Endogenous Labor Market Cycles*

Yunan Li and Cheng Wang†

October 16, 2018

Abstract

This paper shows that in a perfectly stationary physical environment of the labor market, moral hazard and competition in long-term contracts can generate cycles in the tightness of the market, which in turn may induce job creation and destruction, and two period or much longer cycles in employment and output. We claim that the model may shed light on the unemployment volatility puzzle which has inspired many discussions in the literature.

Keywords: endogenous cycles, moral hazard, long-term contract, termination

JEL Classification: E32, D86

*First version: 2010. Earlier versions of this paper were circulated under the title "Endogenous Cycles in a Model of the Labor Market with Dynamic Moral Hazard." For helpful comments and suggestions, we would like to thank Chao Gu, Roger Myerson and the audience at the University of Pennsylvania, the Economic Dynamics Workshop at Fudan University, the 2018 SAET Conference in Taipei, and the 2018 EEA-ESEM Conference in Cologne.

†Li: City University of Hong Kong. Email: *yunanli@cityu.edu.hk*. Wang: Fudan University. Email: *wangcheng@fudan.edu.cn*.

1 Introduction

The search and matching model of the labor market introduces unemployment and aggregate worker flows into the real business cycle theory, but faces difficulties in generating enough cyclical movements in unemployment. This is pointed out by Shimer (2005), who shows that, with commonly used parameter values, the volatility in unemployment that a standard Mortensen and Pissarides model can generate is very small compared to that in the U.S. data. The vacancy-unemployment ratio (the "tightness") in the U.S. data is 20 times as large as what the standard search model predicts. Pissarides (2009) calls this the unemployment volatility puzzle: standard theory cannot explain why unemployment fluctuates so much in the data.

To resolve this puzzle, the baseline search model has been modified to produce more fluctuations in unemployment. So far, the role of sticky wages has been emphasized. Shimer (2005) and Hall (2005) suggest that if wages are sticky, then the economy's response to a productivity shock should be reflected more in the cyclical movements of the model's employment and unemployment measures.[1] Pissarides (2009), however, argues that the empirical evidence on wage rigidity is not consistent with what is needed for the search model to provide a resolution for the unemployment volatility puzzle. Specifically, for search models to generate enough unemployment volatility, wages of new hires must be sufficiently sticky, whereas in the data, it is the wages in ongoing jobs that are sticky, not that of new hires.[2]

This paper aims to contribute to the resolution of the unemployment volatility puzzle. Relative to the literature, however, we take an entirely different approach. Instead of seeking to produce stronger labor market fluctuations as a response to some exogenously given productivity or demand shock, we show that the labor market has, in itself, a natural source of instability in the allocation of market power between firms and workers, and this may drive dynamics in jobs and aggregate output that are consistent with the observed pro-cyclical movements in market tightness. The implication, therefore, is that the large observed pro-

[1] Following this idea, Kennan (2010) and Moen and Rosen (2011) show that private information with regard to match quality can give rise to wage stickiness which, in turn, increases the responsiveness of unemployment to productivity shocks. Costain and Jansen (2010) show that moral hazard, put in a standard search model, could impose a lower bound on the worker's share of match surplus which, in turn, may amplify fluctuations in hiring and make the firm's share of the surplus move pro-cyclically. See Rogerson and Shimer (2011) for a review of this literature.

[2] The same argument is also made by Haefke, Sonntag, and van Rens (2013) and Kudlyak (2014) who provide direct empirical evidence that wages of newly hired workers, unlike the aggregate wage, are as volatile as labor productivity. And they suggest that in order to replicate these findings in a search model, wages must be rigid in ongoing jobs but flexible with new jobs, but this should not affect job creation and thus cannot help resolve the puzzle.

cyclical volatility in labor market tightness, which the search model has so far not been able to provide an adequate account for, may be better explained in a story that does not at all rely on search and the variability in worker productivity.

Specifically, the model economy consists of overlapping generations of workers who each live for two periods. Firms are free in creating new jobs — subject to a cost — and destroying old. Both short-term and long-term contracts may be traded in a competitive labor market. Firms run a stochastic technology that uses worker effort as inputs. Worker effort is not observable to the firm and so there is moral hazard. With such a model, we show that moral hazard and costly job creation generate labor market cycles in vacancies and tightness where the optimal contract, with an efficiently designed termination mechanism, takes the economy from a state with a large measure of vacancies to a state with a small measure of vacancies; from a tight current labor market to a less tight next labor market. In the model, termination generates vacancies. A tight current labor market prescribes contracts that specify small probabilities of termination, and hence a small measure of vacancies or a less tight labor market next period.

These are also cycles in the division of market power between vacant firms and unemployed workers. If firms are given an upper hand in the current labor market, then the contracts they offer will create a condition in the next labor market to let the workers take the upper hand; and if it is the workers who are holding the market power in the current period, then competition in the contracts offered will let the market power be turned over to the firms in the next period. Depending on which side holds the market power, the optimal contract prescribes a lower lifetime compensation and a larger probability of termination, or a larger lifetime compensation and smaller probability of termination, for the worker who takes a long-term contract.

Depending on the environment, labor market cycles could be *pure*, where the economy's employment and output stay constant over time, or they could induce job creation and destruction, and thus movements in employment and aggregate output. In the latter case, a period that starts with a large measure of vacancies would see jobs being destroyed and the economy moving into a recession with a high unemployment rate, whereas a period that starts with a small measure of vacancies would see new jobs created to lead the economy to a boom in employment and total output. Moreover, cycles in employment and output may differ in durations and cyclicality from the cycles in the distribution of market power that generates them. The division of market power between works and firms moves always in two-period cycles, but the model has equilibria where there is large variability in the durations of individual employment and output cycles, resembling that in observed business cycles.

Relative to the search model, there are no productivity shocks in our model, and there are no search and matching frictions. Neither are there any exogenously imposed or endogenously derived wage rigidity. Wages are not bargained periodically between the matched worker and firm; they are, along with an optimally designed termination clock, part of the employment contract whose values for the parties are determined in the competitive equilibrium of the economy.

Related Literature. There is an early literature on endogenous business cycles [see Boldrin and Woodford (1990) for a survey], but the models in that literature do not produce involuntary unemployment or endogenous vacancies. Since the earlier draft of this paper, there has been a growing interest in studying endogenous fluctuations in the labor market. Kaplan and Menzio (2016) and Sniekers (2018) explain labor market fluctuations with externalities in the goods market. In Eeckhout and Lindenlaub (2018), labor market fluctuations are driven by a complementarity between on-the-job search and job creation. More closely related to our work is Golosov and Menzio (2015). In their model, like in ours, there is moral hazard and firms randomize between firing and retaining non-performing workers upon the realized output. In equilibrium, firms use sunspot as a device for coordinating firing and the economy experiences endogenous fluctuations. There are two key assumptions for their results. First, employment contracts are sufficiently incomplete so that firing takes place on the equilibrium path. Second, there is decreasing returns to matching in the labor market.

Relative to Golosov and Menzio (2015), in our model firing arises as part of the optimal complete contract, and unemployed workers and vacant jobs are matched up in a frictionless labor market. In addition, cycles in our model are not driven by self-fulfilling expectations, and there is not a strategic complementarity that coordinates individual actions either. In our model, expectations on future market conditions induce individual actions that work against, rather than reinforce, the cycle.[3] This aspect of our story thus distinguishes it from many of the models in the literature that rely on sunspot or other self-fulfilling mechanisms for generating fluctuations.

Our paper is also related to the literature on endogenous financial market cycles. An earlier contribution in this literature is Suarez and Sussman (1997, 2007), where increased output of the product good in a boom period reduces the price of that good and increases the demand for external finance. With moral hazard, this leads to excessive risk taking and a high failure rate in their investment, pushing the economy into a bust. In Favara (2012), entrepreneurs must costly evaluate and then select a project which generates verifiable income

[3]If other firms prescribe larger probabilities of termination for their new hires, that would imply a tighter labor market next period, discouraging rather than encouraging the individual firm from prescribing also a large probability of termination.

and non-verifiable private benefits. In a boom period, wealthier entrepreneurs rely less on external finance, and select projects with lower productivity and higher private benefits, paving the way for a subsequent bust. Banerji and Wang (2013) consider economies with risk averse agents where cycles are driven by the trade-off between incentives and insurance. In a bust period, young agents take more risk, work harder and are more productive, and this leads to an economic boom. In a boom period, young agents are fully insured, they shirk and are less productive, and this leads to a recession. Myerson (2012) models moral hazard in long-term lending relationships for generating endogenous fluctuations. He shows that the aggregation of the dynamics in individual lending contracts can generate cycles in the credit market. He imposes that the banker is terminated after his investment fails, and because of this, cycles are of the same fixed length as that of the life of the banker. In contrast, in our paper, termination probability is optimally determined and endogenously affected by the tightness of the labor market. Gu, Mattesini, Monnet, and Wright (2013) show that endogenous cycles can arise in credit markets where agents honor debt obligations to avoid exclusion from future credit. Their model is capable of generating a large set of self-fulfilling non-stationary equilibria, including sunspot cycles.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the optimal contracts. Section 4 defines an equilibrium. Section 5 analyzes the stationary equilibria. Section 6 studies the model's equilibrium cycles. Section 7 compares welfare outcomes in the stationary and non-stationary equilibria. Section 8 concludes.

