On the Economic Efficiency of the Combinatorial Clock Auction

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Abstract

Since the 1990s spectrum auctions have been implemented world-wide. This has provided for a practical examination of an assortment of auction mechanisms and, amongst these, two simultaneous ascending price auctions have proved to be extremely successful. These are the simultaneous multiround ascending auction (SMRA) and the combinatorial clock auction (CCA). It has long been known that, for certain classes of valuation functions, the SMRA provides good theoretical guarantees on social welfare. However, no such guarantees were known for the CCA.

In this paper, we show that CCA does provide strong guarantees on social welfare *provided* the price increment and stopping rule are well-chosen. This is very surprising in that the choice of price increment has been used primarily to adjust auction duration and the stopping rule has attracted little attention. The main result is a polylogarithmic approximation guarantee for social welfare when the maximum number of items demanded $\mathcal{C}$ by a bidder is fixed. Specifically, we show that either the revenue of the CCA is at least an $\Omega\left(\frac{1}{\mathcal{C}^2 \log n \log^2 m}\right)$-fraction of the optimal welfare or the welfare of the CCA is at least an $\Omega\left(\frac{1}{\log n}\right)$-fraction of the optimal welfare, where n is the number of bidders and m is the number of items. As a corollary, the welfare ratio – the worst case ratio between the social welfare of the optimum allocation and the social welfare of the CCA allocation – is at most $O(\mathcal{C}^2 \cdot \log n \cdot \log^2 m)$. We emphasize that this latter result requires no assumption on bidders valuation functions. Finally, we prove that such a dependence on $\mathcal{C}$ is necessary. In particular, we show that the welfare ratio of the CCA is at least $\Omega\left(\mathcal{C} \cdot \frac{\log m}{\log \log m}\right)$.

1 Introduction

The question of how best to allocate spectrum dates back over a century. In the academic literature, the case in favor of the sale of bandwidth was first formalized by Coase [14] in 1959. Since the 1990s spectrum auctions have been implemented world-wide. Moreover, for these bandwidth auctions, it has become apparent that “not all markets are alike". Outcomes, in terms of economic efficiency, revenue and the resultant level of competition in the telecommunications industry, are heavily dependent upon the choice of auction mechanism – see [31], [30] and [16] for detailed discussions.

In practice, two simultaneous ascending price type auctions have proved extremely successful: the simultaneous multiround ascending auction (SMRA) and the combinatorial clock auction (CCA). The SMRA was designed by Milgrom, Wilson and McAfee for the 1994 FCC spectrum auction; see [34]. Compared with sealed-bid auctions, such as the VCG, ascending auctions are widely believed to be more suitable for this scenario. For example, ascending auctions induce information transfers that allow prices to more accurately reflect valuations.[1] A consequence is more economically efficient allocations and, potentially, higher revenues. The SMRA has been very successful in practice but it also has drawbacks [16]. Amongst them is the exposure problem: a large set may be desired but such a bid may result in being allocated only a smaller undesirable subset. In auctions with complementarities, such as the spectrum auctions, this can become a serious issue. The CCA, due to Porter et al. [36], was designed to overcome this problem, and its usage has gained substantial momentum recently. Within the last two years, over ten major spectrum auctions have used extensions of the CCA [5, 16, 2] and generated approximately 20 billion dollars in revenue [2].[2]

Whilst, under certain conditions, there are theoretical explanations for the high social welfare (economic efficiency) produced by the SMRA [33, 29, 24, 23], the performance guarantee of the CCA remains elusive. One possible reason for this lack of success is that, upon first examination, no good welfare guarantees seem achievable for the CCA, even for very simple valuation functions. In Appendix A.1, we show that the *welfare ratio* – the worst case ratio between the social welfare of the optimum allocation and the social welfare of the CCA allocation – can be as high as $O(\sqrt{n})$ even for unit-demand bidders. Here n denotes the total number of bidders (we will denote by m the number of items). Moreover, we also show the welfare ratio can be has high as $O(n)$-ratio, even when demand bundles have cardinality at most two. However, a better selection of price increments, allows us to obtain polylogarithmic upper bounds on the welfare ratio for the CCA if the bidders demand is small (e.g. she is only interested in bundles of cardinality at most polylogarithmic in the number of items and bidders). In the Porter et al. CCA mechanism, price increments were fixed to be 1 whenever there is excess demand. We obtain our strong welfare guarantees simply by requiring the price increments to be a function of excess demand. Specifically, if there are k bids containing item j then we increase p_j by $\epsilon \cdot k$, where ϵ is a small constant chosen by the auctioneer. With this modification, we obtain:

Informal Theorem 1. *If all bids have cardinality at most $\mathcal{C}$, then either the revenue of the CCA is at least an $\Omega(\frac{1}{\mathcal{C}^2 \log n \log^2 m})$-fraction of the optimal social welfare, or the welfare of the CCA is at least an $\Omega(\frac{1}{\log n})$-fraction of the optimal welfare.*

This result has two appealing properties. First, it does not make any assumption on the valuations. Hence, it accommodates complementarities, which are common in combinatorial auctions but are typically hard to deal with. Second, it guarantees that either the revenue of the CCA is

[1] Indeed, the price discovery process allows bidders to learn valuations (including their own valuations!). This is particularly important in bandwidth auctions [16].

[2] These auctions were held worldwide, for example, in Austria, Australia, Canada, Denmark, Netherlands, UK, Switzerland, etc.

high or the welfare is close to the optimal. Since the social welfare is always not smaller than the revenue, our theorem directly implies the following upper bound on the welfare ratio.

Informal Theorem 2. *If all bids have cardinality at most $\mathcal{C}$ then the welfare ratio for the CCA is at most $O(\mathcal{C}^2 \log n \log^2 m)$.*

Therefore when bidders only demand sets of cardinality at most k, where $k = O(\text{poly}(\log n, \log m))$, the CCA has polylogarithmic welfare ratio.

So the choice of price-increments is fundamental in guaranteeing good performance. Our results show that the choice of stopping rule is also critical. Indeed, they rely upon usage of the original stopping rule of Porter et al. [36]. However, in recent versions of the CCA this stopping rule has been replaced by the simpler stopping rule used by the SMRA – unfortunately, we show in Appendix A.2 that the SMRA stopping rule is insufficient to guarantee good welfare. Finally, one might wonder if the dependence on $\mathcal{C}$ is necessary. We show in Section 5 that this dependence is unavoidable for general valuations.

Informal Theorem 3. *For any integer $\mathcal{C}$, there exist arbitrarily large integers n and m and an auction with n bidders and m items such that each bidder only bid on sets with cardinality at most $\mathcal{C}$ where the welfare ratio of the CCA is at least $\Omega(\mathcal{C} \cdot \frac{\log m}{\log \log m})$.*

We now provide a brief road-map of our paper. In Section 2, we give an overview of the CCA and SMRA auctions and discuss related work. We give a formal description of the CCA in Section 3. In Section 4.1, we introduce some of the key ideas and techniques required in a simple setting. Namely, we use them to prove a polylogarithmic welfare guarantee for the special case of unit-demand bidders. We generalize this approach to obtain our main results in Section 4.2. Finally, we prove a lower bound of welfare approximation for the CCA in Section 5. In addition, in the appendices, we also provide examples showing that wrong choices of price increment and stopping condition can be detrimental for the performance of the CCA.

2 An Overview of the SMRA and CCA

Both the SMRA and the CCA are based upon the same underlying ascending price mechanism: at time t, each item j has a price p_j^t. At these prices, each bidder i selects her preferred set S_i^t of items. The price of any item that has excess demand then rises in the next time period and the process then is repeated. The first major difference between these auction mechanisms lies in the bidding language. The SMRA uses *item bidding*. The auctioneer views the selection S_i^t as a collection of bids, one bid for every subset of S_i^t. However this leads to the exposure problem. To overcome this problem, the CCA uses *package (combinatorial) bidding*. A package bid is an all-or-nothing proposition: the selection of S_i^t is a bid for exactly S_i^t and nothing else.

The original CCA, due to Porter et al. [36], terminates whenever the last round bids are disjoint and not in conflict with the revenue-optimal allocation; see Section 3 for details. The SMRA (and later versions of the CCA) terminates when there is no excess demand for any item. The auctions differ significantly in how items are allocated. The SMRA utilizes the concept of *standing high bid*. Any item (with a positive price) has a *provisional winner*. That bidder will win the item unless a higher bid is received in a later round. If such a bid is received then the standing high bid is increased and a new provisional winner assigned (chosen at random in the case of a tie). It is not difficult to see that this allocation rule increases the risk of exposure for the bidders. In the CCA, the maximum revenue allocation is output. All bids, regardless of the time they were made, are eligible for this allocation, with the constraint that each bidder has at most one accepted bid.

Social Welfare. In practice these ascending auctions have performed extremely well. A major reason for this is that the associated dynamic processes encourage accurate price discovery [5, 16]. For the SMRA, there are theoretical results that explain this practical performance. These results are driven by the use of (aesthetically unappealing but important) standing high bids which ensure that every item with a positive price is sold, since each such item has a provisional winner. This concept of standing high bids was introduced by Crawford and Knoer [17] to study a simple market matching workers to firms. Their model encompasses an ascending auction with unit-demand bidders which converges, under truthful bidding, to a Walrasian equilibrium that maximizes social welfare. Moreover, Kelso and Crawford [29] showed that welfare-maximizing Walrasian equilibrium are also obtained, under truthful bidding, in auctions where the bidder valuations satisfy the gross substitutes property; see also [24]. The method to select items for price rises in the Kelso-Crawford mechanism differs from the more natural choice made by the SMRA, which increments the prices of all items under excess demand. Milgrom [33], however, showed these results continue to hold for the SMRA. Walrasian equilibrium need not exist for more general valuations, but approximate welfare guarantees can still be obtained for submodular valuation functions [23]. No such theoretical results are know for the CCA and that is the motivation behind this work.

