Global Estimation and Inference of Regression Discontinuity Design with Discrete Ordered or Duration Outcomes*

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Abstract

We consider the regression discontinuity (RD) design with the duration outcome which has discrete support. The parameters of policy interest are treatment effects on unconditional (duration effect) and conditional (hazard effect) exiting probabilities for each discrete level. We find that a flexible separability structure of the underlying continuous-time duration process can be exploited to substantially improve the quality of the fully nonparametric estimator. We propose global series-MLE-based estimators, and associated marginal and simultaneous inference. Simultaneous inference over discrete levels is nonstandard since the asymptotic variance matrix is singular with unknown rank. The peculiarity is delivered by the nature of the RD estimand, and we provide solutions. Random censoring and competing risks can also be allowed in our framework. The standard practice of applying local linear estimators to a sequence of binary outcomes is in general unsatisfactory, which motivates our semi-nonparametric approach. First, it provides poor hazard estimates near the end of the observation period due to small sizes of risk sets (in the neighborhood of the cutoff). Second, it fits each probability separately and thus does not support joint inference. The estimation and inference methods we advocate in this paper are computationally easy and fast to implement, which is illustrated by numerical examples.

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1 Introduction

Regression discontinuity (RD) design has been one of the state-of-the-art empirical methodologies in economics for identifying the effects of a (potentially endogenous) treatment for its simplicity and validity under relatively weak assumptions. Its popularity is witnessed by two recent survey articles on the design (Imbens and Lemieux (2008) and Lee and Lemieux (2010)) which have received, respectively, more than nineteen and seventeen hundreds Google Scholar citations at the time of writing.

In this paper, we consider the RD design with duration outcomes which have discrete support. This work is motivated by the emerging empirical literature on using RD designs with duration outcomes, in which the most active area is probably on sensitivity of job searching behavior to extended/reduced unemployment insurance (UI) benefits. Card, Chetty and Weber (2007) used previous-job-tenure discontinuity in eligibility for both severance pay and extended UI benefits in Austria. Age discontinuity in eligibility for extended UI benefits is exploited by Lalive (2007, 2008) in Austria, and by Schmieder, von Wachter and Bender (2012, 2016) and Caliendo, Tatsiramos and Uhlendorff (2013) in Germany. Lalive (2008) also used discontinuity at the border between eligible regions and control regions. Johnston and Mas (2015) used the calendar-time discontinuity, i.e. a cut of potential UI duration in Missouri in 2011. A related literature on kink designs exploits derivative discontinuities of the benefit schedule at the minimum and maximum benefit levels; see Card, Lee, Pei and Weber (2015), Card, Johnston, Leung, Mas and Pei (2015) and Landais (2015).

In these applications, the (log-)duration treatment effect is typically estimated by treating the outcome as continuous and applying the local linear estimator using observations within a neighborhood of the cutoff, where the neighborhood size is selected in a predetermined or data-dependent way, e.g. the method of Imbens and Kalyanaraman (2012). The hazard effect is less often reported (and at less length), despite its central importance in economic duration analysis. When it is reported, econometric models and methods vary across authors. A practice which
has been adopted in several studies to estimate hazard effects, is to first aggregate outcomes at more coarse levels (e.g. in weeks or months, for unemployment data), and then at each level, a local linear probability model (local LPM) is fitted using the subsample of survivors (up to the said level) within the neighborhood of the cutoff. The LPM-based approach for hazard effects is fully nonparametric, and respects the discrete nature of outcome, but it is not satisfactory for two reasons, in addition to the well cited overshooting-the-unit-interval issue. First, it provides rather poor hazard estimates near the end of the observation period due to small sizes of risk sets (in the neighborhood of the cutoff). We find in realistic settings, through simulations, that the confidence intervals might cover more than half of the unit interval when the sample size is 5000, in the absence of censoring. Second, it fits each probability separately and thus does not support joint inference. For example, it does not speak to joint significance of hazard effects for time intervals, or the shape (e.g. uniformity) of hazard effects across time intervals.

In this paper, we provide a semi-nonparametric discrete-time framework which allows estimation and inference of RD treatment effects on unconditional (i.e. duration effects) and conditional probabilities (i.e. hazard effects) for each discrete time-interval in a unified way. In particular, we find that a separability structure of the underlying continuous-time hazard process can be exploited to produce estimates of RD treatment effects which substantially improve the quality of the estimators based on a fully nonparametric model. The separability structure we exploit ((15) and (16) below) is commonly referred to as the proportional hazard (PH), in which we allow both the baseline hazard function and the risk function to be nonparametric, extending Cox’s (1972) classical specification. For each discrete interval, RD estimands involve both functions. Although only the local behavior of the risk function matters for RD estimands, the baseline hazard function plays a role through a finite number of functionals which do not depend on the running variable and thus can be estimated faster than a nonparametric rate by using the entire sample, due to separability and our discrete-time perspective. This intuition leads to our global approach.

The PH assumption, while it is not costless in restricting the data, underlies the class of perhaps the most commonly used reduced-form hazard-based duration models (van den Berg, 2001). It has been already used the RD literature mentioned above as the primary model to estimate hazard

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1See, e.g. Schmieder et al. (2012, Web Appendix (Section 2)) and Landais (2015).
effects or the alternative model for robustness checks. The version we adopt is relatively weak in the current setting in at least three senses. First, it speaks little to more than the average treatment effects at the cutoff, due to the nonparametric specification of the risk function. Second, it does not improve the convergence rate of the fully nonparametric estimator. Third, it does not identify the underlying continuous-time model from the discrete duration data.

Our point estimators are based on the maximum likelihood with series approximation of unknown functions (Chen, 2007) for its implemental simplicity. We prove asymptotic normality which delivers standard error formulas for duration and hazard effects for each time interval, thus confidence intervals can be constructed. Simultaneous inference across time intervals are nonstandard since the asymptotic variance matrix is singular, with the rank depending on the data generating process. Singularity comes from the different convergence rates for elements of the estimand. The unknown rank of the variance matrix is due to the nature of the RD estimand whose variance matrix is a sum of two (singular) positive semi-definite matrices. The generalized Wald-type inference generally fails in this setting (Andrews, 1987), and we provide two valid alternatives. The MATLAB codes for empirical implementation are available from the author’s website.

The paper is also related to the literature that extends Cox’s classical model to allow a nonparametric risk function; see Fan, Gijbels and King (1997), Chen and Zhou (2007) and Chen et al. (2010), among others. It has been realized that nonparametric extension of this sort is most useful when the covariate is one-dimensional (if no additional structure is added), which is the case in our situation. The results there, however, are not directly adoptable in the current setting for two reasons. First, the literature mostly relies on variants of the Cox’s partial likelihood and continuous-time durations (or with few identical durations (ties)). This is inconvenient in our setting where there are many ties. Second, they mostly focus on estimation and inference of the risk function, which is only an ingredient of the hazard and duration treatment effects of primary policy interest we consider here.

In an earlier paper, Xu (2016) developed a fully nonparametric local multinomial analysis for the RD design with categorical outcomes, as an alternative to the commonly used local LPM-based method. The approach can also be applied in the current setting, with each time-interval as a category, and naturally supports simultaneous inference of (duration or hazard) treatment effects
across time intervals. However, it does not utilize the ordinal structure of the outcome and, of practical importance, becomes computationally burdensome and unstable when the categories are many (which is the case for unemployment discrete-duration applications) or the sample size is large (or both) since it allows a nonparametric function for each time intervals. We instead pursue a parsimonious (yet flexible) specification in the current paper.

Our setting is different from the one considered by van den Berg, Bozio and Costa Dias (2014), which requires the policy change to interrupt duration outcomes and the identification exploits the interrupted spells. For example, individuals are treated (into a program that helps job searching) after being unemployed for a certain time period. Our approach assumes the time-invariant forcing variable (i.e. the treatment status is determined once the individual enters the state), thus is tailored for applications mentioned earlier in this section.

We start in Section 2 by considering estimation and inference of the RD design with ordered discrete outcomes. The analysis is of independent empirical interest, and provides key insight into the statistical issues for the focal case of this paper on discrete duration outcomes, which is outlined in Section 3. We also allow random censoring of the outcome and competing exiting risks. Numerical examples are given in Section 4, and the finite-sample performance of other more commonly-used estimators are compared. Section 5 concludes, and the appendix contains all technical details.

2 RD with ordered outcomes

2.1 The model

For the individual $i$, the outcome $\tilde{Y}_i$ takes $(J + 1)$ mutually exclusive ordered possibilities, $\tilde{Y}_i \in \{1, 2, ..., J, J + 1\}$, e.g. levels of happiness, credit ratings or health statuses. The level $J + 1$ is used as the base category (which will be also convenient in the next section).

Consider the typical (sharp) RD design, in which the treatment $T_i$ is driven by the continuous running variable $X_i$ (taking on values in $\mathcal{X} \subset \mathbb{R}$) together with a cutoff $c$, i.e. $T_i = \mathbb{1}(X_i \geq c)$. Let $\tilde{Y}_i(1)$ and $\tilde{Y}_i(0)$ be two potential outcomes for the treatment and control groups, respectively. We observe $\tilde{Y}_i = T_i\tilde{Y}_i(1) + (1 - T_i)\tilde{Y}_i(0)$. 
For \( j = 1, 2, ..., J + 1 \), the outcome probabilities for two groups are

\[
\begin{align*}
\mathbb{P}(\widetilde{Y}_i(1) = j | X_i = x) &= \mu_{+,j}(x) \\
\mathbb{P}(\widetilde{Y}_i(0) = j | X_i = x) &= \mu_{-,j}(x)
\end{align*}
\]

where continuous functions \( \mu_+(x) = (\mu_{+,1}(x), ..., \mu_{+,J}(x))^{\top} \) and \( \mu_-(x) = (\mu_{-,1}(x), ..., \mu_{-,J}(x))^{\top} \) are unknown, and together with \( \mu_{+,J+1}(x) \) and \( \mu_{-,J+1}(x) \) (the outcome probabilities for the base category \( J + 1 \)) they satisfy \( \sum_{j=0}^{J} \mu_{+,j}(x) = \sum_{j=0}^{J} \mu_{-,j}(x) = 1 \), respectively, for any \( x \in \mathcal{X} \). As in most of the RD literature, we leave functions \( \mu \)'s unspecified.

The objects of primary interest in this paper are

\[
\tau_j = \mu_{+,j}(c) - \mu_{-,j}(c),
\]

for \( j = 1, ..., J \). They are interpreted as the ATEs (average treatment effects) at \( c \) of the treatment \( T_i \) on outcome probabilities under standard RD assumptions; see Imbens and Lemieux (2008). The negatives of \( \tau_j \)'s would be ATEs if the treatment is determined by \( 1 - T_i \). Let \( \tau = (\tau_1, ..., \tau_J)^{\top} \).

To reflect the ordered structure of the outcomes, we use the transformations

\[
\begin{align*}
\mu_{+,j}(x) &= \Phi(\alpha_{+,j} + g_+(x)) - \Phi(\alpha_{+,j-1} + g_+(x)) \\
\mu_{-,j}(x) &= \Phi(\alpha_{-,j} + g_-(x)) - \Phi(\alpha_{-,j-1} + g_-(x))
\end{align*}
\]

for \( 1 \leq j \leq J + 1 \) and \( x \in \mathcal{X} \). The sequences of constants \( \{\alpha_{+,0} < \alpha_{+,1} < ... < \alpha_{+,J+1}\} \) and \( \{\alpha_{-,0} < \alpha_{-,1} < ... < \alpha_{-,J+1}\} \) are such that \( \alpha_{+,0} = \alpha_{-,0} = -\infty \),

\[
\alpha_{+,J} = \alpha_{-,J} = 0
\]

and \( \alpha_{+,J+1} = \alpha_{-,J+1} = \infty \). We write the unknowns \( \alpha_+ = (\alpha_{+,1}, ..., \alpha_{+,J+1})^{\top} \) and similarly for \( \alpha_- \). The functions \( g_+(\cdot), g_-(\cdot) : \mathcal{X} \mapsto \mathbb{R}^1 \) are unknown. The function \( \Phi(\cdot) \) is a strictly increasing CDF, e.g. the logistic function \( (\Phi(\cdot) = (1 + \exp(-\cdot))^{-1}) \), standard normal or complementary log-log CDF (the last of which will be adopted in the next section). The standardization condition (6) is used...
so that \( g_+() \) and \( g_-() \) are unconstrained.\(^2\) The model is semi-nonparametric in the sense of Chen (2007).\(^3\)

If the functions \( g_+ \) and \( g_- \) are linear, the approach we consider reduces to assuming standard ordered outcome models for the treated and control groups in the RD setting. Aitchison and Silvey (1957) and McCullagh (1980) provide early treatment of (parametric) ordered response models.\(^4\)

Note that \( \tau \), the object of interest, depends on both finite-dimensional \((\alpha_+ \text{ and } \alpha_-)\) and infinite-dimensional parameters \((\text{the functions } g_+() \text{ and } g_-())\). We consider estimation and inference of \( \tau \) next.

### 2.2 Estimation

We consider the series estimator. The method is based on approximating \( g_+(x) \), for \( x \in \mathcal{X} \), by the \( p \)-th order polynomial spline function \( g_{+,M}(x) \) with \( K \) knots, \( N_{+,1} < N_{+,2} < \ldots < N_{+,K} \in \text{int}(\mathcal{X}) \cap [c, \infty) \), where \( M = K + p + 1 \). The knots can be set such that a new knot is added each time as \( K \) increases, or be equally spaced (as in common practice). The cubic spline corresponds to the case of \( p = 3 \). The approximation quality presumes that \( K \to \infty \) with \( p \) fixed. Specifically,

\[
g_+(x) = g_{+,M}(x) + r_{+,M}(x),
\]

where \( r_{+,M}(x) \) is the approximation error, and

\[
g_{+,M}(x) = \nu_+(x)^\top \beta_+ = \sum_{j=0}^p \beta_{+,j} x^j + \sum_{k=1}^K \beta_{+,p+k} x^{p+k} I(x \geq N_k)
\]

with \( M \times 1 \) vectors \( \nu_+(x) = (1, x, \ldots, x^p, x^{p+1}(x \geq N_{+,1}), \ldots, x^{p+K}(x \geq N_{+,K}))^\top \) and \( \beta_+ = (\beta_{+,0}, \beta_{+,1}, \ldots, \beta_{+,K+p})^\top \).

Define a sequence of binary outcomes \( Y_{ij} = I(\bar{Y}_i = j) \). Then by mutual exclusion, \( \sum_{j=1}^{J+1} Y_{ij} = 1 \) for any \( i \). Denote \( Y_i = (Y_{i1}, \ldots, Y_{iJ})^\top \).