2 Model

Time is discrete and lasts forever: $t = 1, 2, \ldots.$ There is a single perishable good. The economy contains a sequence of overlapping generations of workers. Each worker, who lives for two periods, is young in the first and old in the second. We normalize the total measure of workers living at any time to one, with each generation containing $1/2$ units of mass. Workers born in period t and alive in periods t and $t+1$ maximize $E_t\{(1-\delta)(c_t - a_t) + \delta(c_{t+1} - a_{t+1})\}$, where E_t denotes expectation taken at the beginning of period t; c_t (c_{t+1}) and a_t (a_{t+1}) are period t $(t+1)$ consumption and effort, respectively; and $\delta \equiv \beta/(1+\beta)$, where $\beta \in (0, 1)$ is workers' discount factor.[4] Consumption must be non-negative: $c_t \geq 0$ for all t. For analytical tractability, assume effort takes one of two values, with the low effort normalized to zero. That is, we assume $a_t \in \{0, \psi\}$ for all t, with $\psi > 0$.

[4]The use of δ normalizes workers' lifetime utilities. This normalization is necessary for the specification of the optimal employment contracting problems which are to appear, as firms in the model live an infinite life whereas workers live only for two periods, and both static and long-term contracts must be considered and their values compared.

The economy also has a positive measure of infinitely lived firms, each of them could employ in each period one worker to produce. There is moral hazard: the effort that an employed worker exerts is observed by herself only. By choosing effort a_t in period t, a worker produces a publicly observed random output in period t that is a function of a_t. Let $\theta^t \in \{\theta_1, \theta_2\}$ denote the realization of this random output, where $\theta_1 < \theta_2$. Let

$$x_i \equiv \text{Prob}\{\theta^t = \theta_i | a = \psi\} \text{ and } x_i' \equiv \text{Prob}\{\theta^t = \theta_i | a = 0\}, \; i = 1, 2.$$

That is, x_i is the probability with which output θ_i is produced if a worker works, and x_i' the probability if she shirks. Assume that $x_i, x_i' > 0$ for both i and $x_1 + x_2 = x_1' + x_2' = 1$. Let $\bar{\theta} \equiv \sum x_i \theta_i$ ($\underline{\theta} \equiv \sum x_i' \theta_i$) denote the mean output produced when a worker works (shirks). We assume that $\bar{\theta} - \underline{\theta}$ is large enough so that $a = 0$ is never desirable for a firm.

At the start of any period, firms are in three different states respectively: incumbent firms (those currently in the labor market) who are with an old worker retained from the prior period, incumbent firms who are currently vacant, and potential entrants. When a period begins, incumbent vacant firms decide whether to exit the market, and potential entrants decide whether to enter the market. Once the entry and exit decisions are made, the labor market opens where vacant firms and unemployed workers match and enter employment contracts. Unlike the standard search and matching theory, we assume that the matching process is frictionless. As it will become clear, we impose this assumption in order to highlight the role played by other frictions in generating economy fluctuations. Finally, production takes place, contracts are carried out, and the period ends.

In order to enter the market an entrant must first make an initial investment, modeled as a fixed entry cost, $C_e/(1-\beta) \geq 0$, which is sunk thereafter. In order to exit the market, however, a vacant firm could simply stop the business and leave, not subject to any costs. In order to stay in the market, each firm must incur an operating cost $C_o > 0$ in each period.

The contract between any firm and any old worker is one-period long. A firm and a young worker have the opportunity to enter a long-term contract which can potentially last for two periods, but may specify a condition under which the worker is fired after a single period. When a worker is fired by a firm, she is free to go back to the labor market to look for new employment, and the firm is also free to go back to the labor market to hire a new worker. As part of the physical environment, we assume that once a worker is fired (i.e., once the contract is terminated), the interaction between the worker and her employer ends. In particular, if a worker is fired by a firm at the end of period t, she will not be able to receive payments from the firm in period $t+1$. We assume that there are no physical costs that either party must incur in the process of any termination.

We assume that a firm is fully committed to the terms of any contract that it is willing to enter. The full commitment assumption also requires a firm to implement any realized outcome of a randomized termination rule. This makes it feasible for the contract to specify a non-degenerate probability of termination. The commitment to a contract from a worker, however, is not assumed.[5]

3 Contracting

Consider the partial problem of optimal contracting for an individual vacant firm, taking as given the current and future states of the market and that the contract must give a worker a given level of expected utility, denoted w.[6] Let the period be t. A *short-term or one-period contract* offered in this period takes the form of $\sigma_s = \{c_1, c_2\}$, where c_i $(i = 1, 2)$ is the worker's compensation in output state i. We omit subscript t for simplicity. For any given expected utility w it must deliver to the worker, the firm seeks to maximize its expected value:

$$V_{s,t}(w) \equiv \max_{\sigma_s} \quad \left\{ \left[(1-\beta) \sum x_i \left(\theta_i - c_i - C_o \right) \right] + \beta V_{t+1} \right\}, \tag{1}$$

$$\text{s.t.} \quad c_1, c_2 \geq 0, \tag{2}$$

$$x_1 c_1 + x_2 c_2 - \psi \geq x_1' c_1 + x_2' c_2, \tag{3}$$

$$x_1 c_1 + x_2 c_2 - \psi = w, \tag{4}$$

where V_{t+1} is the maximized value of a firm who is vacant at the start of period $t+1$, having not made the decision whether to stay in the market. Here, equation (2) is a limited liability condition on compensation; (3) is the incentive compatibility constraint, which says that the worker is better off working than shirking; and (4) is the promise keeping constraint.

Notice that constraint (3) implies $c_2 - c_1 \geq \psi/(x_2 - x_2')$. Combining this with constraints (2) and (4) gives

$$w = c_1 + x_2(c_2 - c_1) - \psi \geq \underline{w}, \text{ where } \underline{w} \equiv \frac{x_2' \psi}{x_2 - x_2'} > 0.$$

[5] In equilibrium, it is optimal for a worker to stay with her current employer. Hence, the result will not change if we also require a worker to be fully committed to a contract.

[6] The problem of dynamic contracting with moral hazard and endogenous termination is studied in Spear and Wang (2005). This problem is embedded in a general equilibrium environment in Wang (2013) to study, in stationary labor market equilibria, the effects of firing costs on aggregate activity and welfare. The idea that the threat of termination can be used as an incentive device for motivating worker effort is modeled in Shapiro and Stiglitz (1984) in a theory of efficiency wages.

Observe that $\underline{w}$ is the minimum promised expected utility w that is consistent with incentive compatibility. When $w \geq \underline{w}$, the firm's value from entering the contract is:

$$V_{s,t}(w) = (1-\beta)(\bar{\theta} - \psi - w - C_o) + \beta V_{t+1}, \ \forall w \geq \underline{w}. \tag{5}$$

Note that the firm's expected value of being vacant is $V_{v,t} \equiv -(1-\beta)C_o + \beta V_{t+1}$.

Let $\underline{C} \equiv \bar{\theta} - \psi - \underline{w}$ denote the maximum one-period value a firm can obtain from entering a one-period contract. Assume $\underline{C} > 0$. That is, conditional on staying in the market in a period, a firm is better off entering a short-term contract than being vacant in that period.

Consider next the problem of optimal long-term contracting. A *long-term or two-period contract* takes the form of $\sigma_l = \{c_{ik}, w_{ik}, p_i, i = 1, 2, k = r, f\}$, where i denotes the state of the first period output; $k = r(f)$ indicates that the worker is retained (fired) after the first period; p_i is the probability with which the worker is fired after the first period in output state i; and c_{ik} and w_{ik} are the worker's first period compensation and second period expected utility respectively in state ik. Note again that we drop subscript t for simplicity.