Bidding Activity Rules. It is important to note that, in practice, accompanying the ascending price mechanisms are a set of *bidding activity rules.* The activity rules are designed to induce truthful bidding in each round. This is extremely important – Cramton [16] states that the "*truthful expression of preferences is what leads to excellent price discovery and ultimately an efficient auction outcome*". For the SMRA, though, the activity rules are quite weak and strategic bidding is common and, often, profitable [16]. In contrast, the CCA incorporates a much stronger set of bidding activity rules for each round than the SMRA. In particular, the CCA applies a set of *revealed preference* (RP) constraints on feasible bids. Suppose at time s we bid for package S and at time $t > s$ we bid for package T. In its weak form, see Ausubel et al. [5], revealed preference produces a constraint $p^t(S) - p^s(S) \geq p^t(T) - p^s(T)$. That is, such bidding behavior can only be rational if the price of package S has risen by at least the rise in the price of package T. If not the bid for package T is forbidden. Moreover, in its general form [26, 1], the revealed preference rules ensure that the only bidding strategies that are admissible correspond to virtual valuation functions. Indeed, even relaxed implementations of the constraints allow only for approximate virtual valuation functions; see Boodaghians and Vetta [11]. If follows that strategic behavior must take a very restricted form in the CCA, essentially consisting of pre-commiting to a virtual valuation function. Given the lack of information and the dynamic nature of the CCA, such a pre-commitment is hard to compute and extremely risky. Consequently, the working assumption in this paper that bidders behave truthfully in a competitive CCA auction is reasonable.

Practical Implications. We show that modifying the price-increments can have fundamental impact on social welfare. This is quite remarkable because, in practice, the choice of price increments has been primarily considered as a matter of fine-tuning. Price increments are seen as a way to affect the length of the auction whilst having minimal effect on the final outcome. Indeed, Ausubel and Baranov [2] state that "*Among all design decisions that need to be made prior to the auction, [the choice of price increments] is considered relatively unimportant and is often overlooked by the design team*". Our results show that the choice of price increments is actually extremely important.

Our results also show that another apparently innocuous aspect of the CCA mechanism is vital in generating high welfare: the choice of *stopping rule.* Recall, the original CCA only terminates when there is no excess demand induced by the bids in the current round *and* the maximum revenue allocation over all rounds is not in conflict with the current round bids. Current implementations of the CCA are based upon the two-stage mechanism of Ausubel, Cramton and Milgrom [5]. There, the ascending price mechanism, as described, is used in the first-phase except that the stopping rule reverts to the simple rule used in the SMRA: the mechanism terminates if bids in the current round are disjoint. The use of this simple stopping rule is unfortunate: its use cannot guarantee high welfare. We present, in Appendix A.2, a simple example with demand sets of cardinality two where, under this stopping rule, the CCA produces arbitrarily poor welfare, even when price-increments are a function of excess demand.

Finally, it remains to discuss computational aspects. The combinatorial allocation problem is notoriously hard to approximate (it contains maximum independent set as a special case). This appears to suggest that the CCA is not implementable in polynomial time. In spectrum auctions, however, bid patterns are highly structured. This has two major effects. First, given a set of prices, bidders do seem able to make bids in a timely manner. Second, the resultant combinatorial allocation problems can be solved almost instantaneously using standard branch and bound optimization techniques. As a consequence, it does not appear that computational constraints are currently a major concern in implementing ascending auctions in practice.

Related Work. An alternative approach for unit-demand auctions was examined by Demange, Gale and Sotomayor [20]. Their ascending auction also outputs a Walrasian equilibrium that maximizes social welfare but without the need for standing high bidders. To achieve this, however, each bidder is now required to submit their entire demand set[3] in each round, rather than just a single bid as in the SMRA and CCA. Given prices, the mechanism tests whether a Walrasian equilibrium can be produced from the demand sets; if not, a set of items under excess demand is obtained based upon Hall's theorem. Gul and Stacchetti [25] showed this approach also generalizes to auctions where bidder valuations satisfy the gross substitutes property. Interestingly, these ascending auctions produce the minimum set of Walrasian prices, which in the case of unit-demands also correspond to VCG payments. However, Gul and Stacchetti [25] proved that it is not possible to implement the VCG mechanism via an ascending auction for general valuations, even those with the gross substitutes property. Thus, truthful bidding does not form an equilibrium in the corresponding direct mechanism. To go beyond the gross substitutes property, De Vries, Schummer and Vohra [38] dropped the requirement of anonymous prices. Instead, each bundle requires a separate price for each bidder.

From the theory of computation side, a relevant and fruitful direction is to approximate the welfare of a combinatorial auction with simple but not necessarily truthful auctions [13, 7, 27, 22, 37, 6], for example, simultaneous single-item auctions, the ascending auction of Gul and Stacchetti, etc. The focus of this sequence of papers is to show the price of anarchy in these games is some small constant for a certain set of valuations. In particular, Babaioff et al. [6] showed that as long as the simple auction maximizes the welfare over the bidders' declared valuations, the price of anarchy is a small constant. Unfortunately, this result cannot be applied to the CCA, which is not a declared welfare maximizer. Indeed, to the best of our knowledge, none the known results directly applies to the CCA. We suspect that the slow progress is largely due to the ignorance of the complicated dynamic behavior of the mechanism. We hope the results and techniques developed may help tackle other research questions related to the CCA, such as its price of anarchy.

3 The Combinatorial Clock Auction

We now give a detailed description of the CCA. To begin with, we present some notations and definitions. Let N be the set of the bidders and M be the set of items. Each bidder i has value $v_i(S)$ for any set of items $S \subseteq M$. The price $p(S)$ for a set of item S is simply the sum of prices

[3] The demand set consists of every bundle that maximizes profits given the prices in that round.

for each item in S. The utility of bidder i for set S is $v_i(S) - p(S)$, where $p(S)$ is the price of S. The CCA outputs an allocation $\mathbf{S} = \{S_1, \ldots, S_n\}$, where the S_i are pairwise-disjoint subsets of the items, and a collection of prices $\{p_j\}_{j\in[m]}$. The *social welfare* of an allocation $\mathbf{S}$ is $\sum_i v_i(S_i)$ and the *revenue* is $\sum_i p(S_i) = \sum_i \sum_{j\in S_i} p_j$. Let $\mathrm{OPT} = \max_{\mathbf{S}} \sum_i v_i(S_i)$ be the optimal social welfare and $\mathbf{R}^*$ be the corresponding allocation. The CCA is then formalized in Procedure 1.

Procedure 1 The Combinatorial Clock Auction

Let $t = 0$ and the initial price p_j^0 be 0 for every item j.
loop
  Each bidder i bids for the set of items S_i^t of largest positive utility (breaking ties arbitrarily). A bidder drops out if she has non-positive utility for every subset of M.
  Let $P_i^t = \sum_{j\in S_i^t} p_j^t$ be the price for set S_i^t in round t.
  for $j = 1$ to n **do**
    $p_j^{t+1} \leftarrow p_j^t \ + \epsilon \cdot \sum_{S_i^t : j \in S_i^t} 1$.
  end for
  if the sets S_i^t are pairwise disjoint **then**
    Amongst all the bids $\{(S_i^{t'}, P_i^{t'})\}_{i\in[n], t'\le t}$ ever made, find the revenue maximizing allocation $\mathbf{S}^* = (S_1^{t_1^*}, \ldots, S_n^{t_n^*})$, that is, $\mathbf{S}^* \in \operatorname{argmax}_{\text{disjoint sets: } S_1^{t_1}, \ldots, S_n^{t_n}} \sum_i \sum_{j\in S_i^{t_i}} p_j^{t_i}$.
    if $S_i^t \cap S_j^{t_j^*} = \varnothing$ for every $i \neq j$ **then**
      Output Allocate the set of items $S_i^{t_i^*}$ to bidder i, and charge her $P_i^{t_i^*}$ as the payment.
    end if
  end if
  $t \leftarrow t + 1$.
end loop

Note that the CCA does terminate. If not, prices will monotonically increase and eventually force every bidder to drop out, then the stopping condition is trivially satisfied. The length of the auction depends on ϵ. In most ascending auctions, the price increment ϵ is a small constant chosen by the auctioneer and the values of bidders are assumed to be integers. Since the CCA is scale invariant[4], for notational simplicity, we set $\epsilon = 1$ and assume every bidder i's value for any set of items S is a multiple of some integer $W \ge n^3 m^2$.

Before proceeding further, we present a few simple but useful facts about the CCA. The *utility* of bidder i in round t, denoted by u_i^t, is her utility for her favorite set of items, given the prices in round t. Formally, $u_i^t = \max_{S\subseteq M}\left(v_i(S) - \sum_{j\in S} p_j^t\right)$. Since S can be $\emptyset$, u_i^t is always non-negative. With this definition, we are ready to show some properties of the CCA. As Fact 2 is self-evident, we do not provide the proof here.

Fact 1. *For any bidder, u_i^t is monotonically non-increasing.*

Proof. Suppose $u_i^t < u_i^{t'}$ for some $t' > t$. Let S be a set of items satisfying $u_i^{t'} = v_i(S) - \sum_{j\in S} p_j^{t'}$. Since prices are non-decreasing, we have $v_i(S) - \sum_{j\in S} p_j^t \ge v_i(S) - \sum_{j\in S} p_j^{t'} > u_i^t$, contradicting the maximality of u_i^t. $\square$

Fact 2. *If bidder i bids on S in round $t > 0$, then $v_i(S) > u_i^t$.*

Fact 3. *If bidder i is still active when the stopping condition is met, then i is allocated a subset of items whose value is at least her utility in the final round.*

[4]If we multiply all the values, bids and prices by a common factor, the execution of the mechanism is the same.