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\(^2\)Recall that in the parametric ordered outcome model, if \( \alpha_1, \ldots, \alpha_J \) are assumed to be unknown, the linear mean function of the latent variable does not contain an intercept term.

\(^3\)By Chen’s classification, Xu (2016) used a nonparametric (multinomial logit) model for the RD design with discrete outcomes which are not necessarily ordered.

The parameters $\alpha_+$ and $\beta_+$ are estimated by

$$\{\hat{\alpha}_+, \hat{\beta}_+\} = \text{arg max}_{\alpha \in \mathbb{R}^{J-1}, \beta \in \mathbb{R}^M} L_{n,+}(\alpha, \beta), \quad (7)$$

where

$$L_{n,+}(\alpha, \beta) = n^{-1} \sum_{i=1}^n \sum_{j=1}^{J+1} Y_{ij} \log \mu_{+,j}(X_i)I_i = n^{-1} \sum_{i=1}^n \ell(\alpha, w_+(X_i)\beta; Y_i)I_i,$$

with $I_i = \mathbb{1}(X_i \geq c)$ and

$$\ell(\alpha, g; y) = \sum_{j=1}^{J+1} y_j \log[\Phi(\alpha_j + g) - \Phi(\alpha_{j-1} + g)]. \quad (8)$$

Denote $\hat{g}_+(c) = w_+(c)\beta_+$. So $\hat{\mu}_{+,j}(c) = \Phi(\hat{\alpha}_{+,j} + \hat{g}_+(c)) - \Phi(\hat{\alpha}_{+,j-1} + \hat{g}_+(c))$. Similarly $\hat{\mu}_{-,j}(c)$ is defined. The ATE estimates are then defined as $\hat{\gamma}_j = \hat{\mu}_{+,j}(c) - \hat{\mu}_{-,j}(c)$.

In practice, the series method is computationally convenient since $\hat{\alpha}_+$ and $\hat{\beta}_+$ (defined in (7), thus $\hat{\mu}_+(c)$) can be obtained in one step through the well-developed routines for the parametric linear-index ordered outcome model. This can be compared with the kernel-based method in which multiple steps are usually needed to estimate $\mu_+(c)$ (Andrews, 1994b).

**Assumption 2.1.**

(a) $\mathcal{X} \subset \mathbb{R}^1$ is compact.

(b) $\{g_+(\cdot), g_-(\cdot)\} \in \mathcal{G}$, where $\mathcal{G} = \{g(\cdot) : \mathcal{X} \to \mathbb{R}, \text{the } \delta\text{-th derivative of } g(\cdot) \text{ is Lipschitz-continuous, sup}_{g \in \mathcal{G}} \sum_{j=0}^{\delta} \sup_{x \in \mathcal{X}} |g^{(j)}(x)| < \infty \text{ for some } \delta > 2\}$.

(c) $\Phi(\cdot)$ is a strictly increasing CDF, with the Lipschitz-continuous second derivative.

(d) $\{\alpha_+, \alpha_-\} \in \mathcal{A}$, where $\mathcal{A} = \{\alpha = (\alpha_1, \alpha_2, ..., \alpha_{J-1}) \in \mathbb{R}^{J-1} : \alpha_1 < \alpha_2 < ... < \alpha_{J-1} < 0\}$ is compact.

(e) $n^{-1}K^4 + nK^{-2\delta} \to 0$.

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5 Caution needs to be used since in the parametric routines a different identification condition from (6) is often used (e.g., the function $g_+(x)$ or $g_-(x)$ contains no intercepts). In this case, suppose the intercept estimates given by the parametric routine (e.g., the mnrfit command in MATLAB) are $\{\hat{\alpha}_{+,1}^P, ..., \hat{\alpha}_{+,J}^P\}$. Then in our context, $\hat{\alpha}_{+,1} = \hat{\alpha}_{+,1}^P - \hat{\alpha}_{+,J}^P, ..., \hat{\alpha}_{+,J-1} = \hat{\alpha}_{+,J-1}^P - \hat{\alpha}_{+,J}^P, \hat{\beta}_{+,0} = \hat{\beta}_{+,J}^P$ in view of (6).

6 A typical kernel-based procedure involves estimation of $\hat{g}_+(x, \alpha_+)$ in the first step for a given $\alpha_+$ (e.g., through the local likelihood method) and then $\hat{\alpha}_+$ is obtained by minimizing some criterion function that depends on $\{\hat{g}_+(x, \alpha_+) : x \in \mathcal{X} \cap [c, \infty)\}$. Then $\hat{g}_+(c)$ is obtained as $\hat{g}_+(c, \hat{\alpha}_+)$. 

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Assumption 2.1 is convenient and not overly restrictive (though stronger than necessary) in applications. In Assumption 2.1 (e), \( n^{-1} K^4 \to 0 \) ensures the uniform convergence rates of \( \tilde{g}_+ (\cdot) \) and \( \tilde{g}_- (\cdot) \) are sufficiently fast (to achieve \( n^{1/2} \)-consistency of \( \tilde{\alpha}_+ \) and \( \tilde{\alpha}_- \)), while \( n K^{-2\delta} \to 0 \) is an undersmoothing condition.

**Theorem 1** (Consistency). Under Assumption 2.1, \( \hat{\tau} \overset{p}{\to} \tau \).

### 2.3 Asymptotic theory

For a column vector \( \mu \), we denote the outer product as \( \mu \otimes 2 = \mu \mu^\top \). Let

\[
V_{\beta,+} = \left\{ \mathbb{E} [ \nabla_g \ell(\alpha_+, g_+(X_i); Y_i)]^2 w_+(X_i) w_+(X_i)^\top I_i \right\}^{-1},
\]

\( V_{g,+} = w_+(c)^\top V_{\beta,+} w_+(c) \) and \( V_{\mu,+} = V_{g,+} [\nabla_g \mu_+(c)] \otimes 2 \), where \( \nabla_g \mu_+(c) = \partial \mu_+(c) / \partial g_+(c) \) is the \( J \times 1 \) vector of derivatives. Similarly \( V_{\beta,-}, V_{g,-} \) and \( V_{\mu,-} \) are defined. The following result implies the asymptotic normality of \( \hat{\mu}_+(c) \) and \( \hat{\mu}_-(c) \).

**Theorem 2** (Asymptotic normality). Under Assumption 2.1, \( V_{g,+} \) and \( V_{g,-} \) are of the same order with \( K \), i.e. \( V_{g,+} \asymp O(K) \) and \( V_{g,-} \asymp O(K) \), and

\[
\begin{align*}
V_{g,+}^{-1/2} [\mu_+(c) - \mu_+(c)] & \overset{d}{\to} \mathcal{N}(0, [\nabla_g \mu_+(c)] \otimes 2), \\
V_{g,-}^{-1/2} [\mu_-(c) - \mu_-(c)] & \overset{d}{\to} \mathcal{N}(0, [\nabla_g \mu_-(c)] \otimes 2).
\end{align*}
\]

The asymptotic normality result, e.g. in (9), follows from the asymptotic distribution of \( \hat{\alpha}_+ \) and \( \hat{\mu}_+(c) \) and a delta-method-like technique. Note that only the Jacobians with respect to \( g_+(c) \) and \( g_-(c) \), but not \( \alpha_+ \) and \( \alpha_- \), are involved in the limit distributions. It is because both \( \alpha_+ \) and \( \alpha_- \) are estimated with a faster rate than \( g_+(c) \) and \( g_-(c) \), i.e.

\[
\hat{\alpha}_+ - \alpha_+ = O_p(n^{-1/2}), \quad \hat{\alpha}_- - \alpha_- = O_p(n^{-1/2}),
\]

thus the estimation effects can be neglected asymptotically, which greatly simplifies inferences.

The benefit comes from the global approach which utilizes the whole range of data. This can be compared with the common practice of localization in most RD applications. This practice
suggests to first trim the observations outside a neighborhood around the cutoff (the width of which is usually determined in a somewhat arbitrary or more automatic data-dependent way), followed by fitting a linear or nonlinear model on each side of the cutoff. Indeed, more often than not, the sample size falling within the chosen neighborhood is often reported in empirical studies instead of the total sample size. This local approach, coupled with the ordered outcome model, estimates both $\alpha$’s and $g$’s with a nonparametric rate (slower than $n^{1/2}$). In contrast, our global approach elaborated in this section estimates the ATEs with better quality (with same convergence rates as the local approach though), since some ingredients of ATEs ($\alpha$’s) are estimated with a faster $n^{1/2}$ rate when observations outside the window are also used.

For the same reason that different parts in ATEs have different convergence rates, it is noteworthy that the asymptotic variance matrices in (9) and (10) are singular. This needs to be taken care of in inference to which we now turn.

### 2.4 Inference

Consider the null hypothesis of $d_q \ (d_q \geq 1)$ linear restrictions $\mathcal{H}_0^q: Q\tau = q$ where $q$ is $d_q \times 1$, and $Q$ is $d_q \times J$ with full rank.

To test $\mathcal{H}_0$, nuisance parameters in (9) and (10) need to be estimated. Let

$$\hat{V}_{\beta,+} = \left\{ n^{-1} \sum_{i=1}^{n} \left[ \nabla_{\beta}(\widehat{\alpha}+, w_+(X_i)\widehat{\beta}+, Y_i) \right] \otimes I_i \right\}^{-1} .$$

Let $\hat{V}_{g,+} = w_+(c)^\top \hat{V}_{\beta,+} w_+(c)$ and $\hat{V}_{\mu,+} = \hat{V}_{g,+} + \left[ \nabla_{\mu}(\widehat{\mu}+, c) \right] \otimes 2$. Similarly $\hat{V}_{\mu,-}$ is defined. Let

$$V_\tau = V_{\mu,+} + V_{\mu,-} \quad \text{and} \quad \hat{V}_\tau = \hat{V}_{\mu,+} + \hat{V}_{\mu,-} .$$

The $t$-test and confidence interval on each $\tau_j$ can then be constructed, and the following theorem provides asymptotic justification.

**Theorem 3** ($t$-tests). For $j = 1, \ldots, J$, let $\hat{V}_{\tau,jj}$ be the $(j,j)$-element of $\hat{V}_\tau$. Then under $\mathcal{H}_0^q$,

$$n^{1/2} V^{-1/2}_{\tau,jj} (\hat{\tau}_j - \tau_j) = n^{1/2} \hat{V}_{\tau,jj}^{-1/2} (\hat{\tau}_j - \tau_j) + o_p(1) \stackrel{d}{\rightarrow} \mathcal{N}(0,1) .$$

The construction of the Wald test of $\mathcal{H}_0$ (for $d_q \geq 2$) based on $\hat{\tau} - \tau$ needs some caution. This sort of tests are useful to determine joint significance of ATEs for all levels (or a subset of levels)
or constancy of ATEs across levels. The standard (generalized-)Wald statistic is defined as

$$W^\tau = n(Q^\tau - q)^\top (Q\hat{V}_\tau Q^\top)^\dagger (Q^\tau - q),$$

where $\dagger$ denotes the Moore-Penrose inverse of a matrix. The generalized inverse is used in the construction since $\hat{V}_\tau$ is singular (a sum of two rank-one matrices). Inference based on $W^\tau$ is not straightforward for two reasons.

First, the limit distribution is generally unknown. Under suitable assumptions, the theory of the generalized Wald test points to the limit distribution with degree of freedom the rank of the (singular) limit variance matrix (Moore, 1977, Andrews, 1987). The rank depends on whether the two vectors $\nabla_g \mu_+(c)$ and $\nabla_g \mu_-(c)$ are linearly independent, which is unknown to the econometrician. Interpretable regularity assumptions are not easy to impose (except for the case of $d_q = 1$ as in Theorem 3). Second, Andrews’s (1987) conditions that the ranks of sample and limit variance matrices are equal, which are necessary for the validity of the generalized Wald test with the limit above, generally fail here. For instance, it happens when in population $\nabla_g \mu_+(c)$ and $\nabla_g \mu_-(c)$ are linearly dependent, e.g. $\nabla_g \mu_+(c) = \nabla_g \mu_-(c)$ under the homogeneity assumption $\alpha_{+j} = \alpha_{-j}$ and $g_+(c) = g_-(c)$, but their sample counterparts $\nabla_g \hat{\mu}_+(c)$ and $\nabla_g \hat{\mu}_-(c)$ are linearly independent (so that the rank of the sample variance matrix is larger than that of the limit variance matrix).

We demonstrate in simulations (Section 4) that the standard generalized Wald test $W^\tau$ can be enormously misleading in an empirically relevant setting.

The difficulty comes from the nature of the RD estimator, which is the difference of two estimators with singular asymptotic variance matrices. The difficulty disappears if we are only interested in $\mu_+(c)$ (say) and the hypothesis $H_{0}^{\tau,+} : Q_+ \mu_+(c) = q_+$, where $Q_+$ is $d_q \times J$ with full rank. Then by (9),

$$W^{\tau,+} = n(Q_+ \hat{\mu}_+(c) - q_+)^\top (Q_+ \hat{V}_{\mu_+,Q_+}^{-1})(Q_+ \hat{\mu}_+(c) - q_+) \overset{d}{\to} \chi^2(1),$$

under $H_{0}^{\tau,+}$, if $\nabla_g \mu_+(c)$ is not a zero vector. Andrews’s (1987) condition of rank equivalence holds

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7 This can be compared with canonical examples of the Wald test with a singular asymptotic variance matrix, in which singularity is often caused by the Jacobian matrix of nonlinear restrictions (e.g. Lütkepohl and Burda, 1997, Dufour, Renault and Zinde-Walsh, 2013, Duplinskiy, 2014, and references therein). In our setting, the singularity occurs even under linear restrictions (the Jacobian matrix is thus of full rank).

8 The derivation of (12) is given in the appendix.
trivially (i.e. \(\mathbb{P}\{\text{rank}(|\nabla g\mu_+(c)|^{\otimes 2}) = \text{rank}(|\nabla g\mu_+(c)|^{\otimes 2}) = 1\} \to 1\), as \(n \to \infty\)).

To test \(H_0\) when \(d_q \geq 2\), we propose two alternative Wald tests to circumvent the difficulty above. The first one is to regularize \(W^r\) by extracting only the first principle component of the sample variance matrix, along the line of Lütkepohl and Burda (1997). The following theorem justifies the regularized Wald test in our semi-nonparametric setting.

