in A$, there is some $b \in \mathcal{B}^n$ with $\theta(b) = x$. We now introduce the axioms that will suffice to identify the Approval mechanism that induce unanimous implementation.

The first axiom deals with the two sort of strategies allowed in an Approval mechanism. Indeed, either a strategy contains finitely many alternatives (zero-dimensional strategy) or infinitely many (one-dimensional strategy). One might argue that zero-dimensional strategies are *stubborn* in the sense that the player is approving of a zero-measure set of the set of available alternatives. Similarly, a one-dimensional strategy is a *compromise* one by opposition to stubborn strategies. The set of stubborn and compromise strategies are respectively labeled by $S(\mathcal{B})$ and $C(\mathcal{B})$ with $\mathcal{B} = S(\mathcal{B}) \cup C(\mathcal{B})$. The Approval Mechanisms in which we will focus give incentives to players to select one-dimensional strategies, in the following sense:

[18]The mechanism $\theta : \mathcal{B}^n \to A$ satisfies Anonymity if for any permutation $\sigma : N \to N$, $\theta(\sigma(b)) = \theta(b)$.

Incentives for Compromise (IC): The mechanism $\theta : \mathcal{B}^n \to A$ satisfies Incentives for Compromise if, for any $i \in N$ and for any $b_i \in S(\mathcal{B})$, $\exists c_i \in C(\mathcal{B})$ with $u_i(\theta(c_i, b_{-i})) > u_i(\theta(b))$, whenever $\theta(b) \neq t_i$.

This axiom ensures that each player has an incentive to submit a compromise strategy rather than a stubborn one, as long as the mechanism does not select his most preferred alternative. The main implication of IC is that there is no equilibrium in which each player announces a singleton as long as the axiom IC holds.

In order to define our second axiom, we introduce the following piece of notation. For each $i \in N$ and any $b_{-i} \in \mathcal{B}^{n-1}$, $\theta(\mathcal{B}, b_{-i})$ denotes the attainable set of player i at b_{-i}; it represents the set of available alternatives that player i can induce when the rest of the players select b_{-i}. Since $\mathcal{B}$ is not finite, the set $\theta(\mathcal{B}, b_{-i})$ need not have a maximum or a minimum. Monotonicity gives precise conditions to characterize the maximum and the minimum of the attainable set when they exist.

Monotonicity (MON): The mechanism $\theta : \mathcal{B}^n \to A$ satisfies Monotonicity if for any $i \in N$ and any $b_{-i} \in \mathcal{B}^{n-1}$, then :

$$b_i^m \in \arg\min \theta(\mathcal{B}, b_{-i}) \text{ if and only if} \exists\ b_i^m = [0, x_i^m] \text{ with } x_i^m = \theta([0, x^m], b_{-i}), \text{and} \tag{1}$$

$$b_i^M \in \arg\max \theta(\mathcal{B}, b_{-i}) \text{ if and only if} \exists\ b_i^M = [x_i^M, 1] \text{ with } x_i^M = \theta([x^M, 1], b_{-i}). \tag{2}$$

That is, when a player attempts to draw the implemented outcome as left as possible it should not be the case that he approves of outcomes to its right and it should not be the case that he does not approve of outcomes to its left, and vice versa.

To define our final two axioms, we consider the following class of strategy profiles. For any $j = 0, 1, \ldots, n$, we define the strategy profile $b^j(x) \in \mathcal{B}^n$ as the strategy profile in which $n - j$ players use the strategy $[0, x]$ and j players use the strategy $[x, 1]$. We let

$$\kappa_j := \{x \in A \mid \theta(b^j(x - \varepsilon)) > \theta(b^j(x)) = x > \theta(b^j(x + \varepsilon)) \text{ for any } \varepsilon > 0\}$$
$$\text{for any } j \in \{0, 1, \ldots, n\}.$$

For simplicity, we say that κ_j denotes the fixed points of $\theta(b^j(x))$ but, more accurately, κ_j is the set of points at which $\theta(b^j(x))$ intersects with x.

For any $b \in \mathcal{B}^n$, we let $\text{Supp}(b) = \bigcup b_i$ denote the support of profile b. The support denotes the set of alternatives that are selected by at least some player. When the support is convex, all alternatives located between the minimum and the maximum of the support are selected. Approval mechanisms are restricted to be continuous in the following sense, as long as they have a convex support.

Continuity (C). The mechanism $\theta : \mathcal{B}^n \to A$ satisfies Continuity if for any $i \in N$, any $b, b^m \in B^n$ with $b^m = (b_i^m, b_{-i})$ such that $\text{Supp}(b), \text{Supp}(b^m)$ are convex,

$$\lim_{m \to \infty} b_i^m = b_i \Longrightarrow \lim_{m \to \infty} \theta(b^m) = \theta(b).$$

This technical axiom introduces a nice property of our mechanism, it should be continuous in each component. This is quite mild since it just applies to the strategy profiles such that $\cup_{j\in N\setminus\{i\}} b_j \in C(\mathcal{B})$.

The final axiom characterizes properties of the fixed points of the Approval mechanisms. We let $h(n) := \frac{n}{2}$ when n is even and $h(n) := \frac{n+1}{2}$ when n is odd, and $G_{g,n} = \{g,..,n-g\}$ when $g \leq \frac{n}{2}$ and $G_{g,n} = \emptyset$ otherwise.

Fixed-Point Monotonicity (FP). The mechanism $\theta : \mathcal{B}^n \to A$ satisfies Fixed-Point Monotonicity if there exists $g \in \{1,...,h(n)\}$ such that: a) κ_j is uniquely defined, interior and strictly increasing in $j \in G_{g,n}$, and b) for any $j < \min\{g, \frac{n}{2}\}$ (resp. $j > \max\{n-g, \frac{n}{2}\}$), $\theta(b^j(x)) < x$ (resp. $\theta(b^j(x)) > x$) when $x \in (0,1)$.