Proof. Let $\hat{t}$ be the final round. Since no item in $S_i^{\hat{t}}$ is allocated to any bidder $j \neq i$ in the CCA (by definition of the stopping condition), the entire set $S_i^{\hat{t}}$ may still be allocated by the mechanism to bidder i. Thus, i must win some items in the revenue-optimal allocation. Let us assume she wins $S_i^{t_i^*} \neq \emptyset$ for some $t_i^* \leq \hat{t}$. By Fact 1, $u_i^{t_i^*} \geq u_i^{\hat{t}}$. Therefore, her value for $S_i^{t_i^*}$ is clearly at least $u_i^{\hat{t}}$. $\square$

4 Social Welfare Guarantees for the CCA

We are now ready to quantitatively analyse the CCA. We begin with the case of unit-demand bidders, and prove that the CCA achieves a polylogarithmic fraction of the optimal social welfare. Whilst the case of unit-demand bidders might seem limited, the techniques developed for this basic case will be important as we then use them to extend our results to general valuation functions.

4.1 The Welfare Ratio for Unit-demand Bidders

We say a bidder is *unit-demand*, if she demands one item at most. Formally, we define bidder i's valuation as $v_i(S) = \max_{j \in S} v_{ij}$ where v_{ij} is i's value for item j. For unit-demand bidders, a feasible allocation is simply a matching between the bidders and the items. In this section, we show the matching selected by the CCA achieves an $\Omega\left(\frac{1}{\log n \log^2 m}\right)$-fraction of the optimal social welfare. But we have already seen two mechanism that maximize social welfare in this special case. So before proving the logarithmic welfare ratio for the CCA, it is informative to understand why the CCA does not achieve optimality. First the Crawford-Knoer mechanism [17] achieves optimality via the use of standing high bids. But the motivation behind the CCA was to allow package bidding on multi-item auctions with complementarities and then standing high bids. The Demange et al. mechanism [20] achieves optimality by requiring that each bidder submits her entire demand set. From a theoretical viewpoint that is exactly the right thing to do, and the CCA losses out by requiring one bid per round only. In practice, however, the Gul and Stacchetti mechanism [25] (which generalizes the Demange et al. mechanism for general demand) would be extremely complicated for bidders to use. Moreover, it is not clear how one could use simple bidding activity rules to incentivize truthful bidding in such a complex auction. In contrast, the CCA is a very simple mechanism that is incentivizable using bidding activity rules. This is important as experiments [10] suggest that simplicity is key if we want to generate welfare and revenue.

Now let's return to analysing the CCA. Whilst it is difficult to directly relate the welfare of the CCA with the optimal social welfare, we establish our result using a greedy allocation as a proxy of the CCA's outcome. We show that there are only two possibilities for the greedy allocation: (i) the revenue of this allocation is at least $\Omega\left(\frac{\mathrm{OPT}}{\log n \log^2 m}\right)$, and since the CCA selects the revenue optimal allocation, its revenue can only be higher; or (ii) the greedy allocation has small revenue, but many bidders still have high utility when the ascending-price phase ends. Combining this property with Fact 3, we can immediately show that the welfare of the allocation selected by the CCA is at least $\Omega\left(\frac{\mathrm{OPT}}{\log n}\right)$.

The greedy allocation method is shown in Procedure 2. Since bidders are unit-demand, we will use v_{ij} to denote bidder i's value for item j, and s_i^t to denote the item bidder i bids on at round t. Our greedy algorithm simply allocates the item to the highest available bid, then removes all bids that conflict with it and repeats. We terminate this procedure when the highest bid is smaller than some predetermined threshold b only to make the analysis cleaner.

Procedure 2 Greedy Allocation Procedure for Unit-demand Bidders

Input: $\mathcal{S} = \{(s_i^t, p_{s_i^t}^t)\}_{i,t}$ the collection of bids made in the CCA and $b \geq n^2$ the threshold.
while $\mathcal{S} \neq \emptyset$ **do**
  Let $(s_i^t, p_{s_i^t}^t)$ be the bid with maximum price (break ties arbitrarily).
  if $p_{s_i^t}^t \geq b$ **then**
    Allocate item s_i^t to bidder i with price $p_{s_i^t}^t$.
    Remove every bid of bidder i and remove every bid (or any bidder) for item s_i^t in $\mathcal{S}$.
  else
    return
  end if
end while

Theorem 4. *Either the revenue of the CCA for unit-demand bidders is at least $\frac{OPT}{480 \log n \log^2 m}$ or the social welfare is at least $\frac{OPT}{24 \log n}$.* [5] *Thus, the welfare ratio is at most $O(\log n \log^2 m)$.*

We will use the following notation. Let X be the set of bidders that have been assigned items in the greedy algorithm and $\tilde{X}$ be the set of items that are allocated to them. Let $k \geq |X|$ be some integer and $c \in [3, \frac{n}{2} - 1]$ be an integer that we will specify later. A key lemma is that if k is small then the utility of bidders that are not in X decreases in a slow rate. Specifically,

Lemma 5 (Time Amplifying for Unit-demand Bidders)**.** *Let S be a set of at least $c \cdot k$ bidders disjoint from X ($|X| \leq k$), such that every bidder $i \in S$ has utility at least $u \geq 2b$ in round $t \geq b + c - 1$. If the greedy algorithm has revenue less than $k \cdot b - mn$, then in any round up to $(c-2)(t+1) - 1$ (the mechanism can terminate before that), there is a subset of at least $|S| - c \cdot k$ bidders of S such that each of them has utility at least $u - 2b$.*

As Lemma 5 is mainly used when $|S| >> c \cdot k$ and $u >> b$, let us consider $|S|$ and u being much larger than k and b. Intuitively, it states that if at round t there exists a large set S, in which all bidders have high utility, then throughout round $c \cdot t$, most bidders in S (at least $|S| - c \cdot k$ bidders) still have high utility ($u - 2b$).

Before providing the formal proof of Lemma 5, let us first sketch the underlying idea. Notice that every bidder in S has utility at least u till round t. Thus, by Facts 1 and 2, every bidder i in S has value at least u for each item in Q_i, where Q_i is the collection of every item that i ever bid on in the first t rounds. For any set $S' \subseteq S$, if all bidders in S' have utility no more than $u - 2b$ at a certain round, then every item in $\cup_{i \in S'} Q_i$ must have price at least $2b$. However, for any item in $\cup_{i \in S'} Q_i - \tilde{X}$, only bidders selected by the greedy algorithm (the ones in X) can bid on it after the price has reached b. If not, then there exists a bidder outside X that should have been chosen by the greedy algorithm because she made a bid on some item not in $\tilde{X}$ at price higher than b. If $|S'| \geq c \cdot k$, then it is easy to argue that $|\cup_{i \in S'} Q_i - \tilde{X}| \approx \frac{c \cdot t \cdot k}{b}$. As each item in $\cup_{i \in S'} Q_i - \tilde{X}$ requires about b bids from bidders in X, these bidders need to make roughly $c \cdot t \cdot k$ many bids. There are at most k bidders in X so, in total, this will take at least $c \cdot t$ rounds. We are now ready to present the formal argument.

Proof of Lemma 5: Let t' be the first round where at least $c \cdot k$ bidders from S have utility at most $u - 2b$, and let S' be the set of these bidders. If t' does not exist, then at least $|S| - c \cdot k$ bidders from S have utility at least $u - 2b$ when the mechanism terminates. Lemma 5 holds.

[5] We remark that we have not attempted to optimize the constants in this theorem.

Because every bidder $i \in S$ has utility u at round t, Facts 1 and 2 ensure that $v_{ij} \geq u$ for any item j that i bids on in the first t rounds. Let M' be the items in $M - \tilde{X}$ that are bid on by some bidder in S' during the first t rounds. Bidders in S' make $|S'| \cdot t$ bids in total. How many bids can they make on $\tilde{X}$? No bidder from S' is allocated an item by the greedy algorithm. Therefore, for any item j in $\tilde{X}$, none of the bids made by bidders in S' can exceed the price p_j that the greedy algorithm sells the item for. So, before the final round, the total number of bids from S' on item j is less than p_j. In the last round, they make at most $|S'| \leq n$ bids. Therefore, the total number of bids made on $\tilde{X}$ is at most the revenue of the greedy allocation plus $n \cdot m$, which is at most $k \cdot b$ by assumption.

As none of the bidders from S' is selected by the greedy algorithm, bidders from S' must stop bidding on any item in $M - \tilde{X}$ after its price reach b. Thus, the total number of bids made by bidders from S' on any item in $M - \tilde{X}$ is at most $b + n$, implying $|M'| \geq \frac{|S'| \cdot t - k \cdot b}{b+n}$. Note that for each item in M', at least one bidder in S' has value at least u for it. Thus, at round t', the price for each of these items must be at least $2b$, otherwise that bidder will have utility greater than $u - 2b$.

In the round when the price of j passes b, its price is at most $b + n$. Bidders from X need to make at least another $b - n$ bids to drive the price up to $2b$. Because $|M'| \geq \frac{|S'| \cdot t - k \cdot b}{b+n}$, they must make at least $(b-n) \cdot \frac{(|S'| \cdot t - k \cdot b)}{b+n}$ bids. As there are at most k bidders in X, we have

$$\begin{aligned}
t' &\geq \frac{b-n}{b+n} \cdot \frac{|S'| \cdot t - k \cdot b}{k} \\
&\geq (1 - \frac{2}{n}) \cdot \frac{|S'| \cdot t - k \cdot b}{k} && (b \geq n^2) \\
&\geq (1 - \frac{2}{n}) \cdot (c \cdot t - b) && (|S'| \geq c \cdot k) \\
&\geq (1 - \frac{2}{n}) \cdot (c-1)(t+1) && (t \geq b + c - 1) \\
&\geq (c-2)(t+1) && (\frac{n}{2} - 1 \geq c)
\end{aligned}$$

By definition of t', it is then straightforward to see that at round $t' - 1 \geq (c-2)(t+1) - 1$ there are at least $|S| - c \cdot k$ bidders from S such that each of them has utility at least $u - 2b$. ■

In order to prove Theorem 4, we need one more Lemma.