**Theorem 4** (Wald test). Suppose \(d_q \geq 2\), and that the decompositions hold (for each \(K\))

\[
QV_rQ^\top = A\Lambda A^\top \quad \text{and} \quad Q\tilde{V}_rQ^\top = \tilde{A}\tilde{\Lambda}\tilde{A}^\top,
\]

where \(A\) and \(\tilde{A}\) are orthogonal matrices, and \(\Lambda = \text{diag}(\lambda_1, \lambda_2, 0, ..., 0)\) and \(\tilde{\Lambda} = \text{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, 0, ..., 0)\) are diagonal matrices with decreasing eigenvalues. Suppose \(\lambda_1 > \lambda_2 \geq 0\) for each \(K\). Let \(A_1 = \text{diag}(\lambda_1, 0, ..., 0)\) and \(\tilde{A}_1 = \text{diag}(\tilde{\lambda}_1, 0, ..., 0)\). Then under \(H_0^r\),

\[
W^r_{\text{Reg}} = n(Q\tilde{V}_r - q)^\top (\tilde{A}\tilde{\Lambda}\tilde{A}^\top)^\dagger (Q\tilde{V}_r - q) \overset{d}{\to} \chi^2(1).
\]

The second proposal is to construct the inference procedure (when \(d_q \geq 2\)) which avoids using the generalized inverse of the singular estimated variance matrix. Instead, to test of \(H_0^r\), we rely on the (non-pivotal) equally-weighted statistic \(W^r_{\text{Eqw}} = n(Q\tilde{V}_r - q)^\top (Q\tilde{V}_r - q)\) and approximate its distribution using Monte Carlo simulations. Inference based on the statistic of this sort, dated at least back to Imhof (1961), was also considered by Francq, Roy and Zakoian (2005, Section 3.4), Duchesne and Francq (2015) and Duplinskiy (2014) to circumvent the singularity problem of the asymptotic variance, but in parametric settings. The rational of our application here is based on the approximation (under \(H_0^r\)), justified by Theorem 2,

\[
n^{1/2}(Q\tilde{V}_r - q) \overset{\text{app.}}{\sim} \mathcal{N}(0, Q\tilde{V}_r Q^\top),
\]

where \(\overset{\text{app.}}{\sim}\) denotes "approximately distributed as (when \(n\) is large)". Then

\[
W^r_{\text{Eqw}} \overset{\text{app.}}{\sim} \varphi_{d_q} Q\tilde{V}_r Q^\top \varphi_{d_q},
\]

where \(\varphi_{d_q} \sim \mathcal{N}(0, I_{d_q})\). Specifically, a 100\(c\)% level test of \(H_0^r\) can be implemented in the following steps:

\[9\] A more recent application of this type of regularized Wald test can be found in Kasahara and Shimotsu (2014) which estimates the number of components in multivariate mixture models.
Algorithm I (Monte Carlo-based Wald test)

1. Compute the statistic $W_{Eqw}$, given the data $\{\tilde{Y}_i, X_i, T_i : i = 1, ..., n\}$
2. Take $B$ random draws from $\mathcal{N}(0, I_d)$, denoted by $\{a_b : b = 1, ..., B\}$.
3. Compute the $1 - \epsilon$ empirical quantile of $\{a_b^T Q \tilde{V}_r Q^T a_b : b = 1, ..., B\}$, denoted by $q_{1-\epsilon}$.
4. Reject $H_0$ at the level $100\%$ if $W_{Eqw} > q_{1-\epsilon}$.

A main advantage of this procedure is simplicity. Bootstrap procedures based on the non-pivotal statistic $W_{Eqw}$ can also be used via, e.g. pairwise bootstrap. However, they require point estimation of $\mu_+(c)$ and $\mu_-(c)$ for each bootstrap sample, and thus can be much more computationally burdensome given that the sample sizes in RD applications are often large.

3 RD with grouped duration outcomes

The results in Section 2 pave the way for the analysis of the RD design with grouped duration outcomes we consider in this section. Duration data recorded in discrete levels, which may not be precise enough to be treated as continuous, are very common in social science in general (Wooldridge, 2010, Section 22.4). Recent studies with outcomes of this type in discontinuity-based designs are mentioned in Section 1.\(^{10}\)

3.1 The latent hazard model and RD estimands

Suppose that survey times (or observation times) at which at least one individual reportedly exit (e.g. find a job) are $\{t_1, t_2, ..., t_J\}.\(^{11}\)$ For individual $i$ ($i = 1, ..., n$), we observe $\tilde{Y}_i$, which is coded as $\tilde{Y}_i = j \in \{1, 2, ..., J + 1\}$. To fix notations, individual $i$ does not exit (thus is right-censored) at the end of the observing period if $\tilde{Y}_i = J + 1$, and the individual has the completed duration at or before $t_j$ if $\tilde{Y}_i = j$ for $1 \leq j \leq J$. (Thus $t_J$ is the observation-ending time.) We here only consider censoring that happens at the end of the observing period (perhaps one of the largest sources of

\(^{10}\)In some applications, even the data are available at a finer level, discrete time data are used for simplicity. For example, in Caliendo et al. (2013), "most of the employment spells start at the beginning of a month and unemployment spells last until the end of a month", thus unemployment durations are aggregated in months (instead of in days).

\(^{11}\)Here the survey times may be evenly spaced, or not, with the latter case arising, e.g. when the durations are recorded in weeks, but with some weeks in which no individuals exit.
censoring). We will discuss Section 3.3 the necessary modification to allow censoring which also happens within the observation period.

Following the earlier studies on grouped durations, we assume an underlying continuous-time duration for each individual, denoted by $R \in \mathbb{R}^+$ (not observable). We only observe that $R$ belongs to (and only belongs to) one of the intervals $(0, t_1], (t_1, t_2], ..., (t_{J-1}, t_J]$ and $(t_J, t_{J+1})$, where we set $t_{J+1} = \infty$. In the RD context, what we observe is $\tilde{Y}_i = T_i \bar{Y}_i(1) + (1 - T_i) \bar{Y}_i(0)$, where $\bar{Y}_i(1) = \lceil R_i(1) \rceil$ (the potential outcome for a treated individual) and $\bar{Y}_i(0) = \lceil R_i(0) \rceil$ (the potential outcome for a control individual), with $\lceil R \rceil = \arg \min_{1 \leq j \leq J+1} \{ t_j : R \leq t_j \}$; see Figure 1 for illustration of the relation between $\tilde{Y}$ and $R$ (e.g. $\tilde{Y}(1)$ and $R(1)$).

For example, if unemployment durations are recorded in weeks and at least one individual exits in every week, we can define $t_j = j$, and $\lceil R \rceil$ is the smallest integer which is larger than or equal to $R$ (round toward infinity) if $R \leq t_J$, and $\lceil R \rceil = J + 1$ otherwise. (i.e. One gets a job and she is not observed to exit until the next time when she is surveyed).

The proportional hazard (PH) models are assumed for $R(1)$ and $R(0)$,

\begin{align}
\lambda_+(r|X) &= \bar{\lambda}_+(r) \exp(g_+(X)), \\
\lambda_-(r|X) &= \bar{\lambda}_-(r) \exp(g_-(X)),
\end{align}

where no parametric specifications are used for baseline hazards $\{\bar{\lambda}_+ (\cdot), \bar{\lambda}_- (\cdot)\}$, or risk functions $\{g_+(\cdot), g_-(\cdot)\}$. The running variable $X$ is assumed to be time-invariant, which is the case in most RD applications, e.g. age, previous job tenure, or calendar time when the UI claim is filed.

The distributions of $R(1)$ and $R(0)$ are then recovered as $F_+(r|X) = 1 - \exp(- \int_0^r \lambda_+(v|X) dv)$.
and \( F_-(r|X) = 1 - \exp(- \int_0^r \lambda_-(v|X)dv) \). For identification\(^{12}\) (from \( \{R_i(1), R_i(0)\} \), together with \( \{X_i, T_i\} \) of \( \{\overline{X}_+, \overline{X}_-\} \) and \( \{g_+, g_-\} \), some standardization assumptions are needed.\(^{13}\) To avoid imposing any artificial constraints on \( \{g_+, g_-\} \), we use the following standardizations on the baseline hazards

\[
\int_0^{t_j} \overline{X}_+(r)dr = 1 \text{ and } \int_0^{t_j} \overline{X}_-(r)dr = 1. \tag{17}
\]

The PH assumptions (15)-(16) extend Cox’s (1972) classical model by allowing a nonparametric risk function. They are useful to generate a parsimonious model if \( J \) is moderate or large, which is common in applications with grouped duration outcomes. The PH assumptions can be dropped if the number of possible spells is small (i.e. \( J \) is small) by using the local multinomial logit analysis (Xu, 2016), e.g. in the survival analysis of very-low-birth-weight newborns by Almond, et al. (2010).

If a continuum of durations could be observed, we would be interested in the hazard effect \( \lambda_+(r|c) - \lambda_-(r|c) \) for \( r \in \mathbb{R}^+ \), and the quantile duration effect which is based on \( F_+(r|c) \) and \( F_-(r|c) \).

We here take a discrete-time approach due to data restriction. Denote \( \mu_{+,j}(X) = P(\overline{Y}(1) = j|X) \) and \( \mu_{-,j}(X) = P(\overline{Y}(0) = j|X) \). Note that \( \mu_{+,j+1}(X) = 1 - \sum_{j=1}^{J} \mu_{+,j}(X) \). A parameter of policy interest is \( \tau \), defined in (3), and interpreted as duration effects, i.e. the RD treatment effects on (unconditional) exiting probabilities for various discrete levels.

The effect on the hazard rate (conditional exiting probability upon survival) is also of interest, as it incorporates the agent’s updated information set thus drives the agent’s economic behavior. For \( j \in \{1,2,\ldots,J\} \), the discrete hazard at the interval \( j \) is,

\[
H_{+,j}(X) = \frac{P(t_{j-1} < R(1) \leq t_j|R(1) > t_{j-1}, X) = P(\overline{Y}(1) = j|\overline{Y}(1) \geq j, X)}{P(\overline{Y}(1) = j|X)} = \frac{P(\overline{Y}(1) = j|X)}{\sum_{k=j}^{J+1} P(\overline{Y}(1) = k|X)} = \frac{\mu_{+,j}(X)}{\sum_{k=j}^{J+1} \mu_{+,k}(X)}, \tag{18}
\]

\(^{12}\)Note below that \( \{\overline{X}_+, \overline{X}_-\} \) can not be identified, while \( \{g_+, g_-\} \) can be, from \( \{\overline{Y}\} \), together with \( \{X_i, T_i\} \). This should not be confused with our estimand of interest which involves discretized hazards, instead of the latent continuous hazards.

\(^{13}\)Recall that in the standard parametric PH model (with the linear risk function \( g \)), the identification is achieved via either assuming that \( g \) contains no intercept (so that \( \overline{X}(r) \) can be an unconstrained parametric hazard, like Weibull), or imposing one constraint on \( \overline{X}(r) \) (so that \( g \) may contain an intercept). The constrains in (17) below are of the second type.
and similarly
\[ H_{-j}(X) = \left( \sum_{k=j}^{J+1} \mu_{-,k}(X) \right)^{-1} \mu_{-,j}(X). \] (19)

The (discrete) hazard rate ATE is then defined as (for \(1 \leq j \leq J\))
\[ \xi_j = H_{+,j}(c) - H_{-,j}(c). \] (20)

They depend on the exiting probabilities \(\mu_{+}(c)\) and \(\mu_{-}(c)\). Let \(\xi = (\xi_1, ..., \xi_J)^T\).

### 3.2 Estimation and inference

To estimate \(\tau\) and \(\xi\), given the data \(\{\tilde{Y}_i, X_i, T_i\}\), we can show that \(\mu_{+,j}(x)\) and \(\mu_{-,j}(x)\) have the same structure as in (4)-(5), where \(\Phi(\cdot) = 1 - \exp(-\exp(\cdot))\) is the complementary log-log (cloglog) CDF, \(\alpha_{+,j} = \log \chi_{+,j}(t_j)\) and \(\alpha_{-,j} = \log \chi_{-,j}(t_j)\), with integrated baseline hazards \(\chi_{+,j}(t_j) = \int_0^{t_j} \chi_{+}(r)dr\) and \(\chi_{-,j}(t_j) = \int_0^{t_j} \chi_{-}(r)dr\), for \(1 \leq j \leq J\).\(^{14}\) Note that the identification assumption (6) is satisfied by (17). Thus the multinomial likelihood and the series approximation in Section 2.2 can be used to obtain \(\hat{\mu}_{+,j}(c)\) and \(\hat{\mu}_{-,j}(c)\) (and thus \(\hat{\tau}\)), and then \(\hat{H}_{+,j}(c)\) and \(\hat{H}_{-,j}(c)\) by (18)-(19). Denote the hazard ATE as \(\hat{\xi}_j = \hat{H}_{+,j}(c) - \hat{H}_{-,j}(c)\). Let \(\hat{\xi} = (\hat{\xi}_1, ..., \hat{\xi}_J)^T\).

The analysis in this section extends Han and Hausman (1990) and Sueyoshi (1995) by allowing a nonparametric specification of the risk function and adopting the RD setting.\(^{15}\)

Inference of \(\tau\) can be done following the analysis in Section 2.4, which also provides basis for the inference of \(\xi\). Denote the Jacobian \(\nabla_\mu H_{+}(c)\) to be the \(J \times J\) matrix with the \((j, j')\)-element \((j, j' = 1, ..., J)\)
\[
(\nabla_\mu H_{+}(c))_{j,j'} = \frac{\partial H_{+,j}(c)}{\partial \mu_{+,j'}} = \begin{cases} 0 & \text{if } j < j' \\ (1 - \sum_{k=1}^{j-1} \mu_{+,k}(c))^{-1} & \text{if } j = j' \\ (1 - \sum_{k=1}^{j-1} \mu_{+,k}(c))^{-2} \mu_{+,j}(c) & \text{if } j > j' \\ 
\end{cases}
\] (21)

and let \(\nabla_\mu H_{-}(c)\) be similarly defined.

\(^{14}\)See Appendix for derivations.

\(^{15}\)In particular, the earlier parametric approaches assume a linear index structure for the risk function and a baseline hazard (e.g. the Weibull or flexible piece-wise constant hazard) for the underlying continuous hazard, and infer about all the parameters. In contrast, here the risk function and the baseline hazard are unspecified, except proportionality, and we focus directly on the value of the discretized hazard.
Theorem 5. Suppose the latent hazards follow (15)-(16), and the assumptions in Theorem 2 hold. Then

\[
n^{1/2}[\hat{H}_+(c) - H_+(c)]/V_{g,+} \xrightarrow{d} \mathcal{N}(0, [\nabla_\mu H_+(c) \nabla_\mu H_+(c)]^{\otimes 2}),
\]

\[
n^{1/2}[\hat{H}_-(c) - H_-(c)]/V_{g,-} \xrightarrow{d} \mathcal{N}(0, [\nabla_\mu H_-(c) \nabla_\mu H_-(c)]^{\otimes 2}).
\]

To test \( \mathcal{H}_0^\xi : Q\xi = q \), where \( q \) is \( d_q \)-dimensional \((d_q \geq 1)\), let \( \hat{V}_{H,+} = \hat{V}_{g,+}[\nabla_\mu \hat{H}_+(c) \nabla_\mu \hat{H}_+(c)]^{\otimes 2} \), where \( \nabla_\mu \hat{H}_+(c) \) is defined in (21) except that \( \mu_+(c) \) is replaced by \( \hat{\mu}_+(c) \). Similarly \( \hat{V}_{H,-} \) is defined. Let \( \hat{V}_\xi = \hat{V}_{H,+} + \hat{V}_{H,-} \). It follows from Theorem 3 that the \( t \)-test on each \( \xi_j \) can then be formed based on

\[
n^{1/2}\hat{V}_{\xi,jj}^{-1/2}(\hat{\xi}_j - \xi_j) \xrightarrow{d} \mathcal{N}(0, 1),
\]

where \( \hat{V}_{\xi,jj} \) is the \((j,j)\)-element of \( \hat{V}_\xi \).