This axiom clearly restricts the class of Approval mechanisms. It is essential to ensure the existence of pure strategy equilibrium and is also behind the uniqueness of the equilibrium outcome.

In order to illustrate which Approval mechanisms satisfy these axioms, we now present two leading examples that will be useful to understand the main intuitions behind our results. Given an approval profile b and an alternative x, we let $s^x(b)$ denote the score of alternative x with $s^x(b) = \#\{i \in N \mid x \in b_i\}$. Thus, any approval profile b generates the function f_b with $f_b(x) = \frac{s^x(b)}{\sum_{i\in N} \lambda_{\dim(b)}(b_i)}$ for any $x \in [0,1]$. As shown by the next lemma, f_b is a well-defined density function for any approval profile b.

Lemma 1. *For any approval profile $b = (b_i, b_{-i})$, f_b is a well-defined density function.*

Proof. : For each profile b, let $V(b,j) \subseteq [0,1]$ be the set such that $V(b,j) = \{x \in [0,1] \mid s^x(b) = j\}$. Moreover,

$$\int_{[0,1]} f_b(x)dx = \frac{1}{\sum_{i\in N} \lambda_{\dim(b)}(b_i)} \int_{[0,1]} s^x(b)dx = \frac{1}{\sum_{i\in N} \lambda_{\dim(b)}(b_i)} \sum_{j=1}^{n} \int_{V(b,j)} j dx.$$

Since $\int_{V(b,j)} j dx = j\lambda_d(V(b,j))$, it follows that $\int_{[0,1]} f_b(x)dx = 1$ as wanted. The previous equality combined with the function $f_b(x)$ being non-negative for any $x \in [0,1]$ concludes the proof. **Q.E.D.**.

The next two Approval mechanisms satisfy the axioms of Continuity, Monotonicity, Fixed Point Coherence and Incentives for Compromise.

Average Approval Mechanism: We let μ_b stand for the mean of the approval profile b with $\mu_b = \int_{[0,1]} x f_b(x)dx$. Note that $\mu_b \in [0,1]$ and hence it always coincides with an alternative. The Average Approval Mechanism associates μ_b to each approval profile b so that $\theta(b) = \mu_b$.

Quantile Approval Mechanism: The cumulative distribution of approvals, $F(x)$, is then given by $F(x) = \int_0^x f_b(t)dt$. The α-Quantile Approval Mechanism associates to each approval profile b the lowest x^* such that $F(x^*) = \alpha$ for some $0 < \alpha < 1$. The median approval mechanism employed in the previous example is a quantile mechanism with $\alpha = \frac{1}{2}$.

5.2. **Sufficiency.** Equipped with the previous results we are now ready to state the sufficient conditions for unanimous implementation.

Theorem 1. *If an Approval Mechanism θ satisfies C, FP, MON and IC, then:*

(1) *there is an equilibrium in pure strategies for every admissible preference profile*
(2) *if $g \leq \frac{n}{2}$, then in every equilibrium b of θ we have $\theta(b) = m(t_1, t_2, \ldots, t_n, \kappa_g, \ldots, \kappa_{n-g})$ and if $g = \frac{n+1}{2}$, then in every equilibrium b of θ we have $\theta(b) = m(t_1, t_2, \ldots, t_n)$,*
(3) *there is an equilibrium b of θ with $\cap_{i=1}^{n} b_i = \theta(b)$.*

Proof. Take some $\theta : \mathcal{B}^n \to [0,1]$ satisfying C, FP, MON and IC. We first notice that there should exist a unique $g \in \{1, \ldots, h(n)\}$ for which FP is satisfied. Throughout the proof we consider that $g \leq \frac{n}{2}$ and in the end we comment why all developed arguments extend to the case in which $g = \frac{n+1}{2}$. For short, we write (t,κ) rather than $(t_1, t_2, \ldots, t_n, \kappa_g, \ldots, \kappa_{n-g})$. The proof first states the existence of equilibrium (**Step A.**), uniqueness of equilibrium outcome (**Step B.**) and finally the existence of a unanimous equilibrium (**Step C.**).

Step A.: There is some equilibrium b of θ with $\theta(b) = m(t,\kappa)$.

Step A. is divided in two cases: either there is no t_h with $t_h = m(t,\kappa)$ (Step A.I.) or there is such t_h to be developed in Step A.II.

Step A.I. $\nexists : t_h$ with $t_h = m(t,\kappa)$. Since there is no t_h with $t_h = m(t,\kappa)$, there must exist $j \in \{g, \ldots, n-g\}$ such that $\kappa_j = m(t,\kappa)$. Therefore, the number of elements located below and above κ_j in (t,κ) is equal to $n-g$, which is equivalent to:

$$\underbrace{\#\{i \in N \mid t_i < \kappa_j\} + (j-g)}_{\text{elements lower than } \kappa_j} = \underbrace{\#\{i \in N \mid t_i > \kappa_j\} + (n-j-g)}_{\text{elements higher than } \kappa_j} = n-g.$$

The previous equalities jointly imply that $\#\{i \in N \mid t_i < \kappa_j\} = n-j$ and $\#\{i \in N \mid t_i > \kappa_j\} = j$. Let $b \in \mathcal{B}^n$ be an approval profile with:

$$b_i = \begin{cases} [0,\kappa_j] & \text{if } t_i < \kappa_j, \\ [\kappa_j, 1] & \text{if } t_i > \kappa_j. \end{cases}$$