Lemma 6. *Let $\boldsymbol{R}^* = (R_1, \ldots, R_n)$ be the allocation that maximizes the social welfare. Then there exists a set B and a real number v^* such that every bidder $i \in B$ has value between $[v^*, 2v^*]$ for the set R_i and the total value for bidders in B is at least $\frac{OPT}{3 \log n}$.*

The proof of this Lemma is fairly standard. We include it in the Appendix B.1 for completeness. With Lemma 5 and 6, we are ready to prove Theorem 4. Again, we first sketch the proof idea. Let B and v^* be the same as in Lemma 6. We set the threshold b to be $\Theta(\frac{v^*}{\log m})$. The goal is to argue that if the revenue of the greedy allocation is less than $\Theta(\frac{\mathrm{OPT}}{\log n \log^2 m})$, then most bidders from B still have utility as high as $\Theta(v^*)$ when the CCA terminates. This implies the social welfare is at least $\Theta(\frac{\mathrm{OPT}}{\log n})$ by Fact 3. First note that if more than $\frac{B}{\log m}$ bidders are selected by the greedy algorithm (*i.e.* bidders in X) then the revenue already meets our target. So we assume this is not the case. This means the number of bidders in $B - X$ is at least $\log m$ times more than the number of bidders in X. Since all these bidders have utility at least $v^* \approx \log m \cdot b$ in the beginning, we can apply Lemma 5 to this set of bidders. The key insight in this proof is that we can repetitively apply Lemma 5 on the set of bidders that still have high utility. Because each application of Lemma 5 decreases the size of the set of bidders and their utilities linearly in $|X|$ and b, we can apply it $\Theta(\log m)$ times provided the CCA has not yet terminated. If this is the case, the CCA runs for at least $\Theta(m \cdot b)$ rounds. Since any active bidder makes a bid each round, it is easy to show that throughout $\Theta(m \cdot b)$ many rounds this bidder has to make a bid on some item outside $\tilde{X}$ (the set of items allocated in the greedy algorithm) at a price larger than b. This gives a contradiction. Hence, the CCA must terminate before that amount of rounds. In this case, we can argue that at least a constant fraction of bidders in $B - X$ still have utility $\Theta(v^*)$ when the CCA terminates, implying high social welfare. We now formalise this argument.

Proof of Theorem 4: Let B and v^* be as in Lemma 6. We use $\mathcal{K}$ to denote the size of B. Let the threshold $b = \frac{v^*}{c_1 \log m}$ and $k = \frac{\mathcal{K}}{c_2 \log m}$ for some constants c_1 and c_2 that will be specified later. We will prove that either the revenue of the greedy algorithm, with our choice of b, is at least $g = \frac{\text{OPT}}{6c_1c_2 \log n \log^2 m}$ or the social welfare of the CCA is at least $\frac{\text{OPT}}{24 \log n}$.

First, by Lemma 6 and our choice of the parameters, it is easy to verify that $k \cdot b \geq g$. So we can assume $|X| \leq k$ from now on. Otherwise the revenue is already at least $|X| \cdot b > g$. Let X' be the set of bidders that items in $\tilde{X}$ are allocated to under $\mathbf{R}^*$ and let $Y = B - X - X'$. Because $|X'| \leq |\tilde{X}| = |X| \leq k$, we have $|Y| \geq (c_2 \log m - 2)k$. Let Z be the set of items that are allocated to bidders in Y in the allocation $\mathbf{R}^*$, and let t_1 be the first round (if it exists) where at least $c \cdot k$ items in Z have prices at least $2b$. We proceed by case analysis.

Case (1): t_1 does not exist. By the definition of t_1, this means that when the CCA terminates at round t, there exists a set $Y' \subseteq Y$ of bidders whose allocated items in $\mathbf{R}^*$ have prices no more than $2b$ at round t and $|Y'| \geq |Y| - ck \geq |Y|/2$. Therefore, $u_i^t \geq v^* - 2b \geq v^*/2$ for any bidder $i \in Y'$. By Fact 3, the social welfare of the greedy allocation is then at least

$$|Y'| \cdot u_i^t \geq \frac{|Y| \cdot v^*}{4} \geq \frac{c_2 \cdot k \cdot \log m \cdot v^*}{8} \geq \frac{\mathcal{K}v^*}{8} \geq \frac{OPT}{24 \log n}.$$

Case (2): t_1 exists. Since Z is disjoint from $\tilde{X}$, only bidders from X can bid on items in Z after their prices reach b. Thus, $t_1 \geq \frac{(b-n)\cdot c\cdot k}{|X|}$, because $k \geq |X|$ and $b \geq n^2$ and $c \geq 3$, $t_1 \geq c \cdot (b - n) > b + c$. By the definition of t_1, we know at round $t_1 - 1$ there are at least $|Y| - c \cdot k$ bidders whose allocated items in $\mathbf{R}^*$ have prices less than $2b$. Let us call this set of bidders Y_1, and clearly they all have utility at least $u_1 = v^* - 2b$ at round $t_1 - 1$. Let t_2 be the first round (if it exists) that at least $c \cdot k$ bidders from Y_1 have utility no greater than $u_2 = u_1 - 2b$, and let Y_2 be the set of bidders from Y_1 that still have utility at least u_2 at round $t_2 - 1$. Now we let Y_1 be S, u_1 be u and $t_1 - 1$ be t and apply Lemma 5 on them, the lemma gives that $t_2 - 1 \geq (c-2)t_1 - 1$. Hence, $t_2 \geq (c-2) \cdot t_1$.

There is nothing special about Y_1, u_1 and $t_1 - 1$. If we recursively define t_i as the first round that at least $c \cdot k$ bidders from Y_{i-1} have utility no greater than $u_i = u_{i-1} - 2b$, and Y_i as the set of bidders from Y_{i-1} that still have utility at least u_i at round $t_i - 1$, we can apply Lemma 5 on Y_{i-1}, u_{i-1} and $t_{i-1} - 1$ as long as they satisfy the conditions in Lemma 5 (and that t_i exists). In that case, we have $t_i \geq (c-2) \cdot t_{i-1}$. How many times can we apply Lemma 5 before the conditions are violated? Since the size of Y_i decreases by at most $c \cdot k$ and u_i decreases by $2b$ in every application of Lemma 5, we have $|Y_i| \geq |Y| - i \cdot c \cdot k$ and $u_i = v^* - 2i \cdot b$. To violate the conditions of Lemma 5, we need $|Y_i| < c \cdot k$ or $u_i < 2b$, hence we can apply Lemma 5 for at least $\ell' = \min\{[\frac{|Y|}{c\cdot k}], [\frac{v^*}{2b}]\} - 1 \geq \min\{[\frac{c_2-1}{c} \cdot \log m], [\frac{c_1-1}{2} \cdot \log m]\}$ times. If we let $c_1 = 8$, $c_2 = 10$ and $c = 4$, we have $\ell' \geq 2 \log m + 6$. Remember that we might not be able to run the recursion till the conditions are violated, because the CCA might terminate before that. We now use case analysis to show that if we take $\ell = \log m + 2$, then our claim holds no matter whether t_ℓ exists or not.

Case (i): The CCA terminates between t_j and t_{j+1} for some $j < \ell$. In this case, there are at least $|Y_j| - c \cdot k$ bidders each of whom has utility at least $u_j - 2b$. Since $\ell \leq \frac{\ell'}{2} + 1$, we have

$|Y_j| - c \cdot k \geq |Y| - c(j+1)k \geq |Y|/2$ and $u_j - 2b \geq v^* - 2(j+1)b \geq v^*/2$. Thus, as in Case (1), the social welfare of the greedy allocation is at least $\frac{OPT}{24 \log n}$.
Case (ii): t_ℓ exists. In this case, we argue that there is a bidder who has made a bid larger than b on $M \setminus X'$ that has not been selected by the greedy algorithm. This results in a contradiction. Note that $t_\ell \geq 2^{\ell-1} \cdot t_1 > 2m \cdot t_1 > (k+m) \cdot b$. The last inequality holds because every bidder in B receives an item, so $k \leq \mathcal{K} \leq m$. Hence, there is a bidder i in Y_ℓ that has made $(k+m) \cdot b$ bids in the ascending-price phase. The revenue of the greedy allocation is less than $k \cdot b$, so i can make at most $k \cdot b$ bids on items in $\tilde{X}$. Therefore, i makes at least $m \cdot b$ bids on $M - \tilde{X}$, which means there must be one item j in $M - \tilde{X}$ that i has bid on with price larger than b. This cannot happen, otherwise i would have been selected by the greedy algorithm. ■

4.2 An Upper Bound on the Welfare Ratio for General Bidders

In this section, we generalize Theorem 4 to accommodate general valuation functions. The idea is similar. Using a greedy algorithm as a proxy for the CCA, we argue that there are only two possibilities: (i) the revenue of the greedy allocation is at least $\Omega\left(\frac{\text{OPT}}{\mathcal{C}^2 \log n \log^2 m}\right)$, and since the CCA selects the revenue optimal allocation, it must achieve no less revenue than the greedy allocation; or (ii) the greedy allocation has small revenue, but many bidders still have high utility at the final round. Using Fact 3, we can immediately show that the welfare of the allocation selected by the CCA is at least $\Omega(\frac{\text{OPT}}{\log n})$. The greedy algorithm and the full proof can be found in Appendix B.

Theorem 7. *If bidders only bid on sets with cardinality at most $\mathcal{C}$, then the either the revenue of the CCA is at least $\frac{OPT}{480\mathcal{C}^2 \log n \log^2 m}$ or the social welfare of the CCA is at least $\frac{OPT}{24 \log n}$. Thus, the welfare ratio is at most $O(\mathcal{C}^2 \cdot \log n \log^2 m)$.*

5 Lower Bounds on the Welfare Ratio

In this section, we present lower bounds on the welfare ratio of the CCA. To begin, we show give a polylogarithmic lower bound for unit-demand bidders.