Since the firm and the worker will not interact with each other after termination, we have $w_{if} = w_{*t+1}$, where w_{*t+1} denotes the expected utility of any unemployed old worker at the start of period $t+1$. With this, the value of a firm who enters a long-term contract that gives the young worker expected utility w at the beginning of period t is

$$V_{l,t}(w) = \max_{\sigma_l} \left\{ \begin{array}{c} \sum x_i[p_i((1-\beta)(\theta_i - c_{if} - C_o) + \beta V_{t+1}) \\ +(1-p_i)((1-\beta)(\theta_i - c_{ir} - C_o) + \beta V_{s,t+1}(w_{ir}))] \end{array} \right\}, \tag{6}$$

$$\text{s.t.} \quad c_{ik} \geq 0, w_{ir} \geq \underline{w}, 0 \leq p_i \leq 1, i = 1, 2, r = k, f,$$

$$\sum_i x_i[(1-p_i)((1-\delta)c_{ir} + \delta w_{ir}) + p_i((1-\delta)c_{if} + \delta w_{*t+1})] - (1-\delta)\psi$$
$$\geq \sum_i x_i'[(1-p_i)((1-\delta)c_{ir} + \delta w_{ir}) + p_i((1-\delta)c_{if} + \delta w_{*t+1})], \tag{7}$$

$$\sum_i x_i[(1-p_i)((1-\delta)c_{ir} + \delta w_{ir}) + p_i((1-\delta)c_{if} + \delta w_{*t+1})] - (1-\delta)\psi = w. \tag{8}$$

Inequality $c_{ik} \geq 0$ is a limited liability condition. Inequality $w_{ir} \geq \underline{w}$ says that if the worker is retained after one period, her second period expected utility must be high enough to support incentive compatibility in the second period. (7) and (8) are, respectively, incentive compatibility and promise-keeping constraints for the first period.

To solve for the optimal long-term contract, consider

Assumption 1 $V_{t+1} - (1-\beta)(\bar{\theta} - \psi - w_{*t+1} - C_o) - \beta V_{t+2} \leq 0$.

This assumption essentially postulates that the firm should minimize rather than maximize the termination probability. To see this, consider an increase in p_i in the above problem of optimal long-term contracting. This change affects the firm's value in two opposite directions. On the one hand, the firm is better off being vacant than staying with the old worker: $V_{t+1} \geq V_{s,t+1}(w_{ir})$, an optimality condition which we will derive later (equation(12)). On the other hand, in order to satisfy the promise keeping constraint the firm must raise w_{ir}, the worker's value of retention. The overall marginal cost of termination is thus $(1-\beta)(\bar{\theta} - \psi - w_{*t+1} - C_o) + \beta V_{t+2} - V_{t+1}$, which is non-negative by Assumption 1. In other words, although there are no physical costs associated with termination, termination is costly from the firm's perspective and the firm should minimize the termination probability.

Observe that in Assumption 1, V_{t+1}, V_{t+2} and w_{*t+1} are all endogenous variables of the model. To resolve this difficulty, we use the following strategy: We first solve for the optimal long-term contract under Assumption 1. We then solve for equilibria in which Assumption 1 does hold in all periods. Finally, we show that the model does not have any equilibria in which Assumption 1 is violated in some period.

Let $w_{A,t} \equiv (1-\delta)\underline{w} + \delta w_{*t+1}$, which, as shown below, is the minimum promised expected utility that is consistent with incentive compatibility in a long-term contract. By Assumption 1 and the optimality condition (12), $w_{*t+1} \leq \underline{w}$. Hence, $w_{A,t} \leq \underline{w}$.

Proposition 1 *Suppose Assumption 1 holds for t. (i) If $w < w_{A,t}$, then w is not attained by any incentive compatible long-term contract. (ii) If $w \in [w_{A,t}, \underline{w})$, then the optimal long-term contract in period t has*

$$p_1^*(w) = \frac{\underline{w} - w}{\delta(\underline{w} - w_{*t+1})}, p_2^*(w) = 0,$$
$$c_{1r}^*(w) = c_{1f}^*(w) = c_{2r}^*(w) = 0,$$
$$w_{1r}^*(w) = \underline{w}, w_{2r}^*(w) = \frac{1}{\delta}\left(w - (1-\delta)\underline{w} + \frac{(1-\delta)\psi}{x_2 - x_2'}\right).$$

(iii) If $w \geq \underline{w}$, the optimal long-term contract in period t has

$$p_1^*(w) = p_2^*(w) = 0,$$
$$c_{1r}^*(w) = c_{2r}^*(w) = 0,$$
$$w_{1r}^*(w) = \frac{w - (1-\delta)\underline{w}}{\delta}, w_{2r}^*(w) = w_{1r}^*(w) + \frac{(1-\delta)\psi}{(x_2 - x_2')\delta}.$$

The proof of the proposition is relegated to Appendix A. Proposition 1 illustrates how termination is used as an incentive device in the optimal long-term contract. Observe that the expected utility of a retained old worker must be at least $\underline{w}$ in order to support incentive compatibility in the second period, but that of an unemployed old worker is equal to w_{*t+1}, where $w_{*t+1} \le \underline{w}$. The gap between w_{*t+1} and $\underline{w}$ implies that termination is a punishment on the worker, and a higher p_1^* means that the worker faces a larger expected punishment for producing the low output. Observe also that termination is costly from the firm's perspective and its use is minimized with the optimal contract. Start now with a sufficiently high expected utility $w(\ge \underline{w})$ that the contract promises to the young worker. Incentive compatibility can then be achieved without using termination. Incentives can be obtained simply by giving the worker more compensation (in expected terms) in the state of high output and less in the state of low output. Suppose next that w is reduced to be below $\underline{w}$. Then a state contingent compensation plan that excludes termination could no longer implement the expected utility promised — the binding constraint being that the worker's expected utilities be at least $\underline{w}$ in all states of retention. A positive probability of termination must then be built into the contract. Moreover, the lower is w, the higher is the probability of termination that the optimal contract must prescribe in the state of low output. That is, p_1^* increases as w decreases. In addition, p_1^* is increasing in w_{*t+1}: A lower w_{*t+1} implies that termination is more effective as an incentive device. As a result, a lower termination probability p_1^* is required in order to support incentive compatibility in the optimal contract.[7]

Notice that it is feasible to set $p_1 = p_2 = 1$ in any long-term contract to make it essentially a short-term contract. Notice also that, as is shown in Appendix A.1, if $p_2 = 1$, the firm can get (weakly) better off by reducing p_2 and c_{2f} simultaneously. Therefore, the firm is (weakly) better off offering the young worker the optimal long-term contract than the optimal one-period contract:

$$V_{l,t}(w) \ge V_{s,t}\left(\frac{w - \delta w_{*t+1}}{1-\delta}\right), \ \forall w \ge w_{A,t}, \tag{9}$$

where the inequality holds strictly if and only if $w_{*t+1} < \underline{w}$. Note that in order to fulfill the promise of a given expected utility w at the beginning of a period t, the one-period contract must give the worker an expected utility of $(w - \delta w_{*t+1})/(1-\delta)$. In particular, inequality (9) implies $V_{l,t}(w_{A,t}) \ge V_{s,t}(\underline{w})$, where the inequality holds strictly if and only if $w_{*t+1} < \underline{w}$.

It follows easily from Proposition 1 that the value of a firm entering the optimal long-term contract for each $w \ge w_{A,t}$ is given by

$$V_{l,t}(w) = \begin{cases} k_0 - kw, & w \in [w_{A,t}, \underline{w}), \\ (1-\beta^2)(\bar{\theta} - \psi - w - C_o) + \beta^2 V_{t+2}, & w \in [\underline{w}, \infty), \end{cases} \tag{10}$$

[7] That the young worker's consumption is always zero is because of risk neutrality. If workers are risk averse, their consumption when young will depend on their life-time utility. Moreover, because productivity is constant in time, our paper is not subject to the criticism of Haefke, Sonntag, and van Rens (2013) and Kudlyak (2014).

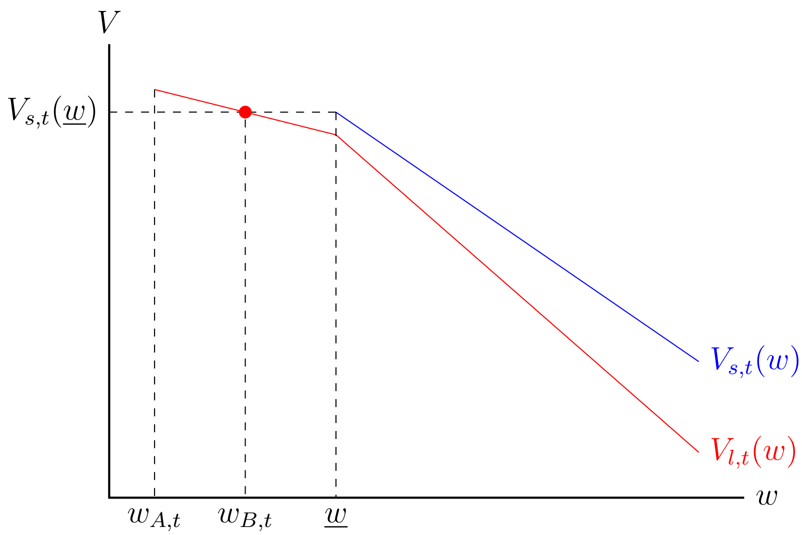


Figure 1: The Value Functions

where

$$k = (1+\beta)x_1 \frac{V_{t+1} - V_{s,t+1}(\underline{w})}{\underline{w} - w_{*t+1}} + (1-\beta^2)x_2 > 0$$

and the inequality holds because of (12). Hence, $V_{l,t}(\cdot)$ is strictly decreasing in w over $[w_{A,t}, \infty)$ and achieves its maximum value at $w_{A,t}$. The relationship between $V_{s,t}(\cdot)$ and $V_{l,t}(\cdot)$ is illustrated in Figure 1.