Since θ is anonymous by assumption, then $\theta(b) = \theta(b^j(\kappa_j))$ so that $\theta(b) = \kappa_j$ due to FP, and hence that $\theta(b) = m(t,\kappa)$. In order to prove that b is an equilibrium, assume that there is some $i \in N$ with a profitable unilateral deviation b_i', so that $\theta(b_i', b_{-i}) \neq \theta(b)$. Assume first that $\theta(b_i', b_{-i}) < \theta(b)$. If $t_i > \kappa_j$ and given that preferences are single-peaked, it follows that $u_i(\theta(b_i', b_{-i})) < u_i(\theta(b_i, b_{-i}))$. In other words, b_i' is not a profitable deviation, entailing a contradiction. If $t_i < \kappa_j$, then by definition $b_i = [0, \kappa_j]$. However, due to MON, b_i is player i's unique best response, which proves that there is no profitable deviation. The same argument applies if $\theta(b_i', b_{-i}) > \theta(b)$, which proves that b is an equilibrium of the game and concludes Step A.I.

Step A.II. $\exists : t_h$ with $t_h = m(t,\kappa)$. If there exists $j \in \{g, \ldots, n-g\}$ such that $\kappa_j = t_h$, then $j = n-h$ or $j = n-h+1$. Using the same line of reasoning as in A.I., one can show that: a) when $j = n-h+1$, $b^{n-h+1}(t_h)$ is an equilibrium with $\theta(b^{n-h+1}(t_h)) = t_h$ and b) when $j = n-h$, $b^{n-h}(t_h)$ is an equilibrium with $\theta(b^{n-h}(t_h)) = t_h$.

If $t_h = m(t,\kappa)$ and $t_h \neq \kappa_j$, there are $n-g$ values smaller than t_h in (t,κ). There are essentially two cases here: a) $t_h \in (\kappa_g, \kappa_{n-g})$ and b) $t_h < \kappa_g$ (the proof for the case $t_h > \kappa_{n-g}$ is symmetric).

a) One can choose j, such that $g < j < n-g$, with $\kappa_j < t_h = m(t,\kappa) < \kappa_{j+1}$. Moreover $\#\{\kappa_l \mid \kappa_l < t_h\} = j-g+1$ and $\#\{i \in N \mid t_i < t_h\} = h-1$ so that: $j-g+1+h-1 = n-g \Longrightarrow j = n-h$. Therefore, $\kappa_{n-h} < t_h < \kappa_{n-h+1}$.

For each $A \in \mathcal{B}$, we define b^A as the approval profile with:

$$b_i^A = \begin{cases} [0,t_h] & \text{if } t_i < t_h, \\ A & \text{if } t_i = t_h, \\ [t_h,1] & \text{if } t_i > t_h. \end{cases}$$

Our objective is to prove that there is at least one b^{A^*} with $\theta(b^{A^*}) = t_h$. Since θ is continuous on a player's strategy, the result immediately follows from the Intermediate Value Theorem provided that there are some C and D with

$$\theta(C, b_{-h}) < t_h < \theta(D, b_{-h}).$$

Let $\phi_{n-h}(x) = \theta(b^{n-h}(x))$ and $\phi_{n-h+1}(x) = \theta(b^{n-h+1}(x))$ for any $x \in [0,1]$. It follows that $\phi_{n-h}(\kappa_{n-h}) = \kappa_{n-h} < t_h$ and $\phi_{n-h+1}(\kappa_{n-h+1}) = \kappa_{n-h+1} > t_h$. Moreover, since θ satisfies FP, it follows that κ_t is a fixed point of ϕ_t with $\phi_t(\kappa_t) = \kappa_t$ whenever $t = n-h, n-h+1$. We know that (i) $\phi_t : [0,1] \to [0,1]$ and $0 < \kappa_t < 1$. Therefore, since κ_t is a fixed point of ϕ_t and ϕ_t is continuous on $(0,1)$, it must be the case that for any $x \in (0,\kappa_t)$, $\phi_t(x) > x$ and for any $x \in (\kappa_t, 1)$, $\phi_t(x) < x$ whenever $t = n-h, n-h+1$. Now, take $C = [0,t_h]$. Then $b^C = b^{n-h}(t_h)$ and $\theta(b^C) = \phi_{n-h}(t_h)$. Similarly, take $D = [t_h, 1]$ so that $b^D = b_{n-h+1}(t_h)$ and $\theta(b^D) = \phi_{n-h+1}(t_h)$. Therefore, since $\kappa_{n-h} < t_h$ and $t_h < \kappa_{n-h+1}$ it must be respectively the case that $t_h > \phi_{n-h}(t_h) = \theta(b^C)$ and $t_h < \phi_{n-h+1}(t_{n-h+1}) = \theta(b^D)$. We can hence conclude that there exists some A^* with $\theta(b^{A^*}) = t_h$.

In order to prove that b^{A^*} (b for short) is an equilibrium, suppose by contradiction that there exists some $i \in N$ with a profitable deviation $b_i^{'}$. Then, it cannot be the player with type t_h since $\theta(b) = t_h$. Suppose then that $\theta(b_i', b_{-i}) < \theta(b)$. Then, $t_i < t_h$; otherwise, if $t_i > t_h$ then $u_i(b_j', b_{-j}) < u_i(b_i, b_{-i})$, a contradiction with $b_i^{'}$ being a profitable deviation. However, any voter with $t_i < t_h$ is playing his unique best response $[0,t_h]$, entailing again a contradiction. A symmetric argument applies when $\theta(b_i', b_{-i}) > \theta(b)$. Therefore b must be an equilibrium concluding a) in Step A.

b) In this case $t_h = m(t,\kappa) < \kappa_g$ and hence $h = n-g+1$. According to FP we have that $\theta(b^{n-h}(x)) < x$ for every $x \in (0,1)$ (because $n-h = g-1 < g$) and $\theta(b^{n-h+1}(x)) = x$ if and only if $x = \kappa_g$ (because $n-h+1 = g$). Therefore, $\theta([0,t_h], b_{-h}) < t_h$ and $\theta([t_h,1], b_{-h}) > t_h$ and, hence, the continuity arguments used in case a) guarantee here the existence of an interval A^* such that $\theta(b^{A^*}) = t_h$, which ensures the existence of an equilibrium as the one described in a), which concludes the proof of step A. .

