Three Is Company, Two's a Crowd:

A Third Rival's Deescalating Effect in the Dollar Auction*

Fredrik Ødegaard[†] Charles Z. Zheng[‡]

February 28, 2018

Abstract

This paper demonstrates a pivotal effect of adding a third rival to bilateral rivalry in the dollar auction, a dynamic model of wars of attrition. Any recursive subgame perfect equilibrium (SPE) involving three rivals on path dynamically Pareto dominates (for the bidders) any recursive SPE involving only two rivals on path, due to a free-rider effect. In any such trilateral-rivalry equilibrium, the gap by which the third-place rival lags behind the frontrunner may collapse or expand, depending on whether the former manages to leapfrog; when the gap reaches the maximum sustainable by the equilibrium, the second-place rival pauses his own costly escalation bid, without conceding to the frontrunner, in the hope to free-ride the third-place bidder's leapfrogging effort to top the frontrunner thereby repeating the trilateral escalation cycle.

*We thank seminar participants at the University of British Columbia, National University of Singapore, Singapore Management University and Lingnan University College at Sun Yatsen University for their comments. The financial support from the Social Science and Humanities Research Council of Canada, Insight Grant R4809A04, is gratefully acknowledged.

[†]Ivey Business School, The University of Western Ontario, London, ON, Canada, N6A 3K7, fodegaard@ivey.uwo.ca.

[‡]Department of Economics, The University of Western Ontario, London, ON, Canada, N6A 5C2, charles.zheng@uwo.ca.

1 Introduction

War of attrition is a prevalent framework to understand escalations in political conflicts and market competitions. However, within the burgeoning war-of-attrition literature only a limited subset incorporates the dynamic aspect of escalation: It is modeled as a dynamic auction game, called the dollar auction, by Shubik [19], studied by O'Neill [17], Leininger [12], Demange [5], Hörner and Sahuguet [10], and applied to political competition by Dekel, Jackson and Wolinsky [4]; it is also treated as a stopping game by Hendricks, Weiss and Wilson [8], and further developed by Gul and Pesendorfer [6] for political contests, and by Damiano, Li and Suen [2, 3] and Meyer-ter-Vehn, Smith and Bognar [13] for intra-committee bargaining. A main assumption in this dynamic literature is that there are only two rivals. Whereas in reality, especially in political competition, a main question is how an additional third rival might alter the course and outcome of an attritional war, say whether a third political party in the United States would exacerbate or mitigate the current partisan escalation between the Democrats and Republicans. The difference between bilateral and trilateral war-of-attrition-type rivalries, despite being seemingly innocuous, is crucial. In both cases, at each step of further escalation, a rival needs to make a sunk cost investment regardless of the outcome; with only two rivals, not making the investment explicitly implies concession. With three contenders, by contrast, one rival may stay put without conceding to his rivals and instead free-ride the other two rivals' escalation efforts. Consequently, having a third rival may lessen the incentive for escalation, which needs to be squared with the opposite effect of bidding intensification due to the larger number of rivals.

Adopting a dynamic framework for wars of attrition, this paper analyzes the dollar auction among multiple, mainly three, bidders. Different from the above-cited studies on the dollar auction, our model adheres to Shubik's original formulation: at each round, the current price can rise only by a small, exogenous increment, so that a bidder cannot preempt competition through initiating the auction with a bid near the full value of the good; no budget constraint or deadline is assumed, so escalation may go on forever.[1] Furthermore, different from the stopping-game literature, in which to remain in the game requires a con-

[1] In addition to the theory literature on the dollar auction, there is an empirical and behavioral literature such as Haupert [7], Morone, Nuzzo and Caferra [14], Murnighan [15], Teger [20], and Waniek, Nieścieruk, Michalak and Rahwan [21], where bid escalation is attributed to psychological factors such as bounded rationality and spiteful bidding (Waniek et al.).

tinuous sunk cost bidding effort, the dollar auction—with more than two bidders—allows a bider to stay put for a while and reenter by leapfrogging to the top. Hence our model allows for the free-rider effect of a third bidder suggested previously.

The ensuing analysis and main results are based on three refinement conditions that exploit the recursive structure of the game. In any subgame equilibrium where the third player no longer participates, the remaining two rivals top each other—potentially indefinitely—such that their expected surplus is completely dissipated. Based on the penal surplus dissipation, there is a continuum of subgame perfect equilibriums where only the two dueling rivals bid at all, with the third player completely inactive. However, all of such *bilateral-rivalry* equilibriums are dynamically Pareto inferior, because at a node along the path the third player can deviate by leapfrogging to the top so that if all three switch to an alternative *trilateral-rivalry* equilibrium, then all three are better-off thereafter and, expecting such a switch, the third player strictly prefers to make the leapfrogging deviation.[2]

Participation of the third rival makes all three better-off thereafter because it reduces the probability of escalation at a critical state when the *underdog*—the bidder furthest from the frontrunner—contemplates conceding. The critical state is defined by the maximum lag to the front that sustains the incentive to remain in the game by leapfrogging to the top. Counter-intuitively, at this critical state the *follower*—the bidder immediately behind the frontrunner—is incentivized to defer the fate of conflict escalation to the underdog by not bidding, the reason being to avoid the dismal surplus dissipating bilateral endgame. While a trilateral-rivalry equilibrium need not generate larger social surplus than a bilateral-rivalry one from the standpoint before the game starts, the above result implies that, when the contest reaches a subgame where we observe escalation between only two dueling rivals, say the more and more radicalized bipartisan rivalry in the United States, it is Pareto improving to have a third party entering the competition.

To check whether all trilateral-rivalry equilibriums have such normative advantage, and whether such advantage may be vacuous in the sense that each such an equilibrium is dominated by another, we characterize all of them. Given the infinite-horizon nature of the game, it is natural to expect multiple equilibriums, one for each configuration of continuation

[2] The proposed "dynamic Pareto dominance" is similar to, but stronger than, Osborne's [18] "upset by a convincing deviation." All three players need to be better-off in our notion, whereas only the deviator needs to be better-off in Osborne's; also see Footnote 9.

values. We find, however, that there are only finitely many trilateral-rivalry equilibriums, each corresponding to an even integer associated with the aforementioned critical state. We find that all such trilateral equilibriums have the aforementioned normative advantage over any bilateral one. Furthermore, there exists a trilateral equilibrium that is not dynamically Pareto dominated. It is the one with the largest critical state and hence its path has the largest sustainable lag between an active underdog and frontrunner.

Our equilibrium characterization captures leapfrogging phenomena observed in real-world attritional wars among more than two rivals. For instance, leapfrogging political candidates, as ranked by the polls, is often seen in the US presidential primary elections, where aspiring candidates need to decide whether to spend a huge amount of campaign money on the state that is about to hold a primary election. Leapfrogging in the form of market leadership rotation and sunk cost bidding efforts is also observed, albeit in a longer time frame, in many R&D-intensive industries, e.g., the cold war era "Concorde fallacy" or more recently the highly competitive global cell-phone market. Finally, internet-based crowdsourcing innovation challenges such as the Netflix Prize[3] and online bidding schemes such as penny auctions[4] also exemplify the dynamics of attritional war and leapfrogging.

The rest of the paper is organized as follows: Section 2 defines the game and provides an intuitive presentation of bilateral-rivalry equilibriums. Section 3, after an informal demonstration of how any bilateral-rivalry equilibrium is dynamically Pareto dominated by a trilateral-rivalry one, presents the formal characterization of trilateral-rivalry equilibriums. Section 4 characterizes the trilateral equilibriums that are not dynamically Pareto dominated

[3] The Netflix Prize offered one million dollars to anyone with a movie recommendation algorithm outperforming Netflix's Cinematch algorithm by at least 10% and those from other contenders. In May 2017, real estate valuation firm Zillow initiated a similar challenge, focusing on home sales price predictions. Like an ascending bidding process, the online challenges openly updated the submissions and their performances ("bids") so that contenders could up their efforts to outperform the frontrunner. Both Netflix and Zillow retained exclusive rights to the submissions thereby becoming the beneficiary of all contenders' sunk efforts. See: http://www.netflixprize.com; https://www.kaggle.com/c/zillow-prize-1; accessed: 2018-02-27.

[4] Augenblick [1], Hinnosaar [9], Kakhbod [11] and Ødegaard and Anderson [16] study online penny auctions, which have a similar sunk bid dynamics as the dollar auction but with two significant differences. First, in online penny auctions, any non-winning bidder, regardless of the lag to the frontrunner, may leapfrog to the top with just a minimal bid increment; whereas in the dollar auction, leapfrogging to the top incurs the accumulated cost proportional to the entire lag. Second, in addition to the sunk bidding cost the winner of an online penny auctions has to pay the final auction price, which is a function of all bids.

by others. Section 5 provides a brief discourse on three extensions: an equilibrium involving four rivals on path, with a trilateral-rivalry equilibrium serving as a subgame play; an alternative tie-breaking rule allowing for multiple frontrunners; and a case with asymmetric information. Section 6 concludes. Proofs are in the appendix, in the order of appearance of the corresponding claims.

2 The Model and Preliminary Observations

2.1 The Dollar Auction

There is one indivisible good and n risk-neutral players. The value of the good, commonly known, is equal to v for every player. The good is to be auctioned off via an ascending-bid procedure with bid increment fixed at a positive constant δ such that $2\delta < v$. In the initial round, all players simultaneously choose whether to bid or stay put; if all stay put then the game ends with the good not sold, else one among those who bid is chosen randomly, with equal probability, to be the *frontrunner*, whose committed payment becomes δ, with everyone else's committed payment being zero, and the current price of the good becomes δ. Suppose that the game continues to any subsequent round, with p being the current price and b_i player i's committed payment ($b_i \leq p$ and strictly so unless i is the frontrunner), all players but the frontrunner simultaneously choose whether to bid or stay put. If all stay put then the game ends, the good is sold to the frontrunner, who pays the price p, and every other player i pays b_i; else the current price becomes $p + \delta$ and one among those who bid in this round is chosen randomly, with equal probability, to be the frontrunner, whose committed payment becomes $p + \delta$, with the committed payments of others unchanged. Then the game continues to the next round. If the game never ends, then each bidder pays the supremum of his committed payment, and the good is randomly assigned to one of those whose supremum committed payments equal infinity.

While most of the paper is based on the above-defined model, Section 5 presents three cases of extensions, including asymmetric information and an alternative tie-breaking rule that allows for multiple frontrunners.

2.2 The Surplus-Dissipating Subgame

Let us start with a subgame equilibrium that plays the role of an endogenous terminal node for other equilibriums. Within any subgame where the price p has risen to at least 2δ, with the top bidder, the frontrunner, having committed a payment p, the second-place bidder, the *follower*, having committed $p-\delta$, and all others having committed at most $p-2\delta$, there is a subgame perfect equilibrium where the top two rivals outbid each other in alternate rounds with a probability $1-2\delta/v$, and all others choose to stay put. This subgame perfect equilibrium results in an expected surplus of zero for every player and hence is called *zero-surplus subgame equilibrium*. Although conflict is escalated to the complete dissipation of surplus in the subgame equilibrium verified above, it only renders an expected revenue of $v-2\delta$ to the seller.

To explain this subgame equilibrium, for each round denote α for the current frontrunner, and β the current follower. The strategy profile prescribes actions that depend only on a player's current position rather than his identity: β bids with probability $1-2\delta/v$ and every other player $i \notin \{\alpha, \beta\}$ chooses to not bid at all (α cannot bid by the rule of the game). In the event that player β ends up bidding, the current price p is incremented by δ, players α and β switch roles, and the strategy profile repeats itself with the roles exchanged. In the off-path event that any other player $i \notin \{\alpha, \beta\}$ bids and becomes the new α, the player who was the α in the previous round, now the new β, bids with probability $1-2\delta/v$ as if it were the on-path event where he was topped by the previous β; whereas the previous β, now 2δ below the current price, chooses to not bid at all, leaving the previous α and the deviating player competing against each other in alternate roles of α and β. Any further off-path event caused by such a unilateral deviation is responded likewise.

We verify this equilibrium in three steps. First, denote V_* for a bidder's continuation value of being the current α player, and M_* that of being the current β. Given the expectation that only the current β player bids at all,

$$V_* = (1 - 1 + 2\delta/v)v + (1 - 2\delta/v)M_* = 2\delta + (1 - 2\delta/v)M_*. \tag{1}$$

In bidding and becoming the next round α, the current β increases his committed payment by 2δ; hence $M_* = (1-2\delta/v)(V_*-2\delta)$. This implies $M_* = (1-2\delta/v)\left(2\delta + (1-2\delta/v)M_* - 2\delta\right) = (1-2\delta/v)^2 M_*$, hence $M_* = 0$, and so (1) implies $V_* = 2\delta$.

Second, it is a best response for the current β player to bid with probability $1-2\delta/v$, and a best response for every other player $i \notin \{\alpha, \beta\}$ to not bid at all: For β, if he bids then he becomes the new α and bears a sunk cost 2δ, hence his expected payoff from bidding is equal to $V_* - 2\delta = 0$; if he does not bid then his payoff is zero as the current α wins. Hence β is indifferent, so bidding with probability $1 - 2\delta/v$ is a best response. For any $i \notin \{\alpha, \beta\}$, with committed payment $b_i \leq p - 2\delta$, the cost $p + \delta - b_i$ that i needs to incur to assume the role of α is larger than 2δ, hence the best response is not to bid at all.

Third, consider an off-path event where a player i other than the α and β in the previous round bids and gets selected to be the current α. In this event, the price committed by the β in the previous round remains to be $p - 2\delta$, with p the current price committed by the new α, and the price committed by the α in the previous round is equal to $p - \delta$. This previous α becoming the current β, the reasoning in the previous paragraph applies and hence he finds it a best response to act as the current β according to the strategy proposed above for this event. The reasoning in the paragraph regarding $i \notin \{\alpha, \beta\}$ now applies to the β in the previous round, as his committed price is 2δ below the current price. Hence his best response is to not bid at all, as in the proposed equilibrium.

2.3 A Continuum of Bilateral Equilibriums

With the zero-surplus subgame equilibrium acting as a penal code to deter conflict escalation, we observe that there is a continuum of subgame perfect equilibriums, each indexed by an $x \in [0, 1]$, such that the equilibrium expected revenue can be as small as 0, when $x = 0$, or as large as v, when $x = 1$. At any such an equilibrium, every player bids with probability x in the initial round, and in the case at least one player bids a frontrunner is selected, who incurrs a sunk cost δ and consequently raising the price to δ. In the second round, every player other than the frontrunner bids with probability $1 - (\delta/v)^{1/(n-1)}$; in the event that someone ends up bidding, the new frontrunner selected thereof commits a sunk cost 2δ, with the previous frontrunner becoming the follower; hence we enter the subgame described in the previous subsection. From this point on the zero-surplus subgame equilibrium is played, where everyone else, except the frontrunner and follower, stays put while competition between the two escalates with a probability, with the follower topping the frontrunner with probability $1 - 2\delta/v$ in any round. Thus, in the second round, anyone other than the frontrunner finds it a best response to bid with only probability $1 - (\delta/v)^{1/(n-1)}$, anticipating the zero-surplus subgame equilibrium should he outbid the frontrunner. In the initial round, where the current price equals zero and no frontrunner has emerged, if a player bids and gets selected as frontrunner, his expected payoff is equal to

$$-\delta + (1 - (1 - (\delta/v)^{1/(n-1)}))^{n-1} v + \left(1 - (1 - (1 - (\delta/v)^{1/(n-1)}))^{n-1}\right) M_* = -\delta + (\delta/v)v,$$

and hence bidding with probability x is a best response for everyone at the initial round.

To summarize: in the initial round all bidders bid with probability $x \in [0, 1]$; in the second round the $n-1$ non-winning bidders all bid with probability $1 - (\delta/v)^{1/(n-1)}$; and in any subsequent round only the current follower bids with probability $1 - 2\delta/v$. While the equilibrium results in a complete depletion of bidder expected surplus, it is anecdotally consistent with the popular classroom exercise of conducting a dollar auction. In a typical classroom dollar auction there tends at the onset to be a general hesitation and "low" participation level (a small x), but once a frontrunner has emerged then in the second round there is a "higher" participation of bidders (a large $1 - (\delta/v)^{1/(n-1)}$), and from then on the bilateral bidding rivalry emerges.

The equilibrium thus verified generates an expected revenue $(1 - (1 - x)^n)\, v$, which, depending on x, ranges from 0 to v. In other words, contrary to the paradox conjectured by Shubik [19] the expected revenue does not exceed the prize's worth v. Furthermore, different from Demange [5] and Hörner and Sahuguet [10], where the difference between surplus dissipation and retention relies on the introduction of asymmetric information or jump bids, here the degree to which the seller extracts surplus depends purely on which equilibrium the bidders happen to play.[5]

3 The Integral Spectrum of Trilateral Equilibriums

The bilateral-rivalry equilibriums observed in the previous section, though generating various surplus for the bidders from the ex ante standpoint, all end with the surplus-dissipating

[5] Note that the proposed equilibrium characterization is not unique. For instance, the following constitutes another set of continuum of equilibriums: in the initial round all bidders bid with probability 1; in the second round the $n-1$ non-winning bidders all bid with probability $x \in \left[0, 1 - (\delta/v)^{1/(n-1)}\right]$; and in any subsequent round only the current follower bids with probability $1 - 2\delta/v$. This equilibrium generates an expected revenue $\delta + \left(1 - (1 - x)^{n-1}\right) v$. Although this equilibrium characterization includes the possibility for strictly positive bidder surplus, with the multiplicity of a continuum, such equilibriums present a severe coordination problem to the contestants in trying to retain surplus via playing one in the continuum.

subgame equilibrium once bilateral rivalry occurs on path. We shall start with an informal demonstration on how any bilateral-rivalry equilibrium at the turning point into the surplus-dissipating subgame is Pareto dominated by a trilateral-rivalry equilibrium. To see if this normative advantage is shared by other trilateral-rivalry equilibriums, we formally characterize all trilateral-rivalry equilibriums by a recursive method. For the remainder of the paper, we restrict the analysis to the case where $n = 3$.

3.1 Dynamic Pareto Improvement due to the Third Rival

Suppose that a bilateral-rivalry equilibrium is being played and the game has continued to the third round, where the price has risen to 2δ and a bilateral rivalry has emerged between two players, the frontrunner having committed 2δ and the follower having committed δ. Suppose contrary to the prescribed equilibrium the third player deviates by bidding. If he gets selected as new frontrunner, the deviator commits 3δ and a trilateral rivalry is formed between him and the previous frontrunner and follower. Furthermore, the rivalry is manifested by a *consecutive configuration* where the distance of committed payments between the current frontrunner and the follower, and that between the follower and the third-place bidder, are each equal to δ. Should the players stick to the status quo, the bilateral-rivalry equilibrium, such a deviation would be unprofitable. However, the deviation could be taken as a call for switching to another equilibrium which, as we will demonstrate later, makes all three rivals better-off; moreover, if the other players follow suit and switch to the new equilibrium conditional on the deviation, the deviator strictly prefers the deviation from the standpoint where he considers it. In other words, the bilateral-rivalry equilibrium is not Pareto perfect: on its path there is a point where it is Pareto dominated by another equilibrium that gives rise to *trilateral rivalry*, and the "renegotiation" can be done tacitly through merely a unilateral deviation.

We will formalize the above claim in subsequent sections but to provide more intuition we first illustrate with a numerical example. It should be intuitive that bidders' expected surplus depends on the relative values of v and δ, and so for this example we focus on the case when $v/\delta \geq 35/2$, and a trilateral-rivalry equilibrium generating 4δ in expected surplus for any frontrunner in the consecutive configuration described above. At this configuration, both the follower and the third-place bidder bid for sure. If the third-place bidder gets selected as the new frontrunner then the consecutive configuration repeats itself, otherwise the *gap* between the current price and the third-place bidder's committed payment widens by δ. In any subsequent round the current follower and the third-place player bid for sure unless the aforementioned gap widens to 3δ. In that event, the follower stays put and the third-place player bids with a probability pinned down by the condition that the frontrunner in the consecutive configuration has surplus 4δ; if the third-place player ends up not bidding then the current frontrunner wins, else the third-place bidder becomes the frontrunner and the trilateral rivalry is back to its consecutive configuration.