Theorem 8. *For any constants $\mathcal{N}$ and $\mathcal{M}$, there exists an instance with $n \geq \mathcal{N}$ unit-demand bidders and $m \geq \mathcal{M}$ items such that the social welfare of the CCA is only $O\left(\frac{\log\log m \cdot OPT}{\log m}\right)$. This implies that the welfare ratio of the CCA is at least $\Omega\left(\frac{\log m}{\log\log m}\right)$.*

The idea behind the lower bound consists in constructing an instance that is essentially tight with respect to the proof of Theorem 4. In particular, in the lower bound instance, we ensure that there are only two bidders who ever make large bids. All the other bidders only make small bids and their largest bids are on items they value the least! Thus, the revenue maximizing allocation selected by the CCA has very poor social welfare. Moreover, we have a similar hardness result for the general case.

Theorem 9. *For any constant $\mathcal{N}$ and $\mathcal{M}$, there exists an instance with $m \geq \mathcal{M}$ items and $n \geq \mathcal{N}$ bidders who only bid on bundles of cardinality at most $\mathcal{C} \leq m^c$, for some absolute constant $c \in (0, \frac{1}{2})$, such that the CCA's social welfare is only $\mathcal{O}\left(\frac{\log\log m \cdot OPT}{\mathcal{C} \cdot \log m}\right)$. This implies that the welfare ratio of the CCA is at least $\Omega\left(\mathcal{C} \cdot \frac{\log m}{\log\log m}\right)$.*

Due to space limitations, we defer the proofs of Theorem 8 and Theorem 9 to Appendix C.

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A Examples

A.1 Welfare Under the Original CCA

In the original CCA model designed by Porter et al. [36], the price increment is always 1 when there is an excess demand. In this section, we show that the welfare under this model could be as low as $O(\frac{\text{OPT}}{\text{poly}(n)})$ even for simple valuations such as additive and unit-demand valuations. The valuation function of bidder i is *additive* if a bidder's value for set S is the sum of the values of the items in S.

`Bad Example I (unit-demand valuations):` Consider a unit-demand auction with items $\{0, 1, 2, \ldots, n\}$. Let t be a real defined below. We have two classes of bidders. In the first class we have n bidders and for any package S, bidder i ($1 \leq i \leq n$) has a value

$$v_i(S) = \max_{j \in S} v_i(\{j\})$$

where

$$v_i(\{j\}) = \begin{cases} t \cdot V & \text{if } j = 0, \\ V & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

In the second class we have $2\sqrt{n}$ bidders. For each $0 \leq \ell \leq \sqrt{n} - 1$ we have two identical bidders. The two identical bidders ℓ have a valuation function

$$v_\ell(S) = \max_{j \in S} v_\ell(\{j\})$$

for any package S, where

$$v_\ell(\{j\}) = \begin{cases} V & \text{if } j \in H_\ell = \{\ell \cdot \sqrt{n} + 1, \ell \cdot \sqrt{n} + 2, \ldots, (\ell + 1) \cdot \sqrt{n}\} \\ 0 & \text{otherwise} \end{cases}$$

Let us examine what happens in this auction. Each pair of bidders in the second class will bid on the cheapest item in the set H_ℓ of cardinality $\sqrt{n}$. In the case of a tie, we may assume they both bid on the smallest index item.[6] It follows that after $V \cdot \sqrt{n}$ rounds the price of every item in H_ℓ will reach V and then the pair of bidders ℓ will drop out of the auction.

[6] This can be enforced by adding a small perturbation to the valuation function.

Meanwhile, bidder i in the first class will continually bid on item 0, at least until its price is $t \cdot V - V$. This will take $(t-1) \cdot V$ rounds. But at this time, provided we set $t \geq \sqrt{n}+1$, the price of item i will be above V. So bidder i will continue to bid on item 0 until its price reaches $t \cdot V$ when it will drop out of the auction.

Given this set of bids what is the optimal allocation? For $0 \leq \ell \leq \sqrt{n}-1$, both bidders of type ℓ in the second class will be allocated one item from the set H_ℓ. Furthermore, exactly one bidder i of the first class, where $1 \leq i \leq n$, will be allocated the item 0. The other bidders of the first class do not make any bid on any other items and thus are not allocated anything. Thus the total welfare of this allocation is

$$2\sqrt{n} \cdot V + t \cdot V = (3\sqrt{n}+1) \cdot V$$

On the other hand the optimal allocation has value

$$n \cdot V + t \cdot V = (n + \sqrt{n} + 1) \cdot V$$

Hence the welfare ratio is at least $\frac{1}{3}\sqrt{n}$.

`Bad Example II (2-demand valuations):` Consider an auction with items $\{0, 1, \ldots, n\}$. Let there be $2n$ bidders. In particular, for each $1 \leq i \leq n$ there are two identical bidders of Type i. Each Type i bidders has a 2-demand function $\mathbf{v}_i$, that is, additive up to 2 items. Each individual item is valued as:

$$v_i(\{j\}) = \begin{cases} V & \text{if } j \in \{0, i\} \\ 0 & \text{otherwise} \end{cases}$$

Here V is a large integer. Now let's run the CCA with a price increment of $\epsilon = 1$ for excess demand items. We begin with all price equal to zero. Then each Type i bidder bids for the pair $\{0, i\}$ as it provides a utility of $2V$. Not that since there are two bidders for each type the every item is in excess demand, so all prices rise to 1. Each Type i bidder then still bids for the pair $\{0, i\}$, and continues to do so until every item has a price of V. The auction then terminates with all prices equal to $V+1$. But now, given these bids, when we find a maximum revenue allocation we may only accept one bid since every submitted bid contained the item 0. Thus we obtain a welfare of $2V$. In contrast, it is easy to see that the optimal welfare is $(n+1) \cdot V$. Hence the welfare ratio is $\frac{1}{2}(n+1)$.

(See [9] for a similar example that shows the poor performance of the Porter et al. mechanism.)

A.2 Bad Examples using the SMRA Stopping Condition

In our CCA and as well as the original model of Porter et.al [36], the stopping condition is to stop when the final allocation and the revenue-optimal allocation are not in conflict. A simpler condition is to stop immediately once the bids become disjoint as in the SMRA. This simpler rule is now used in recent real-world CCA auctions. In the following example, we show that the SMRA stopping condition has arbitrarily bad performance when used in the CCA.

`Example:` Consider an auction with 3 additive bidders and 4 items. Let c be an integer larger than 2. Here are the bidders' valuations

$$v_1(\{j\}) = \begin{cases} V & \text{if } j = 1 \\ 2V & \text{if } j = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$v_2(\{j\}) = \begin{cases} V & \text{if } j = 4 \\ 2V & \text{if } j = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$v_3(\{j\}) = \begin{cases} c \cdot V & \text{if } j \in \{2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

At round 0, every bidder bids on their favorite set. Bidder 1 bids on set $\{1, 2\}$, bidder 2 bids on set $\{3, 4\}$ and bidder 3 bids on set $\{2, 3\}$. It is easy to verify that at round $t \le V$, these are still the sets bidders bid on. Both item 1 and 4 have price t, and both item 2 and 3 have price $2t$. At round $V + 1$, bidder 1 and 2 drop out as they have utility 0, while bidder 3 still bids on set $\{2, 3\}$ with total price $4V + 4$. If we stop at this round, the revenue optimal allocation gives item 1 and 2 to bidder 1 with price $3V$ and item 3 and 4 to bidder 2 with price $3V$. The welfare for this allocation is $6V$.

Since $c > 2$, the welfare maximizing allocation should give bidder 3 both items 2 and 3, give bidder 1 item 1 and give bidder 2 item 4. The welfare of this allocation is $2(c+1) \cdot V$. So the welfare ratio of the CCA with this stopping rule is at least $\frac{c+1}{6} = \Theta(c)$. Since c can be arbitrarily large and does not depend on n and m, the welfare ratio can be arbitrarily bad which achieves the proof.

B Missing Proofs for Upper Bounds on the Welfare Ratio

B.1 Proof of Lemma 6

Proof of Lemma 6: Let there be ℓ non-empty sets in $\mathbf{R}^*$, and we define the value for a set R_i to be $v_i(R_i)$. By definition, $\sum_{i=1}^{\ell} v_i(R_i) = \text{OPT}$. Now we construct $2\lceil \log n \rceil + 1$ bins, for any $1 \le i \le 2\lceil \log n \rceil$ the i-th bin B_i contains all sets with values in $(\frac{\text{OPT}}{2^i}, \frac{\text{OPT}}{2^{i-1}}]$. The last bin contains all the other sets. Since every set in the last bin has value at most $\frac{\text{OPT}}{n^2}$ and contains at most n sets, the total value of the last bin is at most $\frac{\text{OPT}}{n}$. Therefore, $\sum_{i=1}^{2\lceil \log n \rceil} v(B_i) \ge (1 - 1/n)\text{OPT}$, where $v(B_i)$ is the sum of the values from sets in B_i. That means there exists a bin B_{i^*} such that

$$v(B_{i^*}) \ge \frac{(1 - 1/n)\text{OPT}}{2\lceil \log n \rceil} \ge \frac{\text{OPT}}{3 \log n}.$$

We simply let $B = B_{i^*}$. ■

B.2 The Greedy Algorithm for General Valuations

Here we specify the greedy algorithm.

B.3 Time Amplifying Lemma for General Bidders

Let X be the set of bidders that have been assigned items in the greedy algorithm and $\tilde{X}$ be the set of items that are allocated. Let $k \ge |X|$ be some integer. As in the unit-demand case, we will argue a generalization of Lemma 5 for general bidders. With this Lemma, it is straightforward to argue that either at least one bidder not in X must make $b \cdot m$ bids on bundles of items which do not intersect with $\tilde{X}$, or many bidders still have high utility when the price ascending phase ends.