Step B.: Any equilibrium b of θ satisfies $\theta(b) = m(t,\kappa)$. Suppose that, there is some θ that admits an equilibrium b with $\theta(b) > m(t,\kappa)$. We let $L_m := \{i \in N \mid t_i \leq m(t,\kappa)\}$ and $F_m := \{j \in \{g,\ldots,n-g\} \mid \kappa_j \leq m(t,\kappa)\}$ with $\#L_m = i'$ and $\#F_m = j'$. However, by definition, it must be the case that $i' + j' \geq n-(g-1)$ so that $n-(g-1)-i' \leq j'$. Thus, $\kappa_{n-(g-1)-i'} \leq \kappa_{j'} =$

$m(t,\kappa)$ (i). By Monotonicity, the unique best response for any player in S equals $[0,\theta(b)]$ so that $\theta(b) \leq \kappa_{n-(g-1)-i'}$ (ii). Combining both (i) and (ii), it follows that $\theta(b) \leq m(t,\kappa)$ a contradiction with $\theta(b) > m(t,\kappa)$. A symmetric claim delivers also a contradiction whenever $\theta(b) < m(t,\kappa)$, proving that $\theta(b) = m(t,\kappa)$ as wanted.

Step C.: There exists some equilibrium b of θ with $\cap_{i=1}^{n} b_i = \theta(b)$. Note that by construction, the equilibrium built in Step A.I satisfies this claim. In Step A.II, $n-1$ players announce t_h in their equilibrium strategy. Hence it suffices to show that there is some A^* with $t_h \in A^*$. If player h plays $[0,t_h]$, the outcome is lower than t_h whereas if he plays $[t_h,1]$ then the outcome is higher than t_h as proved in Step A.II. Observe that if h plays $[\underline{c},\overline{c}]$ and we start from $\underline{c}=0$ and $\overline{c}=t_h$ and first we start increasing $\overline{b}$ from t_h to 1 and then $\underline{c}$ from 0 to t_h, we should have $i)$ always t_h is included in the interval $[\underline{c},\overline{c}]$ and $ii)$ at some point due to the continuity of outcome in $\underline{c}$ and $\overline{c}$ we should have the outcome being equal to t_h.

To see why all these steps hold for the case in which $g = \frac{n+1}{2}$, notice first that in such a case a) n must be odd and b) $t_h = m(t_1,t_2,\ldots,t_n)$ if and only if $h = \frac{n+1}{2}$. Then observe that from FP we know that $\theta(b^{\frac{n-1}{2}}(t_h)) < t_h$ and $\theta(b^{\frac{n+1}{2}}(t_h)) > t_h$. That is, if players behave according to the profile b^A as presented in step A.II. for $h = \frac{n+1}{2}$, there must exist a strategy $A \in C(\mathcal{B})$ such that $\theta(b^A) = t_h$. This establishes existence of an equilibrium b of θ with $\theta(b) = m(t_1,t_2,\ldots,t_n)$. The arguments that establish uniqueness of equilibrium outcome and existence of a unanimous equilibrium are trivial extensions of steps B and C respectively. **Q.E.D.**

As a by-product of the previous Theorem and the continuity axiom, we can establish the following interesting property of Approval mechanisms: these mechanisms are partially revealing in the sense that any player always has a best response in which he approves of his peak t_i.

Lemma 2 (Partially Revealing). *Let $\theta : \mathcal{B}^n \to A$ satisfy C, FP, MON and IC. For any approval profile b and any $i \in N$, there is some best response $b_i \in \mathcal{B}$ with $t_i \in b_i$.*

The proof is an immediate consequence of Monotonicity whenever $\theta(b) \neq t_i$. If $\theta(b) = t_i$, the claim is a consequence of θ being a deterministic mechanism and of the different axioms.

5.3. **Feasibility.** We now state feasible conditions for unanimous implementation. As we now show the axioms defined in the necessity part are not vacuous in the sense that for any strategy-proof direct mechanism, there exists some Approval mechanism that unanimously implements it.

Let $Z : [0,1] \to [0,1]$ be a differentiable and strictly increasing function and q a non-negative real number with $Z(0) = 0$ and $Z(1) = 1$. For any $b \in \mathcal{B}^n$, consider the approval mechanism $\theta_{q,Z}$ such that:

a) if all voters submit singletons then the median report of the singletons is implemented so that

$$\theta_{q,Z}(b) = m(b_1,\ldots,b_n) \quad \text{if} \quad b_i \in S(\mathcal{B}) \quad \forall i \in N,$$

b) otherwise, there are $m \geq 1$ voters who submit a positive length interval. In this case, we let the density function $f_b^{q,Z}$ be such that:

$$f_b^{q,Z}(x) = \sum_{b_i \in C(b,x)} \frac{\left(\frac{q}{\overline{b_i}-\underline{b_i}} + Z'(x)\right)}{q \times m + \sum_{i \in N}\left(Z(\overline{b_i}) - Z(\underline{b_i})\right)}, \text{ for every } x \in [0,1],$$

where $C(b,x) := \{b_i \in b \mid b_i \in C(\mathcal{B}) \text{ and } x \in b_i\}$. For each such b,

$$\theta_{q,Z}(b) := \min\{x \in [0,1] \mid \int_0^x f_b^{q,Z}(t)dt = \frac{1}{2}\},$$

so that $\theta_{q,Z}$ selects as an outcome the median of the distribution function generated by $f_b^{q,Z}$. Each mechanism $\theta_{q,Z}$ is characterized by the distribution function $f_{q,Z}$ and is called a Generalized Median Approval Mechanism ($GMAM$).