To close this section, remember that Proposition 1 is derived under Assumption 1. In Proposition 6, which is stated and proved in Appendix A.2, we show that if Assumption 1 is violated at t, then for all w with which the problem that defines $V_{l,t}(w)$ has a solution, it is optimal to set $p_i(w) = 1$ for $i = 1, 2$. This is straightforward to see. Suppose Assumption 1 is violated at t, then termination is profitable in period t and the optimal contract must maximize the use of it by setting $p_1^* = p_2^* = 1$. That is, the optimal long-term contract is essentially a short-term contract.

4 Equilibrium

In this section, we formally define a rational expectations equilibrium of the model. We start by looking at the firm's entry/exit or job creation/destruction decisions.

4.1 Job creation and destruction

Before the labor market opens, potential entrants and incumbent vacant firms decide whether to enter and exit it, and they do so taking as given their rationally perceived equilibrium dynamics that will unfold in the current and future markets.

Consider first an incumbent vacant firm. Let V_t denote its value if it decides to stay in the market. Its value is 0 if it exits the market. Let η_t denote the measure of firms in the market in period t after all the entry and exit decisions have been made. In equilibrium, if some incumbent vacant firms choose to stay in the market in any period t, it must hold that $V_t \geq 0$, for otherwise these firms would be better off exiting the market. If there is a positive flow of firms out from the market (i.e., $\eta_t < \eta_{t-1}$), then it must hold that $V_t = 0$ — the outflow would continue until leaving the market no longer entails a positive net gain.

For a potential entrant, the net gains from entering the market are $V_t - C_e$. Since firms are willing to enter the market as long as this value is positive and there is an infinite supply of potential entrants, free entry implies $V_t - C_e \leq 0$. And the equality holds whenever there is a positive inflow of firms (i.e., $\eta_t > \eta_{t-1}$), in which case firms would have entered the market until the net gains from entering are pushed down to zero.

Thus, a *free entry and exit* condition we will impose on the equilibrium of the model, whose full definition to be given shortly, is

$$0 \leq V_t \leq C_e, \ \forall t, \tag{11}$$

where $V_t = 0$ if $\eta_t < \eta_{t-1}$ and $V_t = C_e$ if $\eta_t > \eta_{t-1}$.

4.2 Matching and contracting

Upon the entry and exit of firms, the labor market opens for matching and contracting. An equilibrium in this market will consist of a collection of matched firm-worker pairs and expected utilities offered to the workers within these pairs so that no individual worker or firm can benefit from unilaterally deviating from her/its current status, and no new firm-worker pair can arise to make both the worker and the firm strictly better off.[8]

We focus on equilibria which are non-degenerate (i.e., the labor market does open in each period). For this, we make the following assumption:

Assumption 2 *$C_e < \bar{C} - C_o$, where $\bar{C} \equiv \bar{\theta} - \psi - \frac{w}{(1+x_2\beta)}$.*

Since $C_e \geq 0$, Assumption 2 implies $\bar{C} - C_o > 0$. This ensues the existence of a non-degenerate equilibrium, with $\bar{C}$ being the value a firm attains if it hires a young worker and

[8]This definition of equilibrium is analogous to that of stability in a two-sided matching model with transferable utilities introduced by Shapley and Shubik (1971). The main departure is that we have a continuum of agents to be matched on both sides of the market. In addition, we require that both parties get strictly better off in order to constitute a blocking pair, and this rules out the possibility that a firm and an unemployed worker form a new pair that makes the firm indifferent but the worker strictly better off.

offers her expected utility $w_{A,t}$ — the minimum to support incentives in a long-term contract — in any period t when it is vacant.[9] Furthermore, we require that the entry cost cannot be too high to rule out the trivial case in which no firm would enter the market. Finally, to simplify the analysis, we assume that if firms are indifferent between entering long-term and short-term contracts, they prefer long-term contracts.

Note next, and as is shown in the appendix, that in equilibrium the measure of vacant firms must be no greater than that of unemployed workers (or equivalently $\eta_t \leq 1$ in all t). This is independent of the structure of the optimal contract, neither does it depend on inequality (12).[10]

In equilibrium, expected utilities offered to workers of the same age must be constant across all firm-worker pairs. To see this, suppose there is an equilibrium where one firm-worker pair offers its worker expected utility w_H but another pair offers w_L, with $w_H > w_L$. Then the firm offering w_H and the worker receiving w_L can form a new pair to give the worker an expected utility $w \in (w_L, w_H)$ and make both parties strictly better off, and this breaks the equilibrium. Let $w_{y,t}$ and $w_{o,t}$ denote the prevailing expected utilities offered to young and old workers in equilibrium, respectively.

Now $w_{y,t}$ and $w_{o,t}$ are determined by the measure of vacant firms relative to that of young and unemployed old workers. Suppose the measure of vacant firms is strictly less than that of young workers (1/2) in period t. Then we say that the labor market is *slack* and call period t a *type S period*. In this period, all vacant firms would enter a long-term contract with a young worker and offer her $w_{y,t} = w_{A,t}$. To see why $w_{y,t} = w_{A,t}$, suppose $w_{y,t} > w_{A,t}$. Then any firm that is paired with a young worker can form a new pair with an unemployed young worker and offer her expected utility $w_{A,t}$. With that, both parties of the new pair would be made strictly better off. In addition, no firm would hire an old worker since $V_{l,t}(w_{A,t})$ is (weakly) greater than $V_{s,t}(\underline{w})$ — the highest expected value firms could possibly obtain by entering a short-term contract with an old worker. Obviously, in a type S period, firms capture all the surplus from trading with young workers and the equilibrium value of a vacant firm is $V_t = V_{l,t}(w_{A,t})$.

Suppose the measure of vacant firms strictly exceeds that of young workers (1/2) in period t. We say that the market is *tight* in this period and call it a *type T period*. To see what happens in this period, suppose $w_{*t+1} < \underline{w}$.[11] Then in equilibrium 1/2 units of firms would

[9] In Appendix B.1.1, we show that in equilibrium it holds that $V_t \leq \bar{C} - C_o$ for all t.

[10] See Lemma 7 in Appendix B.1.

[11] The analysis can be easily extended to cover the case of $w_{*t+1} = \underline{w}$. Specifically, when $w_{*t+1} = \underline{w}$, $w_{A,t} = w_{B,t} = \underline{w}$ and $V_{l,t}(w_{A,t}) = V_{s,t}(\underline{w})$. First, we argue that if $\eta_t < 1$, then $w_{y,t} = w_{o,t} = \underline{w}$. Supposing $w_{y,t} > \underline{w}$ (or $w_{o,t} > \underline{w}$), a firm that employs a young worker (or an old worker) and an unemployed worker can form a new pair which offers the worker $\underline{w}$ and both get strictly better off. Second, when $w_{y,t} = w_{o,t} = \underline{w}$, firms are indifferent between entering long-term contracts and short-term contracts. By assumption, in

enter a long-term contract with a young worker and the rest a short-term contract with an old worker. Why? Suppose in equilibrium there are unemployed young workers. Then a firm that is vacant or employs an old worker could pair with an unemployed young worker and offer her an expected utility of $w_{A,t}$. This would make both the firm and the young worker strictly better off since $V_{l,t}(w_{A,t}) > V_{s,t}(\underline{w})$. In this case, $w_{o,t} = \underline{w}$ and $w_{y,t} = w_{B,t}$, where $w_{B,t}$ is such that a firm is indifferent between hiring a young worker and an old worker: $V_{l,t}(w_{B,t}) = V_{s,t}(\underline{w})$ (see Figure 1).[12] Clearly, in a type T period, young workers capture all the surplus from trade.