Lemma 10 (Time Amplifying for General Bidders)**.** *Let S be a set of at least $4\mathcal{C} \cdot k$ bidders disjoint from X ($|X| \le k$), such that every bidder $i \in S$ has utility at least $u \ge 2\mathcal{C} \cdot b$ in round $t \ge 5/2 \cdot b \cdot \mathcal{C}$.*

Procedure 3 Greedy Allocation Procedure for General Bidders

Input: M is the set of items. N is the set of bidders. $\mathcal{S} = \{(S_i^t, P_i^t)\}_{i,t}$ is the collection of bids made in the CCA. $b \geq n^2$ is a threshold on the bid price.

while $\mathcal{S} \neq \varnothing$ **do**
  Let (S_i^t, P_i^t) be a bid of maximum price (break ties arbitrarily).
  if $P_i^t \geq b$ **then**
    Allocate the set of items S_i^t to bidder i with price P_i^t.
  else
    return
  end if
  Remove from $\mathcal{S}$ every bid made by bidder i and every bid for a set of items that are not disjoint from S_i^t.
end while

If the greedy algorithm has revenue g less than $kb - mn$, then in any round up to $2(t+1)-1$ (the mechanism can terminate before this), there is a subset of S with at least $|S| - 4\mathcal{C} \cdot k$ bidders such that each of them has utility at least $u - 2\mathcal{C} \cdot b$.

Proof. Let t' be the first round that at least $4\mathcal{C} \cdot k$ bidders from S have utility at most $u - 2\mathcal{C} \cdot b$, and S' be the set of these bidders. It is not difficult to argue that if t' does not exist, the conclusion holds.

Since every bidder $i \in S$ has utility u in round t, by Fact 1 and 2, we know that for any set S that i bids on in the first t rounds, $v_i(S) \geq u$. Let M' be the subsets of items in $M - \tilde{X}$ that has ever been bid on by some bidder from S' in the first t rounds. The bidders of S' totally make $|S'| \cdot t$ bids. How many of their bids can intersect with $\tilde{X}$? Let us assume the greedy algorithm allocates set $S_i^{r_i}$ to bidder i. Since bidders in S are not selected by the greedy algorithm, they cannot bid on any item $j \in S_i^{r_i}$ after its price has reached $P_i^{r_i}$, so they can make at most $P_i^{r_i} + n$ bids containing j. As the size of $S_i^{r_i}$ is at most $\mathcal{C}$, bidders from S can make at most $\mathcal{C}P_i^{r_i} + \mathcal{C}n$ bids that intersect with $S_i^{r_i}$. Totally, bidders can make at most $\mathcal{C}g + \mathcal{C}nm \leq \mathcal{C}kb$ bids that intersect with $\tilde{X}$, as $\sum_{i \in X} P_i^{r_i} = g \leq kb - nm$. Therefore, bidders from S' make at least $|S'| \cdot t - \mathcal{C}kb$ bids only on subsets of M'.

For any item $j \in M'$, there could be at most $b + n$ bids from bidders in S containing j. As every bid is on a set with size at most $\mathcal{C}$, for any set that has ever been bid on, there can be at most $\mathcal{C}b + \mathcal{C}n$ other bids that intersect with it. Therefore, we can find at least $\frac{|S'| \cdot t - \mathcal{C}kb}{\mathcal{C}b + \mathcal{C}n}$ disjoint bids on M' that are made in the first t rounds by bidders of S'. Let T be the set of items bid by one of these disjoint bids. The total price for T at round t' should be at least $2\mathcal{C}b$, otherwise at least one of the bidders in S' will have utility greater than $u - 2\mathcal{C}b$. On the other hand, for any item $j \in T$, bidders in S' can bid on sets containing it only when its price is less than b or when the set intersects $\tilde{X}$, since otherwise the bid would have been selected by the greedy algorithm. As $|T| \leq \mathcal{C}$, bids from X and bids intersecting $\tilde{X}$ must push up the price for T by at least $\mathcal{C} \cdot (b - n)$. Since there are at least $\frac{|S'| \cdot t - \mathcal{C}kb}{\mathcal{C}b + \mathcal{C}n}$ such disjoint sets and at most $\mathcal{C}kb$ bids can intersect with $\tilde{X}$, the bids from X on M' must increase the total price of items in M' by at least $\frac{b-n}{b+n}(|S'| \cdot t - \mathcal{C}k \cdot b) - \mathcal{C}^2 kb$. Since $|X| \leq k$, we have

$$\begin{aligned}
t' &\geq \frac{1}{\mathcal{C}k}\Big(\frac{b-n}{b+n}(|S'|\cdot t - \mathcal{C}k\cdot b) - \mathcal{C}^2kb\Big) \\
&\geq \frac{b-n}{b+n}\cdot(4t-b) - b\cdot\mathcal{C} && (|S'|\geq 4\mathcal{C}\cdot k) \\
&\geq (1-\frac{2}{n})(4t-b) - b\cdot\mathcal{C} && (b\geq n^2) \\
&\geq 2(t+1) && (t\geq \frac{5}{2}b\cdot\mathcal{C})
\end{aligned}$$

By the definition of t', it is straightforward to see in round $t'-1 \geq 2(t+1)-1$ there are at least $|S| - 4\mathcal{C}\cdot k$ bidders from S such that each of them has utility at least $u - 2\mathcal{C}\cdot b$. □

B.4 Proof for Theorem 7

Proof of Theorem 7: First let us fix the notations. In Lemma 6, we make no assumption on the sets $\{R_i\}_{i\in[n]}$. In particular, we do not assume it is a singleton. Therefore, Lemma 6 also holds in the general case. Let B and v^* be the same as in Lemma 6, and use $\mathcal{K}$ to denote the size of B. Let the threshold $b = \frac{v^*}{c_1\cdot\mathcal{C}\cdot\log m}$ and $k = \frac{\mathcal{K}}{c_2\cdot\mathcal{C}\cdot\log m}$ for some constants c_1 and c_2 that we will specify later. We prove by contradiction that the revenue of the greedy algorithm with our choice of b is at least $g = \frac{\text{OPT}}{6c_1c_2\cdot\mathcal{C}^2\cdot\log n\log^2 m}$ or the social welfare of the CCA is at least $\frac{OPT}{24\log n}$.

As in Theorem 4, we can immediately show that $k\cdot b\geq g$. Thus, if the number of bidders selected by the greedy algorithm $|X|$ exceeds k, the revenue is clearly greater than g. So we can assume $|X|\leq k$. Let X' be the set of bidders that are allocated a subset of items that intersect with $\tilde{X}$ in $\mathbf{R}^*$, and $Y = B - X - X'$. Since $|X'|\leq|\tilde{X}|\leq\mathcal{C}|X|\leq\mathcal{C}k$, we have $|Y|\geq(c_2\cdot\mathcal{C}\cdot\log m - \mathcal{C}-1)k$. Take t_1 (if it exists) to be the first round that at least $4\mathcal{C}\cdot k$ bidders from Y, such that their allocated sets of items in $\mathbf{R}^*$ all have prices at least $2\mathcal{C}\cdot b$. We continue our proof with the following case analysis.

- **Case (1):** t_1 does not exist. Then when the algorithm stops at round t, less than $4\mathcal{C}\cdot k$ subsets of items allocated to bidders from Y in $\mathbf{R}^*$ have prices at least $2\mathcal{C}b$. Let Y' be the bidders of Y such that the total price of the set of items R_i allocated to i is at most $2\mathcal{C}b$ at round t (remind that R_i refers to the set of items allocated to i in $\mathbf{R}^*$). Notice that $|Y'|\geq|Y|-4\mathcal{C}k\geq|Y|/2$ and $u_i^t\geq v^* - 2\mathcal{C}b\geq v^*/2$ for any $i\in Y'$. Fact 3 implies that the social welfare of the CCA is at least
$$\frac{|Y|\cdot v^*}{4}\geq\frac{c_2\mathcal{C}k\cdot\log m\cdot v^*}{8}\geq\frac{\mathcal{K}v^*}{8}\geq\frac{OPT}{24\log n}.$$

- **Case (2):** t_1 exists. Let T be the allocated set of items of one of the $4\mathcal{C}\cdot k$ bidder. As we have argued in Lemma 10, for any item j in T, only bidders from X or bids intersect with $\tilde{X}$ can bid on any set containing j after its price reaches b. A simple calculation shows that bidders from X must make at least $4\mathcal{C}k\cdot(b-n)\cdot\mathcal{C} - \mathcal{C}^2kb$ bids (as we notice in the proof of Lemma 10 that at most $\mathcal{C}^2kb$ of total price increment is due to bids that intersect with $\tilde{X}$) in the first t_1 rounds. Hence, bidders from X must make at least $3(b-n)\mathcal{C}\cdot k - n$ bids, which means t_1 is at least $\frac{3(b-n)\cdot\mathcal{C}\cdot k - n}{|X|}$. Since $k\geq|X|$ and $b\geq n^2$, we have $t_1\geq\frac{5}{2}\mathcal{C}\cdot b$.

  Notice that in round $t_1 - 1$, there are at least $|Y| - ck$ bidders whose allocated items in $\mathbf{R}^*$ have prices less than $2\mathcal{C}\cdot b$. Let us call this set of bidders Y_1, and clearly they all have utility at least $u_1 = v^* - 2\mathcal{C}\cdot b$. Let t_2 (if it exists) be the first round that at least $4\mathcal{C}\cdot k$ bidders from Y_1 have utility no greater than $u_2 = u_1 - 2\mathcal{C}\cdot b$. Let Y_2 be the set of bidders from Y_1 that still have utility at least u_2 in round $t_2 - 1$. Clearly, $|Y_2| \geq |Y_1| - 4\mathcal{C} \cdot k$. Applying Lemma 10 on Y_1, u_1 and $t_1 - 1$ shows that $t_2 - 1 \geq 2t_1 - 1$. Hence, $t_2 \geq 2t_1$.