To see which sort of aggregators are included within this family, we let, for instance, $q = 0$ and $Z(x) = x$, so that

$$f_b^{q,Z}(x) = \frac{\#C(b,x)}{\sum_{i \in N}\left(\overline{b_i} - \underline{b_i}\right)} \text{ for every } x \in [0,1].$$

This density function just differs from the one associated with the Median Approval mechanism, described in Section 4, in zero-measure sets, i.e. the singleton strategies used by the players in $N \setminus C(b,x)$. To see why, notice that for any approval profile b and any alternative x, $\#C(b,x) = s^x(b)$ whenever any $b_i \in C(\mathcal{B})$. Thus, it leads to the same outcome as the Median Approval mechanism.

As in Lemma 1, one can prove that any $f_b^{q,Z}$ is a well-defined density function.

Lemma 3. *For any approval profile $b = (b_i, b_{-i})$, any non-negative q and any $Z : [0,1] \to [0,1]$, $f_b^{q,Z}$ is a well-defined density function.*

Proof. Take any $f_b^{q,Z}$ and note first that $f_b^{q,Z}(x) \geq 0$ for any $x \in [0,1]$. It suffices to show that its integral over [0,1] equals 1, which is equivalent to

$$\int_0^1 f_b^{q,Z}(x)dx = \int_0^1 \sum_{b_i \in C(b,x)} \frac{\left(\frac{q}{\overline{b_i}-\underline{b_i}} + Z'(x)\right)}{q \times m + \sum_{i \in N}\left(Z(\overline{b_i}) - Z(\underline{b_i})\right)} dx = 1.$$

Since $f_b^{q,Z}(x) \geq 0$ for any $x \in [0,1]$, we can express the integral of the sums as the sums of the integrals so that

$$\int_0^1 f_b^{q,Z}(x)dx = \sum_{b_i \in C(b,x)} \int_{\underline{b_i}}^{\overline{b_i}} \frac{\left(\frac{q}{\overline{b_i}-\underline{b_i}} + Z'(x)\right)}{q \times m + \sum_{i \in N}\left(Z(\overline{b_i}) - Z(\underline{b_i})\right)} dx$$
$$= \sum_{b_i \in C(b,x)} \frac{1}{q \times m + \sum_{i \in N}\left(Z(\overline{b_i}) - Z(\underline{b_i})\right)} \int_{\underline{b_i}}^{\overline{b_i}} \left(\frac{q}{\overline{b_i} - \underline{b_i}} + Z'(x)\right)dx = 1,$$

which concludes the proof.

Q.E.D..

The vector of fixed points of each mechanism $\theta_{q,Z}$ is denoted by $\kappa^{q,Z}$ and is defined as follows. Recall that for any $j = 0, 1, \ldots, n$ the strategy profile $b^j(x) \in \mathcal{B}^n$ is the one in which

$n-j$ players use the strategy $[0,x]$ and j players use the strategy $[x,1]$. We let

$$\kappa_j^{q,Z} := \{x \in A \mid \theta_{q,Z}(b^j(x-\varepsilon)) > \theta_{q,Z}(b^j(x)) = x > \theta_{q,Z}(b^j(x+\varepsilon)) \text{ for any } \varepsilon > 0\}$$
$$\text{for any } j \in \{0,1,\ldots,n\}.$$

Therefore, we have for any j,

$$\frac{(n-j)Z(\kappa_j^{q,Z})+(n-j)q}{qn+(n-j)Z(\kappa_j^{q,Z})+j[1-Z(\kappa_j^{q,Z})]} = 1/2 \Leftrightarrow Z(\kappa_j^{q,Z}) = \frac{j+q(2j-n)}{n}.$$

Proposition 1. *Any GMAM satisfies IC, MON, C and FP.*

While the formal proof of this proposition is relegated to the appendix, we not briefly mention the intuition for these axioms to be satisfied by any $GMAM$. IC holds since a player submitting a singleton has no weight in the final decision. MON is satisfied since the maximal influence on the outcome of the mechanism is to include all the alternatives located to the left (to minimize it) or the right of the outcome (to maximize it). C holds since as long as one has convex support, the mechanism is implementing the median of a cumulative distribution. Finally, FP holds almost by construction: indeed, the $GMAM$s are designed to exhibit trackable phantoms.

We are now ready to state the main result of this section.