The Wald test of \( \mathcal{H}_0^\xi \) (for \( d_q \geq 2 \)) would be called for if the interest lies in joint significance of hazard ATEs for time intervals or uniformity of hazard ATEs across time intervals. As in Section 2.4, the naive (generalized) Wald test \( W^\xi \) (which uses \((Q\hat{V}_\xi Q^\top)^\dagger\) as the weighting matrix) is invalid in general, and two simple modifications based on a quadratic form of \( Q\hat{V}_\xi - q \) can be used.

The first one \( W_{\text{Reg}}^\xi \) uses the Moore-Penrose inverse of the first principle component of \( Q\hat{V}_\xi Q^\top \) and critical values from the \( \chi^2(1) \) distribution. The second uses the equally-weighted statistic \( W_{\text{Eqw}}^\xi = n(Q\hat{V}_\xi - q)^\top(Q\hat{V}_\xi - q) \) and critical values obtained through simulations following Steps 2-3 in Algorithm I except for replacing \( \hat{V}_\tau \) by \( \hat{V}_\xi \) in Step 3.

3.3 Allowing for en-route censoring

So far we assume censoring only happens at the end of the observation period. We now allow censoring at any time before that (en-route censoring). As we will see, under assumptions on the censoring mechanism, we only need to modify the likelihood function \( \ell(\alpha, g; y) \) in (8) (to be (22) below), and the corresponding estimation and inference procedures for \( \tau \) and \( \xi \) in last subsections still apply.

For unit \( i \) \((i = 1, \ldots, n)\), we now observe the grouped duration outcome \( \bar{Y}_i \) coupled with the censoring indicator \( D_i \) \((D_i = 1 \text{ means censoring})\). Note that \( D_i = 0 \) if \( \bar{Y}_i = J + 1 \) in our framework.
Suppose that there is a censoring variable \( C_i \in \{1, ..., J\} \), such that \( t_{C_i} \in \{t_1, ..., t_J\} \) is the first survey time when the agent stops responding. The outcome is defined as \( \tilde{Y}_i = T_i \tilde{Y}_i(1) + (T_i - 1) \tilde{Y}_i(0) \), where \( \tilde{Y}_i(1) = \min([R_i(1)], C_i) \) and \( \tilde{Y}_i(0) = \min([R_i(0)], C_i) \). The caveat under such en-route censoring is that we may not observe \( \tilde{Y}_i(1) \) even for a treated individual (similarly for \( \tilde{Y}_i(0) \)). For a censored individual, we only know that \( R_i > t_{C_i} \), through which it contributes to the likelihood function. Correspondingly, such information loss (caused by individuals with censored durations) is reflected by modifying the function \( \ell(\alpha, g; y) \) in (8) to

\[
\ell(\alpha, g; y, d) = (1 - \sum_{j=1}^{J} y_j) \log[1 - \Phi(g)] \\
+ \sum_{j=1}^{J} y_j[(1 - d) \log(\Phi(\alpha_j + g) - \Phi(\alpha_{j-1} + g)) + d \log(1 - \Phi(\alpha_{j-1} + g))].
\] (22)

(22) can be justified under the assumption of conditional independence: \((R_i(0), R_i(1)) \perp C_i|X_i\).

### 3.4 Allowing for competing risks

We can also allow exiting to more than one (say, \( S \geq 1 \)) destination states. If interest is only in one state, exiting to other states can be treated as censoring, and the framework in the last subsection directly applies. In general, all states are of interest, and target parameters are then the duration and hazard treatment effects exiting to destination state \( s \) (for \( 1 \leq s \leq S \)) at time interval \( j \) (for \( 1 \leq j \leq J \)) (23 below). Suppose for now there is no en-route censoring.

We observe the outcome \((\tilde{Y}_i, \tilde{U}_i)\), which are the observed duration and the exiting state respectively. We do not observe the agent’s duration for other states if she exits to a particular state. Let \( \tilde{U}_i = T_i \tilde{U}_i(1) + (1 - T_i) \tilde{U}_i(0) \), where \( \tilde{U}_i(1) = s \) if the agent \( i \) exits to state \( s \) (if treated), and similarly for \( \tilde{U}_i(0) \). Let \( U_i = (U_{i1}, ..., U_{iS})^T \) where \( U_{is} = \mathbb{I}(\tilde{U}_i = s) \) for \( 1 \leq s \leq S \).

Extending previous notations, define

\[
\mu_{+,j,s}(x) = \mathbb{P}(\tilde{Y}_i(1) = j|\tilde{U}_i(1) = s, X_i = x) = \Phi(\alpha_{+,j,s} + g_{+,s}(x)) - \Phi(\alpha_{+,j-1,s} + g_{+,s}(x)), \\
\mu_{-,j,s}(x) = \mathbb{P}(\tilde{Y}_i(0) = j|\tilde{U}_i(0) = s, X_i = x) = \Phi(\alpha_{-,j,s} + g_{-,s}(x)) - \Phi(\alpha_{-,j-1,s} + g_{-,s}(x)),
\]

\[16\] Note that \( C_i \) is observable only when \( D_i = 1 \) (i.e. \( C_i = \tilde{Y}_i \)).

\[17\] For example, if we observe \( \tilde{Y}_i = 1 \) and \( D_i = 1 \) (which implies \( C_i = 1 \)), then no information is provided by this individual.
which hold under the assumption that the latent continuous-time duration for each state $s$ follows a PH model with the interval integrated baseline hazard $\alpha_{+,j,s}$ and the risk function $g_{+,s}(x)$ for the treatment group, and similarly for the control group. The two hazard functions are $H_{+,j,s}(X) = [\sum_{k=j}^{J+1} \mu_{+,k,s}(X)]^{-1} \mu_{+,j,s}(X)$ and $H_{-,j,s}(X) = [\sum_{k=j}^{J+1} \mu_{-,k,s}(X)]^{-1} \mu_{-,j,s}(X)$. The duration and hazard effects (for each pair $(j, s)$) of interest are then defined as

$$
\tau_{j,s} = \mu_{+,j,s}(c) - \mu_{-,j,s}(c) \quad \text{and} \quad \xi_{j,s} = H_{+,j,s}(c) - H_{-,j,s}(c).
$$

They are estimated by approximating each $g_{+,s}(x)$ and $g_{-,s}(x)$ with a set of spline functions, and maximize, e.g. (7) over $\alpha \in \mathbb{R}^{S(J-1)}, \beta \in \mathbb{R}^{SM}$, except replacing $\ell(\alpha, g; y)$ by

$$
\ell(\alpha, g; y, u) = \sum_{j=1}^{J+1} \sum_{s=1}^{S} \sum_{u} \left[ \log(\Phi(\alpha_{j,s} + g_s) - \Phi(\alpha_{j-1,s} + g_s)) + \sum_{v \in \{1,\ldots,S\} \setminus s} \log \sum_{w=1}^{J+1} (\Phi(\alpha_{w,v} + g_v) - \Phi(\alpha_{w-1,v} + g_v)) \right],
$$

under the assumption of conditionally independent risks; we give details in the appendix. It reduces to $\ell(\alpha, g; y)$ in (8) if $S = 1$.

## 4 Simulations

### 4.1 The designs

We now report on the finite-sample performance of the hazard ATE estimates. Materializing (15)-(16), the data-generating designs considered (with nonlinear risk functions) are

$$
\lambda_+(r|X = x) = \gamma_+ r^{\gamma_+ - 1} \exp(b_{+,0} + b_{+,1}x + b_{+,2}x^2 \sin x)
$$

$$
\lambda_-(r|X = x) = \gamma_- r^{\gamma_- - 1} \exp(b_{-,0} + b_{-,1}x + b_{-,2}x^2 \sin x),
$$

where the baseline hazards follow the standardized Weibull. The cutoff is $c = 0$, and $X \sim \text{Unif}(-20, 20)$. Here the time $r$ can be interpreted in weeks, and the data are censored after $J = 28$ weeks.\footnote{(25)-(26) is written in the traditional form of the PH model. Nested in the framework of (15)-(16), $\lambda_+(r, \gamma_+) = J^{-\gamma_+ + 1} \gamma_+ r^{\gamma_+ - 1}$ and $g_+(x) = \gamma_+ \log J + b_{+,0} + b_{+,1}x + b_{+,2}x^2 \sin x$, which satisfy the identification assumptions (17). The intercepts in the ordered outcome model expression (4)-(5) are $\alpha_{+,j} = \gamma_+ (\log j - \log J)$ and $\alpha_{-,j} = \gamma_- (\log j - \log J)$,}
The hazard specification has to be transformed to the distribution which generates the data. The DGP for the duration data follows $R(1) = \exp(Z(1))$ and $Z(1) = \bar{b}_{+;0} + \bar{b}_{+;1}x + \bar{b}_{+;2}x^2 \sin x + \sigma_+ W$ where $W$ has a Type I extreme value distribution and

$$\bar{b}_{+;0} = -\gamma_+^{-1}b_{+;0}, \quad \bar{b}_{+;1} = -\gamma_+^{-1}b_{+;1}, \quad \bar{b}_{+;2} = -\gamma_+^{-1}b_{+;2}, \quad \sigma_+ = \gamma_+^{-1}.$$

Similarly $R(0)$ is generated, and we observe $\hat{Y} = T[R(1)] + (1 - T)[R(0)]$, where $[R]$ is the smallest integer which is larger than or equals $R$ if $R \leq 28$, and $[R] = 29$ otherwise.

We consider the sample size $n \in \{5000, 10000\}$. The parameters are set as $(\sigma_+, \bar{b}_{+;0}, \bar{b}_{+;1}, \bar{b}_{+;2}) = (0.827, 2.757, 0.0252, 0.00064)$ and $(\sigma_-, \bar{b}_{-;0}, \bar{b}_{-;1}, \bar{b}_{-;2}) = (0.757, 2.693, 0.0089, 0.00019)$, which are obtained by fitting an empirical data set on unemployment insurance rules. A realized sample is shown in Figure 2 (a). The probability of falling in each time interval for the treatment/control population is shown in Figure 2 (b). In total, about 15% observations fall in the last category (i.e. are censored).

A thought experiment could be the effect of extending the maximum allowable unemployment insurance benefit duration on the re-employment probability (Schmieder et al., 2012, Caliendo et al., 2013). In some countries, the benefit duration allowed is extended if the job-seeker passes a certain age, so $X$ is the exact age minus the threshold.

The population level probabilities $\mu_{+;j}(c)$ and $\mu_{-;j}(c)$, the level-probability effect $\tau_j$, discrete hazards $H_{+;j}(c)$ and $H_{-;j}(c)$, and the hazard effect $\xi_j$ against the time $j$ are plotted in four panels of Figure 4, for $j = 1, 2, \ldots, 28$. The hazards for the treatment and control groups are both increasing over time, and the effect on the hazard is decreasing over time, being initially positive and then staying negative for most of time periods. The underlying functions $\mu_+(j|x)$, $\mu_-(j|x)$, $H_+(j|x)$ and $H_-(j|x)$ for each $j$ are plotted in Figure 3.

### 4.2 The estimators and the results

We consider three methods of estimation, comparing the global series-based PH estimator proposed in this paper with other two local estimators which use observations within on a pre-specified window of the cutoff. They are referred to as global-PH, local-PH, and local-LPM estimators, for $1 \leq j \leq J$. 

20
respectively.

The first local estimator is based on the idea of fitting the conventional PH model locally, with linear specifications for the risk functions $g_+$ and $g_-$. This estimator arises naturally in the RD design given the design’s local interpretation. It still allows nonparametric modeling of risk functions since linear functions are used only locally. However, this method does not utilize the globality of the baseline hazards.

The second local estimator represents the common practice based on a local linear fit of the linear probability model (LPM) of binary outcomes for treatment and control populations.\textsuperscript{19} For this local-LPM method, to estimate the hazard ATE $\xi_j$, one can either simply apply the local linear estimator to the subsample of survivors for (for the given time-interval $j$), or follow two steps in which all $\mu_j$’s are first estimated and then $H_j$’s (thus $\xi_j$’s) are obtained using (18) and (19). We find both methods produce very similar results.

For all three estimators, we report simulation results for both level-probability ATEs $\tau_j$’s and hazard ATEs $\xi_j$’s.

For the tuning parameters, in the global series-based method we choose the numbers of knots $K_+$ and $K_-$ used in the spline approximations via AIC (Akaike information criterion)-minimization. The proportions (over replications) of selected integers are shown in Figure 5(a). Generally more knots are used on the right sample than the left sample. For local linear LPM-based estimator, we select smoothing bandwidths using the method by Imbens and Kalyanaraman (2012) which assumes the same bandwidth for each side. Since the local linear estimator is level-specific, the smoothing bandwidth varies with $j$. For each $j$, the bandwidth used to estimate the hazard ATE $\xi_j$ (denoted as $h_{\xi_j}$) is generally larger than the one for the level probability ATE $\tau_j$ (denoted as $h_{\tau_j}$) since only a subsample is used. They are shown in Figure 5(b). The values are between 1.5 and 3, which are interestingly close to a bandwidth of two years in age used in the empirical literature. For the local PH estimator, we take advantage of selected bandwidths above and use the single bandwidth $h_{loc-PH} = J^{-1} \sum_{j=1}^{J} h_{\tau_j}$.

Figures 6 and 7 plot the finite-sample biases and standard deviations of $\hat{\tau}_j$ and $\hat{\xi}_j$, for $j = 1, \ldots, 28$. For all three methods, the estimators for $\xi_j$ behave reasonably well for small $j$, however,

\textsuperscript{19}For example, Landais (2014) uses the local-LPM method to estimate the hazard ATEs in a regression kink design.
the estimation quality worsens as \( j \) increases. It results from the smaller number of surviving individuals thus the smaller effective sample size as the time elapses. In contrast, the performance of \( \hat{\tau}_j \) remains stable in general across \( j \).