This equilibrium generates an expected surplus of 4δ for the frontrunner, more than δ for the follower, and $\delta/2$ for the third-place player, when they are in the consecutive configuration. Whereas, at this point, any bilateral-rivalry equilibrium gives only 2δ to the frontrunner and zero to the other two. Hence it is Pareto superior for them to switch to the trilateral equilibrium conditional on the deviation. Furthermore, the switch induces a profit $4\delta - 3\delta = \delta$ for the unilateral deviator from the viewpoint of the previous round.

The previous observation calls for formal construction and characterization of trilateral-rivalry equilibriums. To exploit the recursive structure of the game we restrict the equilibrium concept by three name-independent, Markov perfect conditions. The structure of such equilibriums turns out to be remarkably clean; there are only finitely many of them, each corresponding to an even number.

3.2 The State of the Game and the Equilibrium Concept

The *state* of the game, in any round, consists of the vector $(b_i)_{i=1}^n$ of the payments committed by the players so far, with $\max_{i=1,\ldots,n} b_i$ being the current price, and $\arg\max_{i=1,\ldots,n} b_i$ (singleton by the rule of the game) the current frontrunner. For any subgame perfect equilibrium $\mathscr{E}$ of the game and any state $(b_i)_{i=1}^n$, denote $\mathscr{E}|(b_i)_{i=1}^n$ for the continuation play of $\mathscr{E}$ in any subgame that starts with the state $(b_i)_{i=1}^n$. By *equilibrium* we mean any subgame perfect equilibrium $\mathscr{E}$ of the game that satisfies three conditions:

Symmetry For any two states $(b_i)_{i=1}^n$ and $(b'_i)_{i=1}^n$ such that $b_i = b'_{\psi(i)}$ for all $i \in \{1, \ldots, n\}$ for some permutation ψ on $\{1, \ldots, n\}$, $\mathscr{E}|(b_i)_{i=1}^n$ is isomorphic to $\mathscr{E}|(b'_i)_{i=1}^n$ given the permutation ψ.

Recursion $\mathscr{E}|(b_i)_{i=1}^n$ is equal to $\mathscr{E}|(b'_i)_{i=1}^n$ such that $b'_i = b_i - \min_{j=1,\ldots,n} b_j$ for all $i \in \{1, \ldots, n\}$.

Independence of irrelevant players For any state $(b_i)_{i=1}^n$, if, for some $k \in \{1, \ldots, n\}$ and constant c, at every state $(b'_i)_{i=1}^n$ generated on the path of $\mathscr{E}|(b_i)_{i=1}^n$ we have $b'_k = c < \max_i b'_i$, then $\mathscr{E}|(b_i)_{i=1}^n$ satisfies the previous two conditions such that $\{1, \ldots, n\}$ is replaced by $\{1, \ldots, n\} \setminus \{k\}$.

The symmetry condition requires that the strategy profile in an equilibrium be independent of players' names. The recursion condition says that a player's equilibrium strategy depends not on the amount of payments he has committed so far but rather on the distances between his and others' committed payments, i.e., past bids constitute a sunk cost. The independence condition of irrelevant players says that, if a player k drops out of the race for good according to an equilibrium, then the equilibrium strategy conditional on this subgame should not vary with the position of this player from this point on.

For tractability we specialize to the case where $n = 3$. Consequently, by the symmetry and recursion conditions, we need only to identify the three players by the relative positions of their committed payments, hence denote α for the frontrunner, whose committed payment is the current price p ($b_\alpha = p$), β the follower, whose committed payment is always just δ below the frontrunner's ($b_\beta = p - \delta$), and γ the *underdog*, whose committed payment is the lowest. The discrete state of the game can be represented by the frontrunner-underdog lag

$$s := (p - b_\gamma) / \delta,$$

i.e. $b_\gamma = p - s\delta$. Note that $s \geq 2$, and thus we extend the notation such that the state in the initial round equals zero ($s = 0$), with everyone treated as underdog, and in the second round the state equals one ($s = 1$), with all but the frontrunner being underdog. Then any equilibrium is of the form

$$\left(\pi_0, \pi_1, (\pi_{\beta,s}, \pi_{\gamma,s})_{s=2}^{\infty}\right),$$

with π_0 being every bidder's probability of bidding at the initial round, π_1 the probability of bidding at the second round for everyone but the current α player, and, for every $s \geq 2$ and each $i \in \{\beta, \gamma\}$, $\pi_{i,s}$ being the probability with which the current i player bids. We use π to represent the bidding probability in order to separate the trilateral rivalry equilibrium bidding probabilities from the bilateral rivalry ones.

3.3 The Value Functions

Let any equilibrium $\left(\pi_0, \pi_1, (\pi_{\beta,s}, \pi_{\gamma,s})_{s=2}^{\infty}\right)$ be given. For every $s \geq 2$ and each $i \in \{\beta, \gamma\}$, denote $q_{i,s}$ for the probability with which the current i player becomes the α player in the next round. Note, from the uniform-probability tie-breaking rule, that at any $s \geq 2$

$$q_{i,s} = \pi_{i,s}\left(1 - \pi_{-i,s}/2\right), \tag{2}$$

with $-i$ being the element of $\{\beta, \gamma\} \setminus \{i\}$. Given this equilibrium and any state s, denote V_s for the expected payoff for the current α player, M_s the expected payoff for the current β, and L_s that for the current γ (with $M_1 = L_1$ for every non-α player at the second round). The law of motion is described below:

$$V_1 \longrightarrow \begin{cases} v & \text{prob. } (1-\pi_1)^2 \\ M_2 & \text{prob. } 1-(1-\pi_1)^2, \end{cases} \tag{3}$$

$$M_1 \longrightarrow \begin{cases} 0 & \text{prob. } (1-\pi_1)^2 \\ V_2 - 2\delta & \text{prob. } \pi_1\left(1-\pi_1/2\right) \\ L_2 & \text{prob. } \pi_1\left(1-\pi_1/2\right); \end{cases} \tag{4}$$

and, for each $s \geq 2$:

$$V_s \longrightarrow \begin{cases} v & \text{prob. } 1 - q_{\beta,s} - q_{\gamma,s} \\ M_{s+1} & \text{prob. } q_{\beta,s} \\ M_2 & \text{prob. } q_{\gamma,s}; \end{cases} \tag{5}$$

$$M_s \longrightarrow \begin{cases} 0 & \text{prob. } 1 - q_{\beta,s} - q_{\gamma,s} \\ V_{s+1} - 2\delta & \text{prob. } q_{\beta,s} \\ L_2 & \text{prob. } q_{\gamma,s}; \end{cases} \tag{6}$$

$$L_s \longrightarrow \begin{cases} 0 & \text{prob. } 1 - q_{\beta,s} - q_{\gamma,s} \\ L_{s+1} & \text{prob. } q_{\beta,s} \\ V_2 - (s+1)\delta & \text{prob. } q_{\gamma,s}. \end{cases} \tag{7}$$

3.4 The Dropout State

Since v is finite, at any equilibrium V_2 is finite and hence $V_2 < s\delta$ for all sufficiently large s. Thus, for any equilibrium

$$s_* := \max\{s \in \{1, 2, 3, \ldots\} : V_2 \geq s\delta\}$$

exists and is unique. Call s_* the *dropout state* of the equilibrium. The next lemma, which follows from (7) coupled with the definition of s_*, justifies the appellation.

Lemma 1 *At any equilibrium with dropout state s_*, an underdog (γ player) (i) stays put for sure at state s if and only if $s \geq s_*$, and (ii) bids for sure at state s if $2 \leq s < s_* - 1$.*

For example, in any bilateral-rivalry equilibrium, if the state becomes $s = 2$ then it is already in the zero-surplus subgame equilibrium, where $V_2 = V_* = 2\delta$ (Section 2.2), hence $s_* = 2$. For the illustrative trilateral-rivalry equilibrium sketched at the start of this section, $V_2 = 4\delta$ and so $s_* = 4$, hence an underdog bids with positive probability as long as his lag from the frontrunner is below 4.

By Lemma 1, once the game enters the dropout state or beyond, the player currently in the underdog role will never bid to catch up and only the frontrunner and follower may remain active. A subgame equilibrium henceforth is the zero-surplus one constructed in Section 2.2. By the independence condition of irrelevant players, which deems the position of the underdog irrelevant to any equilibrium projected onto any such subgames, the zero-surplus subgame equilibrium is the only on-path outcome thereafter:

Lemma 2 *At any equilibrium with dropout state $s_* \geq 2$, if $s \geq s_*$ then $V_s = 2\delta$ and $M_s = L_s = 0$.*

Thus, the dropout state of an equilibrium can be viewed as the endogenous terminal node of the game, giving an expected payoff 2δ to the frontrunner, and zero expected payoff to the follower and the underdog.

Reasoning backward from the dropout state s_*, we see that the game does not end if it is in any state $s \leq s_* - 2$, because according to Lemma 1.ii the current underdog bids for sure trying to catch up with the frontrunner. Thus the minimum state at which the game need not continue to the next round is the state $s_* - 1$, at which the underdog need not bid for sure. Furthermore, combining (6), Lemma 2 and the definition of s_* one can show that the follower at the *critical state* $s_* - 1$ would rather be the underdog in the next round, should the game continue, than outbid the frontrunner right now thereby getting into the zero-surplus subgame equilibrium thereupon. Thus, at the critical state $s_* - 1$, the underdog solely determines whether the competition should continue or cease, asserted by the next lemma, which also implies that the frontrunner's equilibrium surplus V_2 in the consecutive configuration is necessarily an integer multiple of the bid increment δ.

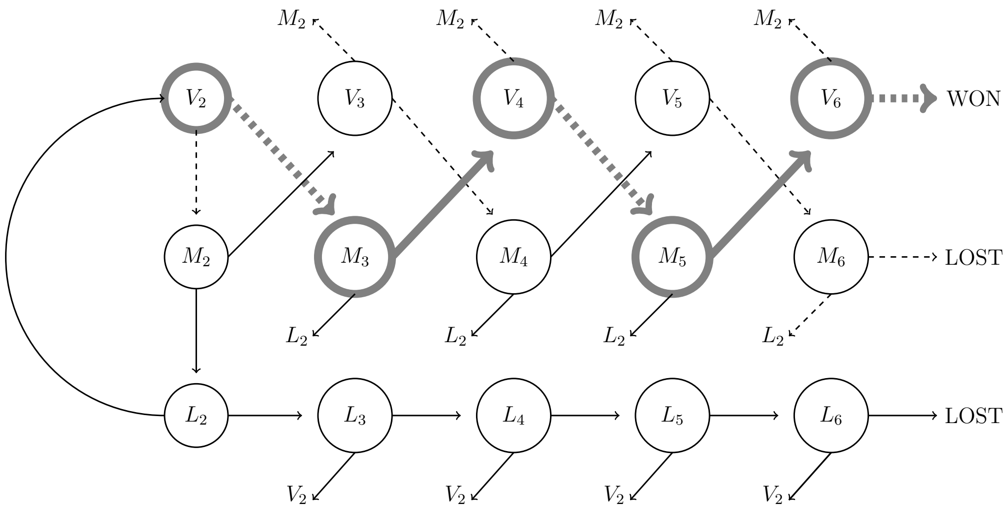

Figure 1: The law of motions and equilibrium winning path if $s_* = 7$.

Lemma 3 *At any equilibrium with dropout state* $s_* \geq 3$*: (i) at the critical state* $s_* - 1$ *the* β *player stays put while the* γ *player bids with a probability in* $(0, 1)$*; and (ii)* $V_2 = s_*\delta$*.*

3.5 Dropout States Can Only Be Even

Lemma 3 implies that on the path of any equilibrium the game ends only when the state is $s_* - 1$, at which only the underdog γ may bid. If he bids (thereby becoming the next α) then the state returns to $s = 2$, else the game ends and the current α wins the good. Thus, in order to win, a player needs to be the α player at the critical state $s_* - 1$. Consequently, if the dropout state s_* is an odd number, then on the path to winning a bidder must in the previous rounds have been the β player for all odd states $s < s_* - 1$, and the α player for all even states $s \leq s_* - 1$. An illustration for $s_* = 7$ is shown in Figure 1. Solid lines represent possible transitions if one bids, and dashed lines if he does not bid. The extra thick gray states and arrows indicate the winning path.

Thus, when s_* is odd, a player who happens to be in the β position at any even state $s < s_* - 1$ would in order to reach the winning path rather become the γ player in state $s = 2$ (through not bidding at all) than become the superfluous α player in the odd state $s + 1$ at the cost of 2δ (through bidding). In particular, in state $s = 2$, the β player would never bid while the γ player would always bid; hence the state $s = 2$ repeats itself, with the players switching roles according to $\gamma \to \alpha \to \beta \to \gamma$, thereby trapping them in an infinite bidding loop. This contradiction, after being formalized, implies the first main finding—

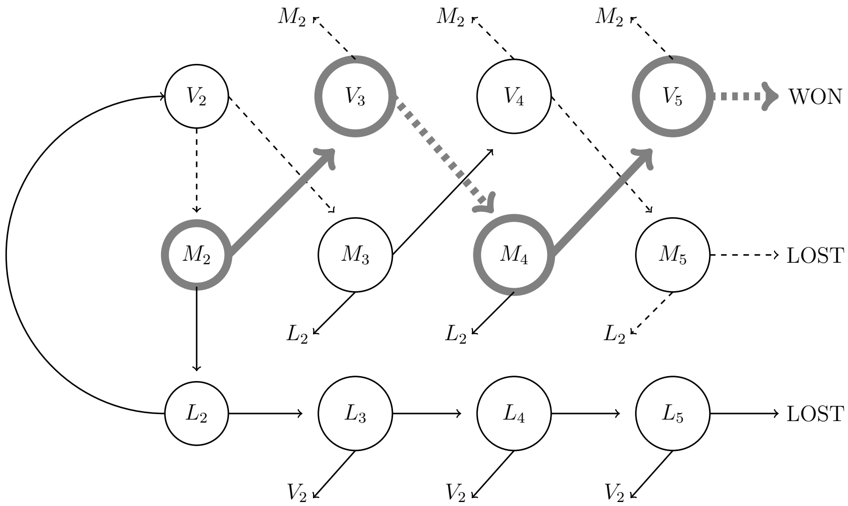

Figure 2: The law of motions and equilibrium winning path if $s_* = 6$.

Theorem 1 *There does not exist any equilibrium whose dropout state s_* is an odd number bigger than 2.*

When the dropout state s_* is an even number, by contrast, a β player is not in the predicament as in the previous case. First, in any even state $s < s_* - 1$ the β player wants to bid in order to stay on the winning path and become the α in the odd state $s + 1$. Second, in any odd state $s < s_* - 1$ the β player would rather bid and become the α in the even state $s + 1$ than stay put thereby becoming the γ player in state 2. With the former option, it takes a cost of 2δ (to become α in $s + 1$) and two rounds for the player to have a chance to become the β player in state $s = 2$ thereby landing on the winning path. With the latter option, it takes a cost of 3δ and three rounds for him to have such a chance of reaching the winning path. In Figure 2, with $s_* = 6$, the situation of this odd-state β player is illustrated by the node M_3, from which the former option (becoming the next α) reaches the winning path state M_2 via $M_3 \to V_4 \to M_2$, while the latter option (being the next γ) reaches M_2 via the more roundabout route $M_3 \to L_2 \to V_2 \to M_2$.[6] Formalizing this intuition we obtain—

Lemma 4 *At any equilibrium with dropout state s_* being an even number and $s_* \geq 4$, at any state $s \in \{1, 2, \ldots, s_* - 2\}$ the β player bids for sure.*

[6] In the more roundabout route, the last step, from V_2 to M_2, is preferable to a player because of a nontrivial Lemma 11, saying that in the consecutive configuration it is better-off to be the follower than the frontrunner.

3.6 Characterization of the Equilibriums

Lemmas 1–4 together have mostly pinned down the strategy profile for any equilibrium with dropout state $s_* > 3$:

(∗) s_* is an even number; at each state $s \in \{1, 2, \ldots, s_* - 2\}$ every non-α player bids for sure; at state $s_* - 1$ the β player does not bid and the γ bids with probability π_{γ,s_*-1}; at any state $s \geq s_*$, the γ player does not bid and the β bids with probability $1 - 2\delta/v$.

By Condition (∗), Eq. (2) and the equal-probability tie-breaking rule,

$$2 \leq s \leq s_* - 2 \Longrightarrow q_{\beta,s} = q_{\gamma,s} = 1/2. \tag{8}$$

Given any $\pi_{\gamma,s_*-1} \in [0, 1]$, the value functions $(V_s, M_s, L_s)_s$ associated to any strategy profile satisfying Condition (∗) can be calculated based on Eq. (8) and the law of motion, (5)–(7). The question is whether such a strategy profile constitutes an equilibrium. The crucial step in answering this question is to verify that, given Condition (∗), bidding is a best response for the β player at every state below $s_* - 1$. Verification for all such states might sound cumbersome, but it turns out that we need only to check two inequalities:

Lemma 5 *For any even number $s_* \geq 4$ and any strategy profile satisfying Condition (∗), bidding is a best response for the β player at state $s \in \{1, 2, \ldots, s_* - 2\}$ if either (i) s is even and $V_3 - 2\delta \geq L_2$, or (ii) s is odd and $V_{s_*-2} - 2\delta \geq L_2$.*

These two sufficient conditions, one can show, are also necessary for any equilibrium. Thus Lemma 5, combined with the previous ones, implies a necessary and sufficient condition for any equilibrium with even-number dropout state $s_* \geq 4$: that the bidding probability π_{γ,s_*-1} at the critical state is determined by the equation $V_2 = s_*\delta$ (Lemma 3.ii), with V_2 as well as other value functions derived from the law of motion (5)–(7) and Condition (∗), such that both $V_3 - 2\delta \geq L_2$ and $V_{s_*-2} - 2\delta \geq L_2$ are satisfied. From this necessary and sufficient condition we obtain a complete characterization of trilateral-rivalry equilibriums, equilibriums with dropout states larger than two:

Theorem 2 *Any $s_* \geq 3$ constitutes an equilibrium if and only if s_* is an even number and—*

i. either $s_* \leq 6$ and the equation

$$\begin{aligned} &\frac{3\mu_* v}{\delta}(1-x)(2-\mu_*) + (2-\mu_*)^2(s_* - 6 + \mu_*) \\ = \quad &(2(1+\mu_*) - 3\mu_* x)\left(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_* x)\right), \end{aligned} \tag{9}$$

where $\mu_* := 2^{-s_*+3}$*, admits a solution for* $x \in [0,1]$*;*

ii. or $s_* \geq 8$ *and Eq. (9) admits a solution for* $x \in [0,1]$ *such that*

$$x \geq 1 - \frac{3(2-\mu_*)}{2(1-2\mu_*)(s_*-4+\mu_*)}. \tag{10}$$

The solution x to equation (9) corresponds to the γ player's bidding probability π_{γ,s_*-1} in the critical state s_*-1. The bifurcated characterization in Theorem 2 is due to a fact, proved in the appendix, that at the solution for $V_2 = s_*\delta$, neither $V_3 - 2\delta \geq L_2$ nor $V_{s_*-2} - 2\delta \geq L_2$ are binding when $s_* \leq 6$, and only one of the inequalities is binding when $s_* \geq 8$.

Contrary to the case of odd-number dropout states, equilibriums with even-number dropout states exist provided that the parameter v/δ is sufficiently large:

Theorem 3 *(i) An equilibrium with* $s_* = 4$ *exists if and only if* $v/\delta > 35/2$*, and that with* $s_* = 6$ *exists if and only if* $v/\delta > 6801/120$*. (ii) For any even number* $s_* \geq 8$*, if*

$$\frac{v}{\delta} \geq \left(\frac{1}{3}s_*^2 + \frac{5}{3}s_* - 8\right)2^{s_*-3} \tag{11}$$

then s_* *constitutes an equilibrium with dropout state equal to* s_**.*

As s_* increases from 8, the right-hand side of Ineq. (11) increases at a rate in the order of 2^{s_*}. Thus, to suffice the equilibrium feasibility of a higher dropout state s_*, Ineq. (11) requires that the parameter v/δ be higher by a magnitude in the order of 2^{s_*}.