Theorem 2. *If the number of voters is even (resp. odd), for any strategy proof mechanism $m(t,\kappa)$ with p different phantoms for any odd p (resp. even), there exists some GMAM that unanimously implements it.*

Proof. By definition, each $GMAM$ $\theta_{q,Z}$ is characterized by a function Z and some non-negative number q. Moreover, since every $\theta_{q,Z}$ satisfies our four axioms, each $\theta_{q,Z}$ unanimously implements a strategy-proof rule f as stated by Theorem 1. Again due to Theorem 1, the game associated to $\theta_{q,Z}$ has a unique equilibrium outcome characterized his vector of fixed points $\kappa^{q,Z} = (\kappa_1^{q,Z},\ldots,\kappa_{n-1}^{q,Z})$. The equilibrium outcome is hence equal to $m(t,\kappa^{q,Z})$. Note that each strategy-proof rule $f = m(t,\kappa)$ is uniquely determined by $\kappa = (\kappa_1,\ldots,\kappa_{n-1})$. Therefore, in order to establish validity of this theorem, it is sufficient to show that for each κ, there exists an admissible θ_Z with a corresponding $\kappa^Z = \kappa$. As previously argued the fixed points of $\theta_{q,Z}$ satisfy the following equation:

$$Z(\kappa_j^{q,Z}) = \frac{j+q(2j-n)}{n}.$$

Note that Z is invertible since, by definition, $Z(x)$ is differentiable and strictly increasing on $[0,1]$ with $Z(0)=0$ and $Z(1)=1$. Thus, for each κ with $n-1$ distinct weights, there exists at least one $GMAM$ $\theta_{q,Z}$ with

$$\kappa^{q,Z} = (\kappa_1^{q,Z},\ldots,\kappa_{n-1}^{q,Z}) \text{ such that } \kappa_j^{q,Z} = Z^{-1}(\frac{j+q(2j-n)}{n}),$$

for each $j \in \{1,\ldots,n-1\}$. In other words, for each κ, there exists an admissible $\theta_{q,Z}$ with a corresponding $\kappa^{q,Z} = \kappa$.

Concerning the fixed points vector κ with less than $n-1$ points, note that if $j < n/2$ then $\frac{j+q(2j-n)}{n} \leq 0$ if and only if $q \geq \frac{j}{n-2j}$ and if $j > n/2$ then $\frac{j+q(2j-n)}{n} \geq 1$ if and only if $q \geq \frac{n-j}{2j-n}$. By increasing the q, one can trim any arbitrary number of extreme $\kappa^{q,Z}$s and then by appropriately choosing Z, one can give to each of the non-trimmed $\kappa^{q,Z}$s any value in [0,1], which concludes the proof. **Q.E.D.**

As an illustration of the previous result, consider the *GMAM* triggered by setting $q = 1$ and $Z(x) = x$. In this case, the density function equals

$$f_b^{q,Z}(x) = \frac{\#C(b,x) + \sum_{b_i \in C(b,x)} \left(\frac{1}{\overline{b_i} - \underline{b_i}}\right)}{m + \sum_{i \in N} \left(\overline{b_i} - \underline{b_i}\right)} \text{ for every } x \in [0,1].$$

In the particular situation in which $N = \{1,2,3\}$ (so that $n = 3$) and the voters' types satisfy $t_1 < t_2 < t_3$, this Approval mechanism implements the pure median rule since:

$$Z(\kappa_j^{q,Z}) = \frac{j + q(2j-n)}{n} \Leftrightarrow \kappa_j^{q,Z} = \frac{3j-3}{3} \Leftrightarrow \kappa_1^{q,Z} = 0 \text{ and } \kappa_2^{q,Z} = 1.$$

To see why this is true, consider the strategy profile b with $b_1 = b_2 = [0,t_2]$ and $b_3 = [t_2,1]$. Then:

$$\theta_{q,Z}(b) = \frac{2t_2+2}{3+2t_2+1-t_2} > \frac{1}{2} \text{ for every } t_2 \in (0,1).$$

If player 2 deviates to $b_2' = [t_2,1]$, then the outcome of $b' = (b_1, b_2', b_3)$ equals:

$$\theta_{q,Z}(b') = \frac{t_2+1}{3+t_2+2(1-t_2)} < \frac{1}{2} \text{ for every } t_2 \in (0,1).$$

That is, if the t_1-voter plays $[0,t_2]$ and the t_3-voter plays $[t_2,1]$ then the t_2-voter by smoothly changing her strategy from $[0,t_2]$ to $[t_2,1]$ can find a strategy that contains t_2 and which leads to $F_{q,Z}(t_2) = 1/2$, that is, to the unanimous implementation of her ideal policy.

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Appendix A. The class of GMAM

To describe the class of *GMAM*s , we start by describing the cumulative distribution of the median approval mechanism described in Section 4.

Suppose that there is just one player in the society os that $n = 1$. If he selects the interval $b_i = [\min b_i, \max b_i]$, the share of approvals lower than x equals $\frac{F(b_i,x)}{\max b_i - \min b_i}$ with

$$F(b_i,x) = \begin{cases} 0, & \text{if } x < \min b_i. \\ x - \min b_i, & \text{if } \max b_i \leq x \leq \min b_i. \\ \max b_i - \min b_i, & \text{if } x > \max b_i. \end{cases}$$

For any $b = (b_i, b_{-i})$, the share of approvals until x equals

$$F(b,x) = \frac{\sum_{i \in N} F(b_i,x)}{\sum_{i \in N}(\max b_i - \min b_i)},$$

and since θ is the Median Approval mechanism,

$$\theta(b) := \min\{x \in [0,1] \mid F(b,x) = \frac{1}{2}\}.$$

The same logic applies to a Generalized Median Approval Mechanism. Let q be a non-negative real number and $Z : [0,1] \to [0,1]$ a continuous and strictly increasing fuction with $Z(0) = 0$ and $Z(1) = 1$. If there is just one player ($n = 1$) and he selects the interval $b_i = [\min b_i, \max b_i]$, so that the share of approvals lower than x given q and Z equals $\frac{F_{q,Z}(b_i,x)}{Z(\max b_i)-Z(\min b_i)+q}$ with

$$F_{q,Z}(b_i,x) = \begin{cases} 0, & \text{if } x < \min b_i. \\ Z(x) - Z(\min b_i) + \frac{(x-\min b_i)}{(\max b_i - \min b_i)}q, & \text{if } \max b_i \le x \le \min b_i. \\ Z(\max b_i) - Z(\min b_i) + q, & \text{if } x > \max b_i. \end{cases}$$

For any $b = (b_i, b_{-i})$ with $b_i \in C(\mathcal{B})$ for every $i \in N$, the share of approvals lower than x equals

$$F_{q,Z}(b,x) = \frac{\sum_{i\in N} F_{q,Z}(b_i,x)}{n \times q + \sum_{i\in N}(Z(\max b_i) - Z(\min b_i))}.$$

The outcome $\theta_{q,Z}$ is the median of this cumulative distribution.