Lastly, suppose the measure of vacant firms is equal to 1/2. Then, similar to that in a type T period, in equilibrium all vacant firms would enter a long-term contract with a young worker. In this scenario, we assume, without loss of generality, that the utility of a newly hired old worker is given by $w_{o,t} = \underline{w}$. Then for firms to be willing to hire a young worker, $w_{y,t}$ must satisfy $V_{l,t}(w_{y,t}) \geq V_{s,t}(\underline{w})$, which implies $w_{y,t} \in [w_{A,t}, w_{B,t}]$. We call such a period a *type I period*.

The above discussion also suggests that in equilibrium all vacancies are filled with probability one and the following inequality holds:

$$V_t = \max \{V_{s,t}(w_{o,t}), V_{l,t}(w_{y,t})\} \geq V_{s,t}(\underline{w}). \tag{12}$$

4.3 Equilibrium: formal definition

Before defining an equilibrium, we first describe the law of motion for the aggregate states of the model. Let $\hat{p}^*_{it} \equiv p^*_i(w_{y,t})$ for simplicity. Let L_t denote the measure of all unemployed workers at the beginning of period t and $\alpha_{y,t}$ ($\alpha_{o,t}$) denote the job finding probability of an unemployed young (old) worker in period t. Then the evolution of η_t, L_t, $\alpha_{y,t}$ and $\alpha_{o,t}$ must satisfy

$$L_t = 1 - \frac{1}{2}\alpha_{y,t-1}[1 - (x_1\hat{p}^*_{1t-1} + x_2\hat{p}^*_{2t-1})], \tag{13}$$

$$\frac{1}{2}\alpha_{y,t} + (L_t - \frac{1}{2})\alpha_{o,t} = \eta_t - \frac{1}{2}\alpha_{y,t-1}[1 - (x_1\hat{p}^*_{1t-1} + x_2\hat{p}^*_{2t-1})]. \tag{14}$$

equilibrium 1/2 units of firms would enter a long-term contract with a young worker and the rest a short-term contract with an old worker. Similarly, if $\eta_t = 1$, then all workers would be employed. In this case, $w_{o,t} \in [\underline{w}, \overline{\theta} - \psi]$ and $w_{y,t}$ satisfies $V_{l,t}(w_{y,t}) = V_{s,t}(w_{o,t})$. For ease of exposition, we assume $w_{o,t} = \underline{w}$.

[12] If $\eta_t = 1$, then any $w_{o,t} \in [\underline{w},\ \overline{\theta} - \psi]$ and a $w_{y,t}$ that satisfy $V_{l,t}(w_{y,t}) = V_{s,t}(w_{o,t})$ would meet the equilibrium requirement specified at the beginning of Section 4.2. Now $w_{o,t} > \underline{w}$ would lead to a violation of inequality (12), which we have used for establishing Assumption 1. However, $w_{o,t} > \underline{w}$ together with $\eta_t = 1$ does imply that Assumption 1 holds with equality. In this case, it holds with the optimal contract that $w_{A,t-1} > \underline{w}$ and $\hat{p}^*_{1t-1} = \hat{p}^*_{2t-1} = 0$. We leave these cases, which are not important for our purpose, not explicitly discussed and assume instead $w_{o,t} = \underline{w}$ throughout our analysis, for ease of exposition.

Equation (13) says, in particular, that the measure of unemployed workers in any period depends on the firm's firing policy in the prior period. Equation (14) says that in equilibrium the measure of vacant firms in any period is equal to that of unemployed workers who obtain employment in that period. It then follows from equation (13) that the measure of vacant firms in period t satisfies $\eta_t - \frac{1}{2}\alpha_{y,t-1}[1-(x_1\hat{p}^*_{1t-1}+x_2\hat{p}^*_{2t-1})] = \eta_t - (1-L_t)$.

We are now ready to define a rational expectation equilibrium of the model.

Definition 1 *A rational expectations equilibrium of the model is a sequence*

$$\left\{\eta_t, L_t, \alpha_{o,t}, \alpha_{y,t}, w_{*t}, V_t; \sigma^*_{s,t}, \sigma^*_{l,t}, w_{y,t}, w_{o,t}\right\}_{t\geq 1}$$

where for all t, the following conditions hold.

1. *Given w_{*t+1}, $\sigma^*_{s,t}$ solves (1)-(4) and $\sigma^*_{l,t}$ solves (6)-(8), and*
   (a) *If $\eta_t - (1-L_t) < \frac{1}{2}$ (the period is type S),*
   $$w_{o,t} = \underline{w},\ w_{y,t} = w_{A,t}, \tag{15}$$
   $$0 < \alpha_{y,t} < 1,\ \alpha_{o,t} = 0, \tag{16}$$
   $$V_t = V_{l,t}(w_{y,t}); \tag{17}$$
   (b) *If $\eta_t - (1-L_t) > \frac{1}{2}$ (the period is type T),*
   $$w_{o,t} = \underline{w},\ w_{y,t} = w_{B,t}, \tag{18}$$
   $$\alpha_{y,t} = 1,\ \alpha_{o,t} > 0, \tag{19}$$
   $$V_t = V_{l,t}(w_{y,t}) = V_{s,t}(\underline{w}); \tag{20}$$
   (c) *If $\eta_t - (1-L_t) = \frac{1}{2}$ (the period is type I),*
   $$w_{o,t} = \underline{w}, w_{y,t} \in [w_{A,t}, w_{B,t}], \tag{21}$$
   $$\alpha_{y,t} = 1,\ \alpha_{o,t} = 0, \tag{22}$$
   $$V_t = V_{l,t}(w_{y,t}) \geq V_{s,t}(\underline{w}). \tag{23}$$
2. *The expected utility of an unemployed old worker at the beginning of period t is given by $w_{*t} = \alpha_{o,t}w_{o,t}$.*
3. *The aggregate variables η_t, $\alpha_{y,t}$, $\alpha_{o,t}$ and L_t satisfy (13) and (14).*
4. *Firms make optimal entry and exit decisions, i.e., V_t satisfies (11).*

The analysis in Appendix B.1 shows that Assumption 1 holds for all t in any equilibrium of the model (see Proposition 7). In other words, it is indeed legitimate to focus on equilibria that satisfy Assumption 1. Recall that an increase in the probability of termination affects a firm's value in two ways. First, termination allows the firm to go back to the labor market to do better than continuing with the current contract: $V_t \geq V_{s,t}(\underline{w})$. Second, to compensate the worker for the utility loss that results from the higher probability of termination, the firm must raise her expected utility in the state of retention, w_{ir}, which is costly for the firm. With free entry of firms into the market, in equilibrium V_t can never be sufficiently high so that the benefit of termination dominates its cost. This is consistent with Assumption 1.

5 Stationary equilibria

We say an equilibrium is *stationary* if in that equilibrium all of the model's aggregate states are constant in time. In this section, we study such equilibria.

5.1 Costly job creation: $C_e > 0$

Suppose firms must incur positive costs in entering the market to create a new job. Then multiple stationary equilibria exist in the model, as the following result states.

Proposition 2 *Suppose Assumption 2 holds. Then there exist $\eta_s^-(C_o, C_e)$ and $\eta_s^+(C_o)$ with $(1+x_2)/2 < \eta_s^-(C_o, C_e) < \eta_s^+(C_o) \leq 1$, and are such that for each $\eta \in [\eta_s^-, \eta_s^+]$ the model has a stationary equilibrium in which all periods are type I, the optimal long-term contract is as described in Proposition 1(ii) if $\eta < 1$ and Proposition 1(iii) if $\eta = 1$, and the equilibrium values of the aggregate variables satisfy*

$$\eta_t = \eta, \ L_t = \frac{3}{2} - \eta,$$
$$\alpha_{y,t} = 1, \ \alpha_{o,t} = 0,$$
$$w_{o,t} = \underline{w}, \ w_{y,t} = \underline{w} - \frac{2\beta(1-\eta)(\underline{w} - w_{*t+1})}{(1+\beta)x_1}, \ w_{*,t+1} = 0,$$
$$V_t = \bar{\theta} - \psi - \frac{[x_1(1+\beta) - 2\beta(1-\eta)]\underline{w}}{x_1[1+\beta-2\beta(1-\eta)]} - C_o.$$

Given Proposition 2, several intuitive results emerge, all straightforward to verify. First, as η increases from η_s^- to η_s^+, a firm's expected value V_t, which is constant in t, falls from C_e to 0. Second, both η_s^- and η_s^+ are decreasing in the operating cost C_o, indicating that if the cost to stay in the market is higher, then in equilibrium a smaller measure of firms will stay in the market. Third, for any given level of C_o, η_s^- decreases as C_e, the cost of job creation, increases. Obviously, a larger C_e impedes job creation and allows the market to support a smaller measure of firms who then each enjoy a higher value.