Contrary to the bilateral-rivalry equilibriums, which constitute a continuum, there are only finitely many trilateral-rivalry ones, as the next theorem asserts. That is because the parameter v/δ, through the facts $v \geq V_2$ and $V_2 = s_*\delta$, implies an upper bound for equilibrium-feasible dropout states, which can only be integers, and given each dropout state Eq. (9) admits at most two solutions for x (i.e., π_{γ,s_*-1}), which in turn determines the equilibrium strategy profile uniquely.

Theorem 4 *There are at most finitely many equilibriums with dropout states* $s_* \geq 3$*.*

3.7 Numerical Illustration

To illustrate the formal results in Theorems 2 and 3, we fix $\delta = \$1$, vary v from \$0 to \$1,000 and consider the cases $s_* = 4, 6, 8, 10$. Figure 3 shows the γ player's (underdog)

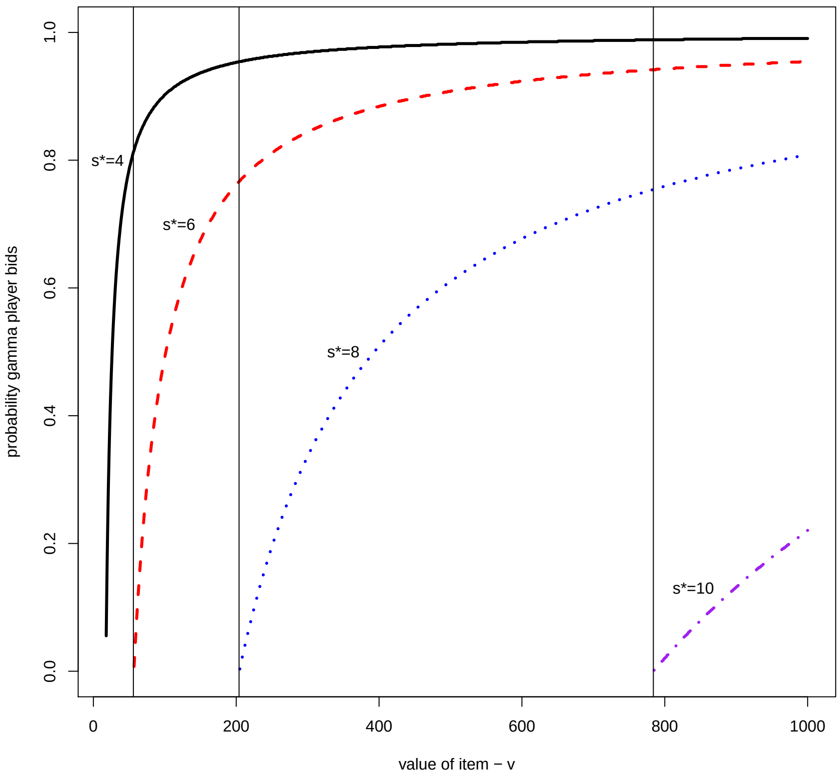

Figure 3: Equilibrium bidding probability for the underdog in the critical state $s_* - 1$; $\delta = 1$.

equilibrium bidding probability in the critical state $s_* - 1$ as a function of the underlying value v. The vertical lines indicate the point at which additional equilibriums for $s_* > 4$ are admitted. For instance, starting at $v = \$57$ ($\approx 6801/102$) the equilibrium corresponding to the dropout state $s_* = 6$ is permissible. We observe that within each equilibrium the bidding probability is increasing in the underlying value v (or v/δ as δ is fixed at one in this example). On the other hand, and a bit surprisingly, when a new equilibrium with a higher dropout state becomes permissible the corresponding equilibrium bidding probability drastically reduces. Furthermore, each additional equilibrium requires an order of magnitude increase in v, somewhat confirming our previous remark on the right-hand side of Ineq. (11).

4 The Unique Pareto Perfect Dropout State

Now we formalize the notion of dynamic Pareto dominance, which was used informally to motivate trilateral-rivalry equilibriums. Based on the full characterization of such equilibriums in the previous section, we observe that a trilateral equilibrium with the largest dropout state dominates all equilibriums, bilateral or trilateral, with smaller dropout states, and the equilibrium itself is not dominated. Not only does this observation help to reduce the multiplicity of equilibriums in the dollar auction, but it also implies that the normative advantage of trilateral-rivalry equilibriums over bilateral-rivalry ones, in terms of dynamic Pareto dominance, is not vacuous.

For any equilibriums $\mathscr{E}$ and $\bar{\mathscr{E}}$ of the dollar auction game, equilibrium $\mathscr{E}$ is said *dynamically Pareto dominated* by equilibrium $\bar{\mathscr{E}}$ if and only if there exists a state $s \in \{0, 1, 2, \ldots\}$ such that—

a. s is off the path of $\mathscr{E}$ and on the path of $\bar{\mathscr{E}}$;

b. starting from state s, each player has higher expected payoff from $\bar{\mathscr{E}}|s$ than from $\mathscr{E}|s$;

c. at $\mathscr{E}$, the state s is reached by a player's unilateral deviation from a state t that is on the path of $\mathscr{E}$, and the unilateral deviation, from the standpoint of t, is profitable for the deviating player provided that $\bar{\mathscr{E}}|s$ is the subgame play starting from s.

An equilibrium is said *weakly Pareto perfect* if and only if it is not dynamically Pareto dominated by another equilibrium.

To switch away from a dynamically Pareto dominated equilibrium, interpretation-wise, the players do not need pre-play communication at the outset or "renegotiation" after a deviation. Rather, the choice of one equilibrium over the other can be instigated by a single player's unilateral deviation from the status quo.

Theorem 5 *Any bilateral equilibrium is dynamically Pareto dominated by any trilateral-rivalry equilibrium. Furthermore, if $v/\delta > 35/2$ then the weakly Pareto perfect equilibriums are exactly the trilateral-rivalry ones with a unique dropout state equal to the maximum among the even numbers s_* that satisfy condition (i) or (ii) in Theorem 2.*

In other words, among the spectrum of equilibriums, not only are the bilateral-rivalry equilibriums dynamically Pareto dominated by the trilateral-rivalry ones (which exist by the hypothesis $v/\delta > 35/2$ and Theorem 3), but also are the trilateral-rivalry equilibriums except the one(s) with the maximum dropout state. Section 3.1 has sketched how any bilateral-rivalry equilibrium with positive probability of occurrence of bilateral rivalry is dominated by the trilateral-rivalry one with $s_* = 4$. The same argument also implies that any such bilateral equilibrium is dominated by *any* trilateral-rivalry one, because among trilateral equilibriums $s_* = 4$ is the lowest dropout state and hence the V_2 in any other trilateral one can only be larger than 4δ.[7] The only bilateral-rivalry equilibrium left to address is the no-conflict equilibrium, where the probability with which any rivalry occurs is zero, i.e., every non-frontrunner in the second round refrains from topping the frontrunner, expecting the surplus-dissipating subgame equilibrium in case of deviation. One can easily show that this equilibrium is also dynamically Pareto dominated.[8]

For any trilateral-rivalry equilibrium whose dropout state s_* is not the maximum one, it is dominated by another trilateral-rivalry one with a higher dropout state say s'_*. At state $s_* - 1$, when the β player is supposed to stay put and receive zero surplus at the former equilibrium, β can deviate by bidding. In the event that he deviates and becomes the next frontrunner, if the former equilibrium is played then the deviator-turned frontrunner gets 2δ and the other two get zero surplus, with the deviator and the previous-round frontrunner engaged in the surplus-dissipating bilateral rivalry, and the underdog dropping out; if they switch to the latter equilibrium, by contrast, the trilateral rivalry is prolonged since the larger dropout state has not been reached, and one can show that each gets positive surplus, which for the one who has just deviated is at least $s'_*\delta$, larger than the 2δ that he would get from the former equilibrium conditional on his deviation, and also larger than the additional payment 2δ that he commits in making the deviation.[9]

[7] This means in the consecutive configuration the frontrunner is better-off in any trilateral-rivalry equilibrium than in any bilateral-rivalry one; the two non-frontrunners are both better-off as well, as M_2 and L_2 are both positive in any trilateral equilibrium, and are both zero in any bilateral one.

[8] The no-conflict equilibrium is dominated at the second round, when a non-frontrunner, who is supposed to not bid, deviates by committing to pay 2δ thereby topping the frontrunner. Conditional on the deviation, one can show that any trilateral-rivalry equilibrium yields positive surplus for all three players, with a surplus larger than 4δ for the deviator specifically, while the original equilibrium, now running according to the zero-surplus subgame equilibrium, yields only 2δ for the deviator-turned frontrunner and zero surplus for the other two. Provided that the equilibrium switch is made conditional on the deviation, it generates a profit at least $4\delta - 2\delta$ for the deviator from the standpoint before he deviates.

[9] Another interpretation of this weak notion of Pareto perfection is to iteratively eliminate equilibriums by

5 Extensions

The above analysis and main results are based on complete information, the tie-breaking rule as prescribed by Shubik's original model, and a trilateral rivalry. While a comprehensive relaxation of these assumptions is beyond the scope of this paper, this section presents partial extensions of these assumptions individually. Subsection 5.1 considers a case of four players and constructs a quadrilateral-rivalry equilibrium based on trilateral-rivalry ones. Subsection 5.2 considers an alternative tie-breaking rule where in each round all bidders who simultaneously bid incur the sunk cost and become frontrunners simultaneously, and the game ends once no more bids are submitted, with the prize awarded only if the current frontrunner is unique. Subsection 5.3 considers a two-player model with asymmetric information and constructs a perfect Bayesian equilibrium that resembles a purification of the bilateral-rivalry equilibrium.

5.1 A Quadrilateral Rivalry Equilibrium

The analysis above has highlighted two effects of adding a third rival to an otherwise bilateral rivalry: first, once a play has reached the bilateral-rivalry subgame, it is Pareto dominated by any trilateral-rivalry equilibrium (Theorem 5); second, participation of the third rival eliminates the bilateral-rivalry in the sense that in any trilateral equilibrium, the bilateral rivalry can only be an off-path event (Lemmas 1 and 3). Then can the participation of a *fourth* rival have an analogous effect on trilateral-rivalry? Here we construct an example to show that the answer is No.

Let there be four players, denoted as frontrunner (α), follower (β), underdog (γ_1), and "bottomdog" (γ_2) for the fourth-place bidder. Let $t \in \{0, 1, 2, 3, \dots\}$ denote the gap between the current price and the committed payment of the current lowest player, i.e.. the bottomdog's lag from the frontrunner. In the quadrilateral-rivalry equilibrium constructed below, the dropout state for the bottomdog is when $t = 4$. In the subgame once this dropout state is reached, the other three players play the trilateral-rivalry equilibrium whose dropout

forward induction, starting from the equilibrium with the second-highest dropout state. For any equilibrium with non-maximum dropout state say s_*, conditional on a deviation at critical state $s_* - 1$, the equilibrium with the maximum dropout state $\bar{s}_*$ is the only one that Pareto dominates the one with s_*, since either $\bar{s}_* = s_* + 2$ or any equilibrium with dropout states between s_* and $\bar{s}_*$ have been eliminated in this manner. Hence the deviation can be taken as a signal that the deviator is to play the equilibrium with $\bar{s}_*$.

state is $s_* = 4$. Specifically, the quadrilateral strategy profile is:

a. In the initial round ($t = 0$), everyone bids for sure.

b. In the second ($t = 1$) and third ($t = 2$) rounds, every non-frontrunner bids for sure.

c. In any round where $t = 3$:

   i. if the current configuration is

$$\begin{bmatrix} \alpha & p \\ \beta & p-\delta \\ \gamma_1 & p-2\delta \\ \gamma_2 & p-3\delta \end{bmatrix} \tag{12}$$

   for some $p \geq 3\delta$, then β, γ_1 and γ_2 each bid for sure;

   ii. else then it is the fourth round and the configuration is in the form

$$\begin{bmatrix} \alpha & 3\delta \\ \beta & 2\delta \\ \varnothing & \delta \\ \{\gamma_1, \gamma_2\} & 0 \end{bmatrix}, \tag{13}$$

   then the play mimics the above-specified trilateral-rivalry equilibrium at the critical state $s_* - 1 = 3$: β stays put, and γ_1 and γ_2 each bid with probability $1 - (1 - \pi_{\gamma,3})^{1/2}$; where $\pi_{\gamma,3}$, as defined by Condition $(*)$ in subsection 3.6, is the trilateral equilibrium probability that escalation continues.

d. If $t \geq 4$, then γ_2 quits from now on, and the other players play the above-specified trilateral-rivalry equilibrium; in the off-path event where γ_2 leapfrogs to the top, then he and the previous frontrunner and follower constitute a consecutive 3-player configuration, and the three play the trilateral-rivalry equilibrium from now on, with γ_1 as the new bottomdog quitting from now on.

Appendix A.9 proves that the above strategy profile constitutes a subgame perfect equilibrium. Note that trilateral rivalry occurs on path with positive probability in this equilibrium. Thus, adding a fourth rival does not eliminate trilateral rivalry as adding a third rival does to bilateral rivalry. To check if this quadrilateral equilibrium dominates any trilateral one, define, given the consecutive configuration (12), A to be the continuation value for α, B to be the continuation value for β, C for γ_1, and D for γ_2. By Appendix A.9,

$$4\delta < A < 5\delta, \quad 5\delta < B < 6\delta, \quad 3\delta/8 < C < 2\delta/5, \quad \delta/9 < D < 5\delta/27.$$

In the configuration (12), if the players stick to the trilateral-rivalry equilibrium with dropout state $s_* = 4$, then α gets $V_2 = 2\delta$, β gets M_2, which by Eqs. (43) and (44) in Appendix A.9 is equal to $\left(8 - \frac{1}{2}\pi_{\gamma,3}\right)\delta$, γ_1 gets $L_2 = \delta/2$, and γ_2, not supposed to participate, gets zero. By contrast, if they switch to the quadrilateral-rivalry equilibrium from now on, α gets $A > 4\delta = V_2$, β gets B, γ_1 gets C, and γ_2 gets $D > 0$. While the change of equilibrium would make α and γ_2 better-off, it would make β and γ_1 worse-off: By Eq. (55) in Appendix A.9, $M_2 - B > 2\delta$, and by Eq. (54) in Appendix A.9, $L_2 - C > \delta/10$. Thus, given configuration (12), the quadrilateral equilibrium does not Pareto dominate the trilateral one. In any other configuration where a γ_2's leapfrogging is commonly seen as a deviation from the trilateral equilibrium, the position of γ_2 can only be lower (i.e., $t \geq 4$) and hence the leapfrog would cost him at least 5δ; since $A < 5\delta$, it is an unprofitable deviation for γ_2 even if the deviation could switch the equilibrium to the quadrilateral one. Thus, this quadrilateral equilibrium does not dynamically Pareto dominate the trilateral one.

The above example suggests that the effect of adding a third rival to bilateral rivalry might be more critical than adding an $(n+1)$th rival to an n-bidder play: while the bilateral-rivalry subgame equilibrium is surplus-dissipating, n-rivalry subgame equilibriums need not be so detrimental to the bidders.

5.2 Alternative Tie-Breaking with Multiple Frontrunners

In this section we consider an alternative tie-breaking rule: Suppose the auctioneer allows multiple frontrunners but that the item is only awarded when there is a single frontrunner and no more bids are placed; in cases with multiple frontrunners but no more bids are placed the auction ends without a winner; we denote this state as Λ. In other words, given current price p, if k bidders simultaneously bid then all k bidders become frontrunners and all are committed to paying $p + \delta$. As this tie-breaking rule generates further complexities to the game dynamics we restrict the discussion to cases with two and three bidders.

Bilateral Rivalry with Ties We start by considering the simplest case when there are only two bidders. Without loss of generality, label the bidders as Bidder 1 and 2, and consider the analysis from Bidder 1's perspective. Note that by the rule restricting the frontrunner from preemptively bidding, once the two bidders are arranged sequentially (i.e. a frontrunner and an immediate follower), they will never revert back to a situation where they are both frontrunners. However, if at any stage during the auction the two bidders are both frontrunners and both decide to bid, then they both remain frontrunners. We denote the state where both bidders are frontrunners as the *bilateral-tie* state.

From our previous results we have that in situations with two bidders arranged sequentially, the follower should bid with a probability of $1 - 2\delta/v$. This entails the frontrunner to have an expected value $V_* = 2\delta$, and the follower to have expected value $M_* = 0$. Consequently, Bidder 1's expected surplus in the bilateral-tie state is,

$$\begin{aligned} V_{\text{tie}} &= \rho^2(V_{\text{tie}} - \delta) + \rho(1-\rho)(V_* - \delta) + (1-\rho)\rho M_* + (1-\rho)^2\Lambda \\ &= \rho^2(V_{\text{tie}} - \delta) + \rho(1-\rho)\delta, \end{aligned}$$

where ρ denotes the bidding probability for each bidder in the bilateral-tie state at any symmetric equilibrium. Thus,

$$V_{\text{tie}} = \frac{\rho(1-2\rho)\delta}{1-\rho^2}. \tag{14}$$

At equilibrium, it is necessary that $0 < \rho < 1$. If $\rho = 0$, a bidder would rather deviate by bidding thereby getting a positive payoff instead of ending the game with zero payoff. If $\rho = 1$, then the bilateral-tie state is repeated in every round, rendering each bidder's payoff $-\infty$, again a contradiction to the equilibrium condition. Now that $0 < \rho < 1$, a bidder is indifferent between bidding and not bidding given a bilateral-tie state. If he does not bid, his expected payoff is equal to $M_* = 0$. If he bids, his expected payoff is equal to

$$\rho V_{\text{tie}} + (1-\rho)V_* - \delta = \rho V_{\text{tie}} + \delta - 2\rho\delta.$$

Thus, at any symmetric equilibrium,

$$\rho\left(V_{\text{tie}} - 2\delta\right) + \delta = 0.$$

This, combined with Eq. (14), implies

$$\rho\left(\frac{\rho(1-2\rho)\delta}{1-\rho^2} - 2\delta\right) + \delta = 0,$$

which is equivalent to $\rho = 1/2$, which implies $V_{\text{tie}} = 0$ by Eq. (14). Thus, the subgame starting from a bilateral-tie state admits a unique symmetric equilibrium, where each tying bidder bids with probability 1/2 and gets zero expected payoff. Consequently, from any bilateral-tie state there is a 1/4 chance the bilateral-tie state repeats, a 1/2 chance a frontrunner and follower emerges (and the auction subsequently will end with a winner), and a 1/4 chance the auction terminates without a winner.