For ease of exposition, we introduce the following notation and focus on profiles with $b_i \in C(\mathcal{B})$ for every $i \in N$. The similar argument applies if some player(s) select singletons.

For any b with convex support, we let $\eta_{-i}(b,q,Z) := (n-1) \times q + \sum_{j\in N\setminus\{i\}}(Z(\max b_j) - Z(\min b_j))$. Note that

$$F_{q,Z}(b,x) = \frac{Z(x) - Z(\min b_i) + \frac{(x-\min b_i)}{(\max b_i - \min b_i)}q + \sum_{j\neq i} F_{q,Z}(b_j,x)}{Z(\max b_i) - Z(\min b_i) + q + \eta_{-i}(b,q,Z)}.$$

Since $F_{q,Z}(b_j,x)$ is a cumulative distribution, note that

$$\sum_{j\neq i} F_{q,Z}(b_j,x) < \eta_{-i}(b,q,Z)$$

for any b, q, Z and any $i \in N$. The next proposition show how the outcome varies when a player varies the lower and upper bound of his strategy.

Lemma 4. *Consider any profile $b = (b_i, b_{-i})$ with convex support. Then,*

(1) *if $\min b_i < \max b_i < \theta(b)$, then*

$$\frac{\partial}{\partial \min b_i}\theta(b_i,b_{-i}) > 0 \text{ and } \frac{\partial}{\partial \max b_i}\theta(b_i,b_{-i}) < 0.$$

(2) *if $\min b_i < \theta(b) < \max b_i$, then*

$$\frac{\partial}{\partial \min b_i}\theta(b_i,b_{-i}) > 0 \text{ and } \frac{\partial}{\partial \max b_i}\theta(b_i,b_{-i}) > 0.$$

(3) *if $\theta(b) < \min b_i < \max b_i$, then*

$$\frac{\partial}{\partial \min b_i}\theta(b_i,b_{-i}) < 0 \text{ and } \frac{\partial}{\partial \max b_i}\theta(b_i,b_{-i}) > 0.$$

Proof. Consider first the case with $\min b_i < \max b_i < \theta(b)$. Consider x such that $F_{q,Z}(b,x) = 1/2$. Note that

$$\frac{\partial}{\partial \max b_i} F_{q,Z}(b_i,x) = \frac{\Big(\eta_{-i}(b,q,Z) - \sum_{j\neq i} F_{q,Z}(b_j,x))\Big) Z'(\max b_i)}{(Z(\max b_i) - Z(\min b_i) + \eta_{-i}(b,q,Z) + q)^2} > 0.$$

That is as $\max b_i$ increases $\theta(b)$ has to decrease for the median to be still equal to $\frac{1}{2}$ so that $\frac{\partial}{\partial \max b_i}\theta(b_i, b_{-i}) < 0$, as wanted. As far as varying the lower bound of b_i, notice that

$$\frac{\partial}{\partial \min b_i} F_{q,Z}(b_i,x) = \frac{\Big(\sum_{j\neq i} G_{q,Z}(b_j,x) - \eta_{-i}(b,q,Z)\Big) Z'(\min b_i)}{(Z(\max b_i) - Z(\min b_i) + \eta_{-i}(b,q,Z) + q)^2} < 0.$$

Again, since $\min b_i$ increases $\theta(b)$ has to increase for the median to be still equal to $\frac{1}{2}$ so that $\frac{\partial}{\partial \min b_i}\theta(b_i, b_{-i}) > 0$, as wanted. The case in which $\theta(b) < \min b_i < \max b_i$ is symmetric and hence is omitted.

Consider now the case with $\min b_i < \theta(b) < \max b_i$. One can check that

$$\frac{\partial}{\partial \max b_i}\left(Z(x) - Z(\min b_i) + q\frac{x - \min b_i}{\max b_I - \min b_i} + \sum_{j\neq i} F_{q,Z}(b_i,x)\right) = \frac{q(\min b_i - x)}{(\min b_i - \max b_i)^2} < 0,$$

whereas

$$\frac{\partial}{\partial \max b_i}\Big(Z(\max b_i) - Z(\min b_i) + q + \eta_{-i}(b,q,Z)\Big) = Z'(\max b_i) > 0.$$

Thus,

$$\frac{\partial}{\partial \max b_i} F_{q,Z}(b_i,x) < 0,$$

so that as $\max b_i$ increases x has to increase, showing that so that $\frac{\partial}{\partial \max b_i}\theta(b_i, b_{-i}) > 0$, as wanted. Symmetrically one can show that as $\min b_i$ increases $\theta(b)$ has to increase for $F_{q,Z}(b,x)$ to be still equal to $\frac{1}{2}$. **Q.E.D.**

Once we have proved this key property of *GMAM*, we prove that each *GMAM* satisfies the different axioms used in the characterization.