Among the three competing methods, the global PH estimator works remarkably well and is the least sensitive as \( j \) increases. It clearly dominates the other two in terms of both finite-sample biases and variances, even it is driven by the nonparametric rate (like the other two estimators) as shown theoretically earlier in the paper. The superior performance is explained by better estimation of the global parameters that are involved in the hazard effects (i.e. integrated baseline hazards for treated and controlled groups).

The local linear estimators for hazard ATEs \( \hat{\xi}_j \) appear hardly acceptable for large \( j \) for the extremely large variances (even given that the selected bandwidths are larger than the ones for smaller \( j \)).\(^{20}\) For example, for \( j \geq 20 \), the standard deviation is over 0.1 when \( n = 5000 \). For \( j \geq 26 \), the standard deviation exceeds 0.13, and the associated 95% confidence interval is expected to cover more than half of the unit interval. A significantly larger sample seems compulsory to obtain reasonable precision. This observation motivates well our consideration of the PH structure. The standard deviation of the global PH estimator is about one-fourth of that of the local linear estimator. Lastly, the local PH estimator, which utilizes the structure of proportional hazards but only with local observations, performs in between the global PH and the local linear estimators.

4.3 Inference

To examine the finite-sample performance of inference associated with the global series-based estimator, we consider two data generating processes, referred to as HOMO and HETERO. The HETERO design is the same as the one described in Section 4.1. The HOMO design is the same as HETERO except that the same hazard function (25) is used for the left and right populations. For the Wald test, we consider the null \( \mathcal{H}_0^\xi : \xi = \xi_{\text{true}} \). The true value vector for the design HETERO is plotted in Figure 4(d), and is zero for the design HOMO.

Figure 8 shows the finite-sample coverage probability of 95% confidence interval for each \( \xi_j \)

\(^{20}\)We emphasize that the situation could be even worse if we encounter en-route censoring which is common in practice. Of course, all estimators would be affected, and the results here shed light on comparative performance if that happens.
under two designs. For small or moderate $j$, the coverage is quite accurate (especially for the design HOMO). Under-coverage starts to show up as $j$ increases. To a large extent it is due to effects of small effective sample sizes for large $j$’s, as mentioned in simulation results for estimation. The length for each confidence interval is plotted in Figure 9. The length is generally shorter than the finite-sample standard deviation (as shown in Figure 7(b)) times $2 \times 1.96$ for large $j$’s, reflecting the under-estimation of the standard error, which in turn leads to confidence interval undercoverage for these $j$’s.

Figures 10(a) and 11(a) plot the finite-sample distributions (in histograms) of the conventional Wald statistic $W^\xi$ and the regularized Wald statistic $W^\xi_{Reg}$ for two designs respectively. The values of $W^\xi$ are very large in general, and the upper quantiles of its distribution far exceed any practical critical value of $\chi^2(1)$ or $\chi^2(2)$ distribution. In sharp contrast, the regularized statistic $W^\xi_{Reg}$ (which simply uses the first of the two nonzero principle components of the sample variance matrix) greatly stabilizes $W^\xi$, and its distribution matches $\chi^2(1)$ quite closely. The rejection rates of $W^\xi_{Reg}$ are plotted in Figures 10(b) and 11(b) against the nominal sizes between 0.01 and 0.1. In the same graphs we also plot the rejections rates for the simulation-based test $W^\xi_{Eqw}$. Both tests generally preform well, with minor size distortions. In particular, we observe some over-rejection for $W^\xi_{Eqw}$, which again can be attributed to under-estimation of standard errors of $\tilde{\xi}_j$’s for large $j$’s that are involved in simultaneous testing, and some under-rejection for $W^\xi_{Reg}$, which might be explained by information loss caused by regularization. The test $W^\xi_{Eqw}$ has slightly larger power than $W^\xi_{Reg}$; the power and size-adjusted power functions are plotted in Figure 12 when the null true of $\xi$ has a scalar deviation from $\xi_{true}$.

5 Conclusion

In this paper we respond to the increasing empirical literature which involves using the regression discontinuity design with duration outcomes, and we focus on those with discrete support. We provide new estimators and associated inferential procedures for the duration and hazard effects under the design.

In the RD design, the treatment effect of interest is identified locally at the threshold of the forcing variable, which rationalizes most of the literature that only uses local observations and
avoids imposing any global restriction. This paper aims to bring to attention that in some cases local methods yield poor estimates and a global restriction may serve to produce a strong alternative. This happens if the interest is on the effect on the hazard rate, which is identified only by the subpopulation of survivors. The quality of the estimate declines as time elapses, which manifests itself strongly if only observations in the neighborhood are used. The same concern also arises if continuous durations are observed. In this case, the fully nonparametric hazard effect estimate is a function of nonparametric CDF and density estimates. The separability structure considered in this paper will be also useful. In this sense, the paper is also related to the strand of literature that strengthens RD assumptions to identify a global effect (e.g. Angrist and Rokkanen, 2015).

The extension to the kink design is worth a separate treatment.

6 Appendix: Proofs

In this section we provide proofs of Theorems 1-4 in the main text. In particular, we first establish consistency and pointwise asymptotic normality results for the series M-estimator of a general semi-nonparametric model. Theorems 1 and 2 then follow from application of these results to the treatment and control populations.

6.1 Series M-estimator: nonparametric model

In this subsection, we establish the local asymptotic normality results for the series M-estimator, extending the existing ones which are mostly focused on series the least squares estimator (Newey (1997), Huang (2003), Chen (2007, Section 3.4)). The goal of this subsection is to establish local asymptotic normality (Lemma 3).

Suppose the smooth criterion (or likelihood) function \( \ell_i \) is known up to a scalar function \( g(x) : X \mapsto \mathbb{R} \), where \( X \subset \mathbb{R} \). The identification of \( g \) will be discussed below. In the case of conditional mean function \( g(x) = \mathbb{E}(Y|X = x) \), \( \ell_i(g) = (Y_i - g)^2 / 2 \). In the maximum likelihood case (as in the main text),

\[
\ell_i(g) = - \sum_{j=1}^{J+1} Y_{ij} \log[\Phi(\alpha_j + g) - \Phi(\alpha_{j-1} + g)].
\]  

(27)

In the nonparametric model we consider in this subsection, assume that \( \alpha_j \)'s are known (which will
be relaxed in the next subsection).

Ideally, \( g(\cdot) \) is estimated by \( \hat{g}_{\text{inf}} = \arg \min_{g \in \mathcal{G}} \sum_{i=1}^{n} \ell_i(g(X_i)) \), where \( \mathcal{G} \) is a compact space of smooth functions. We use the polynomial (cubic) spline approximation \( g(X_i) = Z_i^T \beta \) where \( Z_i = z(X_i) \) is \( M \times 1 \), with \( M = K + 4 \) and \( K \) being the number of knots. Let \( \hat{\beta} \) be the approximate minimizer of \( n^{-1} \sum_{i=1}^{n} \ell_i(Z_i^T \beta) \), i.e.

\[
    n^{-1} \sum_{i=1}^{n} \ell_i(Z_i^T \hat{\beta}) \leq \inf_{\beta \in \mathbb{R}^M} n^{-1} \sum_{i=1}^{n} \ell_i(Z_i^T \beta) + O_p(n^{-1/2}).
\]

Then \( \hat{g}(x) = z(x)^T \hat{\beta} \). Without loss of generality, we assume \( \mathbb{E}(Z_i Z_i^T) = I_M \). This will alter the definition of \( \beta \) but will not affect the definition of \( g(x) \), which is of primary interest.

Let \( C, C_1 \) and \( C_2 \) be generic positive constants. To simplify notations, in many places we do not distinguish \( g \) from its true value \( g^* \), i.e. \( g \) is understood as its true value unless it is explicitly stated as a generic element of the (infinite-dimensional) parametric space or sieve spaces. We follow the similar convention for \( \alpha \) and \( \alpha^* \) (in the next subsection).

**Assumption ID.** \( \mathbb{E}[^{\ell}_1'(g(X_i))|X_i] = 0 \), a.s.

**Assumption INFO_EQ.** \( \mathbb{E}(^{\ell}_1''(g(X_i))|X_i) = \mathbb{E}(^{\ell}_1'(g(X_i))^2|X_i) \), a.s.

**Assumption EIGEN.** \( 0 < C_1 \leq \lambda_{\min}(\mathbb{E}[^{\ell}_1'(g(X_i))^2 Z_i Z_i^T]) \leq \lambda_{\max}(\mathbb{E}[^{\ell}_1'(g(X_i))^2 Z_i Z_i^T]) \leq C_2 < \infty \).

**Assumption BD.** (i) \( \mathbb{E}[^{\ell}_1'(g(X_i))^2 Z_i^T Z_i] = O(K) \). (ii) \( \mathbb{E}[^{\ell}_1''(g(X_i))^2 Z_i Z_i^T] = O(K) \).

**Assumption LIP.** (i). \( \mathbb{E}[^{\ell}_1'(g(X_i)|X_i = x] \) and \( \mathbb{E}[^{\ell}_1'(g)^2|X_i = x] \) are Lipschitz continuous in \( g \) for all \( x \). (ii). \( \mathbb{E}[(^{\ell}_1'(g_1) - ^{\ell}_1'(g_2))^2|X_i = x] < C|g_1 - g_2|^2 \) for some \( C \) for all \( x \). \( \mathbb{E}[(^{\ell}_1''(g_1) - ^{\ell}_1''(g_2))^2|X_i = x] < C|g_1 - g_2|^2 \) for some \( C \) for all \( x \).

**Assumption SM.** The true function \( g(x) \) has the continuous \( \delta \)-th derivative (\( \delta > 2 \)) on a compact support \( \mathcal{X} \).

**Assumption K.** \( n^{-1} K^2 \rightarrow 0 \).

We now show the assumptions are reasonable when \( \ell_i(g) \) takes the form of (27). Assumptions
ID and INFO_EQ hold trivially.\textsuperscript{21} INFO_EQ is imposed to simplify the algebra. When it is not satisfied, arguments we use should hold similarly with some modification (e.g. the variance would have a sandwich form).

Assumption EIGEN is satisfied provided

\begin{equation}
\lambda_{\text{min}}(\mathbb{E}[Z_iZ_i^\top \mathbb{E}(\ell_i'(g(X_i))^2|X_i)]) \geq C_1 > 0
\end{equation}

\begin{equation}
\lambda_{\text{max}}(\mathbb{E}[Z_iZ_i^\top \mathbb{E}(\ell_i'(g(X_i))^2|X_i)]) \leq C_2 < \infty.
\end{equation}

(29) holds from \(\mathbb{E}[\ell_i'(g(X_i))^2 Z_i Z_i^\top] = \mathbb{E}[Z_i Z_i^\top \mathbb{E}(\ell_i'(g(X_i))^2|X_i)] \geq \mathbb{E}[Z_i Z_i^\top \inf_{x \in \mathcal{X}} \mathbb{E}(\ell_i'(g(X_i))^2|X_i = x)]\) (in matrix sense), \(\lambda_{\text{min}}(\mathbb{E}(Z_i Z_i^\top)) = 1\) and

\[
\inf_{x \in \mathcal{X}} \mathbb{E}(\ell_i'(g(X_i))^2|X_i = x) = \inf_{x \in \mathcal{X}} \sum_{j=1}^{J+1} \left[ \frac{\varphi'(\alpha_j + g(x)) - \varphi'(\alpha_{j-1} + g(x))}{\Phi(\alpha_j + g(x)) - \Phi(\alpha_{j-1} + g(x))} \right]^2 \geq C > 0,
\]

which holds from mutually distinctive \(\alpha\)'s and Assumption SM. (30) follows similarly.

BD (ii) holds by \(\mathbb{E}[\ell_i''(g(X_i))^2 Z_i Z_i^\top] \leq \mathbb{E}[Z_i Z_i^\top \sup_{x \in \mathcal{X}} \mathbb{E}(\ell_i''(g(X_i))^2|X_i = x)] = O(K)\), which holds from mutually distinctive \(\alpha\)'s and Assumption SM. BD (i) holds similarly.

LIP holds by the forms of \(\ell_i'(g)\) and \(\ell_i''(g)\).\textsuperscript{21}

\begin{align*}
\ell_i'(g) & = \frac{\varphi(\alpha_j + g) - \varphi(\alpha_{j-1} + g)}{\Phi(\alpha_j + g) - \Phi(\alpha_{j-1} + g)} \\
\ell_i''(g) & = \frac{\varphi'(\alpha_j + g) - \varphi'(\alpha_{j-1} + g)}{\Phi(\alpha_j + g) - \Phi(\alpha_{j-1} + g)}
\end{align*}

where \(\Phi(g) = 1 - e^{-eg}, \varphi(g) = e^{g-e^g}\) and \(\varphi'(g) = e^{g-e^g}(1 - e^g)\). The Bartlett identity can be verified:

\[
\mathbb{E}[\ell_i''(g)] = -\sum_{j=1}^{J+1} \left\{ \frac{\varphi'(\alpha_j + g) - \varphi'(\alpha_{j-1} + g)}{\Phi(\alpha_j + g) - \Phi(\alpha_{j-1} + g)} \right\} = 0,
\]

where \(\varphi(\alpha_j + g) = e^{g-e^g}\). [no negative sign, since \(\ell_i\) is the negative of the likelihood]
Lemma 1. Define $\beta_K$ as in the Stone–Weierstrass approximation:

$$\beta_K = \arg \min_{\beta \in \mathbb{R}^M} \sup_{x \in \mathcal{X}} |z(x) \mathbf{T} \beta - g(x)|.$$ 

Then $(\hat{\beta} - \beta_K)^\mathbf{T} (\hat{\beta} - \beta_K) = O_p(n^{-1} K)$.

Lemma 2. $\sup_{x \in \mathcal{X}} |\hat{g}(x) - g(x)| = O_p(n^{-1/2} K) + O_p(K^{-\delta})$. (variance term+bias term)

Lemma 3. Suppose we are interested in $g(x)$ at a point, where $x \in \mathcal{X}$. Let $\hat{g}(x) = a^\top \hat{\beta}$, where $a = z(x_0)$. Let

$$v = a^\top [\mathbb{E} \ell'_i (g(X_i))^2 Z_i Z_i^\top]^{-1} a.$$  \hspace{1cm} (31)

Then under the undersmoothing condition

$$nK^{-2\delta} = o(1),$$  \hspace{1cm} (32)

we have

$$n^{1/2} (\hat{g}(x) - g(x)) \hat{v}^{-1/2} = n^{1/2} (\hat{g}(x) - g(x)) v^{-1/2} + o_p(1) \xrightarrow{d} \mathcal{N}(0, 1),$$

where $\hat{v} = a^\top [n^{-1} \sum_{i=1}^n \ell'_i (Z_i^\top \hat{\beta})^2 Z_i Z_i^\top]^{-1} a$.