Trilateral Rivalry with Ties Now consider the case with three bidders. Although there is only one *trilateral-tie* state with all three bidders as frontrunners, $[\alpha, \alpha, \alpha]$, and only one *follower-tie* state with two followers exactly one increment behind the frontrunner, $[\alpha, \beta, \beta]$, there is a multitude of tie-states with two frontrunners and the underdog behind by some lag. Furthermore, the various tie-states may all reoccur even from states where the three bidders are aligned in a consecutive configuration.[10] As above, without loss of generality, we label the bidders as Bidder 1, 2, and 3, and consider the analysis from Bidder 1's perspective. The following five states define the most pertinent tie-states (Bidder 1's position in encircled),

$$
\begin{aligned}
\text{State 0} &:= [\textcircled{\alpha}, \alpha, \alpha] \\
\text{State 1} &:= [\textcircled{\alpha}, \alpha, \beta] \\
\text{State 2} &:= [\textcircled{\alpha}, \beta, \beta] \\
\text{State 3} &:= [\alpha, \alpha, \textcircled{\beta}] \\
\text{State 4} &:= [\alpha, \textcircled{\beta}, \beta]
\end{aligned}
$$

The possible transitions from these five states are:

$$
\begin{aligned}
&[\textcircled{\alpha}, \alpha, \alpha] \Rightarrow && [\textcircled{\alpha}, \alpha, \alpha]; \quad [\textcircled{\alpha}, \alpha, \beta]; \quad [\textcircled{\alpha}, \beta, \beta]; \quad [\alpha, \alpha, \textcircled{\beta}]; \quad [\alpha, \textcircled{\beta}, \beta]; \quad \Lambda \qquad (15) \\
&[\textcircled{\alpha}, \alpha, \beta] \Rightarrow && [\textcircled{\alpha}, \alpha, \alpha]; \quad [\textcircled{\alpha}, \alpha, \beta]; \quad [\alpha, \alpha, \textcircled{\beta}]; \quad [\alpha, \textcircled{\beta}, \beta]; \quad [\textcircled{\alpha}, \alpha, \gamma]; \quad [\textcircled{\alpha}, \beta, \gamma]; \quad [\alpha, \textcircled{\beta}, \gamma]; \quad \Lambda \\
&[\textcircled{\alpha}, \beta, \beta] \Rightarrow && [\alpha, \alpha, \textcircled{\beta}]; \quad [\alpha, \textcircled{\beta}, \gamma]; \quad Winner \\
&[\alpha, \alpha, \textcircled{\beta}] \Rightarrow && [\textcircled{\alpha}, \alpha, \alpha]; \quad [\textcircled{\alpha}, \alpha, \beta]; \quad [\textcircled{\alpha}, \beta, \beta]; \quad [\alpha, \beta, \textcircled{\gamma}]; \quad [\alpha, \alpha, \textcircled{\gamma}]; \quad \Lambda \\
&[\alpha, \textcircled{\beta}, \beta] \Rightarrow && [\textcircled{\alpha}, \alpha, \beta]; \quad [\textcircled{\alpha}, \beta, \gamma]; \quad [\alpha, \beta, \textcircled{\gamma}]; \quad \Lambda
\end{aligned}
$$

[10] To see this, consider a scenario where the three bidders are aligned consecutively as $[\alpha, \beta, \gamma]$, and both the follower and underdog decide to bid, then the next configuration would be $[\alpha, \alpha, \beta]$. From this state all three bidders are permitted to bid and hence the trilateral-tie state $[\alpha, \alpha, \alpha]$ may occur.

Note that in state 2, Bidder 1 does not have a choice over bidding or not bidding, and cannot influence the transitions or chance of becoming the winner.

To use the result in the main paper and in the previous section, let us restrict attention, for now, to only bilateral-rivalry equilibriums such that—

> At the state $[\alpha, \alpha, \beta]$, the β player does not bid, and each of the two α-players bids with probability $1/2$, thereby engaged in the bilateral-rivalry subgame equilibrium characterized above.

The strategy profile described below constitutes such a subgame perfect equilibrium in the three-player game provided that $v/\delta \leq 8$ (proved in Section A.10):

a. At the state $[\alpha, \alpha, \alpha]$, each player bids with probability $1 - (\delta/v)^{1/2}$.

b. At the state $[\alpha, \alpha, \beta]$, the β player does not bid, and each of the two α-players bids with probability $1/2$, thereby engaged in the bilateral-rivalry subgame equilibrium characterized in the previous section.

c. At any state $[\alpha, \beta, \gamma]$, with the lag between γ and β bigger than or equal to δ, the γ player does not bid, and α- and β-bidders play the bilateral-rivalry subgame equilibrium characterized in our main paper.

d. At any state, a non-α player does not bid unless he is the β player in any state $[\alpha, \beta, \gamma]$ described in provision (c).

The above equilibrium relies on the assumption that $v/\delta \leq 8$. If $v/\delta > 8$, the β player in the state $[\alpha, \alpha, \textcircled{\beta}]$ would deviate to bidding, as in that case the Ineq. (65) is not true. We conjecture, and leave for future research to formally analyze, that a symmetric equilibrium would necessarily involve trilateral rivalry.

5.3 Asymmetric Information in Bilateral Rivalry

In the dollar auction we have been considering, assume that there are only two bidders. For each $i \in \{1, 2\}$, bidder i's type is drawn from a commonly known distribution F_i, absolutely continuous and strictly increasing on its support $[a_i, z_i]$, with $z_i > a_i \geq 0$. The realized type t_i of bidder i is i's private information at the outset; if b_i is the highest level among bidder i's committed bid, then i's payoff from the game is equal to $v - b_i/t_i$ if he wins the prize, and equal to $-b_i/t_i$ if he does not win it. Recall that δ denotes the exogenous increment price ascension. The tie-breaking rule is: if no one bids in the initial round, then the game ends with the good not sold; if exactly one player bids in the initial round, then the game ends with the good sold to the only bidder at the price equal to δ; else one of the two bidders is selected randomly with probability 1/2 to be the frontrunner in the second round, after which no tie will occur.

A perfect Bayesian equilibrium The idea is that at each round where a player is supposed to make a move, he bids if and only if his type is above a cutoff in the support of the posterior belief about his type, and the cutoff is so chosen that his opponent, now the frontrunner, would have been indifferent about bidding in the previous round if the opponent's type is equal to the opponent's cutoff in the previous round.

The cutoffs for the initial round: Let $(s_1^0, s_2^0) \in (a_1, z_1) \times (a_2, z_i)$ satisfy

$$\forall i \in \{1, 2\} : \frac{\delta}{v s_{-i}^0} = F_i(s_i^0). \tag{16}$$

For example, if F_i is the uniform distribution on $[0, 1]$ for each i, then $s_1^0 = s_2^0 = \sqrt{\delta/v}$ constitutes such a pair.

The cutoffs for the second round: For each $i \in \{1, 2\}$, define $s_i^1 \in (s_i^0, z_i)$ by

$$\frac{F_i(s_i^1) - F_i(s_i^0)}{1 - F_i(s_i^0)} = \frac{\delta}{v s_{-i}^0}. \tag{17}$$

The cutoffs for any round after the second one: For any $(s_1, s_2) \in (a_1, z_1) \times (a_2, z_i)$, if $[s_i, z_i]$ is the support of the posterior distribution of i's type at the start of this round for each $i \in \{1, 2\}$, then define the cutoff $s_i' \in (s_i, z_i)$ for each player i in this round by

$$\frac{F_i(s_i') - F_i(s_i)}{1 - F_i(s_i)} = \frac{2\delta}{v s_{-i}}. \tag{18}$$

The equilibrium: Initialize $s_i := a_i$ for each $i \in \{1, 2\}$.

a. In the initial round, for each player i of type t_i, i bids if and only if $t_i \geq s_i^0$. If player i bids then the posterior about i becomes $F_i(\cdot)/(1 - F_i(s_i^0))$, hence his infimum type is updated to $s_i := s_i^0$; else the game ends, with the good either sold at price δ to the other bidder if the latter has bid, or not sold if neither has bid, and hence there is no need for updating.

b. In the second round, with the frontrunner α selected among those who bid in the initial round, the follower β of type t_β bids if and only if $t_\beta \geq s^1_\beta$. If β does bid, then the posterior about bidder β becomes $F_\beta(\cdot)/(1 - F_\beta(s^1_\beta))$, hence his infimum type is updated to $s_\beta := s^1_\beta$; else the game ends and there is no need for updating. If β does not bid, the game ends and there is no need for updating.

c. If the game continues to any round after the second round, with s_i denoting the updated infimum type of player i at the start of the current round (hence the posterior distribution about i is $F_i(\cdot)/(1 - F_i(s_i))$), the current follower β of type t_β bids if and only if $t_\beta \geq s'_\beta$, where s'_β is derived from (s_1, s_2) by Eq. (18). If β does bid then the posterior about bidder β becomes $F_\beta(\cdot)/(1 - F_\beta(s'_\beta))$, hence his infimum type is updated to $s_\beta := s'_\beta$; else the game ends and there is no need for updating.

This equilibrium exhibits two interesting features. First, the allocation is not ex post efficient, as the winner need not be the bidder with the stronger realized type. Second, in the case wher $z_1 < a_2$, it is commonly known ex ante that bidder 1 is weaker than bidder 2, yet at the equilibrium their bidding competition may escalate for many rounds, especially when their realized types are near the corresponding supremums.

Verification of the equilibrium At the start of any round after the initial one, let (s_1, s_2) denote the pair of current updated type infimums of the two players and, for any $i \in \{1, 2\}$, let $M_i(t_i|s_i, s_{-i})$ denote the expected payoff for player i of type t_i if i is the current follower, given the continuation equilibrium described above. Then

$$M_i(t_i|s_i, s_{-i}) = \max\left\{0, -\frac{2\delta}{t_i} + V_i(t_i|s'_i, s_{-i})\right\}, \tag{19}$$

note that $V_i(t_i|s'_i, s_{-i})$ denotes i's expected payoff from being the frontrunner in the next round, with his type infimum updated to s'_i, derived from (s_i, s_{-i}) by Eq. (18). In general, $V_i(t_i|s_i, s_{-i})$ denotes the expected payoff for player i of type t_i if i is the current frontrunner in any round after the second one such that the updated type infimums at the start of the current round are s_i and s_{-i} respectively. Then

$$\begin{aligned} V_i(t_i|s_i, s_{-i}) &= \frac{F_{-i}(s'_{-i}) - F_{-i}(s_{-i})}{1 - F_{-i}(s_{-i})} v + \frac{1 - F_{-i}(s'_{-i})}{1 - F_{-i}(s_{-i})} M_i(t_i|s_i, s'_{-i}) \\ &= \frac{2\delta}{s_i} + \left(1 - \frac{2\delta}{s_i}\right) M_i(t_i|s_i, s'_{-i}), \end{aligned} \tag{20}$$

where s'_{-i} is derived from (s'_i, s_{-i}) by Eq. (18), and the second line follows from Eq. (18), with the roles of i and $-i$ switched. By contrast, if, after both players bid in the initial round, player i is selected the frontrunner in the second round, then i's expected payoff, viewed at the start of this round, is equal to

$$\begin{aligned} V_i^0(t_i|s_i^0, s_{-i}^0) &= \frac{F_{-i}(s_{-i}^1) - F_{-i}(s_{-i}^0)}{1 - F_{-i}(s_{-i}^0)} v + \frac{1 - F_{-i}(s_{-i}^1)}{1 - F_{-i}(s_{-i}^0)} M_i(t_i|s_i^0, s_{-i}^1) \\ &= \frac{\delta}{s_i^0} + \left(1 - \frac{\delta}{s_i^0}\right) M_i(t_i|s_i^0, s_{-i}^1), \end{aligned} \tag{21}$$

where the second line follows from Eq. (17), with the roles of i and $-i$ switched. Based on the Bellman equations (19), (20) and (21), one can prove that the above-described bidding strategy and updating rule constitute a perfect Bayesian equilibrium (Section A.11).

6 Conclusion

Wars of attrition are suitable to model R&D races, lobbying, bargaining, and escalating conflicts between political parties and between superpowers, but the extant literature on dynamics within such games has been restricted by the two-rival assumption. Removing this assumption, this paper analyzes a dynamic war of attrition, the dollar auction, and demonstrates a pivotal effect of adding a third rival: In any equilibrium where only two rivals are expected to emerge, once they emerge the play is locked into a surplus-dissipating escalation between the two; such a detrimental outcome can be avoided if the players switch to any equilibrium where the escalation involves all three players. Not only does the latter equilibrium Pareto dominate the former thereafter, it also gives the third-place rival a strict incentive to leapfrog to the top, at the onset of the detrimental bilateral rivalry, thereby to convince the other rivals to switch away from the surplus-dissipating bilateral rivalry. This normative result suggests that adding a viable third political party to a two-party system such as the United States may help to mitigate the more and more acute conflict between the two sides.

To obtain the above normative result, this paper fully characterizes the trilateral-rivalry equilibriums with a recursive method. Any such trilateral equilibrium exhibits three interesting, dynamic features. First, as the trilateral escalation continues, the gap between the frontrunner and the third-place rival may collapse or expand, depending on whether the third-place rival manages to leapfrog thereby replacing the frontrunner. Second, the escalation may end only when this gap reaches its maximum that the equilibrium can sustain, at which point it is the current third-place rival that decides, through his decision on the leapfrog, whether the escalation shall continue. Third, at the onset of a trilateral rivalry, the ideal position for a player is to be the follower, wedged in between the frontrunner and the third-place rival, rather than the frontrunner (Footnote 6).

While the paper focuses on the three-rival case, our trilateral-rivalry equilibriums can be used as subgame plays for the construction of equilibriums involving more than three rivals. This we illustrate with a quadrilateral-rivalry equilibrium as an extension case. In that example, adding a fourth rival to trilateral rivalry does not have the pivotal effect as a third rival to bilateral rivalry. It is an open question for future research What is the general effect of adding an $(n+1)$th rival to an n-rival escalation.

For simplicity the paper uses the tie-breaking rule in Shubik's original formulation of the dollar auction, which allows for only one frontrunner at each round. This paper suggests with an example on how our recursive method may be applied to an alternative tie-breaking rule that allows for multiple frontrunners, which may be more realistic in applications to lobbying and R&D races.

To focus on the three-rival dynamics, this paper assumes complete information. Our recursive method, however, is at least partially extendable to asymmetric information. This we illustrate in the paper with a perfect Bayesian equilibrium in a two-rival case where a rival's private information is his marginal cost of monetary payments.

A Proofs

A.1 Lemmas 1 and 2

Lemma 1 By definition of L_s, the equilibrium expected payoff for an underdog whose lag from the frontrunner is s, we know that $L_s = 0$ for all $s \geq v/\delta$. Starting from any such s and use backward induction towards smaller s, together with the law of motion (7) and the fact $V_2 - (s+1)\delta < 0$ for all $s \geq s_*$ due to the definition of s_*, we observe that $L_s = 0$ for all $s \geq s_*$. At any state $s \geq s_*$, by (7), an underdog gets zero expected payoff if he does not bid; if he bids then by Eq. (2) there is a positive probability with which he gets a negative payoff $V_2 - (s+1)\delta$; hence his best response is uniquely to not bid at all. Hence

$$s \geq s_* \implies L_s = 0 \text{ and } \pi_{\gamma,s} = q_{\gamma,s} = 0, \tag{22}$$

which proves Claim (i) of the lemma. Apply backward induction to (7) starting from $s = s_*$ and we obtain

$$2 \leq s \leq s_* - 1 \Longrightarrow V_2 - (s+1)\delta \geq L_s \geq L_{s+1} \geq 0, \tag{23}$$

with the inequality $L_s \geq L_{s+1}$ being strict whenever $s < s_* - 1$. Thus, for any $s < s_* - 1$, $V_s - (s+1)\delta > L_{s+1} \geq 0$; hence Eqs. (2) and (7) together imply that an underdog's best response is uniquely to bid for sure:

$$2 \leq s < s_* - 1 \implies L_s > 0 \text{ and } \pi_{\gamma,s} = 1, \tag{24}$$

which proves Claim (ii) of the lemma. ■

Lemma 2 Take any equilibrium, with dropout state s_* and value functions V_s, M_s and L_s. By Lemma 1.i, at any state $s \geq s_*$ the player who is the current underdog stays put for all future rounds, and hence the independence condition of irrelevant players implies that in any subgame give s the equilibrium strategy profile for the remaining two players, the current frontrunner α and the follower β, satisfies the symmetry and recursion conditions as if the two constitute the entire set of players. The two conditions together imply that in any such subgame a remaining player's strategy depends only on his current role as either the α or the β, regardless of his name or the amount of his committed payment. Thus, there exist constants $(V_*, M_*, y) \in \mathbb{R}^2 \times [0,1]$ such that $V_* = V_s$, $M_* = M_s$ and $y = \pi_{\beta,s}$ for all $s \geq s_*$. Then the Bellman equations are

$$\begin{aligned} V_* &= (1-y)v + yM_*, \\ M_* &= y(-2\delta + V_*). \end{aligned}$$

Note that $y > 0$, otherwise $V_* = v$ and $M_* = 0$; with $v > 2\delta$ by assumption, the current β player would bid for sure, so the two players are trapped in an infinite bidding loop and each get zero payoff. Also note $y < 1$, otherwise $V_* = 0$ and $M_* = -2\delta$, violating individual rationality. Now that $0 < y < 1$, the β player is indifferent about bidding, hence $M_* = 0$. This combined with the Bellman equations uniquely pins down the subgame equilibrium as $V_* = 2\delta$, $M_* = 0$ and $y = 1 - 2\delta/v$, which is exactly the zero-surplus subgame equilibrium. Hence $V_s = V_* = 2\delta$ and $M_s = M_* = 0$. Since (22) implies $L_s = 0$, the lemma is proved. ■

A.2 Lemma 3 and Theorem 1

To prove Lemma 3 we make several observations first. By (7) and (23), L_2 is a convex combination between L_3 and $V_2 - 3\delta$, with $V_2 - 3\delta \geq L_3$ when $s_* \geq 3$. Thus,

$$s_* \geq 3 \Longrightarrow L_2 \leq V_2 - 3\delta. \tag{25}$$

Lemma 2, combined with (6) and (7), implies

$$M_{s_*-1} = q_{\gamma,s_*-1} L_2 \overset{(25)}{\leq} (V_2 - 3\delta)^+ . \tag{26}$$

Lemma 6 *There does not exist an equilibrium with dropout state $s_* = 3$*

Proof Suppose, to the contrary, that $s_* = 3$. Hence $0 \leq V_2 - 3\delta < \delta$. Thus, by Eq. (26), $M_2 < \delta$. Then (3) requires that $\pi_1 < 1$, otherwise $V_1 = M_2 < \delta$, implying a contradiction that no one would bear the sunk cost δ to become the initial α player. Now consider the decision of any non-α player at the state $s = 1$, as depicted by (4). Since $V_2 - 2\delta > V_2 - 3\delta \geq L_2$, with the second inequality due to (25), each non-α player at $s = 1$ would maximize the probability of becoming the α in the next round, i.e., $\pi_1 = 1$, contradiction. ■

Lemma 7 *At any equilibrium with dropout state $s_* \geq 4$, $V_3 - 2\delta \geq M_2 \geq L_2 > 0$.*

Proof Suppose that $V_3 - 2\delta < L_2$. Then, by the fact $\pi_{\gamma,2} = 1$ (Lemma 1.ii and $s_* \geq 4$) and Eq. (2), the β player at state $s = 2$ would rather stay put than bid, hence $\pi_{\beta,2} = 0$. This, combined with (5) in the case $s = 2$ and the fact $\pi_{\gamma,2} = 1$, implies that $V_2 = M_2$. Since $V_3 - 2\delta < L_2$ coupled with (6) implies $M_2 \leq L_2$, we have a contradiction $V_2 \leq L_2 < V_2$, with the last inequality due to (7). Thus we have proved $V_3 - 2\delta \geq L_2$. Therefore, with M_2 a convex combination between $V_3 - 2\delta$ and L_2 (since $\pi_{\gamma,2} = 1$), $V_3 - 2\delta \geq M_2 \geq L_2$. Finally, to show $L_2 > 0$, note from the hypothesis $s_* \geq 4$ and definition of s_* that $V_2 - 3\delta > 0$. This positive payoff the underdog at state $s = 2$ can secure with a positive probability through bidding. Hence $L_2 > 0$ follows from (7). ■

Lemma 8 *At any equilibrium with dropout state $s_* \geq 4$, $\pi_{\gamma,s_*-1} > 0$.*

Proof Suppose, to the contrary, that $\pi_{\gamma,s_*-1} = 0$ at equilibrium. Then $M_{s_*-1} = 0$ according to (6), with $s = s_* - 1$, and the fact $V_{s_*} - 2\delta = 0$ by Lemma 2. Consequently, (5) applied to the case $s = s_* - 2$, coupled with the fact $\pi_{\gamma,s_*-2} = 1$ (Lemma 1.i), implies that $V_{s_*-2} \leq M_2$, which in turn implies, by (6) in the case $s = s_* - 3$ and the fact $\pi_{\gamma,s_*-3} = 1$, that $M_{s_*-3} \leq \max\{M_2 - 2\delta, L_2\} \leq M_2$, with the last inequality due to Lemma 7. That in turn implies $V_{s_*-4} \leq M_2$ by (5) and the fact $\pi_{\gamma,s_*-4} = 1$. Thus $V_{s_*-2} \leq M_2$, $V_{s_*-4} \leq M_2$ and $M_{s_*-3} \leq M_2$.