There is nothing special about Y_1, u_1 and $t_1 - 1$. If we recursively define t_i (if it exists) as the first round that at least $4\mathcal{C} \cdot k$ bidders from Y_{i-1} have utility no greater than $u_i = u_{i-1} - 2\mathcal{C} \cdot b$, and define Y_i as the set of bidders from Y_{i-1} that still have utility at least u_i in round $t_i - 1$, we can apply Lemma 10 on Y_{i-1}, u_{i-1} and $t_{i-1} - 1$ as long as they satisfy the conditions of Lemma 10 and the CCA does not terminate before t_i happens. In that case, we have $t_i \geq 2t_{i-1}$. How many times can we apply this Lemma before the conditions are violated? Since every time the size of Y_i decrease by at most $4\mathcal{C} \cdot k$ and u_i decrease $2\mathcal{C} \cdot b$, we can apply Lemma 10 for at least $\ell' = \min\{[\frac{|Y_1|}{4\mathcal{C} \cdot k}], [\frac{u_1}{2\mathcal{C} \cdot b}]\} - 1 \geq \min\{[\frac{c_2 - 1}{4} \cdot \log m], [\frac{c_1 - 1}{2} \cdot \log m]\}$. If we let $c_1 = 8$ and $c_2 = 10$, we have $\ell' \geq 2[\log m] + 8$. Let $\ell = \log m + 2$. Remember that we might not be able to run the recursion till the conditions are violated, because the CCA might terminate before that. We now use case analysis to show that if we take $\ell = \log m + 2$, then our claim holds no matter the CCA terminates before t_ℓ or not.

- **Case (i):** The CCA terminates between t_j and t_{j+1} where $j < \ell$. By our choice of the parameters, we have $|Y_i| - 4\mathcal{C}k \geq |Y|/2$ and $u_i - 2\mathcal{C}b \geq v^*/2$. Thus if t_{j+1} does not exist, Fact 3 implies that the welfare of the CCA is at least $(|Y_i| - 4\mathcal{C}k) \cdot (u_i - 2\mathcal{C}b) \geq \frac{|Y| \cdot v^*}{4} \geq \frac{OPT}{24 \log n}$.
- **Case (ii)** t_ℓ exists. It is straightforward to see that $t_\ell \geq 2^{\ell-1} \cdot t_1 > 2m \cdot t_1 > (\mathcal{C}k + m) \cdot \mathcal{C} \cdot b$, which means there is a bidder i in Y_ℓ that has made at least $(\mathcal{C}k + m) \cdot \mathcal{C} \cdot b$ bids in the price ascending phase. Since the revenue of the greedy allocation is less than $k \cdot b - nm$, so as we have argued in Lemma 10, i can make at most $\mathcal{C}^2 \cdot k \cdot b$ bids on sets of items intersecting $\tilde{X}$. Therefore, i makes at least $\mathcal{C} \cdot m \cdot b$ bids contained in $M - \tilde{X}$, then there must be a set $S \subseteq M - \tilde{X}$ that i has bid on with price at least b. This is a contradiction, because i would have been selected by the greedy algorithm.

■

C Proofs for Upper Bounds of the CCA'a Social Welfare

Proof of Theorem 8: Let k, ℓ be two integers. We will show that for sufficiently large k and ℓ we can construct an instance that has low social welfare as promised. To avoid cumbersome notations, we assume that the price increment of the CCA is 1 when two bidders bid on it. Let us denote by $(u_n)_n$ the sequence such that $u_0 = 1$, $u_1 = k + 1$ and $u_p = k \cdot \left(\sum_{i=0}^{p-1} u_i \right)$ for any $p \geq 2$.

Let us consider the following instance of the CCA. There are $2k+2$ bidders $s_0, s'_0, s_1, s'_1 \ldots, s_k, s'_k$. For each bidder s_i with $i \geq 1$ and for every $v \in [\ell]$, create a set of items X^i_v such that there are $u_{\ell - v}$ items in X^i_v. In other words, bidder s_i has one item of value ℓ, $k+1$ items of value $\ell - 1$, u_2 items of value $\ell - 2$, etc.. Let $X_v = \cup_{i=1}^k X^i_v$ and we will refer to it as *the set of items of value v*[7]. Moreover, we assume that bidder s_i has the following preference rule over the items: if there are a few items with the same utility, the one with the lowest value is preferred. This preference rule can also be simulated by making a small modification to the valuation function of s_i. We will stick with the preferences rule, as we believe it makes the proof cleaner. The bidders $s'_1, \ldots, s'_k$ are respectively

[7] In Section 3, we assume all bidders' values are multiples of some large integer W. We can easily modify our hard instances in this section by multiplying all values with W, and the same conclusion still holds.

copies of bidders $s_1, \dots, s_k$. Since they have the same valuation functions and the same preference rules, we can assume that at any round they will bid on the same item.

Let us now describe the valuation function of s_0. For every $v \leq \ell$, the value of s_0 for any item in X_v is v. Finally, we create a new item which is added in X_ℓ and we assume that s_0 has value ℓ for this item. We further assume that, this new item is the one preferred by s_0 among all items of value ℓ. We also assume that s_0 has the following preference rule: if there are a few items with the same utility, the one with the highest value is preferred. Note that the preference rule ensures that if an item of X_v^i has price p and an item of X_{v-1}^i has price $(p-1)$, bidder s_i bids on the item in X_{v-1}^i while s_0 bids on the item in X_v^i. Again we create a bidder s'_0 as a copy of s_0.

The fact that bidder s_i prefers the item with the lowest value (and the fact that X_{v-1}^i is large compared to X_v^i) will ensure that the welfare of the allocation of the CCA is small compared to the optimal one. The key step is to prove the following statement:

Claim 1. *None of the bidders $s_1, s'_1, \dots, s_k, s'_k$ makes a bid of price at least 2. Moreover, all their bids of price 1 are performed on items of X_1.*

Proof. At round $t = 0$, all the bidders $s_1, s'_1 \dots, s_k, s'_k$ bid on their favorite items (the items of value ℓ for their own valuation functions). Both bidders s_0 and s'_0 bid on their favorite item, which is the special item in X_ℓ. So the prices of all these items increase by one (by definition of the price increment). Now, all items of X_ℓ have price 1 and all the other items have price 0.

By construction of our preference rules, bidders $s_1, s'_1 \dots, s_k, s'_k$ start bidding on items of value $\ell - 1$ and s_0, s'_0 bid on items of value ℓ. Recall that since s_i, s'_i are copies of the same bidder, they bid on the same item at any round. Since, for every $i \leq k$, $X_{\ell-1}^i$ contains $k+1$ items, bidder s_i and s'_i need to spend $k+1$ rounds to increase the prices of all the items in $X_{\ell-1}^i$ from 0 to 1. Since there are $k+1$ items in X_ℓ, bidders s_0, s'_0 also need $k+1$ rounds to increase the prices of all the items of value ℓ from 1 to 2. So by the end of round $t = k+2$, all items in X_ℓ have price 2 and all items in $X_{\ell-1}$ have price 1. All the other items have price 0.

We now repeat this argument by induction. More precisely, we will show that for every v, there exists a round t_v such that:

- All items in X_{v-p} have price 0 by the end of round t_v for any $p > 0$.
- All items in X_{v+p} by the end of round t_v have price $p+1$ for any $p \geq 0$.

We have already showed that $t_\ell = 1$ and $t_{\ell-1} = k+2$. Let us prove that the existence of t_v implies the existence of t_{v-1}. Let us study the structure of the bids at round t_v. By construction of the preference rule, bidders $s_1, s'_1, \dots, s_k, s'_k$ prefer bidding on items of value $(v-1)$ at price 0. On the other hand, the bidders s_0, s'_0 prefer bidding on items of value between v and ℓ (more precisely they prefer items in X_ℓ, then $X_{\ell-1}$ and so on). Since by definition of the sequence $(u_n)_n$, the number of items in X_{v-1}^i is is equal to the number of items in $\cup_{v'=v}^{\ell} X_{v'}$. So, for every i, the number of rounds needed by the bidders s_i and s'_i to increase the prices of all the items of X_{v-1}^i from 0 to 1 is exactly the number of rounds needed by s_0 and s'_0 to increase all the prices of all the items in $X_{v'}$ for any $v \leq v' \leq \ell$ by one.

Consider finally by the end of round t_1. All items in X_v have price v for any $v \leq \ell$. The preference rule ensures that, bidders $s_1, s'_1 \dots, s_k, s'_k$ will increase the prices of items of X_1[8]. On the other hand, bidders s_0 and s'_0 increase the prices of all the items in X_v for $v \geq 2$ before increasing the prices of items in X_1. Since the size of X_1^i is precisely the size of $\cup_{v=2}^{\ell} X_v$ for every i, after $|X_1^i|$ rounds the price for each item has increased by one. Now, all items in X_v have price $v+1$ for every $v \leq \ell$. As a result, all bidders have negative utility and they drop out. □

[8]We assume bidders still participate in the mechanism even if they have utility 0. An alternative approach to force this behavior is to add a tiny ϵ to all the values.

Finally, we compare the optimal welfare with the welfare of the allocation of the CCA. The optimal solution allocates to each of $s_0, s_1, \dots, s_k$ her favorite item of value ℓ. Then we can allocate to $s'_0, s'_1, \dots, s'_k$ an item in $X^i_{\ell-1}$ for every i. The welfare of the optimal allocation is $\Theta(k\ell)$. The allocation of the CCA allocates an item of X_ℓ at price ℓ to both s_0 and s'_0. All the other bidders only bid on items at price 0 except for items in X_1, on which they make bids at price 1, so the CCA allocates an item of value 1 to each of $s_1, s'_1, \dots, s_k, s'_k$. Therefore, the welfare of the CCA is $2\ell + 2k$. Note that the number of items in this construction is $m = \Theta(k^\ell)$ (the number of items of value $v-1$ is essentially k times the number of items of value v). Now if we choose $k = \Theta(\log m)$, and the number of bidders is $n = \Theta(k)$. If we let $k = \Theta(m)$, then $\ell = \Theta\left(\frac{\log m}{\log\log m}\right)$ and $n = \Theta(\log m)$. The welfare of the CCA is $\mathcal{O}\left(\frac{\log\log m \cdot \mathrm{OPT}}{\log m}\right) = \mathcal{O}\left(\frac{\log n \cdot \mathrm{OPT}}{n}\right)$. ■

Proof of Theorem 9: Let $k, \ell, \mathcal{C}$ be three integers. We fix $\mathcal{C}$ to be an odd constant, and we will show that for sufficiently large k and ℓ we can construct an instance that has low social welfare as promised. The proof uses the same ingredient as the proof of Theorem 8 but is slightly more involved. We assume that the price increment for any item is 1/2 times the number of bidders bidding on it. Let u_n be the following sequence: $u_0 = 1$, $u_1 = k \cdot \frac{\mathcal{C}-1}{2} + 1$ and $u_n = k \cdot \frac{\mathcal{C}-1}{2} \cdot \sum_{i=0}^{n-1} u_i + 1$. Note that since $\mathcal{C}$ is odd, u_n is a sequence of integers.