We now prove Lemmas 1-3. For a $M \times 1$ vector $Z$, denote the inner product $Z^\otimes 2 = Z^\top Z$, and the outer product $Z^\otimes 2 = ZZ^\top$. The matrix norm is in Frobenius sense, i.e. $||A|| = (\sum_{jk} A_{jk}^2)^{1/2}$.

Proof of Lemma 1. Following the argument in the proof of Lemma 4 (in the next subsection), we have the consistency result

$$\sup_{x \in \mathcal{X}} |z(x)^\top \hat{\beta} - g(x)| = o_p(1).$$  \hspace{1cm} (33)

By the definition of $\beta_K$ and the Stone–Weierstrass theorem,

$$\sup_{x \in \mathcal{X}} |z(x)^\top \beta_K - g(x)| \leq O(K^{-\delta}).$$  \hspace{1cm} (34)

Then $\sup_{1 \leq i \leq n} |Z_i^\top \hat{\beta} - Z_i^\top \beta_K| \leq \sup_{1 \leq i \leq n} |Z_i^\top \hat{\beta} - g(X_i)| + \sup_{1 \leq i \leq n} |g(X_i) - Z_i^\top \beta_K| = o_p(1)$ by
(33) and (34). It permits us to perform the Taylor expansion below

\[ 0 = \sum_{i=1}^{n} \ell_i'(Z_i^T \beta)Z_i = \sum_{i=1}^{n} [\ell_i'(Z_i^T \beta_K) + \ell_i''(Z_i^T \beta)Z_i^T (\beta - \beta_K)]Z_i \]

\[ = \sum_{i=1}^{n} \ell_i'(Z_i^T \beta_K)Z_i + \sum_{i=1}^{n} \ell_i''(Z_i^T \beta)Z_iZ_i^T (\beta - \beta_K), \]

where \( \beta \) is between \( \beta \) and \( \beta_K \). Thus

\[ \widehat{\beta} = \beta_K = \left( \frac{1}{n} \sum_{i=1}^{n} \ell_i''(Z_i^T \beta)Z_iZ_i^T \right)^{-1} \frac{1}{n} \sum_{i=1}^{n} \ell_i'(Z_i^T \beta_K)Z_i. \]  

(35)

Then

\[ \begin{align*}
(\widehat{\beta} - \beta_K)^\top (\widehat{\beta} - \beta_K) \\
= \frac{1}{n} \sum_{i=1}^{n} \ell_i'(Z_i^T \beta_K)Z_i^\top \left( \frac{1}{n} \sum_{i=1}^{n} \ell_i''(Z_i^T \beta)Z_iZ_i^T \right)^{-2} \frac{1}{n} \sum_{i=1}^{n} \ell_i'(Z_i^T \beta_K)Z_i \\
\leq \left[ \lambda_{\max} \left( \left( \frac{1}{n} \sum_{i=1}^{n} \ell_i''(Z_i^T \beta)Z_iZ_i^T \right)^{-1} \right) \right]^2 \left[ \frac{1}{n} \sum_{i=1}^{n} \ell_i'(Z_i^T \beta_K)Z_i \right]^2 \\
\end{align*} \]

by the inequality for a quadratic form.\(^{22}\) Lemma 1 is proved provided

\[ \lambda_{\max} \left( \left( \frac{1}{n} \sum_{i=1}^{n} \ell_i''(Z_i^T \beta)Z_iZ_i^T \right)^{-1} \right) = O_p(1), \]  

(36)

\[ \left( \frac{1}{n} \sum_{i=1}^{n} \ell_i'(Z_i^T \beta_K)Z_i \right)^{\odot 2} = O_p(n^{-1} K). \]  

(37)

(36) follows from Assumption EIGEN,

\[ \left\| \frac{1}{n} \sum_{i=1}^{n} \ell_i''(Z_i^T \beta)Z_iZ_i^T - E\ell_i''(g(X_i))Z_iZ_i^T \right\| = o_p(1), \]  

(38)

and

\[ E\ell_i''(g(X_i))Z_iZ_i^T = E\ell_i'(g(X_i))^2 Z_iZ_i^T, \]  

(39)

which follows from LIE and Assumption INFO_EQ.

\(^{22}\) For a positive semi-definite matrix \( A \), we have \( \lambda_{\min}(A)x^\top x \leq x^\top Ax \leq \lambda_{\max}(A)x^\top x \). It holds from the spectral decomposition of \( A : A = Q\text{diag}(\lambda_1, ..., \lambda_K)Q^\top \), where \( 0 \leq \lambda_{\min}(A) = \lambda_1 \leq ... \leq \lambda_K = \lambda_{\max}(A) \) are eigenvalues of \( A \), and \( Q \) is such that \( QQ^\top = Q^\top Q = I_K \).
(38) holds by (40) and (41) below:

\[
\frac{1}{n} \sum_{i=1}^{n} \ell''(Z_i^\top \beta) \frac{Z_i Z_i^\top}{n} - \frac{1}{n} \sum_{i=1}^{n} \ell''(g(X_i)) Z_i Z_i^\top = o_p(1) \tag{40}
\]

\[
\frac{1}{n} \sum_{i=1}^{n} \ell''(g(X_i)) Z_i Z_i^\top - \mathbb{E} \ell''(g(X_i)) Z_i Z_i^\top = o_p(1). \tag{41}
\]

The following algebraic results (which are obviously true in scalars) will be useful:

\[
\mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^{n} A_i \right\|^2 = n^{-1} \mathbb{E} ||A_i||^2, \tag{42}
\]

\[
\mathbb{E} ||A_i - \mathbb{E} A_i||^2 \leq \mathbb{E} ||A_i||^2, \tag{43}
\]

\[
||aa^\top||^2 = (a^\top a)^2, \tag{44}
\]

where \( A_i \) are iid matrices and \( a \) is a vector.\(^{23}\)

Consider (41) first. Note that

\[
\mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^{n} \ell''(g(X_i)) Z_i Z_i^\top - \mathbb{E} \ell''(g(X_i)) Z_i Z_i^\top \right\|^2 = n^{-1} \mathbb{E} \left\| \ell''(g(X_i)) Z_i Z_i^\top - \mathbb{E} \ell''(g(X_i)) Z_i Z_i^\top \right\|^2
\]

\[
\overset{(42)}{=} n^{-1} \mathbb{E} \left\| \ell''(g(X_i)) Z_i Z_i^\top - \mathbb{E} \ell''(g(X_i)) Z_i Z_i^\top \right\|^2
\]

\[
\overset{(43)}{\leq} n^{-1} \mathbb{E} \left\| \ell''(g(X_i)) Z_i Z_i^\top \right\|^2
\]

\[
\overset{(44)}{=} n^{-1} \mathbb{E} \ell''(g(X_i))^2 (Z_i^\top Z_i)^2
\]

\[
\leq n^{-1} \sup_{x \in \mathcal{X}} z^\top(x) z(x) \cdot \mathbb{E} \ell''(g(X_i))^2 Z_i^\top Z_i = O(n^{-1} K^2) = o(1),
\]

where we used

\[
\sup_{x \in \mathcal{X}} z^\top(x) z(x) = O(K) \tag{45}
\]

\(^{23}\)(42), (43) and (44) follow from, respectively,

\[
\mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^{n} A_i \right\|^2 = n^{-2} \mathbb{E} \sum_{j,k=1}^{M} \sum_{i=1}^{n} (A_{i,jk})^2 = n^{-1} \sum_{j,k=1}^{M} \mathbb{E} A_{i,jk}^2 = n^{-1} \mathbb{E} ||A_i||^2,
\]

\[
\mathbb{E} ||A_i - \mathbb{E} A_i||^2 = \mathbb{E} \sum_{j,k=1}^{M} \sum_{i=1}^{n} (A_{i,jk} - \mathbb{E} A_{i,jk})^2 \leq \sum_{j,k=1}^{M} \mathbb{E} \text{Var}(A_{i,jk}) \leq \sum_{j,k=1}^{M} \mathbb{E} A_{i,jk}^2 = \mathbb{E} ||A_i||^2,
\]

\[
||aa^\top||^2 = \sum_{j,k=1}^{M} a_j^2 a_k^2 = \sum_{k=1}^{M} a_k^4 = (a^\top a)^2.
\]

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(as shown in Newey, 1997, for polynomial splines) together with Assumptions BD(ii) and K. So (41) holds.

Consider (40). Note that

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}[\ell''_i(Z_i^T \beta) - \ell''_i(g(X_i))]Z_iZ_i^T\right]^2$$

\[= n^{-1}\mathbb{E}\left[\|\ell''_i(Z_i^T \beta) - \ell''_i(g(X_i))\|_2^2\right]^2 \overset{(44)}{=} n^{-1}\mathbb{E}[\ell''_i(Z_i^T \beta) - \ell''_i(g(X_i))]^2(Z_i^T Z_i)^2 \]

\[\overset{\text{LIE}}{=} n^{-1}\mathbb{E}\{((Z_i^T Z_i)^2)\mathbb{E}[(\ell''_i(Z_i^T \beta) - \ell''_i(g(X_i)))^2|X_i]\}
\]

\[\overset{\text{LIP}(ii)}{\leq} Cn^{-1}\mathbb{E}\{(Z_i^T Z_i)^2|Z_i^T \beta - g(X_i)|^2\}
\]

\[\leq Cn^{-1}\mathbb{E}\{(Z_i^T Z_i)^2(\sup_i |Z_i^T \beta - Z_i^T \beta_K| + \sup_i |Z_i^T \beta_K - g(X_i)|)^2\}
\]

\[\overset{(33)\&(34)}{=} Cn^{-1}\mathbb{E}(Z_i^T Z_i)^2(o_p(1) + O(K^{-2\delta}))^2 = Cn^{-1}O(K^2)[o_p(1) + O(K^{-2\delta})]
\]

\[= o(n^{-1}K^2) + O(n^{-1}K^{2-2\delta}) \overset{K}{=} o_p(1).
\]

So (40) holds. Thus (38), and consequently, (36) is proved.

Consider (37) now. Note that

$$n^{-1}\mathbb{E}\{[\ell'_i(Z_i^T \beta_K)^2 - \ell'_i(g(X_i))^2]Z_i^T Z_i\}
\]

\[\overset{\text{LIE}}{=} n^{-1}\mathbb{E}\{\mathbb{E}[\ell'_i(Z_i^T \beta_K)^2 - \ell'_i(g(X_i))^2|X_i]Z_i^T Z_i\}\overset{\text{LIP}(ii)}{\leq} n^{-1}C[\mathbb{E}Z_i^T Z_i(Z_i^T \beta_K - g(X_i))^2]
\]

\[\leq n^{-1}C[\mathbb{E}Z_i^T Z_i(\sup_i |Z_i^T \beta_K - g(X_i)|)^2] \leq n^{-1}C[\mathbb{E}Z_i^T Z_i(\sup_x |z(x)^T \beta_K - g(x)|)^2]
\]

\[\overset{(34)}{=} n^{-1}C\mathbb{E}Z_i^T Z_i \cdot O(K^{-2\delta}) = O(n^{-1}K^{1-2\delta}),
\]

and thus

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\ell'_i(Z_i^T \beta_K)Z_i\right]^2 = n^{-1}\mathbb{E}\ell'_i(Z_i^T \beta_K)^2Z_i^T Z_i \overset{(46)}{=} n^{-1}\mathbb{E}\ell'_i(g(X_i))^2Z_i^T Z_i + O(n^{-1}K^{1-2\delta}) \overset{\text{BD}(i)}{=} O(n^{-1}K).
\]

Thus (37) holds by Markov inequality. The proof is complete. □
Proof of Lemma 2. It follows from

\[
\sup_{x \in \mathcal{X}} |\hat{g}(x) - g(x)| = \sup_{x \in \mathcal{X}} |z(x)^\top (\hat{\beta} - \beta_K)| + \sup_{x \in \mathcal{X}} |z(x)^\top \beta_K - g(x)|
\]

\[
(34) \quad \sup_{x \in \mathcal{X}} z(x)^\top (\hat{\beta} - \beta_K) z(x)]^{1/2} + O_p(K^{-\delta})
\]

Lemma 1

\[
O_p(n^{-1/2} K^{1/2}) \left[ \sup_{x \in \mathcal{X}} z(x)^\top z(x) \right]^{1/2} + O_p(K^{-\delta}) = O_p(n^{-1/2} K) + O_p(K^{-\delta}).
\]

□

Proof of Lemma 3. The variance is of the same order with \( K \) asymptotically, i.e. \( v \asymp O(K) \), since \( O(K) = \lambda_{\min}([\mathbb{E} \ell_i'(g(X_i))^2 Z_i Z_i^\top]^{-1}) a^\top a \leq v \leq \lambda_{\max}([\mathbb{E} \ell_i'(g(X_i))^2 Z_i Z_i^\top]^{-1}) a^\top a = O(K) \) using (45) and Assumption EIGEN. Note that

\[
n^{1/2}v^{-1/2}(\hat{g}(x) - g(x)) = n^{1/2}v^{-1/2}(a^\top \hat{\beta} - g(x))
\]

\[
(34) \quad n^{1/2}v^{-1/2}a^\top (\hat{\beta} - \beta_K) + O_p(n^{1/2}v^{-1/2}K^{-\delta})
\]

\[
(32) \quad n^{1/2}v^{-1/2}a^\top (\hat{\beta} - \beta_K) + o_p(1).
\]

By (35),

\[
n^{1/2}v^{-1/2}a^\top (\hat{\beta} - \beta_K) = -v^{-1/2}a^\top \left( \frac{1}{n} \sum_{i=1}^{n} \ell_i''(Z_i^\top \beta) Z_i Z_i^\top \right)^{-1} \frac{1}{n^{1/2}} \sum_{i=1}^{n} \ell_i'(Z_i^\top \beta_K) Z_i.
\]