The supposition $\pi_{\gamma,s_*-1} = 0$, coupled with the fact that $\pi_{\gamma,s} = 0$ at all $s > s_* - 1$ (Lemma 1.ii), also implies that α drops out of the race starting from the state $s_* - 1$. Thus, by the independence condition of irrelevant players, $V_{s_*-1} = V_{s_*}$, hence Lemma 2 implies $V_{s_*-1} = 2\delta$. Then (6) applied to the case $s = s_* - 2$, coupled with the fact $\pi_{\gamma,s_*-2} = 1$, implies $M_{s_*-2} \leq L_2$. Thus, by (5) and the fact $\pi_{\gamma,s_*-3} = 1$, we have $V_{s_*-3} \leq \max\{L_2, M_2\} \leq M_2$, the last inequality again due to Lemma 7. With $V_{s_*-3} \leq M_2$, (6) implies $M_{s_*-4} \leq \max\{M_2 - 2\delta, L_2\} \leq M_2$. Thus $V_{s_*-1} = 2\delta$, $V_{s_*-3} \leq M_2$, $M_{s_*-2} \leq L_2 \leq M_2$ and $M_{s_*-4} \leq M_2$.

Repeat the above reasoning on (5) and (6) for smaller and smaller s and we obtain the fact that $V_{s_*-1} = 2\delta$, $V_s \leq M_2$ and $M_s \leq M_2$ for all $s \leq s_* - 2$. Thus, $V_3 \leq \max\{M_2, 2\delta\}$, which contradicts Lemma 7. ■

Proof of Lemma 3 Since $s_* \geq 4$ by Lemma 6, $L_2 > 0$ by Lemma 7. Thus, for the β player at $s = s_* - 1$, depicted by (6), given the fact $V_{s_*} - 2\delta = 0$ by Lemma 2 and the fact that $L_2 > 0$ and $\pi_{\gamma,s_*-1} > 0$ (Lemma 8), it is the unique best response to not bid at all, i.e., $\pi_{\beta,s_*-1} = 0$. Thus, the β player stays put for sure at state $s_* - 1$, as the lemma asserts.

Next we show that $0 < \pi_{\gamma,s_*-1} < 1$. The first inequality is implied by Lemma 8 since $s_* \geq 4$. To prove $\pi_{\gamma,s_*-1} < 1$, suppose to the contrary that $\pi_{\gamma,s_*-1} = 1$. Then by the fact $\pi_{\beta,s_*-1} = 0$ and (5) applied to the case $s = s_* - 1$, we have $V_{s_*-1} = M_2$. Consequently, by (6) applied to the case $s = s_* - 2$, $M_{s_*-2} \leq \max\{M_2 - 2\delta, L_2\} \leq M_2$, with the last inequality due to Lemma 7. The supposition $\pi_{\gamma,s_*-1} = 1$ also implies $M_{s_*-1} = L_2$, which in turn implies, via (5) in the case $s = s_* - 2$, that $V_{s_*-2} \leq \max\{L_2, M_2\} \leq M_2$, the last inequality again due to Lemma 7. Then (6) for the case $s = s_* - 3$ implies $M_{s_*-3} \leq \max\{M_2 - 2\delta, L_2\} \leq M_2$, and (5) implies $V_{s_*-3} \leq \max\{M_{s_*-2}, M_2\} \leq M_2$. Repeat the above reasoning on smaller s and we prove that $V_s \leq M_2$ for all $s \leq s_* - 1$. Hence $V_3 \leq M_2$, which contradicts Lemma 7. Thus we have proved that $\pi_{\gamma,s_*-1} < 1$.

With $\pi_{\gamma,s_*-1} < 1$, bidding is not the unique best response for the γ player at state $s_* - 1$, hence $V_2 \leq s_*\delta$ (otherwise the bottom branch of (7) in the case $s = s_* - 1$ is strictly positive and, by (22), is strictly larger than the middle branch, so the γ player would strictly prefer to bid). By definition of s_*, $V_2 \geq s_*\delta$. Thus $V_2 = s_*\delta$. ∎

Proof of Theorem 1 Suppose, to the contrary, that there is an equilibrium with dropout state s_* an odd number. Since $s_* = 3$ is impossible by Lemma 6 and $s_* = 1$ meaningless in our model, $s_* \geq 5$. By Lemma 7, $L_2 \leq M_2 \leq V_3 - 2\delta$. Consider (5) in the case $s = s_* - 2$ together with the facts that $\pi_{\gamma,s_*-2} = 1$ (thereby ruling out $V_{s_*-2} \to v$) due to Lemma 1.ii and $s_* \geq 5$, that $M_{s_*-1} \leq L_2$ due to (26), and that $M_2 \leq V_3 - 2\delta$. Thus we have $V_{s_*-2} \leq V_3 - 2\delta$. Then consider the decision of the β player at state $s = s_* - 3$, depicted by (6), to observe that M_{s_*-3} is between L_2 and $V_3 - 4\delta$. Thus, by (5) applied to the case $s = s_* - 4$, together with the facts $\pi_{\gamma,s_*-4} = 1$ and $M_2 \leq V_3 - 2\delta$, we have $V_{s_*-4} \leq V_3 - 2\delta$. Since s_* is an odd number and $s_* \geq 5$, this procedure of backward reasoning eventually reaches V_3, i.e., $3 = s_* - 2m$ for some positive integer m. Hence we obtain the contradiction $V_3 \leq V_3 - 2\delta$. ∎

A.3 Lemma 4

Lemma 4 follows from Lemmas 10 and 12, the former showing that bidding is a follower's unique best response to an equilibrium at even-number states, and the latter, odd-number states. We start with—

Lemma 9 *At any equilibrium with dropout state an even number $s_* \geq 4$, $L_2 < V_2 \leq V_3 - 2\delta$.*

Proof Since $\pi_{\beta,s_*-1} = 0$ (Lemma 3), $M_{s_*-1} \leq L_2$. Thus, since $\pi_{\gamma,s_*-2} = 1$ (Lemma 1.ii), V_{s_*-2} is a convex combination between M_{s_*-1}, which is less than L_2, and M_2, which is a convex combination between $V_3 - 2\delta$ and L_2, as $\pi_{\gamma,2} = 1$. Thus V_{s_*-2} is between L_2 and $V_3 - 2\delta$. Consequently, M_{s_*-3}, a convex combination between L_2 and $V_{s_*-2} - 2\delta$ (since $\pi_{\gamma,s_*-3} = 1$), is between L_2 and $V_3 - 2\delta$. Repeating this reasoning, with s_* being an even number, we eventually reach $2 = s_* - 2m$ for some integer $m \geq 1$, and obtain the fact that V_2 is a number between L_2 and $V_3 - 2\delta$. Thus, $L_2 < V_3 - 2\delta$, otherwise the fact $L_2 < V_2$ by (7) would be contradicted. Hence $L_2 < V_2 \leq V_3 - 2\delta$. ∎

A.3.1 Bidding at Even States

Lemma 10 *At any equilibrium with any even number dropout state $s_* \geq 4$, $\pi_{\beta,s} = 1$ if $2 \leq s \leq s_* - 2$ such that s is an even number.*

Proof First, by Lemma 9, $L_2 < V_3 - 2\delta$. Thus at state $s = 2$ the β player strictly prefers to bid, i.e., $\pi_{\beta,2} = 1$. Second, pick any even number s such that $4 \le s \le s_* - 2$ and suppose, to the contrary of the lemma, that $\pi_{\beta,s} < 1$, which means that the β player at state s does not strictly prefer to bid. Thus $M_s \le L_2$ (as the transition $M_s \to 0$ is ruled out by the fact $\pi_{\gamma,s} = 1$). Consequently, V_{s-1}, a convex combination between M_s and M_2, is weakly less than M_2, as $L_2 \le M_2$ by Lemma 7. Furthermore, M_{s-2}, a convex combination between $V_{s-1} - 2\delta$ and L_2, is less than M_2, and that in turns implies $V_{s-3} \le M_2$. Repeating this reasoning, with s an even number, we eventually obtain the conclusion that $V_3 \le M_2$, which contradicts Lemma 7. Thus, $\pi_{\beta,s} = 1$. ■

At any equilibrium with any even number dropout state $s_* \ge 4$, since $\pi_{\gamma,s} = 1$ for all $s \le s_* - 2$ (Lemma 1.ii), Eq. (2) and the equal-probability tie-breaking rule together imply

$$\forall s \in \{2, 3, 4, \ldots, s_* - 2\} : [\pi_{\beta,s} = 1 \Longrightarrow q_{\beta,s} = q_{\gamma,s} = 1/2] \,. \tag{27}$$

By Lemma 10,

$$2 \le s \le s_* - 2 \text{ and } s \text{ is even} \Longrightarrow q_{\beta,s} = q_{\gamma,s} = 1/2. \tag{28}$$

A.3.2 Bidding at Odd States

In the following, we extend the summation notation by defining, for any sequence $(a_k)_{k=1}^{\infty}$,

$$i > j \Longrightarrow \sum_{k=i}^{j} a_k := 0. \tag{29}$$

In particular, $\sum_{k=1}^{0} a_k = 0$ according to this notation.

Lemma 11 *At any equilibrium with any even number dropout state $s_* \ge 4$, $M_2 > V_2 + \delta/2$.*

Proof Let $m := \min\{k \in \{0, 1, 2 \ldots\} : V_{2k+4} - 2\delta \le L_2\}$. Note that m is well-defined because $s_*/2 - 2$ belongs to the set, as $V_{s_*} - 2\delta = 0 \le L_2$ (Lemma 2). At any odd state $2k + 1 \le 2m + 1$ (hence $k - 1 < m$) we have $V_{2k+2} - 2\delta = V_{2(k-1)+4)} - 2\delta > L_2$, with the last inequality due to the definition of m; hence by (5) in the state $s = 2k + 1$ the β player bids for sure, i.e. $\pi_{\beta,2k+1} = 1$. Thus, (27) implies that $q_{\beta,s} = q_{\gamma,s} = 1/2$ at any such odd state. Coupled with (28), that means the transition at every state s from 2 to $2m + 2$ is that the current β and γ players each have probability 1/2 to become the next α player. Thus,

$$V_2 = M_2 \sum_{k=0}^{m} 2^{-2k-1} + L_2 \left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2} z_m \right) - 2\delta \sum_{k=1}^{m} 2^{-2k}, \tag{30}$$

where $z_m := 1$ if $2m+2 < s_*-2$, and $z_m := 2\pi_{\gamma,s_*-1}-1$ if $2m+2 = s_*-2$; and the last series $\sum_{k=1}^{m}$ on the right-hand side uses the summation notation defined in (29) when $m = 0$.

To understand the term for M_2 on the right-hand side, note that M_2 enters the calculation of V_2 at the even states $s = 2, 4, 6, \ldots, 2m-2$, and upon entry at state s and in every round transversing from states s to 2, the M_2 is discounted by the transition probability $1/2$. The term for L_2 is similar, except that L_2 enters at the odd states $s = 3, 5, 7, \ldots, 2m-1$, and that the transition probability for the L_2 at the last state $2m-1$ is equal to one if $2m-1 < s_*-1$, and equal to π_{γ,s_*-1} if $2m-1 = s_*-1$. That is why the last two terms within the bracket for L_2 are

$$2^{-2m-2}+2^{-2m-2}z_m = \begin{cases} 2^{-2m-2}+2^{-2m-2} = 2^{-2m-1} & \text{if } z_m = 1 \\ 2^{-2m-2}+2^{-2m-2}\left(2\pi_{\gamma,s_*-1}-1\right) = 2^{-2m-1}\pi_{\gamma,s_*-1} & \text{if } z_m = 2\pi_{\gamma,s_*-1}-1. \end{cases}$$

The term for -2δ is analogous to that for M_2.

With $s_* \geq 4$, $V_2 - 4\delta \geq 0$. Thus, by the above-calculated transition probabilities,

$$L_2 = \frac{1}{2}(L_3+V_2-3\delta) \leq \frac{1}{2}(V_2-4\delta+V_2-3\delta) = V_2 - \frac{7}{2}\delta.$$

This, combined with Eq. (30) and the fact $z_m \leq 1$ due to its definition, implies that

$$\begin{aligned} V_2 &\leq M_2\sum_{k=0}^{m}2^{-2k-1} + \left(V_2-\frac{7}{2}\delta\right)\left(\sum_{k=0}^{m}2^{-2k-2}+2^{-2m-2}\right) - 2\delta\sum_{k=1}^{m}2^{-2k} \\ &< M_2\sum_{k=0}^{m}2^{-2k-1} + V_2\left(\sum_{k=0}^{m}2^{-2k-2}+2^{-2m-2}\right) - \frac{7}{8}\delta. \end{aligned}$$

Thus, the lemma is proved if

$$1-\left(\sum_{k=0}^{m}2^{-2k-2}+2^{-2m-2}\right) = \sum_{k=0}^{m}2^{-2k-1}, \tag{31}$$

as the left-hand side of this equation is clearly strictly between zero and one. To prove (31), we use induction on m. When $m = 0$, (31) becomes $1-2^{-2}-2^{-2} = 2^{-1}$, which is true. For any $m = 0, 1, 2, \ldots$, suppose that (31) is true. We shall prove that the equation is true when m is replaced by $m+1$, i.e.,

$$1-\left(\sum_{k=0}^{m+1}2^{-2k-2}+2^{-2(m+1)-2}\right) = \sum_{k=0}^{m+1}2^{-2k-1}. \tag{32}$$

The left-hand side of (32) is equal to

$$\begin{aligned}
& 1-\left(\sum_{k=0}^{m} 2^{-2k-2}+2^{-2m-2}\right)+2^{-2m-2}-2^{-2(m+1)-2}-2^{-2(m+1)-2} \\
= & \sum_{k=0}^{m} 2^{-2k-1}+2^{-2m-2}-2^{-2(m+1)-1} \quad \text{(the induction hypothesis)} \\
= & \sum_{k=0}^{m} 2^{-2k-1}+2^{-2m-3},
\end{aligned}$$

which is equal to the right-hand side of (32). Thus (31) is true in general, as desired. ■

Lemma 12 *At any equilibrium with any even number dropout state $s_* \geq 4$ and at any state $1 \leq s \leq s_* - 2$ such that s is an odd number, $\pi_{\beta,s} = 1$.*

Proof Pick any odd number s such that $s \leq s_* - 2$. It suffices to prove that $V_{s+1} - 2\delta > L_2$. Since $s+1$ is even, it follows from (28) that

$$V_{s+1} = \frac{1}{2}\left(M_2 + M_{s+2}\right) \geq \frac{1}{2}\left(M_2 + L_2\right),$$

with the inequality due to the fact $M_{s+2} \geq L_2$, which in turn is due to the fact that the β player at state $s+2$ can always secure the payoff L_2 through not bidding at all. Thus,

$$\begin{aligned}
V_{s+1} - 2\delta - L_2 & \geq \frac{1}{2}\left(M_2 + L_2\right) - 2\delta - L_2 \\
& = \frac{1}{2}M_2 - \frac{1}{2}L_2 - 2\delta \\
& = \frac{1}{2}M_2 - \frac{1}{2}\left(\frac{1}{2}L_3 + \frac{1}{2}(V_2 - 3\delta)\right) - 2\delta \\
& \geq \frac{1}{2}M_2 - \frac{1}{2}\left(\frac{1}{2}(V_2 - 4\delta) + \frac{1}{2}(V_2 - 3\delta)\right) - 2\delta \\
& = \frac{1}{2}M_2 - \frac{1}{2}V_2 - \frac{1}{4}\delta,
\end{aligned}$$

with the second inequality due to the definition of L_s and the fact $V_2 - 4\delta \geq 0$ ($s_* \geq 4$). Since $\frac{1}{2}M_2 - \frac{1}{2}V_2 - \frac{1}{4}\delta > 0$ by Lemma 11, $V_{s+1} - 2\delta - L_2 > 0$, as desired. ■

A.4 Lemma 5

All lemmas in this subsection assume the hypotheses in Lemma 5, that $s_* \geq 4$ is an even number and a strategy profile $\left(\pi_0, \pi_1, (\pi_{\beta,s}, \pi_{\gamma,s})_{s=2}^{\infty}\right)$ satisfying Condition $(*)$ is given, with the associated value functions $\left(V_s, M_s, L_s\right)_s$ derived from (5)–(7) and Eq. (8).

Lemma 13 *For any positive integer m such that $2m+1 \le s_* - 1$, if $V_{2m+1} - 2\delta \le L_2$ then $V_3 - 2\delta < L_2$.*

Proof Pick any m specified by the hypothesis such that $V_{2m+1} - 2\delta \le L_2$. Suppose, to the contrary of the lemma, that $V_3 - 2\delta \ge L_2$. Thus, the law of motion (5) in the case $s = 2$, with $\pi_{\gamma,2} = 1$, implies that M_2 is between L_2 and $V_3 - 2\delta$, hence $V_3 - 2\delta \ge M_2 \ge L_2$. By the law of motion (6) in the case $s = 2m$, M_{2m} is a convex combination among zero, $V_{2m+1} - 2\delta$ and L_2. Thus the hypothesis implies that $M_{2m} \le L_2$. Consequently, the law of motion (5) in the case $s = 2m - 1$, together with $\pi_{\gamma,2m-1} = 1$ and $M_2 \ge L_2$, implies that $V_{2m-1} \le M_2$ and hence $V_{2m-1} - 2\delta \le M_2 - 2\delta$. Then (6) in the case $s = 2m - 2$ implies $M_{2m-2} \le L_2$. Repeating this reasoning backward, with 3 being odd, we eventually reach state $s = 3$ and obtain $V_3 \le M_2$. But since $V_3 - 2\delta \ge M_2$, we have a contradiction $V_3 - 2\delta \ge M_2 \ge V_3$. ■

Lemma 14 *Denote $x := \pi_{\gamma,s_*-1}$. For any integer m such that $1 \le m \le s_*/2 - 1$,*

$$M_{s_*-(2m-1)} = -\delta \sum_{k=1}^{m-1} 2^{-2k+2} + M_2 \sum_{k=1}^{m-1} 2^{-2k} + L_2 \left(\sum_{k=1}^{m-1} 2^{-2k+1} + 2^{-2(m-1)} x \right), \tag{33}$$

$$V_{s_*-2m} = -\delta \sum_{k=1}^{m-1} 2^{-2k+1} + M_2 \sum_{k=1}^{m} 2^{-2k+1} + L_2 \left(\sum_{k=1}^{m-1} 2^{-2k} + 2^{-2m+1} x \right), \tag{34}$$

$$V_{s_*-(2m-1)} = -\delta \sum_{k=1}^{m-1} 2^{-2k+1} + 2^{-2(m-1)}(1-x)v + L_2 \sum_{k=1}^{m-1} 2^{-2k} + M_2 \left(\sum_{k=1}^{m-1} 2^{-2k+1} + 2^{-2(m-1)} x \right), \tag{35}$$

$$M_{s_*-2m} = -\delta \sum_{k=0}^{m-1} 2^{-2k} + 2^{-2m+1}(1-x)v + L_2 \sum_{k=1}^{m} 2^{-2k+1} + M_2 \left(\sum_{k=1}^{m-1} 2^{-2k} + 2^{-2m+1} x \right), \tag{36}$$

$$L_2 = \delta \left(s_* - 4 + 2^{-s_*+3} \right). \tag{37}$$

Proof First, we prove Eqs. (33) and (34). When $m = 1$, Eq. (33), coupled with the summation notation defined in (29), becomes $M_{s_*-1} = xL_2 = \pi_{\gamma,s_*-1} L_2$, which follows from (6) and the fact that $V_s = 2\delta$ and $M_s = 0$ for all $s \ge s_*$, due to Condition $(*)$. This coupled with Eq. (8) implies that

$$V_{s_*-2} = (M_{s_*-1} + M_2)/2 = M_2/2 + xL_2/2,$$

which is Eq. (34) when $m = 1$ (using again the summation notation in (29)). Suppose, for any integer m' with $1 \leq m' \leq s_*/2 - 2$, that Eqs. (33) and (34) are true with $m = m'$. By the induction hypothesis of (34) and Eq. (8),

$$\begin{aligned} M_{s_*-(2m'+1)} &= \frac{1}{2}\left(V_{s_*-2m'} - 2\delta + L_2\right) \\ &= -\delta\left(1 + \frac{1}{2}\sum_{k=1}^{m'-1} 2^{-2k+1}\right) + \frac{M_2}{2}\sum_{k=1}^{m'} 2^{-2k+1} + \frac{L_2}{2}\left(1 + \sum_{k=1}^{m'-1} 2^{-2k} + 2^{-2m'+1}x\right), \end{aligned}$$

which is Eq. (33) when $m = m' + 1$. By the above calculation of $M_{s_*-(2m'+1)}$ and Eq. (8),

$$\begin{aligned} V_{s_*-(2m'+2)} &= \frac{1}{2}\left(M_{s_*-(2m'+1)} + M_2\right) \\ &= -\frac{\delta}{2}\left(1 + \frac{1}{2}\sum_{k=1}^{m'-1} 2^{-2k+1}\right) + \frac{M_2}{2}\left(1 + \sum_{k=1}^{m'} 2^{-2k+1}\right) \\ &\quad + \frac{L_2}{4}\left(1 + \sum_{k=1}^{m'-1} 2^{-2k} + 2^{-2m'+1}x\right), \end{aligned}$$

which is Eq. (34) in the case $m = m' + 1$. Thus Eqs. (33) and (34) are proved.