The instance of the CCA has $k \cdot \mathcal{C} + 2$ bidders. They are denoted by :$s^i_1, \dots, s^i_{\mathcal{C}}$ for every $i \le k$ and two special bidders denoted by s_0 and s'_0. Let B_i be the set of bidders $s^i_1, \dots, s^i_{\mathcal{C}}$. Before describing the construction, we first introduce the v-gadgets.

v-gadgets. We say a set K of $\binom{\mathcal{C}}{2}$ items form a v-gadget for a set of bidders $\{s_1, \dots, s_{\mathcal{C}}\}$, if we can use edges of a clique on $\mathcal{C}$ vertices to encode K, such that for any bidder s_i only the subset of items corresponding to all edges incident to the vertex i has value $\mathcal{C} \cdot v$ to her, while all the other subsets have value 0. Note that, for any item of the v-gadget, there are exactly two bidders containing it in their interested bundles (since each edge has two endpoints).

Instance. Before introducing formally the instance, let us briefly describe it. As in the matching case, all the bidders only bid on items with price 0 till the last round except bidders s_0 and s'_0. At any round, bidders in B_i for every i bid on a disjoint v-gadget. We will create the appropriate number of v-gadgets (and an appropriate preference rule) to ensure that none of the bids is positive except on items with low value. Therefore, we can argue the revenue optimal allocation selected by the CCA has low social welfare.

For every $v \in [\ell]$ and every $i \in [k]$, we create a set X^i_v of $u_{\ell-v}$ copies of the v-gadgets for bidders in B_i. Additionally, for every bidder s^i_j, we create a special item r^i_j. Each bidder s^j_i has value 1/2 for her special item and $\mathcal{C} \cdot v$ for any bundle of items corresponding to the set of all edges of the i-th vertex in some v-gadget of X^i_v for all $v \in [\ell]$, but has value 0 for any other bundle. We use X_v to denote the union of all X^i_v and refer to it as the set of items with value v.

To simplify the proof, we assume that all bidders in B_i obey the following preference rule: (i) all bidders have the same preference order over the v-gadgets of the same utility, *i.e.* if two v-gadgets have exactly the same prices[9], then all bidders prefer the same v-gadget; (ii) each bidder s^i_j prefers to bid on the set of items with lower value when there is a tie; (iii) a bidder will not bid on any v-gadget with utility 0, but will still bid on her special item at price $\frac{1}{2}$. As we mentioned in the proof of Theorem 8, the preference rule above can be easily simulated with slight perturbations on the valuation functions.

[9]Each item and its counterpart in the other v-gadget shares the same price.

Now we describe the valuation function of s_0. We first create a special item r_0 with value $1/2$, and add a set of $\mathcal{C}$ new items to X_ℓ such that the whole set has value $\mathcal{C} \cdot v$. For any v-gadget, we partition it into $\frac{\mathcal{C}-1}{2}$ disjoint subsets of size $\mathcal{C}$, such that v_0's value for each of these subsets is $\mathcal{C} \cdot v$. Except the bundles mentioned above, all other bundles have value 0. We also assume that s_0 obeys a preference rule: (i) among all bundles of value $\mathcal{C} \cdot \ell$, the bundle of $\mathcal{C}$ new items is preferred; (ii) s_0 prefers the set of items with higher value when there is a tie. Finally, we create s_0' as a copy of bidder s_0, such that she has the same valuation function and same preference over bundles.

Our preference rule ensures that the CCA's welfare is small. The key step of our proof is to show the following statement:

Claim 2. *None of the bidders $s_1^j, \dots, s_\mathcal{C}^j$ for any $j \leq k$ makes positive bids except on her special item on which she makes a bid at price $\frac{1}{2}$.*

Proof. The proof is similar to Claim 1.

At round $t = 0$, all bidders $s_1^j, \dots, s_\mathcal{C}^j$ bid on their favorite bundles (the unique bundle of value $\mathcal{C} \cdot \ell$ in the ℓ-gadget). Moreover s_0 and s_0' bid on the bundle of new items in X_ℓ (the set of $\mathcal{C}$ items we create when we define the valuation function of s_0). So by the end of the first round, the prices of all items of X_ℓ are increased to 1, while the prices for all the other items remain at 0.

By definition of our preference rules, bidders $s_1^i, \dots, s_\mathcal{C}^i$ start bidding on $(\ell-1)$-gadgets and s_0, s_0' continue to bid on the ℓ-gadgets. Since there are $k \cdot \frac{\mathcal{C}-1}{2} + 1$ different $(\ell-1)$-gadgets in $X_{\ell-1}^i$ for every $i \in [k]$, $s_1^i, \dots, s_\mathcal{C}^i$ need $k \cdot \frac{\mathcal{C}-1}{2} + 1$ rounds to increase the prices of all the items in $X_{\ell-1}$ from 0 to 1. Since there are $k \cdot \binom{\mathcal{C}}{2} + \mathcal{C}$ items in X_ℓ and these items are partitioned into subsets of size $\mathcal{C}$, bidders s_0, s_0' also need $k \cdot \frac{\mathcal{C}-1}{2} + 1$ rounds to increase the prices of all these items of value ℓ from 1 to 2. So by the end of round $t = k \cdot \frac{\mathcal{C}-1}{2} + 2$, all the items in X_ℓ have price 2 and all the items in $X_{\ell-1}$ have price 1. All the other items still have price 0.

We now repeat this argument by induction. More precisely, let us prove that, for every v, there exists a round t_v such that:

- All items in X_{v-p} have price 0 by the end of round t_v for any $p > 0$.
- All items in X_{v+p} by the end of round t_v have price $p+1$ for any $p \geq 0$.

We have already showed that $t_\ell = 1$ and $t_{\ell-1} = k \cdot \frac{\mathcal{C}-1}{2} + 2$. Now we will prove that the existence of t_v implies the existence of t_{v-1}. Let us first understand the structure of the bids at round t_v. By construction of the preference rule, $s_1^i, \dots, s_\mathcal{C}^i$ prefer bidding on bundles of the $(v-1)$-gadgets in X_{v-1}^i which have price 0 for every $i \in [k]$. Since they have the same preference over the gadgets, they bid on the same $(v-1)$-gadget. In the meantime, the bidders s_0, s_0' are bidding on v'-gadgets with $v \leq v' \leq \ell$ (more precisely, they first bid on ℓ-gadgets, then on $(\ell-1)$-gadgets, etc.). By the definition of the sequence $(u_n)_n$, the number of $(v-1)$-gadgets in X_{v-1}^i equals to $\frac{|\cup_{v'=v}^{\ell} X_{v'}|}{\mathcal{C}}$. Thus, the number of rounds needed by the bidders $s_1^i, \dots, s_\mathcal{C}^i$ to increase the prices of all the items in X_{v-1}^i from 0 to 1 is exactly the number of rounds needed by s_0 and s_0' to increase the prices of all the items in $X_{v'}$ for any $v \leq v' \leq \ell$ by 1. As a result, t_{v-1} exists and equals to $t_v + |X_{v-1}^i|$.

By the end of round t_1, every bidder has utility 0 on any item in X_v for any $v \in [\ell]$. In the next round, all bidders bid on their special items. By the end of round $t_1 + 1$, each items r_j^i has price $\frac{1}{2}$ and r_0 has price 1. Thus, bidders s_0 and s_0' drop out (since the last item reach price 1) and the other bidders will bid on their special items for another round at price 1/2. By the end of round $t_1 + 2$, the mechanism terminates as the stopping condition is met. Namely, all bids are disjoint and the revenue optimal allocation does not conflict with the final allocation. $\square$

Now we are ready to compare the welfare of the CCA to the optimal social welfare. Consider the following allocation (which is not optimal but sufficient to show our result): for every i, assign s_1^i its bundle in the ℓ-gadget, each of the other bidders her bundle in a distinct $(\ell-1)$-gadget (assuming $k \geq 2$). The welfare of this allocation is larger than $\Theta(k \cdot \mathcal{C}^2 \cdot \ell)$. In the revenue optimal allocation selected by the CCA, two disjoints sets of value $\mathcal{C} \cdot \ell$ are allocated to s_0 and s'_0, and all the other bidders only receive their special items of value $\frac{1}{2}$. The welfare of the CCA is only $2\mathcal{C} \cdot \ell + k \cdot \mathcal{C}/2$. Note that the number of items in this construction is $\Theta((k \cdot \mathcal{C})^{\ell})$ (the number of items of value $v-1$ is essentially $k \cdot \mathcal{C}$ times the number of items of value v) and the number of bidders is $\Theta(\mathcal{C} \cdot k)$. Since $\log \mathcal{C} \leq c \cdot \log m$ for some absolute constant c, we choose k to be $\Theta(\log m)$, then $\ell = \Theta\left(\frac{\log m}{\log \mathcal{C} + \log\log m}\right)$ and $n = \Theta(\mathcal{C} \cdot \log m)$. The welfare of the CCA is $\mathcal{O}\left(\frac{(\log\log m + \log \mathcal{C}) \cdot \mathrm{OPT}}{\mathcal{C} \cdot \log m}\right) = \mathcal{O}\left(\frac{\log n \cdot \mathrm{OPT}}{n}\right)$. ■