Let \( \eta_i = a^\top (\mathbb{E} \ell_i'(g(X_i))Z_i Z_i^\top)^{-1} \ell_i'(Z_i^\top \beta_K)Z_i \). Then Lemma 3 follows from

\[
v^{-1/2} \frac{1}{n^{1/2}} \sum_{i=1}^{n} \eta_i \xrightarrow{d} \mathcal{N}(0, 1),
\]

and

\[
A_n = v^{-1/2}a^\top \left[ \left( \frac{1}{n} \sum_{i=1}^{n} \ell_i''(Z_i^\top \beta) Z_i Z_i^\top \right)^{-1} - (\mathbb{E} \ell_i'(g(X_i))Z_i Z_i^\top)^{-1} \right] \frac{1}{n^{1/2}} \sum_{i=1}^{n} \ell_i'(Z_i^\top \beta_K)Z_i = o_p(1).
\]
Consider (48) first. Let \( \bar{\eta}_i = a^\top (E \ell'''(g(X_i)) Z_i Z_i^\top) - \ell'_i(g(X_i)) Z_i \). Note that

\[
\begin{align*}
\mathbb{E}[v^{-1/2}n^{-1/2} \sum_{i=1}^n (\eta_i - \bar{\eta}_i)] \\
= v^{-1/2}n^{1/2} \mathbb{E}(\eta_i - \bar{\eta}_i) \\
= v^{-1/2}n^{1/2} \mathbb{E}\{a^\top (E \ell''(g(X_i)) Z_i Z_i^\top) - \ell'_i(g(X_i)) Z_i\} \\
\stackrel{(51)}{=} v^{-1/2}O(n^{1/2}K^{1/2-\delta}) \stackrel{(32)}{=} o(1),
\end{align*}
\]

since

\[
(\mathbb{E}\{a^\top (E \ell''(g(X_i)) Z_i Z_i^\top) - \ell'_i(g(X_i)) Z_i\})^{1/2}
\]

\begin{align*}
\text{Cauchy} & \leq \mathbb{E}\{a^\top (E \ell''(g(X_i)) Z_i Z_i^\top) - \ell'_i(g(X_i)) Z_i\}^{1/2} \cdot \mathbb{E}[\ell'_i(Z_i^\top \beta_K) - \ell'_i(g(X_i)) Z_i^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} a] \\
\text{LIE} & \leq \mathbb{E}\{a^\top (E \ell''(g(X_i)) Z_i Z_i^\top) - \ell'_i(g(X_i)) Z_i\}^{1/2} \cdot \mathbb{E}[\ell'_i(Z_i^\top \beta_K) - \ell'_i(g(X_i)) Z_i^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} a] \\
\text{LIP(ii)} & \leq C \mathbb{E}\{a^\top (E \ell''(g(X_i)) Z_i Z_i^\top) - \ell'_i(g(X_i)) Z_i\}^{1/2} \cdot \mathbb{E}[\ell'_i(Z_i^\top \beta_K) - \ell'_i(g(X_i)) Z_i^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} a] \\
\text{EIGEN} & \leq C K^{-2\delta} \mathbb{E}\{a^\top (E \ell''(g(X_i)) Z_i Z_i^\top) - \ell'_i(g(X_i)) Z_i\}^{1/2} \cdot \mathbb{E}[\ell'_i(Z_i^\top \beta_K) - \ell'_i(g(X_i)) Z_i^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} a] \\
\end{align*}

(51)

The bias of \( n^{1/2}v^{-1/2}(\hat{g}(x) - g(x)) \), which is contributed from (47) and (50), has the dominate term (50) (on which the undersmoothing condition (32) is imposed). We also have

\[
\begin{align*}
\mathbb{E}\left[v^{-1/2}n^{-1/2} \sum_{i=1}^n (\eta_i - \bar{\eta}_i)^2\right] \\
= v^{-1} \mathbb{E}(\eta_i - \bar{\eta}_i)^2 \\
= v^{-1} a^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} \cdot \mathbb{E}[\ell'_i(Z_i^\top \beta_K) - \ell'_i(g(X_i)) Z_i^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} a] \\
\leq \lambda_{\max}\left\{\mathbb{E}[\ell'_i(Z_i^\top \beta_K) - \ell'_i(g(X_i)) Z_i^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} a]\right\} \\
\cdot v^{-1} a^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} \cdot \mathbb{E}[\ell'_i(Z_i^\top \beta_K) - \ell'_i(g(X_i)) Z_i^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} a] \\
\end{align*}
\]

INFO_EQ_psd \leq \lambda_{\max}\left\{\mathbb{E}[\ell'_i(Z_i^\top \beta_K) - \ell'_i(g(X_i)) Z_i^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} a]\right\} \\
\leq \lambda_{\max}\left\{\mathbb{E}[\ell'_i(Z_i^\top \beta_K) - \ell'_i(g(X_i)) Z_i^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} a]\right\} \\
\cdot v^{-1} a^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} \cdot \mathbb{E}[\ell'_i(Z_i^\top \beta_K) - \ell'_i(g(X_i)) Z_i^\top (E \ell''(g(X_i)) Z_i Z_i^\top)^{-1} a] \\
\leq o(1),
\end{align*}
\]

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We now examine the three terms:

\[
\left\| \mathbb{E}[\ell'_i(Z_i^\top \beta_K) - \ell'_i(g(X_i)]^2 Z_i Z_i^\top \right\|
\]

\[
\overset{\text{LIE}}{=} \left\| \mathbb{E}Z_i Z_i^\top [\mathbb{E}(\ell'_i(Z_i^\top \beta_K) - \ell'_i(g(X_i)]^2 |X_i] \right\| \quad \overset{\text{LIP(ii)}}{\leq} \sup_x |z(x)^\top \beta_K - g(x)|^2 \cdot \left\| \mathbb{E}Z_i Z_i^\top \right\|
\]

\[
\leq O(K^{-2\delta}) \left\| \mathbb{E}Z_i Z_i^\top \right\| = O(K^{1-2\delta}) = o(1).
\]

So

\[
v^{-1/2} \frac{1}{n^{1/2}} \sum_{i=1}^n (\eta_i - \bar{\eta}_i) = o_p(1). \tag{52}
\]

By Assumption ID, \( \mathbb{E}\bar{\eta}_i = 0 \). By Assumption INFO_EQ, \( \mathbb{E}(v^{-1/2}n^{-1/2} \sum_{i=1}^n \tilde{\eta}_i)^2 = v^{-1}E\tilde{\eta}_i^2 \rightarrow 1 \). So \( v^{-1/2}n^{-1/2} \sum_{i=1}^n \tilde{\eta}_i \xrightarrow{d} \mathcal{N}(0, 1) \) and thus (48) holds by (52).

Consider (49) now. Note that

\[
|A_n| = v^{-1/2} \left| a^\top \left( \frac{1}{n} \sum_{i=1}^n \ell''_{i}(Z_i^\top \beta)Z_i Z_i^\top \right)^{-1} \left[ \mathbb{E}(\ell''_{i}(g(X_i))Z_i Z_i^\top) - \frac{1}{n} \sum_{i=1}^n \ell''_{i}(Z_i^\top \beta)Z_i Z_i^\top \right] \right|
\]

\[
\cdot \left[ \mathbb{E}(\ell''_{i}(g(X_i))Z_i Z_i^\top) \right]^{-1} \frac{1}{n^{1/2}} \sum_{i=1}^n \ell'_i(Z_i^\top \beta_K)Z_i
\]

\[
\leq v^{-1/2} |A_{1n}| \cdot |A_{2n}| \cdot |A_{3n}|,
\]

where

\[
A_{1n} = a^\top \left[ n^{-1} \sum_{i=1}^n \ell''_{i}(Z_i^\top \beta)Z_i Z_i^\top \right]^{-1}
\]

\[
A_{2n} = \mathbb{E}(\ell''_{i}(g(X_i))Z_i Z_i^\top) - n^{-1} \sum_{i=1}^n \ell''_{i}(Z_i^\top \beta)Z_i Z_i^\top
\]

\[
A_{3n} = (\mathbb{E}(\ell''_{i}(g(X_i))Z_i Z_i^\top))^{-1} n^{-1/2} \sum_{i=1}^n \ell'_i(Z_i^\top \beta_K)Z_i.
\]

We now examine the three terms \( A_{1n}, A_{2n} \) and \( A_{3n} \). Note that \( |A_{2n}| = o_p(1) \) by (38).

\[
|A_{1n}|^2 = \text{Trace}(a^\top \left( n^{-1} \sum_{i=1}^n \ell''_{i}(Z_i^\top \beta)Z_i Z_i^\top \right)^{-2} a)
\]

\[
\leq \text{Trace}(\lambda_{\max} \left( n^{-1} \sum_{i=1}^n \ell''_{i}(Z_i^\top \beta)Z_i Z_i^\top \right)^{-2} a^\top a) = O_p(K)
\]
by (36). For the term \(A_{3n}\), \(||A_{3n}||^2 = \text{Trace}(\tilde{A}_{3n}) = O(1)\), where

\[
\tilde{A}_{3n} = (\mathbb{E}(\ell''(g(X_i))Z_iZ_i^\top))^{-1} \left[ n^{-1/2} \sum_{i=1}^n \ell''(Z_i^\top \beta_K)Z_i \right] \otimes (\mathbb{E}(\ell''(g(X_i))Z_iZ_i^\top))^{-1}
\]

and \(||\mathbb{E}\tilde{A}_{3n}|| = ||(\mathbb{E}(\ell''(g(X_i))Z_iZ_i^\top))^{-1}|| + s.o. = O(1)\) by Assumptions EIGEN and INFO_EQ. Thus \(||A_n|| = o_p(1)\) by \(v \asymp O(K)\). So (49) holds.

Finally,

\[
\frac{\tilde{v} - 1}{v} = \frac{\tilde{v} - v}{v} = \frac{a^\top \left\{ [n^{-1} \sum_{i=1}^n \ell'(Z_i^\top \beta)^2 Z_iZ_i^\top]^{-1} - [\mathbb{E} \ell'(g(X_i))^2 Z_iZ_i^\top]^{-1} \right\} a}{a^\top [\mathbb{E} \ell'(g(X_i))^2 Z_iZ_i^\top]^{-1} a} \leq \frac{o_p(a^\top a)}{\lambda_{\min}(\mathbb{E} \ell'(g(X_i))^2 Z_iZ_i^\top)^{-1}) a^\top a = o_p(1),
\]

which follows from the proofs of (40) and (41), and Assumptions EIGEN and INFO_EQ. So

\[
n^{1/2}(\hat{g}(x) - g(x))\tilde{v}^{-1/2} = n^{1/2}(\hat{g}(x) - g(x))(\tilde{v}/v)^{-1/2}v^{-1/2} = n^{1/2}(\hat{g}(x) - g(x))v^{-1/2} + o_p(1).
\]

### 6.2 Series M-estimator: semi-nonparametric model

In this subsection, we show that Lemma 3 continues to hold if we extend the model above to include an additional finite-dimensional unknown parameter \(\alpha\); c.f. Lemma 6 below. In such a semi-nonparametric model, \(\alpha\) and \(g\) are estimated simultaneously by using a sieve approximation of \(g\). Let

\[
\{\hat{\alpha}, \hat{g}\} = \arg \min_{\alpha \in \mathcal{A}, g \in \mathcal{G}_K} n^{-1} \sum_{i=1}^n \ell_i(\alpha, g(X_i)),
\]

where \(\mathcal{G}_K\) is the linear space of cubic splines with \(K\) knots.

Let \(\Theta = \mathcal{A} \times \mathcal{G}\) and define the distance \(d(\theta_1, \theta_2) = ||\alpha_1 - \alpha_2|| + ||g_1 - g_2||_{\infty}\). The sieve spaces are denoted as \(\Theta_K = \mathcal{A} \times \mathcal{G}_K\).

**Assumptions.**

(i). Under \(d(\cdot, \cdot)\), \(\Theta\) is compact and \(\mathbb{E}\ell_i(\alpha, g(X_i))\) is continuous on \(\Theta\).

(ii). \(\mathbb{E}\ell_i(\alpha, g(X_i))\) is uniquely maximized at \(\{\alpha^*, g^*\}\) and \(\mathbb{E}\ell_i(\alpha^*, g^*(X_i)) > -\infty\).
(iii). \( \Theta_K \subseteq \Theta_{K+1} \subseteq \Theta \) for all \( K \), and there exists \( \pi K \theta^* \in \Theta_K \) such that \( d(\pi K \theta^*, \theta^*) \to 0 \) as \( K \to \infty \).

(iv). For each \( K \geq 1 \), \( \Theta_K \) is compact under \( d(\cdot, \cdot) \).

(v). For each \( K \geq 1 \), \( \sup_{\theta \in \Theta_K} |n^{-1} \sum_{i=1}^n \ell_i(\alpha, g(X_i)) - \mathbb{E} \ell_i(\alpha, g(X_i))| = o_p(1) \).

(vi). \( \mathbb{E} \nabla_\alpha \ell_i(\alpha^*, g^*(X_i)) = 0 \).

(vii). \( n^{1/2} \mathbb{E} \nabla_\alpha \ell_i(\alpha^*, \hat{g}(X_i)) = o_p(1) \).

(viii). \( \nu_1(\alpha, g) \) is stochastically equicontinuous at \( g^* \), where \( \nu_1(\alpha, g) = n^{-1/2} \sum_{i=1}^n [\nabla_\alpha \ell_i(\alpha, g(X_i)) - \mathbb{E} \nabla_\alpha \ell_i(\alpha, g(X_i))] \).

(ix). \( \sup_{\theta \in \Theta} n^{-1/2} |\nu_1(\alpha, g)| = o_p(1) \).

(x). \( \sup_{\theta \in \Theta} n^{-1/2} |\nu_1(\alpha, g)| = o_p(1) \), where \( \nu_2(\alpha, g) = n^{-1/2} \sum_{i=1}^n [\nabla_{\alpha \alpha} \ell_i(\alpha, g(X_i)) - \mathbb{E} \nabla_{\alpha \alpha} \ell_i(\alpha, g(X_i))] \).

(xi). Assumptions in Lemma 3 hold, where all assumptions are interpreted under \( \alpha = \alpha^* \).

(xii). \( n^{-1} K^4 \to 0 \).

**Lemma 4.** Under Assumptions (i-v), \( \hat{\alpha} \to^p \alpha \) and \( \sup_x |\hat{g}(x) - g(x)| = o_p(1) \).

**Lemma 5.** Under Assumptions (i-x), \( n^{1/2}(\hat{\alpha} - \alpha) = O_p(1) \).

**Lemma 6.** Let \( \mu(x) = \psi(\alpha, g(x)) \) be \( J \times 1 \), where \( \psi(\alpha, g) \) has continuous derivatives \( \nabla_\alpha \psi(\alpha, g) \) and \( \nabla_g \psi(\alpha, g) \). Let \( \hat{\mu}(x) = \psi(\hat{\alpha}, \hat{g}(x)) \). Then under Assumptions (i-xi), \( n^{1/2} \tilde{v}^{-1/2}[\hat{\mu}(x) - \mu(x)] = n^{1/2} v^{-1/2}[\mu(x) - \mu(x)] + o_p(1) \overset{d}{\to} \mathcal{N}(0, [\nabla_g \psi(\alpha, g(x))]^\otimes 2) \), where \( v \) and \( \tilde{v} \) are defined in Lemma 3.