Next we prove Eqs. (35) and (36). When $m = 1$, Eq. (35), coupled with the notation $\sum_{k=1}^{0} a_k = 0$, becomes $V_{s_*-1} = (1-x)v + xM_2$, which is true by definition of x and the fact $\pi_{\beta, s_*-1} = 0$ (Condition $(*)$). Then by Eq. (8)

$$M_{s_*-2} = \left(V_{s_*-1} - 2\delta + L_2\right)/2 = \left((1-x)v + xM_2 - 2\delta + L_2\right)/2,$$

which is Eq. (36) when $m = 1$ (again using the notation $\sum_{k=1}^{0} a_k = 0$). Suppose, for any integer m' with $1 \leq m' \leq s_*/2 - 2$, that Eqs. (35) and (36) are true with $m = m'$. By the induction hypothesis and Eq. (8),

$$\begin{aligned} V_{s_*-(2m'+1)} &= \frac{1}{2}\left(M_{s_*-2m'} + M_2\right) \\ &= -\frac{\delta}{2}\sum_{k=0}^{m'-1} 2^{-2k} + 2^{-1}2^{-2m'+1}(1-x)v + \frac{L_2}{2}\sum_{k=1}^{m'} 2^{-2k+1} \\ &\quad + M_2\left(2^{-1} + 2^{-1}\sum_{k=1}^{m'-1} 2^{-2k} + 2^{-1}2^{-2m'+1}x\right), \end{aligned}$$

which is Eq. (35) in the case $m = m' + 1$. By the above calculation and Eq. (8),

$$\begin{aligned} M_{s_*-(2m'+2)} &= \frac{1}{2}\left(V_{s_*-(2m'+1)} - 2\delta + L_2\right) \\ &= -\delta\left(1 + \frac{1}{2}\sum_{k=1}^{m'} 2^{-2k+1}\right) + 2^{-1}2^{-2m'}(1-x)v \\ &\quad + L_2\left(\frac{1}{2} + 2^{-1}\sum_{k=1}^{m'} 2^{-2k}\right) + \frac{M_2}{2}\left(\sum_{k=1}^{m'} 2^{-2k+1} + 2^{-2m'}x\right), \end{aligned}$$

which is Eq. (36) in the case $m = m' + 1$. Hence Eqs. (35) and (36) are proved.

Finally we prove Eq. (37). Applying Eq. (8) to (7) recursively we obtain, for any integer $s_* \geq 4$, that

$$\begin{aligned} L_2 &= \frac{1}{2}\left(V_2 - 3\delta + \frac{1}{2}\left(V_2 - 4\delta + \frac{1}{2}\left(\cdots + \frac{1}{2}\left(V_2 - (s_* - 1)\delta\right)\right)\right)\right) \\ &= \frac{\delta}{2}\left(s_* - 3 + \frac{1}{2}\left(s_* - 4 + \frac{1}{2}\left(\cdots + \frac{1}{2}\cdot 1\right)\right)\right) \\ &= \delta\left(\frac{1}{2}(s_* - 3) + \frac{1}{2^2}(s_* - 4) + \frac{1}{2^3}(s_* - 5) + \cdots + \frac{1}{2^{s_*-3}}\right), \end{aligned}$$

which is equal to the right-hand side of (37). In the above multiline calculation, the first and second lines are due to $V_2 = s_*\delta$ (Lemma 3.ii). ■

Lemma 15 $V_{s_*-2} - 2\delta \geq L_2 \Longrightarrow \forall m \in \{1, \ldots, s_*/2 - 1\} : V_{s_*-2m} - 2\delta \geq L_2$.

Proof By the law of motion and Eq. (8), Eqs. (33), (34), (35), (36) and (37) hold. Denote

$$\begin{aligned} \mu(m) &:= 2^{-2m+1}, \\ \mu_* &:= 2^{-s_*+3}. \end{aligned}$$

With the fact $\sum_{k=1}^{m-1} 2^{-2k} = (1 - 2^{-2m+2})/3$, Eq. (34) becomes

$$V_{s_*-2m} = -\delta \cdot \frac{2}{3}(1 - 2\mu(m)) + M_2\left(\frac{2}{3}(1 - 2\mu(m)) + \mu(m)\right) + L_2\left(\frac{1}{3}(1 - 2\mu(m)) + \mu(m)x\right).$$

Hence

$$\begin{aligned} V_{s_*-2m} - 2\delta - L_2 &= -\delta\left(\frac{2}{3}(1 - 2\mu(m)) + 2\right) + M_2\left(\frac{2}{3}(1 - 2\mu(m)) + \mu(m)\right) \\ &\quad - L_2\left(1 - \frac{1}{3}(1 - 2\mu(m)) - \mu(m)x\right) \\ &= -\frac{4}{3}(2 - \mu(m))\delta + \frac{1}{3}(2 - \mu(m))M_2 \\ &\quad - (s_* - 4 + \mu_*)\,\delta\left(\frac{2}{3}(1 + \mu(m)) - \mu(m)x\right), \end{aligned}$$

with the second equality due to (37). Thus, $V_{s_*-2m} - 2\delta \geq L_2$ is equivalent to

$$\frac{1}{3}(2-\mu(m))M_2 \geq \delta\left(\frac{4}{3}(2-\mu(m)) + (s_* - 4 + \mu_*)\left(\frac{2}{3}(1+\mu(m)) - \mu(m)x\right)\right),$$

i.e.,

$$\frac{M_2}{\delta} \geq 4 + \frac{2(1+\mu(m)) - 3\mu(m)x}{2-\mu(m)}(s_* - 4 + \mu_*). \tag{38}$$

Since $s_* - 4 \geq 0$ by hypothesis, and

$$\begin{aligned}\frac{d}{d\mu(m)}\left(\frac{2(1+\mu(m)) - 3\mu(m)x}{2-\mu(m)}\right) &= \frac{(2-\mu(m))(2-3x) + 2(1+\mu(m)) - 3\mu(m)x}{(2-\mu(m))^2} \\ &= \frac{6(1-x)}{(2-\mu(m))^2} \geq 0,\end{aligned}$$

the right-hand side of (38) is weakly increasing in $\mu(m)$, which in turn is strictly decreasing in m. Thus the right-hand side of (38) is weakly decreasing in m. Consequently, $V_{s_*-2m} - 2\delta - L_2 \geq 0$ is satisfied for all m if the inequality holds at the minimum $m = 1$, i.e., if $V_{s_*-2} - 2\delta - L_2 \geq 0$, as claimed. ■

Proof of Lemma 5 Let $s \in \{1, 2, \ldots, s_*-2\}$. If s is even and $V_3 - 2\delta \geq L_2$, then Lemma 13 implies $V_{s+1} - 2\delta > L_2$; thus, by (6) and by the fact that $\pi_{\gamma,s} = 1$ due to Condition $(*)$, the β player at s gets L_2 if he does not bid, and $\frac{1}{2}(V_{s+1} - 2\delta) + \frac{1}{2}L_2$ if he does. Hence bidding is the unique best response for β at s. If s is odd and $V_{s_*-2} - 2\delta \geq L_2$, then Lemma 15 implies that $V_{s+1} - 2\delta \geq L_2$; thus, by the same token as in the previous case, the β player at s weakly prefers to bid. ■

A.5 Theorem 2

Lemma 16 *For any even $s_* \geq 4$, if Eqs. (8) and (37) hold and $M_2 \geq V_2 = s_*\delta$, then at the initial and second rounds each player strictly prefers to bid.*

Proof First, consider the second round, which means $s = 1$. For each non-α player, becoming the next α player gives him an expected payoff $V_2 - 2\delta = (s_* - 2)\delta$ by the hypothesis $V_2 = s_*\delta$, whereas staying put gives payoff L_2, which is less than $(s_* - 3)\delta$ by Eq. (37). Thus, each non-α player strictly prefers to bid at state one, hence $s = 2$ occurs for sure given $s = 1$. Second, consider the initial state. Based on the analysis of the previous step (from $s = 1$ to $s = 2$), becoming the first α yields the expected payoff $-\delta + M_2$, whereas staying put yields

$\frac{1}{2}(V_2 - 2\delta + L_2)$. Since $M_2 \geq V_2$ by hypothesis and $V_2 - 2\delta > L_2$ by the previous analysis, each player strictly prefers to become the first α player. ∎

Lemma 17 *Any integer $s_* \geq 3$ constitutes an equilibrium if and only if s_* is an even number and there exists $(M_2, x, L_2) \in \mathbb{R}^3_+$ such that—*

a. $(M_2, x, L_2) \in \mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+$ and it solves simultaneously Eq. (34) in the case $m = s_*/2 - 1$, Eq. (36) in the case $m = s_*/2 - 1$ such that $V_2 = s_*\delta$, and Eq. (37);

b. $M_2 \geq s_*\delta$;

c. Ineq. (38) is satisfied in the case $m = 1$.

Proof The necessity that s_* is even for an equilibrium follows from Theorem 1. In Condition (a), the necessity of $V_2 = s_*\delta$ follows from Lemma 3, and the rest from Lemma 14, which in turn follows from Condition ($*$), necessary due to Lemmas 1–4. With $V_2 = s_*\delta$, the condition $M_2 \geq s_*\delta$ is equivalent to $M_2 \geq V_2$; hence the necessity of Condition (b) follows from Lemma 11. The necessity of Condition (c) follows from Lemma 4, which implies the necessity of $V_{s_*-2m} - 2\delta \geq L_2$, which as shown in the proof of Lemma 15 requires Ineq. (38).

To prove that these conditions together suffice an equilibrium, pick any even number $s_* \geq 4$ and assume Conditions (a)–(c). Consider the strategy profile such that everyone bids in the initial round, each non-α player bids in the second round and, in any future round, acts according to Condition ($*$). This strategy profile implies Eq. (8), which allows calculation of the value functions $(V_s, M_s, L_s)_{s=2}^{s_*}$ via the law of motions. By Conditions (a) and (b), $M_2 \geq V_2 = s_*\delta$, hence Lemma 16 implies that bidding at the initial round is a best response for each player, and bidding at second rounds a best response for each non-α player. The incentive for each player to abide by the strategy profile at any state $s \geq s_*$ is the same as in the two-bidder equilibrium. At the state $s_* - 1$, bidding with probability x is a best response for the γ player because he is indifferent about bidding, since $V_2 - s_*\delta = 0 = L_{s_*}$, and not bidding at all is the best response for the β player because $V_{s_*} - 2\delta = 0 < L_2$. At any state s with $2 \leq s \leq s_* - 2$, bidding is the best response for the γ player because $V_2 - (s+1)\delta > L_{s+1}$ (by Eq. (7)); Condition (c) by Lemma 15 suffices the incentive for the β player at every odd state to bid. To incentivize the β player at every even state $s \leq s_* - 2$ to bid, Lemma 13 says that it suffices to have $V_3 - 2\delta \geq L_2$, which is equivalent to $M_2 \geq L_2$

since, by the law of motion and Eq. (8), M_2 is the midpoint between $V_3 - 2\delta$ and L_2. Since $L_2 < s_*\delta$ by Eq. (37), the condition $M_2 \geq L_2$ is guaranteed by Condition (b), $M_2 \geq s_*\delta$. ■

Lemma 18 *For any $s_* \geq 4$, Condition (c) in Lemma 17 implies Condition (b) in Lemma 17.*

Proof Condition (c) in Lemma 17 is Ineq. (38) in the case $m = 1$, i.e., when $\mu(m) = 2^{-2m+1} = 1/2$. Hence the condition is equivalent to

$$\frac{M_2}{\delta} \geq 4 + (2-x)(s_* - 4 + \mu_*). \tag{39}$$

To prove that this inequality implies Condition (b), i.e., $M_2/\delta \geq s_*$, it suffices to show

$$4 + (2-x)(s_* - 4 + \mu_*) > s_*,$$

i.e.,

$$(1-x)(s_* - 4) + \mu_*(2-x) > 0,$$

which is true because $s_* \geq 4$, $\mu_* = 2^{-s_*+3} > 0$ and $x \leq 1$. ■

Lemma 19 *Condition (a) in Lemma 17 is equivalent to the existence of an $x \in [0,1]$ that solves Eq. (9).*

Proof Condition (a) requires existence of $(M_2, x, L_2) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R}_+$ that satisfies Eqs. (34), (36) and (37) in the case of $m = s_*/2 - 1$ and $V_2 = s_*\delta$. Combine (34) with (37) and use the notation $\mu_* := 2^{-s_*+3}$ and the fact $\sum_{k=1}^{m-1} 2^{-2k} = (1 - 2^{-2m+2})/3$ to obtain

$$s_*\delta = V_2 = -\delta \cdot \frac{2}{3}(1-2\mu_*) + M_2\left(\frac{2}{3}(1-2\mu_*) + \mu_*\right) + \underbrace{\delta(s_* - 4 + \mu_*)}_{L_2}\left(\frac{1}{3}(1-2\mu_*) + \mu_* x\right),$$

i.e.,

$$\frac{M_2}{\delta} = \frac{1}{2-\mu_*}\left(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_* x)\right). \tag{40}$$

By the same token, (36) coupled with (37) is equivalent to

$$M_2\left(1 - \frac{1}{3}(1-2\mu_*) - \mu_* x\right) = -\delta\left(1 + \frac{1}{3}(1-2\mu_*)\right) + (1-x)\mu_* v + \delta(s_* - 4 + \mu_*)\left(\frac{2}{3}(1-2\mu_*) + \mu_*\right),$$

i.e.,

$$\frac{M_2}{\delta}\left(2(1+\mu_*) - 3\mu_* x\right) = \frac{3\mu_* v}{\delta}(1-x) + (2-\mu_*)(s_* - 6 + \mu_*). \tag{41}$$

Plug (40) into (41) and we obtain Eq. (9). ■

Lemma 20 *For any even number $s_* \geq 4$, suppose that Eq. (40) holds. Then Condition (c) in Lemma 17 is equivalent to Ineq. (10), which is implied by $x \geq 0$ if and only if $s_* \leq 6$.*

Proof Condition (c) in Lemma 17 has been shown to be equivalent to Ineq. (39). Provided that Eq. (40) is satisfied, Ineq. (39) is equivalent to

$$4 + (2 - x)(s_* - 4 + \mu_*) \leq \frac{1}{2 - \mu_*} \left(3s_* + 2(1 - 2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_* x)\right).$$

This inequality, given the fact $1 - 2\mu_* \geq 0$, is equivalent to

$$x \geq \frac{1}{2(1 - 2\mu_*)} \left(5 - 4\mu_* - \frac{3(s_* - 2)}{s_* - 4 + \mu_*}\right),$$

i.e., Ineq (10). Given Condition (a) in Lemma 17, which implies $x \geq 0$, Ineq. (10) is redundant if and only if the right-hand side of (10) is nonpositive, i.e.,

$$\frac{3(2 - \mu_*)}{2(1 - 2\mu_*)(s_* - 4 + \mu_*)} \geq 1,$$

i.e.,

$$s_* \leq 4 - 2^{-s_*+3} + \frac{3\left(2 - 2^{-s_*+3}\right)}{2\left(1 - 2^{-s_*+4}\right)}.$$

This inequality is satisfied when $s_* \in \{4, 6\}$, as its right-hand side is equal to ∞ when $s_* = 4$, and $61/8$ when $s_* = 6$. The inequality does not hold, by contrast, when $s_* \geq 8$, as

$$\begin{aligned} s_* \geq 8 \quad &\Rightarrow \quad 2^{-s_*+2} \leq 2^{-6} \quad \Rightarrow \quad \frac{1 - 2^{-s_*+2}}{1/4 - 2^{-s_*+2}} \leq \frac{1 - 2^{-6}}{1/4 - 2^{-6}} = \frac{63}{15} \\ &\Rightarrow \quad 4 - 2^{-s_*+3} + \frac{3\left(2 - 2^{-s_*+3}\right)}{2\left(1 - 2^{-s_*+4}\right)} < 4 + \frac{3}{2} \cdot \frac{2}{4} \cdot \frac{63}{15} < 8 \leq s_*. \end{aligned}$$

Thus, for all even numbers $s_* \geq 4$, Ineq. (39) follows if and only if $s_* \leq 6$. ■

Proof of Theorem 2 The theorem follows from Lemma 17, where Condition (a) has been characterized by Lemma 19, Condition (b) by Lemmas 18 can be dispensed with, and Condition (c), by Lemma 20, can be dispensed with when $s_* \leq 6$ (hence Claim (i) of the theorem) and is equivalent to Ineq (10) when $s_* > 6$ (hence Claim (ii) of the theorem). ■

A.6 Theorem 3

Lemma 21 *If $x = 1$, the left-hand side of (9) is less than the right-hand side of (9).*

Proof When $x = 1$, the left-hand side of (9) is equal to $(2 - \mu_*)^2(s_* - 6 + \mu_*)$, and the right-hand side equal to

$$\begin{aligned} & (2(1+\mu_*) - 3\mu_*)\left(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_*)\right) \\ = \; & (2-\mu_*)\left(2s_* + 6 - \mu_* - \mu_* s_* - \mu_*^2\right). \end{aligned}$$

Thus, the lemma follows if

$$(2-\mu_*)(s_* - 6 + \mu_*) < 2s_* + 6 - \mu_* - \mu_* s_* - \mu_*^2,$$

i.e., $9\mu_* < 18$, which is true because $\mu_* = 2^{-s_*+3}$. ■

Lemma 22 *$s_* = 4$ constitutes an equilibrium if and only if $v/\delta > 35/2$, and $s_* = 6$ constitutes an equilibrium if and only if $v/\delta > 6801/120$ ($= 56.675$).*

Proof By Theorem 2, with $s_* \leq 6$ the necessary and sufficient condition for equilibrium is that Eq. (9) admits a solution for $x \in [0, 1]$. By Lemma 21, the left-hand side of that equation is less than its right-hand side when $x = 1$. Thus, it suffices to show that the left-hand side is greater than the right-hand side when $x = 0$, i.e.,

$$\frac{3\mu_* v}{\delta}(2-\mu_*) + (2-\mu_*)^2(s_* - 6 + \mu_*) > 2(1+\mu_*)\left(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1-2\mu_*)\right),$$

which is equivalent to

$$\frac{v}{\delta}(2-\mu_*) > s_*(4+\mu_*) + (6-\mu_*)(2/\mu_* - 2 - \mu_*).$$

Since μ_* is equal to $1/2$ when $s_* = 4$, and equal to $1/8$ when $s_* = 6$, the above inequality is equivalent to $v/\delta > 35/2$ when $s_* = 4$, and $v/\delta > 6801/120$ when $s_* = 6$. ■