Lemma 5. *Any GMAM satisfies IC.*

Proof. Take some b with $\theta(b) \neq t_i$ and $b_i \in C(\mathcal{B})$. Let $t_i < \theta(b)$ w.l.o.g. Applying Lemma 4, it is simple to see that $\theta([t_i, t_i + \delta], b_{-i}) < \theta(b)$ so that $\exists\, c_i \in C(\mathcal{B})$ with $u_i(\theta(c_i, b_{-i})) > u_i(\theta(b))$, as desired. **Q.E.D.**

Lemma 6. *Any GMAM satisfies MON.*

Proof. We now prove that for any *GMAM* θ, the equivalence (1) holds. A similar proof applies to the characterization of the maximum of the attainable set.

1. Sufficiency. Take some $i \in N$ and assume that there is some $b_i^* \in \arg\min \theta(\mathcal{B}, b_{-i})$ with $b_i^* \neq b_i^m$. Let $x^* = \theta(b_i^*, b_{-i})$. Since $b_i^* \neq b_i^m$, this means that either $b_i^* \cap [0, x^*] \neq \emptyset$ (1.a.) or $b_i^* \cap [x^*, 1] \neq \emptyset$ (1.b) or both (1.c). In each of these cases, Lemma 4 directly implies that $\theta([0, x^*], b_{-i}) < \theta(b_i^*, b_{-i})$, a contradiction with $b_i^* \in \arg\min \theta(\mathcal{B}, b_{-i})$.

2. Necessity. Take some $i \in N$ and assume that there is some b_i^m with $b_i^m = [0, x_i^m]$ and $x_i^m = \theta([0, x^m], b_{-i})$. Assume that $b_i^m \notin \arg\min \theta(\mathcal{B}, b_{-i})$, so that there is some b_i^* with $\theta(b_i^*, b_{-i}) < \theta([0, x^m], b_{-i})$. By definition, it must be the case that this means that either $b_i^* \cap [0, x_i^m] \neq \emptyset$ or $b_i^* \cap [x_i^m, 1] \neq \emptyset$ or that both inequalities hold simultaneously. However, Lemma 4 again directly. proves that for any $b_i^* \in \mathcal{B}$, $\theta(b_i^*, b_{-i}) \geq \theta([0, x^m], b_{-i})$, entailing a contradiction. **Q.E.D.**

Lemma 7. *Any GMAM satisfies C.*

Proof. : Take any *GMAM* θ with density function f_b. Take some $i \in N$ and any pair $b, b^m \in B^n$ with $b^m = (b_i^m, b_{-i})$ such that $\text{Supp}(b), \text{Supp}(b^m) \in C(\mathcal{B})$ Assume moreover that $\lim_{m\to\infty} b_i^m = b_i$. It follows that

$$\lim_{m\to\infty} f_{b^m}(x) = f_b(x) \qquad \text{for any } x \in \text{Supp}(b).$$

We let $F_b(x)$ and $F_{b^m}(x)$ respectively denote the cumulative distribution of f_b and f_{b^m}. Since $\text{Supp}(b), \text{Supp}(b^m) \in C(\mathcal{B})$, $F_b(x)$ and $F_{b^m}(x)$ are strictly increasing and continuous (hence invertible) on $\text{Supp}(b)$ and $\text{Supp}(b^m)$. The respective inverse functions are denoted by $F_b^{-1} : [0,1] \to \text{Supp}(b)$ and $F_{b^m}^{-1} : [0,1] \to \text{Supp}(b^m)$. Therefore,

$$\lim_{m\to\infty} F_{b^m}^{-1}(x) = F_b^{-1}(x) \qquad \text{for any } x \in \text{Supp}(b).$$

Since for any b with invertible F_b, $\theta(b) = F_b^{-1}(\frac{1}{2})$, it follows that $\lim_{m\to\infty} \theta(b^m) = \theta(b)$, as wanted. **Q.E.D.**

Lemma 8. *Any GMAM satisfies FP.*

Proof. The vector of fixed points of each mechanism $\theta_{q,Z}$ is denoted by $\kappa^{q,Z}$ and is defined as follows. Recall that for any $j = 0, 1, \ldots, n$ the strategy profile $b^j(x) \in \mathcal{B}^n$ is the one in which $n - j$ players use the strategy $[0, x]$ and j players use the strategy $[x, 1]$. We let

$$\kappa_j^{q,Z} := \{x \in A \mid \theta_{q,Z}(b^j(x - \varepsilon)) > \theta_{q,Z}(b^j(x)) = x > \theta_{q,Z}(b^j(x + \varepsilon)) \text{ for any } \varepsilon > 0\} \text{ for any } j \in \{0, 1, \ldots, n\}.$$

Each such $\kappa^{q,Z}$ must satisfy for any $j \in \{1, ..., n-1\}$, $\kappa_j^{q,Z}$ the following equivalence:

$$\frac{Z(\kappa_j^{q,Z}) \times (n-j) + (n-j) \times q}{q \times n + Z(\kappa_j^{q,Z}) \times (n-j) + [1 - Z(\kappa_j^{q,Z})] \times j} = 1/2 \Leftrightarrow Z(\kappa_j^{q,Z}) = \frac{j + q(2j - n)}{n}.$$

Note that by assumption Z is continuous and strictly increasing. It is hence invertible so that for any j

$$\kappa_j^{q,Z} = Z^{-1}\left(\frac{j + q(2j - n)}{n}\right).$$

If $j < n/2$ then $\frac{j+q(2j-n)}{n} \leq 0$ if and only if $q \geq \frac{j}{n-2j}$ and if $j > n/2$ then $\frac{j+q(2j-n)}{n} \geq 1$ if and only if $q \geq \frac{n-j}{2j-n}$. By increasing the q we can "trim" any arbitrary number of extreme $\kappa^{q,Z}$s. Moreover, by appropriately choosing Z, we can give to each of the non-trimmed $\kappa^{q,Z}$s any value between zero and one as required. **Q.E.D.**