Proposition 2 indicates that, in any stationary equilibrium, only young workers are offered employment and only long-term contracts are traded. Once laid off, a worker never regains employment. Under $C_e > 0$, one-period contracts are not offered for two reasons. (i) Relative to the long-term contract, they are less profitable to the firm — or at least weakly so. (ii) In any stationary equilibrium of the model, the measure of vacant firms, measured at the beginning of any period, cannot be strictly larger than 1/2, and so there is always a young worker there to take a long-term contract if it is offered. Given these, naturally and as the analysis in Section 4.2 shows, in equilibrium all vacant firms would be matched with and employ a young worker, and no old workers would be hired.

To see why (ii) is the case, suppose otherwise. Then competition for young workers ensures that in equilibrium young workers find jobs with probability one and would be retained with probability one after one period. This, given (by Lemma 7 in the appendix) that the equilibrium measure of firms is no more than one unit, in turn implies that the measure of vacant firms next period must be less than or equal to 1/2, a contradiction to the stationarity assumption.

5.2 Costless job creation: $C_e = 0$

Suppose entering the market or creating a job imposes no costs on the firm: $C_e = 0$. Then the model has a unique equilibrium which is stationary.

To see this, notice that with $C_e = 0$, the free entry and exit condition dictates $V_t = 0$ for all t, which then implies that in equilibrium all firms in the market must be just breaking even in each period: their expected profits must be zero in all periods.

Given this, suppose $\underline{C} - C_o < 0$. That is, suppose vacant firms are strictly better off exiting the market than staying in the market to hire an old worker. Then in equilibrium all periods are type S or type I. In this case, since $w_{*t} = 0$ and $V_t = 0$ for all t, it holds that

$$V_{l,t}(w) = (1-\beta^2)(\underline{C} - C_o + \underline{w} - w) - x_1\beta(1-\beta)(\underline{C} - C_o + \underline{w})\frac{\underline{w} - w}{\underline{w}}, \ \forall w \in [w_{A,t}.\underline{w}].$$

It is straightforward to verify that $V_{l,t}(w_{A,t}) \geq (1-\beta)(1+x_2\beta)(\overline{C} - C_o) \geq 0$ and $V_{l,t}(\underline{w}) = (1-\beta^2)(\underline{C} - C_o) < 0$, and thus there is a unique $w_{y,t} \in [w_{A,t}, \underline{w})$ to satisfy $V_{l,t}(w_{y,t}) = 0$. This implies that $w_{y,t}$ and therefore the corresponding $\hat{p}^*_{1t}$ are constant over time. It also implies that in equilibrium all periods are type I in which young workers find jobs with probability one ($\alpha_{y,t} = 1$). With these, (13) and (14) can be solved for the equilibrium η_t and L_t, which are also constant in time. That is, the model has a unique equilibrium, and that equilibrium is stationary. Moreover, as is easy to verify, the equilibrium is characterized in Proposition 2 with $\eta_t = \eta_s^+$.

Suppose instead $\underline{C} - C_o > 0$. If $\eta_t < 1$ for some t, then vacant firms in the market have $V_t \geq (1-\beta)(\underline{C} - C_o) > 0$, as they can always hire an unemployed worker with a one-period contract, contradicting $V_t = 0$. Now since, by Lemma 7, in equilibrium it must hold that $\eta_t \leq 1$ for all t, the stationary equilibrium must thus have $\eta_t = 1$ for all t. And it then follows that all other endogenous variables of the model are also time invariant in equilibrium.[13]

As is with $C_e > 0$, under $C_e = 0$ again old workers never gain or regain employment in the stationary equilibrium of the model. Here, free entry into the market forces all individual contracts offered to earn a zero profit, rendering long-term contracts the only contracts offered — one-period contracts would earn a strictly negative profit if they were offered.

5.3 Stationarity as a constraint on the outcome of the model

Obviously, stationarity imposes a constraint on the solution to the model. With $C_e = 0$, however, this constraint is not binding. In this case, free entry into the market squeezes out any positive profits from the firms, forcing them to settle in a state where all surplus from contracting is given to the worker, resulting in a unique equilibrium of the model which is stationary.

With $C_e > 0$, the requirement of stationarity does put an active constraint on the variables whose movements drive or characterize the model's outcomes, including the measure of vacancies created in a period — they cannot exceed the measure of young workers, for otherwise too few vacancies would be created next period to break the stationarity requirement — and hence the tightness of the market. That old workers never gain or regain employment is another key outcome that results from the stationarity requirement. Stationarity rules out the possibility that firms are willing to incur a loss by employing an old worker in the current market, anticipating that the market condition would improve to bring to them a positive value next period.

In the section to follow, we show that once the model is free from the stationarity requirement, it is able to achieve a larger set of equilibria, some of which displaying cycles over which the tightness of the labor market — and thus the allocation of market power between workers and firms — move dynamically over time, and old workers may gain or regain employment with a positive probability.

[13]This result continues to hold when $\underline{C} = C_o$. A formal analysis on this is in Lemma 11 in the appendix.

6 Cycles

We now study the model's non-stationary equilibria where the aggregate states of the economy move dynamically in time. As was shown in Section 5.2, if $C_e = 0$, then, generically, the model has a unique equilibrium which is stationary. We therefore assume in this section $C_e > 0$. To streamline notation, call a period type SI if it is type S or type I, and a period type TI if it is type T or type I.

Let η_t^V denote the measure of vacant firms in period t after the firms have made their entry and exit decisions. The following law of motion holds for η_t^V :

$$\eta_{t+1}^V \equiv \eta_{t+1} - (1 - L_{t+1}) = \eta_{t+1} - \eta_t + \eta_t - \min\left\{\eta_t^V, 1/2\right\}(1 - x_1\hat{p}_{1t}^*).$$

That is, the measure of vacant firms in period $t+1$ (η_{t+1}^V) equals the inflow of firms in period $t+1$ ($\eta_{t+1} - \eta_t$) plus the measure of incumbent firms that have just become vacant $\left[\eta_t - \min\left\{\eta_t^V, 1/2\right\}(1 - x_1\hat{p}_{1t}^*)\right]$, where the latter equals the measure of active firms in period t (η_t) minus that of those who have retained their current worker $\left[\min\left\{\eta_t^V, \frac{1}{2}\right\}(1 - x_1\hat{p}_{1t}^*)\right]$ into period $t+1$.

As was shown earlier in Section 4.2, η_t^V plays a key role in defining the condition of the current labor market and thus in generating market dynamics. First, all else equal, a larger measure of vacant firms in the current market means that (weakly) more firms will enter a long-term contracts with a young worker (a larger $\min\left\{\eta_t - (1 - L_t), \frac{1}{2}\right\}$), resulting in a smaller measure of vacant firms in the next period. Second, when the market has more vacant firms in the current period, the equilibrium long-term contract is more likely to prescribe a lower probability of termination ($\hat{p}_{1t}^*$) which, in turn, implies a smaller measure of vacant firms in the next period. This gives us

Lemma 1 *Suppose Assumptions 1 and 2 hold. Then in equilibrium there cannot be two successive type TI periods with $w_{y,t} = w_{B,t}$.*

To illustrate the idea, suppose for example both the current and next periods are type T. Then competition for young workers in the current market would imply that all firms whose vacancies are filled in the current period would retain their worker after the period ends. But this would result in too few vacant firms in the labor market next period to make the period type T, which is a contradiction.

Lemma 2 *Suppose Assumptions 1 and 2 hold. Then in equilibrium, for any t, if both periods $t-1$ and t are type SI, period $t+1$ is also type SI.*

To illustrate the intuition, consider an equilibrium where there is no entry and exit of firms. Let periods $t-1$ and t be type SI. In these periods, the market condition is favorable to firms, which allows them to hire young workers at a low expected utility and fire them with a high probability. With no entry and exit of firms, vacancies in any period must be from two sources: those who have just fired a young worker and those who employed an old worker in the prior period. In period $t+1$, the measure of vacant firms from the first source is bounded from above by $x_1\hat{p}^*_{1t}/2$ (there are no more than 1/2 of vacant firms in period t), while that from the second is limited by the high probability of termination that the optimal contract prescribes at period $t-1$ (so that a small measure of old workers are retained in period t). As a result, the measure of vacant firms is bounded from above by 1/2, and so period $t+1$ must be type SI.

Combining Lemmas 1 and 2 gives us

Proposition 3 *Suppose Assumptions 1 and 2 hold. Then in equilibrium there exists $\hat{t} \geq 0$ such that for all $t \geq \hat{t}$, either all periods are type SI, or type SI and type T periods alternate.*

This offers directions for where to look for the model's dynamic equilibria. In the rest of the section, three special classes of the model's dynamic equilibria are constructed to illustrate how cycles may emerge in the labor market. The first is a class of two-period cycles that feature entry and exit of firms. The second is a class of equilibria where the economy's stock of firms is constant in time, while the tightness of the market, together with wages and termination probabilities, fluctuates in two-period cycles. The third is a class of equilibria that display cycles in employment and output that are longer than two periods, in fact the length of each individual cycle could be any even number. In these equilibria, each individual cycle starts with an expansion which is followed by a recession whose duration could be viewed as being randomly generated.