Proof of Theorem 3 Claim (i) of the theorem is just Lemma 22. To prove Claim (ii), pick any even number $s_* \in \{8, 10, 12, \ldots\}$. By Theorem 2.ii, s_* constitutes an equilibrium if Eq. (9) admits a solution for $x \in [0, 1]$ that satisfies Ineq. (10). By Lemma 21, the left-hand side of (9) is less than its right-hand side when $x = 1$. Thus, it suffices to show that the left-hand side is greater than the right-hand side when x is equal to some number greater than or equal to the right-hand side of Ineq. (10). To that end, note from $s_* \geq 8$ that $\mu_* = 2^{-s_*+3} \leq 1/32$, hence $2 - \mu_* \geq 63/32$ and $1 + \mu_* < 33/32$. Thus, the left-hand side of (9) is greater than

$$\frac{3\mu_* v}{\delta}(1-x)\frac{63}{32} + \left(\frac{63}{32}\right)^2 (s_* - 6),$$

and the right-hand side of Ineq. (10)

$$1-\frac{3(2-\mu_*)}{2(1-2\mu_*)(s_*-4+\mu_*)}<1-\frac{3\times 63/32}{2\times 1\times(s_*-3)}.$$

Therefore, it suffices, for s_* to constitute an equilibrium, to have

$$\frac{3\mu_* v}{\delta}(1-x)\frac{63}{32}+\left(\frac{63}{32}\right)^2(s_*-6)$$

greater than or equal to the right-hand side of (9) when

$$x=x_*:=1-\frac{3\times 63}{64(s_*-3)}.$$

To that end, denote $\phi(s_*,x)$ for the right-hand side of (9), i.e.,

$$\phi(s_*,x):=\left(2(1+\mu_*)-3\mu_* x\right)\left(3s_*+2(1-2\mu_*)-(s_*-4+\mu_*)(1-2\mu_*+3\mu_* x)\right)$$

(recall that $\mu_*=2^{-s_*+3}$). Note, from $0<x<1$, that $-1/32<\mu_*(2-3x)<1/16$. Hence

$$\begin{aligned}\frac{63}{32}=2-\frac{1}{32} &< 2(1+\mu_*)-3\mu_* x < 2+\frac{1}{16}=\frac{33}{16},\\ \frac{15}{16}=1-\frac{1}{16} &< 1-2\mu_*+3\mu_* x < 1+\frac{1}{32}=\frac{33}{32}.\end{aligned}$$

Thus, the first factor $2(1+\mu_*)-3\mu_* x$ of $\phi(s_*,x)$ is positive for all $x\in(0,1)$. If the second factor of $\phi(s_*,x)$ is nonpositive when $x=x_*$ then $\phi(s_*,x_*)\le 0$ and we are done, as the left-hand side of (9) is positive. Hence we may assume, without loss of generality, that

$$3s_*+2(1-2\mu_*)-(s_*-4+\mu_*)(1-2\mu_*+3\mu_* x_*)>0.$$

Consequently, $\phi(s_*,x_*)$ can only get bigger if we replace its first factor by the upper bound $33/16$, and the term $1-2\mu_*+3\mu_* x$ in the second factor by its lower bound $15/16$ (note that, in the second factor, $s_*-4+\mu_*>0$ because $s_*\ge 8$). I.e., $\phi(s_*,x_*)$ is less than

$$\begin{aligned}\frac{33}{16}\left(3s_*+2(1-2\mu_*)-\frac{15}{16}(s_*-4+\mu_*)\right) &= \frac{33}{16}\left(\frac{33}{16}s_*+\frac{23}{4}-\frac{79}{16}\mu_*\right)\\ &< \frac{33}{16}\left(\frac{33}{16}s_*+\frac{23}{4}\right)\\ &< 5s_*+12.\end{aligned}$$

Therefore, the above observations put together, we are done if

$$\frac{3\mu_* v}{\delta}(1-x_*)\frac{63}{32}+\left(\frac{63}{32}\right)^2(s_*-6)\ge 5s_*+12$$

In other words, it suffices to have

$$\frac{3\mu_* v}{\delta} \cdot \frac{3 \times 63}{64(s_* - 3)} \cdot \frac{63}{32} + \left(\frac{63}{32}\right)^2 (s_* - 6) \geq 5s_* + 12,$$

i.e.,

$$\frac{3^2 \mu_* v}{\delta} \geq -2(s_* - 6)(s_* - 3) + \frac{32 \times 64}{63^2} (5s_* + 12)(s_* - 3).$$

With $\frac{32 \times 64}{63^2} \approx 0.516$, the above inequality holds if

$$\frac{3^2 \mu_* v}{\delta} \geq -2(s_* - 6)(s_* - 3) + (5s_* + 12)(s_* - 3),$$

i.e.,

$$\frac{9\mu_* v}{\delta} \geq 3s_*^2 + 15s_* - 72,$$

which is equivalent to Ineq. (11). Thus the theorem is proved. ■

A.7 Theorem 4

Since $v/\delta \geq s_*$ for any dropout state of any equilibrium, there are only finitely many equilibrium-feasible dropout states. Given any dropout state of any equilibrium, each player's action at every state is uniquely determined, according to Lemmas 1–4 and 16, except the γ player's bidding probability π_{γ,s_*-1} at the critical state. Thus, it suffices to prove that for each dropout state s_* there are only finitely many compatible π_{γ,s_*-1} at the equilibrium. To that end, since π_{γ,s_*-1} is determined by Eq. (9) given s_*, we need only to show that for each s_* Eq. (9) admits at most two solutions for x, the shorthand for π_{γ,s_*-1}. To show that, note that the left-hand side of Eq. (9) is a linear function of x, whereas the right-hand side is strictly convex in x: The derivative of the right-hand side with respect to x is equal to

$$\begin{aligned} &-3\mu_* \left(3s_* + 2(1 - 2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_* x)\right) \\ &+ \left(2(1 + \mu_*) - 3\mu_* x\right)\left(-(s_* - 4 + \mu_*)3\mu_*\right), \end{aligned}$$

whose derivative with respect to x is equal to

$$3\mu_*(s_* - 4 + \mu_*)3\mu_* + 3\mu_*(s_* - 4 + \mu_*)3\mu_* = 18\mu_*(s_* - 4 + \mu_*) > 0,$$

with the inequality due to the fact that $s_* - 4 + \mu_* = s_* - 4 + 2^{-s_*+4} > 0$ as $s_* \geq 4$ (Theorem 1). Thus, Eq. (9) admits at most two solutions for x, as desired. ■

A.8 Theorem 5

By the hypothesis $v/\delta > 35/2$ and Theorem 3, a trilateral-rivalry equilibrium exists. Hence bilateral-rivalry equilibriums are dynamically dominated by a trilateral-rivalry one, as is explained around the statement of the theorem. We need only to prove that, for any two trilateral-rivalry equilibriums with dropout states s'_* and s''_* such that $s'_* < s''_*$, the one with s'_* is dynamically Pareto dominated by the one with s''_*, while the converse is not true. (If there are multiple trilateral-rivalry equilibriums of the same dropout state, none of them is dynamically dominated by the other, because they reach exactly the same set of states.)

Hence pick any two trilateral-rivalry equilibriums with dropout states $s'_* < s''_*$. As both s'_* and s''_* are even numbers, $s'_* \leq s''_* - 2$. Label the value functions in the equilibrium with dropout state s'_* by $(V'_1, M'_1, (V'_s, M'_s, L'_s))_{s=2}^{\infty}$, and those in the equilibrium s''_* by $(V''_1, M''_1, (V''_s, M''_s, L''_s))_{s=2}^{\infty}$. First note that s''_* is not dynamically Pareto dominated by s'_*. That is because $s''_* > s'_*$, hence Condition (a) of compelling Pareto dominance is not satisfied, as any state that can be reached on the path of s'_*, i.e., any integer $s \in \{1, 2, \ldots, s'_* - 1\}$ by Condition ($*$), can also be reached on the path of s''_*, includes any integer up to $s''_* - 1$.

Second, we show that s'_* is dynamically Pareto dominated by s''_*. To that end, consider the state s'_*. Since $s'_* \leq s''_* - 1$, it can be reached on path of the equilibrium s''_*, but cannot be reached on the path of equilibrium s'_* (since the game according to the latter equilibrium either ends or collapses back to state 2 at the critical state $s'_* - 1$; c.f. Condition ($*$)). Hence Condition (a) is met. To verify Condition (b), let us compare the players' expected payoffs from the two equilibriums conditional on state s'_* being reached.

i. For the current α player (who was the previous β and deviated at state $s'_* - 1$), the status quo equilibrium s'_* gives him a payoff equal to $V'_{s'_*}$, which is equal to 2δ (Lemma 2). Whereas the new equilibrium s''_* would give him $V''_{s'_*}$; since $s'_* \leq s''_* - 2$, the incentive condition in equilibrium s''_* implies that $V''_{s'_*} - 2\delta \geq L''_2 > 0$ (Condition ($*$) and Lemma 7) and hence $V''_{s'_*} > 2\delta = V'_{s'_*}$.

ii. For the current β player at state s'_*, the status quo equilibrium s'_* yields a payoff $M'_{s'_*}$, which equals zero (Lemma 2); while the new equilibrium s''_* yields $M''_{s'_*}$, which by (6) and (8) is equal to the average between $V''_{s'_*+1} - 2\delta$ and L''_2, each strictly positive. Hence $M''_{s'_*} > M'_{s'_*}$.

iii. For the current γ player, the status quo equilibrium s'_* gives $L'_{s'_*} = 0$ (Lemma 2); while the new equilibrium s''_* gives him $L''_{s'_*} > 0$ because $s'_* \leq s''_* - 2$ (c.f. (24)).

Thus Condition (b) is satisfied. To verify Condition (c), note that in the status quo equilibrium s'_*, the state s'_* is reached by the unilateral deviation of the β player at state $s'_* - 1$, without whose bid the game would either end (when the γ player at state $s'_* - 1$ does not bid) or collapse back to state 2 (when γ bids). We still need to check the deviation incentive for this β player. His deviation is pivotal only when he becomes the next α player, at the state s'_*. Hence it suffices to compare his payoffs when he becomes the next α players versus otherwise. When he gets to become the next α player, his expected payoff becomes $V''_{s'_*} - 2\delta$ provided that the other two players abides by the new equilibrium s''_* from now on. When he does not get to be the next α player, the game does not reach the off-path state s'_* and hence the status quo equilibrium remains at place, which gives him a payoff $\pi'_{\gamma,s'_*-1}L'_2$. As explained previously, $V''_{s'_*} - 2\delta \geq L''_2$, and

$$\begin{aligned} L''_2 \overset{(37)}{=} \left(s''_* - 4 + 2^{-s''_*+3}\right)\delta &\geq \left(s'_* + 2 - 4 + 2^{-s''_*+3}\right)\delta \\ &= \left(\left(s'_* - 4 + 2^{-s'_*+3}\right) + (2 - 2^{-s'_*+3} + 2^{-s''_*+3})\right)\delta \\ &\overset{(37)}{=} L'_2 + \left(2 - 2^{-s'_*+3} + 2^{-s''_*+3}\right)\delta \\ &> L'_2 + \delta, \end{aligned}$$

with the first inequality due to $s''_* \geq s'_* + 2$, and the last due to the fact $2^{-s'_*+3} \leq 1/2$. Thus $V''_{s'_*} - 2\delta \geq L''_2 > \pi'_{\gamma,s'_*-1}L'_2$, and Condition (c) is met. ∎

A.9 Verification of the Quadrilateral Equilibrium in Section 5.1

First, from our characterization of trilateral-rivalry equilibriums, one can obtain the associated value function for the trilateral-rivalry equilibrium with dropout state $s_* = 4$:

$$V_2 = 4\delta, \tag{42}$$

$$V_3 = (16 + 3/2 - \pi_{\gamma,3})\,\delta, \tag{43}$$

$$M_2 = V_3/2 - 3\delta/4, \tag{44}$$

$$L_2 = \delta/2, \tag{45}$$

$$M_3 = \pi_{\gamma,3}L_2 = \pi_{\gamma,3}\delta/2, \tag{46}$$

$$L_3 = 0. \tag{47}$$

Second, recall the notation A, B, C and D: In the consecutive configuration (12) such that $m = 2$, let A denote the continuation value for α, B the continuation value for β, C for γ_1, and D for γ_2. By Provision (c.i) of the proposed strategy profile, players β, γ_1 and γ_2 each bid for sure, with others staying put and hence omitted. Thus the configuration in the next round is one of the following three, each with probability 1/3:

$$[\beta, \alpha, \Box, \gamma_1, \gamma_2], \quad t = 4; \tag{48}$$

$$[\gamma_1, \alpha, \beta, \Box, \gamma_2], \quad t = 4; \tag{49}$$

$$[\gamma_2, \alpha, \beta, \gamma_1], \quad t = 3. \tag{50}$$

If it is (48) or (49), player γ_2 quits and the other three play the trilateral-rivalry equilibrium with $s_* = 4$; if it is (50) then each non-frontrunner bids for sure, as in (12).

Given any consecutive configuration in the of (12), let A denote the continuation value for α, B the continuation value for β, C for γ_1, and D for γ_2. Then

$$D = \frac{1}{3}(A - 4\delta) \tag{51}$$

because γ_2 quits, thereby getting zero payoff, unless (50) happens. Since (48), (49) and (50) each happen with probability 1/3,

$$A = \frac{1}{3}(M_3 + M_2 + B), \tag{52}$$

$$B = \frac{1}{3}\left((V_3 - 2\delta) + L_2 + C\right) \overset{(45)}{=} \frac{1}{3}\left(V_3 - \frac{3}{2}\delta + C\right), \tag{53}$$

and

$$\begin{aligned} C &= \frac{1}{3}(L_3 + (V_2 - 3\delta) + D) \\ &= \frac{1}{3}\left(\delta + \frac{1}{3}(A - 4\delta)\right) \quad \text{by (42), (47) and (51)} \\ &= \frac{1}{9}\left(-\delta + \frac{1}{3}(M_3 + M_2 + B)\right) \quad \text{by (52)} \\ &= \frac{1}{27}\left(-3\delta + \frac{\pi_{\gamma,3}}{2}\delta + \left(\frac{1}{2}V_3 - \frac{3}{4}\delta\right) + \frac{1}{3}\left(V_3 - \frac{3}{2}\delta + C\right)\right) \quad \text{by (46), (44) and (53).} \end{aligned}$$

Thus,

$$\begin{aligned} C &= \frac{1}{80}\left(\frac{5}{2}V_3 + \left(-9 + \frac{1}{4}(6\pi_{\gamma,3} - 15)\right)\delta\right) \\ &\overset{(43)}{=} \frac{1}{80}\left(\frac{5}{2}\left(16 + \frac{3}{2} - \pi_{\gamma,3}\right)\delta + \left(-9 + \frac{1}{4}(6\pi_{\gamma,3} - 15)\right)\delta\right) \\ &= \frac{1}{80}(31 - \pi_{\gamma,3})\delta. \end{aligned} \tag{54}$$

This plugged into Eq. (53) gives

$$\begin{aligned} B &= \frac{1}{3}\left(V_3 - \frac{3}{2}\delta + \frac{1}{80}(31-\pi_{\gamma,3})\,\delta\right) \\ &\overset{(43)}{=} \frac{1}{3}\left(\left(16+\frac{3}{2}-\pi_{\gamma,3}\right)\delta - \frac{3}{2}\delta + \frac{1}{80}(31-\pi_{\gamma,3})\,\delta\right) \\ &= \frac{1}{3}\left(16+\frac{31}{80}-\left(1+\frac{1}{80}\right)\pi_{\gamma,3}\right)\delta. \end{aligned} \tag{55}$$

Plugging this into Eq. (52), we have

$$\begin{aligned} A &= \frac{1}{3}\left(M_3+M_2+\frac{1}{3}\left(16+\frac{31}{80}-\left(1+\frac{1}{80}\right)\pi_{\gamma,3}\right)\delta\right) \\ &\overset{(46),(44)}{=} \frac{1}{3}\left(\frac{\pi_{\gamma,3}}{2}\delta+\left(\frac{1}{2}V_3-\frac{3}{4}\delta\right)+\frac{1}{3}\left(16+\frac{31}{80}-\left(1+\frac{1}{80}\right)\pi_{\gamma,3}\right)\delta\right) \\ &\overset{(43)}{=} \frac{1}{3}\left(\frac{\pi_{\gamma,3}}{2}\delta+\frac{1}{2}\left(16+\frac{3}{2}-\pi_{\gamma,3}\right)\delta-\frac{3}{4}\delta+\frac{1}{3}\left(16+\frac{31}{80}-\left(1+\frac{1}{80}\right)\pi_{\gamma,3}\right)\delta\right) \\ &= \frac{1}{9}\left(40+\frac{31}{80}-\left(1+\frac{1}{80}\right)\pi_{\gamma,3}\right)\delta \end{aligned} \tag{56}$$

Then Eq. (51) implies

$$D = \frac{1}{27}\left(4+\frac{31}{80}-\left(1+\frac{1}{80}\right)\pi_{\gamma,3}\right)\delta. \tag{57}$$

Recall that $0 < \pi_{\gamma,3} < 1$, which plugged into Eqs. (54), (55), (56) and (57) implies

$$\tfrac{3}{8}\delta < C < \tfrac{32}{80}\delta = \tfrac{2}{5}\delta, \tag{58}$$

$$5\delta = \tfrac{1}{3}\cdot 15\delta < B < \tfrac{1}{3}\cdot 17\delta < 6\delta, \tag{59}$$

$$4\delta < \tfrac{1}{9}\cdot 39\delta < A < \tfrac{1}{9}\cdot 41\delta < 5\delta, \tag{60}$$

$$\tfrac{1}{9}\delta = \tfrac{3}{27}\delta < D < \tfrac{5}{27}\delta. \tag{61}$$

We verify the equilibrium conditions through backward induction. Let us start with any subgame with state $t \geq 4$. Expecting the trilateral-rivalry equilibrium to be played in the subgame (Provision (e)), the current frontrunner, follower and third-place bidder each finds it a best response to abide by it, as verified in our paper. The lowest-place bidder γ_2 cannot profit from the deviation of leapfrogging to the top: being on the top gives him a continuation value equal to $V_2 = 4\delta$ because he and the previous frontrunner and follow form a consecutive trilateral configuration, while the leapfrog costs him a payment at least as large as 5δ. Thus, the proposed strategy profile is an equilibrium in this subgame.

Next consider any subgame with $t = 3$, i.e., the consecutive configuration (12). For player γ_2: if he does not bid then the state becomes $t = 4$ next round, in which he will quit and get zero; whereas if he bids now and becomes the next frontrunner, he gets a payoff $A - 4\delta$, which is positive by (60); thus bidding is his best response. For player γ_1: if he becomes the next frontrunner (through bidding), then the next configuration becomes (49), giving him a payoff equal to $V_2 - 3\delta = \delta$; whereas, if γ_1 does not bid, the next round is either (48), giving him a payoff $L_3 = 0$, or (50), giving him a payoff $D < 5\delta/27$ by (61). Thus, bidding is the best response for player γ_1. For player β: if he bids and becomes the next frontrunner, the next configuration is (48), giving him a payoff equal to $V_3 - 2\delta$, which is larger than 14δ by Eq. (43); whereas, if β does not bid, then the next configuration is either (49), giving him a payoff $L_2 = \delta/2$, or (50), giving him a payoff $C < 2\delta/5$ by (58). Thus bidding is the best response for player β. Therefore, the proposed strategy profile constitutes an equilibrium in any subgame that starts with the consecutive configuration (12).