**Proof of Lemma 4.** It follows from Chen (2007, p.5591, where Conditions 3.1”, 3.2, 3.4 and 3.5(i) are satisfied). □

**Proof of Lemma 5.** Note that \( \hat{\alpha} \) can be viewed as defined by \( \hat{\alpha} = \arg \min_{\alpha \in \mathbb{R}^d} n^{-1} \sum_{i=1}^n \ell_i(\alpha, \hat{g}(X_i); Y_i) \), where \( \hat{g}(\cdot) \) is the simultaneous series estimator of \( g(\cdot) \) defined in (53). It thus falls in the MINPIN framework of Andrews (1994b). The \( n^{1/2} \)-consistency of \( \hat{\alpha} \) follows by noting that conditions in Andrews (1994b) are satisfied under Assumptions (i-x). □

**Proof of Lemma 6.** By Taylor expansion and Assumption 2.1(c),

\[
\hat{\mu}(x) - \mu(x) = \nabla_\alpha \psi(\hat{\alpha}, \hat{g})(\hat{\alpha} - \alpha) + \nabla_g \psi(\hat{\alpha}, \hat{g})(\hat{g}(x) - g(x)) \tag{54}
\]
where \((\hat{\alpha}^\top, \hat{g})\) is between \((\tilde{\alpha}^\top, \tilde{g}(x))\) and \((\alpha^\top, g(x))\). Thus

\[
\begin{align*}
n^{1/2}v^{-1/2}[\hat{\mu}(x) - \mu(x)] & = n^{1/2}v^{-1/2}\nabla_{\alpha^\top} \psi(\tilde{\alpha}, \tilde{g})(\tilde{\alpha} - \alpha) + n^{1/2}v^{-1/2}\nabla_g \psi(\tilde{\alpha}, \tilde{g})(\tilde{g}(x) - g(x)) \\
\text{Lemma 5} & \Rightarrow o_p(1) + n^{1/2}v^{-1/2}\nabla_g \psi(\hat{\alpha}, g(x))(\hat{g}(x) - g(x)) \\
\text{Lemma 4} & \Rightarrow o_p(1) + n^{1/2}v^{-1/2}\nabla_g \psi(\alpha, g(x))(\hat{g}(x) - g(x)) \\
\xrightarrow{d} & \mathcal{N}(0, [\nabla_g \psi(\alpha, g(x))]^\otimes 2),
\end{align*}
\]

where the last line is by Lemma 3 and the \(O_p(n^{-1/2})\) term in (28). \(\square\)

We now discuss the Assumptions (i-xi) when \(\ell_i(\alpha, g)\) takes the form of (27).

(ii) holds by identification of \(\alpha\) and \(g(\cdot)\), discussed in the text before Section 2.1.

(iii) holds by Stone–Weierstrass approximation theorem and the smoothness of \(g\).

(v) holds by uniform LLN and smoothness of \(\ell_i\).

(vi) holds by identification of \(\alpha\) and \(g(\cdot)\).

(vii) holds by \(\sup_{x\in\mathcal{X}} |\tilde{g}(x) - g(x)| = o_p(n^{-1/2}K) = o_p(n^{-1/4})\) (if we assume \(n^{-1}K^4 \to 0\) as in (xii)), which can be shown slightly extending the framework in the previous subsection to include an additional finite-dimensional unknown parameter.

To show (viii-x), note that stochastic equicontinuity of \(\nu_{1n}(\alpha, g)\) and \(\nu_{2n}(\alpha, g)\) over \(\mathcal{A} \times \mathcal{G}\) follows from Andrews (1994a, Theorems 1 and 2) since \(\{\ell_i(\alpha, g(X_i)) : \alpha \in \mathcal{A}, g \in \mathcal{G}\}\) belongs to the Type III class of functions (Andrews, 1994a, p.2271) by Assumptions 2.1 (a)-(c), and it has a constant envelope function by Assumptions 2.1 (a)-(d) and \(Y_{ij}\) being binary. To show (ix) and (x), note that given \(\alpha \in \mathcal{A}\) and \(g \in \mathcal{G}\), \(n^{-1/2}\nu_{1n}(\alpha, g) = o_p(1)\) and \(n^{-1/2}\nu_{2n}(\alpha, g) = o_p(1)\) by the pointwise WLLN. The set \(\mathcal{A} \times \mathcal{G}\) is totally bounded by Assumptions 2.1 (b) and (d). Then uniform convergence of \(n^{-1/2}\nu_{1n}(\alpha, g)\) and \(n^{-1/2}\nu_{2n}(\alpha, g)\) over \(\mathcal{A} \times \mathcal{G}\) follows from Andrews (1992, Theorem 1).

### 6.3 Proofs of Theorems

**Proof of Theorem 1.** It follows from Lemma 4. \(\square\)

**Proof of Theorem 2.** It follows from Lemma 6. \(\square\)

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Proof of Theorem 3. Consider testing $H_0'$ when $d_q = 1$. Write $\mu_+(x) = \psi_+(\alpha_+, g_+(x))$ and $\mu_-(x) = \psi_-(\alpha_-, g_-(x))$. Apply (54) to $\hat{\mu}_+(c) - \mu_+(c)$ and $\hat{\mu}_-(c) - \mu_-(c)$, we have

$$
\hat{\tau} - \tau = \nabla_{\alpha^+} \psi_+(\alpha_+, g_+)(\alpha_+ - \alpha_+) - \nabla_{\alpha^-} \psi_-(\alpha_-, g_-)(\alpha_- - \alpha_-) + \nabla_g \psi_+(\alpha_+, g_+)(\hat{g}_+(c) - g_+(c)) - \nabla_g \psi_-(\alpha_-, g_-)(\hat{g}_-(c) - g_-(c)),
$$

(55)

where $(\alpha_+^+, g_+)$ is between $(\hat{\alpha}_+^+, \hat{g}_+(x))$ and $(\alpha_+, g_+(x))$, and $(\alpha_-^+, g_-)$ is between $(\hat{\alpha}_-^+, \hat{g}_-(x))$ and $(\alpha_-, g_-(x))$. For a vector of constants $Q \in \mathbb{R}^J$, by (55),

$$
\text{Proof of (12)}.
$$

It follows that, using (9) and Moore (1977, Lemma 2(b)),

$$
n(\hat{Q}_+ - Q_+)(\hat{Q}_+ - Q_+)^\top(Q_+ \hat{V}_{\mu,+} + Q_+^\top(\hat{Q}_+ - Q_+))
\quad = \quad n(\hat{Q}_+ - Q_+)\hat{V}_{\mu,+} + (\hat{Q}_+ - Q_+)Q_+^\top(\hat{Q}_+ - Q_+)
\quad = \quad \chi^2(\text{rank}(\hat{Q}_+ - Q_+)),
$$

if $[\nabla_g \mu_+(c)]^\otimes 2 \neq 0$. Replacing $V_{\mu,+}$ with $\hat{V}_{\mu,+}$ does not change the limit distribution by Lemma 6 and Andrews's condition. \(\square\)

Proof of Theorem 4. Let $A_1$ be the first column of $A$, and $\tilde{A}_1$ be the first column of $\tilde{A}$. We have

$$
n^{1/2}A_1^\top (\hat{Q}_- - q) \quad \overset{d}{\to} \quad N(0, 1),
$$

following the proof of Theorem 3, since $\lambda_1^{-1} A_1^\top QV_\tau Q^\top A_1 = \lambda_1^{-1} A_1^\top A \Lambda A^\top A_1 = \lambda_1^{-1} (1, 0, ..., 0)^\top A(1, 0, ..., 0)^\top = 37$
1. Then \( n^{1/2}(A_1^j)^{1/2}A^\top(Q\hat{\tau} - q) = [n^{1/2}\lambda_1^{1/2}A_1^j(Q\hat{\tau} - q), 0, ..., 0]^d \overset{d}{\rightarrow} \mathcal{N}(0, \text{diag}(1, 0, \ldots, 0)) \). So

\[
n(Q\hat{\tau} - q)^\top(AA powerful)_{j=1}^{j}(Q\hat{\tau} - q) \overset{d}{\rightarrow} \chi^2(1).
\] (56)

Looking at \( \mathcal{W}^{(k)}_{\text{Reg}} \), note that \( AA_1a_{1} = \lambda_1A_1A_1^j \) and \( \tilde{A}A_1A_1^j = \tilde{\lambda}_1\tilde{A}_1A_1^j \). We have \( K^{-1}\tilde{\lambda}_1\tilde{A}_1A_1^j - K^{-1}\lambda_1A_1A_1^j \overset{p}{\rightarrow} 0 \) since \( \tilde{V}_r/K - V_r/K = o_p(1) \), and \( \lambda_1 \) is a simple root and \( A_1 \) is the corresponding eigenvector. Thus \( (K^{-1}\tilde{\lambda}_1\tilde{A}_1A_1^j)^{j} - (K^{-1}\lambda_1A_1A_1^j)^{j} \overset{p}{\rightarrow} 0 \) by Andrews (1987, Theorem 2). Then (13) follows from (56). \( \Box \)

**Proof of (4) (under (15)).** For \( j = 2, \ldots, J \), it follows from

\[
\mu_{+j}(X) = \mathbb{P}(\tilde{Y}(1) = j|X) = \mathbb{P}(|R(1)| = j|X) = \mathbb{P}(t_{j-1} < R(1) \leq t_j|X)
\]

\[
= \exp\left(-\int_0^{t_{j-1}} \lambda_+(r|X)dr\right) - \exp\left(-\int_0^{t_j} \lambda_+(r|X)dr\right)
\]

\[
(15) \quad \exp(-\exp(\log A_+(t_{j-1}) + g_+(X))) - \exp(-\exp(\log A_+(t_j) + g_+(X)))
\]

\[
= \Phi(\log A_+(t_j) + g_+(X)) - \Phi(\log A_+(t_{j-1}) + g_+(X))
\]

\[
= \Phi(\alpha_{+j} + g_+(X)) - \Phi(\alpha_{+j-1} + g_+(X)). \quad \Box
\]

**Proof of (24).** Suppose \( \tilde{Y}(1, s) \) is the duration (if treated) to exit to state \( s \). Similarly define \( \tilde{Y}(0, s) \). Then \( \tilde{Y}(1) = \min_{s \in \{1, \ldots, S\}} \tilde{Y}(1, s) \) and \( \tilde{Y}(0) = \min_{s \in \{1, \ldots, S\}} \tilde{Y}(0, s) \). Like in Section 3.1, define \( R(1, s) \) and \( R(0, s) \) be corresponding latent continuous-time duration for each type of exiting, and their hazards follow

\[
R(1, s) : \lambda_{+s}(r|X) = \lambda_{+,s}(r) \exp(g_{+,s}(X)),
\]

\[
R(0, s) : \lambda_{-s}(r|X) = \lambda_{-,s}(r) \exp(g_{-,s}(X)).
\]

We assume conditionally independent risks (CIR): \( \mathbb{P}(\cap_{s \in \{1, \ldots, S\}} \{R(1, s) < a_s\}|X) = \Pi_{s \in \{1, \ldots, S\}} \mathbb{P}(R(1, s) < a_s|X) \) for any \( a_s \in \mathbb{R}^+ \). Similarly for \( R(0, s) \). Then \( \tilde{Y}(1, s) = [R(1, s)] \) and \( \tilde{Y}(0, s) = [R(0, s)] \) as
in Section 3.1. The log-likelihood for \( i \) (if treated) is

\[
\begin{align*}
\mathbb{P}(\widetilde{Y}_i(1) = j, \widetilde{U}_i = s \mid X) &= \mathbb{P}(\tilde{Y}(1, s) = j, \tilde{Y}(1, v) \geq j \text{ for all } v \neq s \mid X) \\
&= \mathbb{P}(t_{j-1} < R_i(1, s) \leq t_j, R_i(1, v) \geq t_{j-1} \text{ for all } v \neq s \mid X) \\
&= \mathbb{P}(t_{j-1} < R_i(1, s) \leq t_j \mid X) \prod_{v \in \{1, \ldots, S\} \setminus s} \mathbb{P}(R_i(1, v) \geq t_{j-1} \mid X) \\
&= \mu_{+,j,s}(X) \prod_{v \in \{1, \ldots, S\} \setminus s} \sum_{w = j}^{J+1} \mu_{+,w,v}(X).
\end{align*}
\]

Similarly for \( i \) in the control group. \( \square \)

References


and \{2\}-inverses–with Applications," *Statistics*, 49, 475-496.


Figure 2: (a) A sample \( (n = 5000) \); (b) The population proportion (for the control/treatment (left/right) group, respectively) for each discrete time interval \( j \).

Figure 3: (a) \( \mu_+(j|x) \) and \( \mu_-(j|x) \) against \( x \) (for \( j = 1, 2, 3 \) and \( 28 \)); (b) \( H_+(j|x) \) and \( H_-(j|x) \) against \( x \).
Figure 4: Population level probabilities (Panel (a)), level probability ATEs (Panel (b)), hazards (Panel (c)), and hazard ATEs (Panel (d)) (all at the cutoff) across $j$ in the simulation design.
Figure 5: (a) AIC-selected numbers of knots for the global PH-based estimator; (b) IK-selected local smoothing bandwidths (for level probability ATEs and hazard ATEs, respectively) (averaging over replications) for the local LPM-based estimators.

Figure 6: The finite-sample biases of $\hat{\tau}_j$ (in Panel (a)) and $\hat{\xi}_j$ (in Panel (b)), where $1 \leq j \leq 28$, using the global PH (proportional hazards)-based, local-linear LPM (linear probability model)-based, and local PH-based methods. $n = 5000$. 

Figure 7: The finite-sample standard deviations of $\hat{\tau}_j$ (in Panel (a)) and $\hat{\xi}_j$ (in Panel (b)), where $1 \leq j \leq 28$, using the global PH (proportional hazards)-based and local-linear LPM (linear probability model)-based, and local PH-based methods. $n = 5000$.

Figure 8: The finite-sample coverage probability of the 95% CI $\hat{\xi}_j$ for each $j$ under (a) DGP: HOMO and (b) DGP: HETERO, for $n \in \{5000, 10000\}$
Figure 9: The length of the 95% CI $\xi_j$ for each $j$ under (a) DGP: HOMO and (b) DGP: HETERO, for $n \in \{5000, 10000\}$

Figure 10: (a) Histograms of the conventional Wald statistic (divided by 10) and regularized Wald statistic (DGP: HOMO), $n = 5000$. (b) Finite sample size of the regularized Wald test and simulation-based Wald test (DGP: HOMO), $n \in \{5000, 10000\}$.
Figure 11: (a) Histograms of the conventional Wald statistic (divided by 10) and regularized Wald statistic (DGP: HETERO), $n = 5000$. (b) Finite sample size of the regularized Wald test and simulation-based Wald test (DGP: HETERO), $n \in \{5000, 10000\}$.

Figure 12: (a) Power functions of the 5% tests $W_{\text{Reg}}$ and $W_{\text{Eqw}}$; (b) Size-adjusted power functions. Design: HOMO. $n = 5000$. 

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