6.1 Two-period cycles with employment and output fluctuations

The model has a class of equilibria that display two-period cycles where the economy's aggregate values, including the stock of firms, employment, unemployment, output, as well as the job finding and separation probabilities, rise and fall over the cycle. Specifically,

Proposition 4 *Suppose Assumptions 1 and 2 hold and $C_e > 0$. For all (C_o, C_e) such that*

$$C_o > \underline{C} \text{ and } 0 < C_e \leq \frac{(1-\beta)(1+x_2\beta)(\bar{C}-C_o)}{1-x_2\beta^2},$$

the model has an equilibrium in two-period cycles where all periods are type I, the optimal long-term contract offered is as described in Proposition 1(ii), and the values of the economy's aggregates, $\{L^{(j)}, \alpha_y^{(j)}, \alpha_o^{(j)}, w_y^{(j)}, w_o^{(j)}, w_*^{(j)}, V^{(j)}; j = 1, 2\}$, where j denotes the jth period in the cycle, are given as follows.

1. *For the first period of the cycle,*

$$\eta^{(1)} = 1 - \frac{1}{2}x_1\hat{p}_1^{*(2)},\ L^{(1)} = \frac{1}{2} + \frac{1}{2}x_1\hat{p}_1^{*(2)},$$
$$\alpha_y^{(1)} = 1,\ \alpha_o^{(1)} = 0,$$
$$w_o^{(1)} = \underline{w},\ w_y^{(1)} = (1 - \delta\hat{p}_1^{*(1)})\underline{w},\ w_*^{(1)} = 0,$$
$$\bar{V}^{(1)} = C_e.$$

2. *For the second period of the cycle,*

$$\eta^{(2)} = 1 - \frac{1}{2}x_1\hat{p}_1^{*(1)},\ L^{(2)} = \frac{1}{2} + \frac{1}{2}x_1\hat{p}_1^{*(1)},$$
$$\alpha_y^{(2)} = 1,\ \alpha_o^{(2)} = 0,$$
$$w_o^{(2)} = \underline{w},\ w_y^{(2)} = (1 - \delta\hat{p}_1^{*(2)})\underline{w},\ w_*^{(2)} = 0,$$
$$\bar{V}^{(2)} = 0.$$

Here $\hat{p}_1^{*(1)}$ and $\hat{p}_1^{*(2)}$ are defined by

$$\hat{p}_1^{*(1)} = \frac{(1-\beta^2)[C_e - (\underline{C} - C_o)]}{-x_1\beta[(1-\beta)(\bar{\theta} - \psi - \underline{w}/x_1 - C_o) + \beta C_e]},$$
$$\hat{p}_1^{*(2)} = \frac{-(1-\beta^2)(\underline{C} - C_o)}{x_1\beta[C_e - (1-\beta)(\bar{\theta} - \psi - \underline{w}/x_1 - C_o)]}.$$

The proof of the proposition, which is in Appendix B.3, is by construction. So the first period of the cycle can be viewed as a boom period. At its start, the market condition is favorable to firms, with the measure of vacant firms less than that of young workers: $\eta^{(2)} - (1 - L^{(1)}) < 1/2$. If no new firms enter the market, the value of a vacant firm, who would enter a long-term contract with a young worker, is higher than the cost of entering the market to create a new job: $(1-\beta)(1+x_2\beta)(\bar{C} - C_o) + x_2\beta^2 C_e > C_e$. This generates an inflow of firms into the market to push the value of the vacant firm down to C_e, where the measure of vacant firms in the market is just equal to that of young workers ((1) in Figure 2). Thus the average job finding probability in this market is $\frac{1}{2}/\left(\frac{1}{2} + \frac{1}{2}\hat{p}_1^{*(2)}\right)$, where $(1/2) \times \hat{p}_1^{*(2)}$ is the measure of the unemployed old workers — those newly laid off from a long-term contract. As such, in equilibrium, young workers are all hired at a low expected

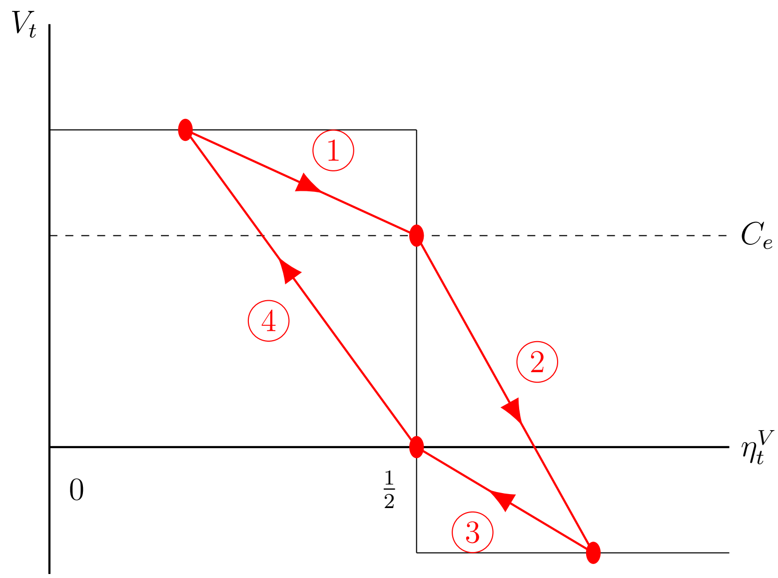


Figure 2: Two-period cycles with entry and exit of firms

utility $w_{y,t}$ and to be laid off with a high probability $\left(\hat{p}_1^{*(1)} > \hat{p}_1^{*(2)}\right)$ after one period, and there are no newly hired old workers. This will then create a large measure of vacant firms in the next (or second) period of the cycle (② in Figure 2).

The second period of the cycle can be viewed as a recession period. At the start of this period when recession begins, the market condition has become favorable to workers: $\eta^{(1)} - (1 - L^{(2)}) > 1/2$. If no firms exit the market, then each of the vacant firms would be of a negative value. This generates an outflow of firms, until the remaining vacant firms are indifferent between exiting and staying in the market, which happens only when the measure of the remaining firms is just equal to that of the young workers (③ in Figure 2). After the outflow ends, the average job finding probability is $\frac{1}{2}/\left(\frac{1}{2} + \frac{1}{2}\hat{p}_1^{*(1)}\right)$, lower than that in the boom period. In addition, relative to the boom period, in the recession period young workers are all hired at a higher expected utility and to be laid off with a lower probability after producing the low output. This generates a smaller measure of vacant firms in the next period (i.e., relative to what happens in the prior period), moving the distribution of market power in a direction that favors firms (④ in Figure 2).

The boom period features a lower unemployment rate and a higher (average) job finding probability.[14] The higher job finding probability is for two reasons. First, new jobs are created through an inflow of firms. Firms flow in because the period starts with a labor market condition favoring vacant firms, which in turn results from a tight labor market in

[14] In this equilibrium, all unemployed workers are old and the job finding probability for young workers is one.

the prior period that generated fewer new vacancies. Second, fewer workers are unemployed at the beginning of a boom period, because the tight labor market in the prior period generated fewer layoffs.

Similarly, in a recession period, the unemployment rate is higher and the job finding probability is lower. This is because (i) some of the existing jobs have been destroyed, and (ii) the *slack* labor market in the prior boom period generated more layoffs and hence more unemployed workers in the current period.

To summarize, in these equilibria, unemployment is determined by job creation and job destruction. Vacancies or jobs are destroyed in periods of recessions to give rise to a larger unemployment rate, and they are created in booms to produce a lower unemployment rate. Recessions always start with a large measure of vacancies, and booms always start with a small measure of vacancies.

6.1.1 Unemployment volatility

In these equilibria of the model, job creation and job destruction also produce pro-cyclical movements in the tightness of the labor market. Specifically, in the model, the tightness ratio for any given period t is measured by

$$\frac{\eta_t^V}{\eta_t^V + (1 - E_t)}, \tag{24}$$

where η_t^V is the measure of total jobs posted when the labor market opens (that is after all job creation and destruction take place), and E_t is total employment (i.e., the measure of workers who will be employed in the period after the labor market closes) and thus $\eta_t^V + (1 - E_t)$ is the measure of unemployed workers who are looking for jobs in the current labor market. Now η_t^V is constant at $1/2$ across booms and recessions in these equilibria, and E_t is high in booms but low in recessions, and so tightness moves pro-cyclically with E_t in two period cycles.