We next consider any subgame with $t = 2$, which means the third round, with configuration in the form

$$\begin{bmatrix} \alpha \\ \beta \\ \{\gamma_1, \gamma_2\} \end{bmatrix}. \tag{62}$$

For player β: if he becomes the next frontrunner, then according to Provision (c.ii) he gets

$$V_3 - 2\delta \overset{(43)}{=} \left(16 + \frac{3}{2} - \pi_{\gamma,3}\right)\delta - 2\delta = \left(14 + \frac{3}{2} - \pi_{\gamma,3}\right)\delta;$$

by contrast, if he does not bid then, since both of the γ players are bidding according to Provision (b), β in the next round will become the third-place bidder in the consecutive configuration and hence gets payoff $C < 2\delta/5$ by (58). Thus bidding is the best response for β. For each of the γ players: if he does not bid then he becomes the fourth-place player in the next round thereby getting only $D < 5\delta/27$ by (61); whereas bidding and becoming the next frontrunner gives him $A - 3\delta > \delta$ by (60); thus bidding is the best response for each of γ_1 and γ_2. Hence the proposed strategy profile constitutes an equilibrium in any subgame with $t = 2$.

Next consider any subgame with $t = 1$, which means the second round, at which the configuration is in the form

$$\begin{bmatrix} \alpha \\ \{\gamma_1, \gamma_2, \gamma_3\} \end{bmatrix}. \tag{63}$$

If a γ player does not bid, he in the third round will become one of the two γ players in the configuration (62) and hence his payoff will be equal to

$$\frac{1}{3}\cdot 0+\frac{1}{3}(A-3\delta)+\frac{1}{3}D \overset{(60),(61)}{<} \frac{1}{3}(5\delta-3\delta)+\frac{5}{81}\delta=\frac{59}{81}\delta<\delta,$$

where the zero term on the left-hand side is his payoff in the event where the β player in the third round gets to become the next frontrunner thereby starting the subgame equilibrium according to Provision (c.ii), rendering zero expected payoff for both bottom-row players, this γ one of them. By contrast, if this γ player bids and becomes the next frontrunner, then in the third round he will be the α in configuration (62), which in the fourth round will become one of the following, each with probability 1/3:

$$\begin{bmatrix} \beta \\ \alpha \\ \varnothing \\ \{\gamma_1,\gamma_2\} \end{bmatrix}, \begin{bmatrix} \gamma_1 \\ \alpha \\ \beta \\ \gamma_2 \end{bmatrix}, \begin{bmatrix} \gamma_2 \\ \alpha \\ \beta \\ \gamma_1 \end{bmatrix}; \tag{64}$$

thus, when $t=1$, his expected payoff from bidding and becoming the next frontrunner is

$$\frac{1}{3}M_3+\frac{2}{3}B-2\delta \overset{(59)}{>} \frac{2}{3}\cdot 5\delta-2\delta=\frac{4}{3}\delta>\delta.$$

Hence bidding is the best response for the player. Thus, the proposed strategy constitutes an equilibrium in any subgame with $t=1$.

Finally, consider $t=0$, i.e., the initial round. Consider any bidder i. If i bids and becomes the frontrunner (in the second round), then he will become the β in the configuration (62) in the third round, and then in the fourth round, his position one of the three configurations in (64) occupied by the β there. Thus his payoff from bidding in the initial round and becoming the first frontrunner is equal to

$$-\delta+\frac{1}{3}(V_3-2\delta)+\frac{2}{3}C \overset{(43)}{=} -\delta+\frac{1}{3}\left(\left(16+\frac{3}{2}-\pi_{\gamma,3}\right)\delta-2\delta\right)+\frac{2}{3}C>\frac{23}{6}\delta+\frac{2}{3}\cdot\frac{3}{8}\delta=\frac{49}{12}\delta.$$

If i does not bid in the initial round, he becomes in the second round one of the γ players in the configuration (63); then he either (i) pays 2δ to become the frontrunner in the third round (and become the second-place player in the fourth round to get either M_3 or B), or (ii) becomes one of the two γ players in the configuration (62) in the third round. In Case (ii), as shown in the previous step on $t=2$, the best outcome for the bidder is to get $A-3\delta$.

Since $M_3 < \delta/2$ by (46) and $B - 2\delta < 4\delta$ by (59), Case (i) renders less than 4δ for him; since $A - 3\delta < 2\delta$ by (60), Case (ii) gives him less than 2δ. Thus, either case in the alternative of not bidding produces less than 4δ, while bidding and becoming the initial frontrunner yields more than 4δ. Thus, bidding is the best response for each player i in the initial round. Therefore, the quadrilateral-rivalry strategy profile is a subgame perfect equilibrium.

A.10 Proof of the Subgame Perfect Equilibrium in Section 5.2

Denote W_s for the expected payoff of bidder 1 when he is the circled player in the tie-state s listed in Section 5.2 for the case with three players ($s = 0, 1, 2, 3, 4$). Due to the strategy profile,

$$\begin{aligned} W_1 &= 0, \\ W_2 &= v, \\ W_3 &= 0, \\ W_4 &= 0, \end{aligned}$$

with the first line due to the calculation in the previous section. If π_t denotes each player's probability of bidding at the trilateral-tie state $[\alpha, \alpha, \alpha]$, then, by the above calculations and (15),

$$\begin{aligned} W_0 &= \pi_t^3 W_0 + 2\pi_t^2(1-\pi_t)0 + \pi_t(1-\pi_t)^2 v + \pi_t^2(1-\pi_t)0 + 2\pi_t(1-\pi_t)^2 0 - \pi_t\delta \\ &= \pi_t^3 W_0 + \pi_t(1-\pi_t)^2 v - \pi_t\delta. \end{aligned}$$

Thus, when $\pi_t = 1 - (\delta/v)^{1/2}$ as prescribed by provision (a) of the strategy profile described in Section 5.2,

$$(1-\pi_t^3)W_0 = \pi_t\left((1-\pi_t)^2 v - \delta\right) = \pi_t\left(((\delta/v)^{1/2})^2 v - \delta\right) = 0$$

and hence

$$W_0 = 0.$$

Next we show that each provision in the strategy profile described in Section 5.2 is a best response. First, consider provision (a), the strategy given state 0. The next state for a player who bids now is state 0 or 1 or 2, among which only state 2 yields nonzero expected payoff v according to the above calculation. Thus, at state 0, a player's expected payoff from bidding is equal to

$$(1-\pi_t)^2 v - \delta = \left((\delta/v)^{1/2}\right)^2 v - \delta = 0.$$

If the player does not bid now, then the next round will either be the end of the game or put him as non-α player, either way he will get zero expected payoff. Thus, the player is indifferent about bidding and hence the randomization in provision (a) is a best response.

Second, consider provision (b), the strategy profile given state $[\alpha, \alpha, \beta]$. With the β player expected to not bid at all, the subgame to the two α players is equivalent to the bilateral-rivalry subgame calculated in the previous section, hence it is a best response for each of them to abide by the subgame equilibrium in the previous section, including bidding in the current round with probability 1/2. To check the incentive of the β player, note that if he does not bid then the state for him will either be the end of the game or his being a γ player, giving him zero expected payoff either way. If the β player bids, then his state will be $[@, \alpha, \alpha]$ or $[@, \alpha, \beta]$ or $[@, \beta, \beta]$, among which only $[@, \beta, \beta]$ yields a nonzero expected payoff, equal to v. Since $[@, \beta, \beta]$ occurs with probability $(1/2)^2$ by the strategy of the two α players in provision (b), the β-player's expected payoff from bidding is equal to

$$-2\delta + (1/2)^2 v \leq 0 \tag{65}$$

because we assume $v/\delta \leq 8$. Hence it is a best response for the β player to not bid given state $[\alpha, \alpha, \beta]$.

Third, provision (c) is justified in the same way as it is justified for the bilateral-rivalry subgame equilibrium in the main paper.

Finally, consider provision (d), the strategy profile given any state that has at least a non-α player. Other than the states already covered by provision (c), there are only two such states: $[\alpha, \alpha, \textcircled{\beta}]$ and $[\alpha, \textcircled{\beta}, \beta]$. Given the state $[\alpha, \alpha, \textcircled{\beta}]$, the justification of provision (b) has already explained why it is a best response for the β player to not bid. Given the state $[\alpha, \textcircled{\beta}, \beta]$, not bidding gives a β-player zero expected payoff, as the other β-player is expected to not bid. Alternatively, if a β-player bids, he bears a sunk cost 2δ to land himself in the state $[@, \beta, \gamma]$ (with the other β-player become the γ with a lag equal to 3δ from him) and hence gets the expected payoff $V_* = 2\delta$. Hence it is a best response to not bid, as prescribed by provision (d).

A.11 Verification of the Perfect Bayesian Equilibrium in Section 5.3

First, consider any round after the second one, and let (s_1, s_2) be the current updated pair of infimum types. Let i be the current follower. If i does not bid now, the game ends and he gets zero, with the cost of the payment he has committed in the past already sunk. If i bids then he adds 2δ to his committed payment and becomes the next frontrunner, with his infimum type updated to s'_i; thus, given type t_i, his expected payoff from bidding is equal to

$$\begin{aligned} -\frac{2\delta}{t_i} + V_i(t_i|s'_i, s_{-i}) &\overset{(20)}{=} -\frac{2\delta}{t_i} + \frac{2\delta}{s'_i} + \left(1 - \frac{2\delta}{s'_i}\right) M_i(t_i|s'_i, s'_{-i}) \\ &\overset{(19)}{=} -\frac{2\delta}{t_i} + \frac{2\delta}{s'_i} + \left(1 - \frac{2\delta}{s'_i}\right) \max\left\{0, -\frac{2\delta}{t_i} + V_i(t_i|s''_i, s'_{-i})\right\}, \end{aligned} \tag{66}$$

where s'_{-i} is derived from (s'_i, s_{-i}) by Eq. (18), and s''_i from (s'_i, s'_{-i}) analogously; and Eq. (20) is applicable to the first line because $V_i(t_i|s'_i, s_{-i})$ is player i's expected payoff from being the frontrunner in the next round, which is after at least the second round.

Claim: For any $t_i \in [a_i, z_i]$ there exists an integer $N_i(t_i)$ such that at the start of the $N_i(t_i)$th round $M_i(t_i|s_i^{N_i(t_i)}, s_{-i}^{N_i(t_i)}) = 0$, with (s_i^n, s_{-i}^n) denoting the updated infimums at the start of the nth round. Otherwise, Eq. (66), applied iteratively, implies that for any $n = 1, 2, 3, \ldots$

$$0 < -\frac{2\delta}{t_i} + V_i(t_i|s'_i, s_{-i}) = -\frac{2\delta}{t_i}n + \left(1 - \frac{2\delta}{s_i^n}\right) M_i(t_i|s_i^n, s_{-i}^n) \leq -\frac{2\delta}{t_i}n + v,$$

which is impossible given v a finite constant.

The claim established above, coupled with the first line of (66), implies that, at the start of the $N_i(t_i)$th round, if the updated infimums thereof are denoted by (s'_i, s'_{-i}), then

$$-\frac{2\delta}{t_i} + V_i(t_i|s'_i, s_{-i}) \begin{cases} > 0 & \text{if } t_i > s'_i \\ \leq 0 & \text{if } t_i \leq s'_i. \end{cases} \tag{67}$$

Then, at the start of the $(N_i(t_i) - 1)$th round, Eq. (19) implies

$$M_i(t_i|s_i, s_{-i}) \begin{cases} > 0 & \text{if } t_i > s'_i \\ = 0 & \text{if } t_i \leq s'_i. \end{cases} \tag{68}$$

In other words, in the $(N_i(t_i) - 1)$th round, at the start of which the updated infimum types are (s_i, s_{-i}), and the current follower i would bid if and only if his type is above s'_i (i.e., if and only if the continuation value of currently being the follower is positive).

At the start of the $(N_i(t_i)-2)$th round, the updated infimum of player i, the frontrunner now and soon to become the follower next, is still s_i, while that of player $-i$ is some s_{-i}^{-1} such that her updated infimum s_{-i} at the $(N_i(t_i)-1)$th round is derived from (s_i, s_{-i}^{-1}) by Eq. (18); by Eq. (20),

$$V_i(t_i|s_i, s_{-i}^{-1}) = \frac{2\delta}{s_i} + \left(1 - \frac{2\delta}{s_i}\right) M_i(t_i|s_i, s_{-i}^{-1}) \geq \frac{2\delta}{s_i},$$

with the inequality due to Eq. (19). Thus,

$$t_i > s_i \Longrightarrow -\frac{2\delta}{t_i} + V_i(t_i|s_i, s_{-i}^{-1}) \geq -\frac{2\delta}{t_i} + \frac{2\delta}{s_i} > 0; \tag{69}$$

if $t_i \leq s_i$ then $t_i < s_i'$, as $s_i < s_i'$ by Eq. (18), then Eq. (68) implies $M_i(t_i|s_i, s_{-i}^{-1}) = 0$ and hence

$$t_i \leq s_i \Longrightarrow -\frac{2\delta}{t_i} + V_i(t_i|s_i, s_{-i}^{-1}) = -\frac{2\delta}{t_i} + \frac{2\delta}{s_i} \leq 0. \tag{70}$$

Thus, (67) is extended from the $N_i(t_i)$th round to the $(N_i(t_i)-2)$th round. Thus, in the $(N_i(t_i)-3)$th round, where the current follower i contemplates whether or not to bid, (69) and (70) together imply that player i would bid if and only if his type is above s_i, as prescribed by the proposed equilibrium.

We can repeat the above reasoning, thereby extending (67) backward round by round, as long as the value function V_i obeys Eq. (20). Hence by backward induction we extend (67) down to the third round, with (s_i', s_{-i}) denoting the updated infimum types at the start of the third round. That means, in the second round, the follower i finds it a best response to bid if and only if his type is above s_i', as prescribed by the proposed equilibrium.

Thus we need only to justify the equilibrium strategy for the initial round. Consider the decision of any player $i \in \{1, 2\}$ in the initial round. If player i bids and becomes the frontrunner in the second round, then he commits the first increment δ; if the other player $-i$ does not bid, then player i wins and gets the payoff

$$-\frac{\delta}{t_i} + v;$$

if player $-i$ also bids in the initial round (and fails to be selected the frontrunner), player i's expected payoff is equal to

$$-\frac{\delta}{t_i} + V_i^0(t_i|s_i^0, s_{-i}^0) \stackrel{(21)}{=} -\frac{\delta}{t_i} + \frac{\delta}{s_i^0} + \left(1 - \frac{\delta}{s_i^0}\right) M_i(t_i|s_i^0, s_{-i}^1). \tag{71}$$

If player i bids but is not selected the frontrunner, then his expected payoff is equal to

$$M_i(t_i|s_i^0, s_{-i}^0) = \max\left\{0, -\frac{2\delta}{t_i} + V_i(t_i|s_i^1, s_{-i}^0)\right\}.$$

If player i stays put, then the game ends, either with no sale if player $-i$ also stays put, or with $-i$ bidding and winning the good at price δ; in either case player i gets zero. Ineq. (67), applied to the case where $(s_i', s_{-i}) = (s_i^1, s_{-i}^0)$ in the third round, means

$$-\frac{2\delta}{t_i} + V_i(t_i|s_i^1, s_{-i}^0) \begin{cases} > 0 & \text{if } t_i > s_i^1 \\ \leq 0 & \text{if } t_i \leq s_i^1. \end{cases}$$

Thus, in the second round, Eq. (19) implies

$$M_i(t_i|s_i^0, s_{-i}^0) \begin{cases} > 0 & \text{if } t_i > s_i^1 \\ = 0 & \text{if } t_i \leq s_i^1. \end{cases}$$

If $t_i \leq s_i^0$, $t_i < s_i^1$ and hence $M_i(t_i|s_i^0, s_{-i}^0) = 0$, so Eq. (71) implies

$$-\frac{\delta}{t_i} + V_i^0(t_i|s_i^0, s_{-i}^0) = -\frac{\delta}{t_i} + \frac{\delta}{s_i^0} + \left(1 - \frac{\delta}{s_i^0}\right) M_i(t_i|s_i^0, s_{-i}^1) = -\frac{\delta}{t_i} + \frac{\delta}{s_i^0} \leq 0;$$

by contrast, if $t_i > s_i^0$,

$$-\frac{\delta}{t_i} + V_i(t_i|s_i^0, s_{-i}^0) = -\frac{\delta}{t_i} + \frac{\delta}{s_i^0} + \left(1 - \frac{\delta}{s_i^0}\right) M_i(t_i|s_i^0, s_{-i}^1) \geq -\frac{\delta}{t_i} + \frac{\delta}{s_i^0} \geq 0.$$

Thus, as long as $-\delta/s_i^0 + v \geq 0$, it is a best response for player i to bid in the initial round if and only $t_i > s_i^0$. Note that $-\delta/s_i^0 + v \geq 0$ is equivalent to $\delta/s_i^0 \leq v$, which is guaranteed by Eq. (16), because Eq. (16), with the roles of i and $-i$ switched, implies

$$\frac{\delta}{s_i^0} = vF_{-i}(s_{-i}^0) \leq v.$$

Hence $-\delta/s_i^0 + v \geq 0$ is true, so the strategy in the initial round prescribed by the proposed equilibrium is a best response for i.

References

[1] Ned Augenblick. The sunk-cost fallacy in penny auctions. *Review of Economic Studies*, 83(1):58–86, 2016. 4

[2] Ettore Damiano, Li Hao, and Wing Suen. Optimal deadlines for agreements. *Theoretical Economics*, 7:357–393, 2012. 1

[3] Ettore Damiano, Li Hao, and Wing Suen. Optimal delay in committees. Mimeo, May 2, 2017. 1

[4] Eddie Dekel, Matthew O. Jackson, and Asher Wolinsky. Vote buying: Legislatures and lobbying. *Quarterly Journal of Political Science*, 4:103–128, 2009. 1

[5] Gabrielle Demange. Rational escalation. *Ann. Econ. Stat.*, 25(26):227–249, 1992. 1, 2.3

[6] Faruk Gul and Wolfgang Pesendorfer. The war of information. *Review of Economic Studies*, 79:707–734, 2012. 1

[7] Michael J. Haupert. Sunk Cost and Marginal Cost: An Auction Experiment. Web Posting, 1994. 1

[8] Ken Hendricks, Andrew Weiss, and Charles Wilson. The war of attrition in continuous time with complete information. *International Economic Review*, 29:633–680, 1988. 1

[9] Toomas Hinnosaar. Penny auctions. *International Journal of Industrial Organization*, 48:59–87, 2016. 4

[10] Johannes Hörner and Nicolas Sahuguet. A war of attrition with endogenous effort levels. *Economic Theory*, 47:1–27, 2011. 1, 2.3

[11] Ali Kakhbod. Pay-to-bid auctions: To bid or not to bid. *Operations Research Letters*, 41(5):462–467, 2013. 4

[12] Wolfgang Leininger. Escalation and cooperation in conflict situations: The dollar auction revisited. *Journal of Conflict Resolution*, 33(2):231–254, 1989. 1

[13] Moritz Meyer-ter-Vehn, Lones Smith, and Katalin Bognar. A conversational war of attrition. *Review of Economic Studies*, 2017. Forthcoming. 1

[14] Andrea Morone, Simone Nuzzo, and Rocco Caferra. The Dollar Auction Game: A laboratory comparison between individuals and groups. SSRN Working Paper, SSRN 2912914, Feb. 7, 2017. 1

[15] J. Keith Murnighan. A very extreme case of the dollar auction. *Journal of Management Education*, 26(1):56–69, 2002. 1

[16] Fredrik Ødegaard and Chris K. Anderson. All-pay auctions with pre- and post-bidding options. *European Journal of Operational Research*, 239:579–592, 2014. 4

[17] Barry O'Neill. International escalation and the dollar auction. *Journal of Conflict Resolution*, 30(1):33–50, 1986. 1

[18] Martin J. Osborne. Signaling, forward induction, and stability in finitely repeated games. *Journal of Economic Theory*, 50:22–36, 1990. 2

[19] Martin Shubik. The dollar auction game: A paradox in noncooperative behavior and escalation. *Journal of Conflict Resolution*, 15(1):109–111, 1971. 1, 2.3

[20] Allan I. Teger, Mark Cary, Aaron Katcher, and Jay Hillis. *Too Much Invested to Quit*. New York: Pergamon Press, 1980. 1

[21] Marcin Waniek, Agata Nieścieruk, Tomasz Michalak, and Talal Rahwan. Spiteful bidding in the dollar auction. *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence*, pages 667–673, 2015. 1