As discussed earlier in the paper, Shimer (2005) observes that search models produce volatility measures for the tightness ratio that are too small relative to the data. Our model suggests a mechanism for generating pro-cyclical movements in labor market tightness. Could this theory have potential in contributing to the resolution of the unemployment volatility puzzle?

6.1.2 Wages

Are wages pro or counter-cyclical? There has been much debate on the answer to this question. Standard RBC models produce unambiguously pro-cyclical real wages — a positive technology shock that results in an economic boom is also the cause of increased labor productivity and wages. Empirical evidence, however, sends mixed messages.[15] While our theory with two-period lived overlapping generations should not be taken too seriously for interpreting labor market data which are typically in frequencies that are not compatible with the model, the logic of the model about wages and cycles may shed light on the issue that has attracted so much interest from economists.

Specifically, with a time invariant production function and no economy wide demand or productivity shocks, our model offers a novel perspective on this question. It says that wages rise or fall and that need not be because labor is intrinsically more or less productive, or the external good market assigns a higher or lower value for the output it produces. Wages may rise or fall because the state of the competition is giving more or less power to the workers across individual periods.

In the equilibria discussed here, starting wages are acyclical — they are constant at zero across booms and recessions. Wages of retained workers are pro-cyclical — hired in recessions, their wages are high in booms; hired in booms, wages of continuing workers are low in recessions. Average wages are, therefore, pro-cyclical.

These results hold because compensation is backloaded with the optimal contract and young worker's lifetime utility/compensation moves counter-cyclically over the cycle. Backloading is optimal because it helps minimize the use of termination, which is costly, as an incentive device — allowing more to be paid in the state of retention/high-output allows the contract to reduce the probability of termination in the state of low output. Lifetime compensation is counter-cyclical because, in equilibrium, periods of recessions always start with a large measure of vacancies that tightens competition for workers and increases their lifetime utilities; and booms occur in periods that start with a small measure of vacancies that loosens the competition for workers and lowers their lifetime utilities.

Note, obviously, that the starting wage of a young worker does not move cyclically with employment and output is not consistent with the data. This property of the model is derived because workers are assumed risk neutral in the model. Supposing workers are risk averse, the outcome of the model should have both a backloading in compensation over the worker's lifetime, and a starting wage for young workers that moves counter-cyclically over the cycle. And this may offer a novel insight on why the cyclicality in observed wages is not

[15] For more recent work on this, see for example Rotemberg (2006) and Otrok and Pourpourides (2017).

as strong or as unambiguous as standard models have predicted.

6.2 Pure labor market cycles

The model has a class of equilibria where the labor market displays cycles in tightness, wages and termination probabilities, but with no active entry and exit of firms — a sufficiently large cost of job creation discourages new entrants into the market and a sufficiently low C_o keeps existing firms in — and there is a constant stock of firms to give the economy constant employment and output over time.[16]

We call labor market fluctuations pure if they do not induce fluctuations in employment and output, or business fluctuations in the commonly understood sense.

Proposition 5 *Suppose Assumptions 1 and 2 hold and $C_e > 0$. Let (C_o, C_e) be such that*

$$\underline{C} < C_o < \underline{C} + \frac{x_2 \beta^2 \underline{w}}{1+\beta} \text{ and } C_e \geq \frac{(1-\beta)(C_o - \underline{C})}{\beta}. \tag{25}$$

Then there exists an interval $[\eta_c^-, \eta_c^+]$ such that for any η in this interval, the model has an equilibrium where $\eta_t = \eta$ for all $t \geq 1$, type S and type T periods alternate, the optimal long-term contract is described by Proposition 1, and the values of the economy's aggregates $\left\{L^{(j)}, \alpha_y^{(j)}, \alpha_o^{(j)}, w_y^{(j)}, w_o^{(j)}, w_*^{(j)}, V^{(j)}; j = 1,2\right\}$ are given as follows.

1. *In the first period (type S) of a cycle,*

$$L^{(1)} = \frac{1}{2} + \frac{1}{2} x_1 \hat{p}_1^{*(2)},$$
$$\alpha_y^{(1)} = 2\eta - 1 + x_1 \hat{p}_1^{*(2)}, \ \alpha_o^{(1)} = 0,$$
$$w_o^{(1)} = \underline{w}, \ w_y^{(1)} = (1-\delta)\underline{w} + \delta w_*^{(2)}, \ w_*^{(1)} = 0,$$
$$\bar{V}^{(1)} = \bar{\theta} - \psi - \underline{w} + \frac{x_2 \beta}{1+\beta}(\underline{w} - w_*^{(2)}) - C_o.$$

2. *In the second period (type T) of a cycle,*

$$L^{(2)} = 1 - x_2(\eta - \frac{1}{2} + \frac{1}{2} x_1 \hat{p}_1^{*(2)}),$$
$$\alpha_y^{(2)} = 1, \ \alpha_o^{(2)} = \frac{w_*^{(2)}}{\underline{w}},$$
$$w_o^{(2)} = \underline{w}, \ w_y^{(2)} = (1 - \delta \hat{p}_1^{*(2)})\underline{w}, \ w_*^{(2)} = \underline{w} - \frac{\hat{p}_1^{*(2)}}{\beta(1 - x_1 \hat{p}_1^{*(2)})}\underline{w},$$

[16] These equilibria are a subset of a larger class of equilibria where the market oscillates between type SI and type T periods. Lemma 13 in Appendix B.3 gives a general characterization of all such equilibria.

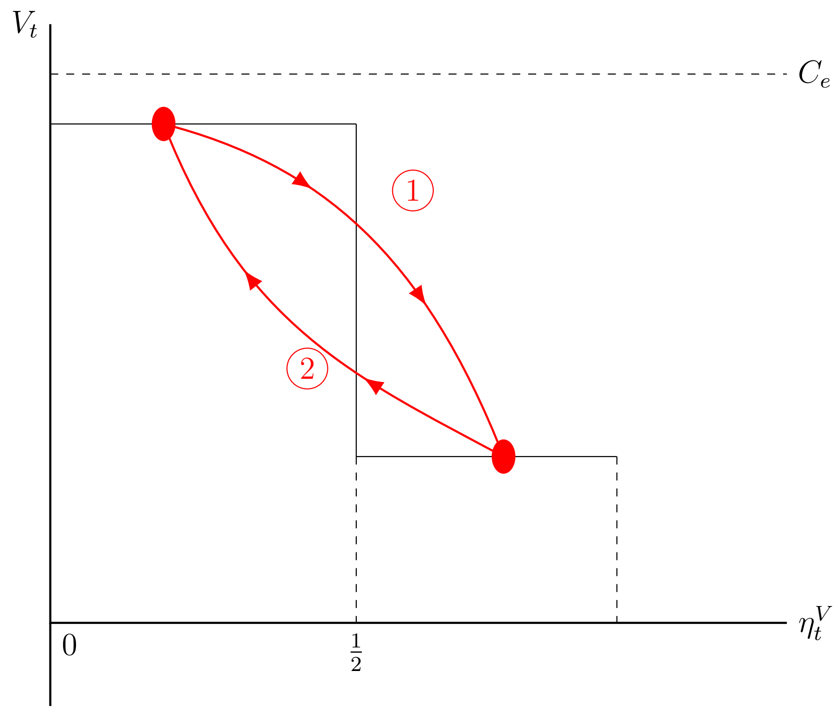


Figure 3: Pure labor market cycles

$$\bar{V}^{(2)} = \bar{\theta} - \psi - \underline{w} + \frac{x_2\beta^2}{1+\beta}(\underline{w} - w_*^{(2)}) - C_o.$$

Here $\hat{p}_1^{*(2)}$ *is the smaller root of*

$$\frac{1}{2}x_1x_2p_1^2 - \left[\frac{1}{2}x_1 + (x_2 + x_1\beta)(1-\eta)\right]p_1 + (1-\eta)\beta = 0. \tag{26}$$

At the start of a type S period, the market condition favors firms, with the equilibrium job finding probability being strictly less than one for young workers and zero for old workers. As such, in equilibrium all young workers are hired at a low $w_{y,t}$ and to be fired after one period with a high probability ($\hat{p}_1^{*(1)} > \hat{p}_1^{*(2)}$). In addition, in contrast to a type T period, fewer firms enter long-term contracts with young workers: $\eta - 1/2 + x_1\hat{p}_1^{*(2)}/2 < 1/2$. Both of these help create more vacancies in the next period, which is type T (① in Figure 3).

In a type T period, with a large measure of vacancies, the market condition favors workers, the equilibrium job finding probability is one for young workers and strictly positive for old workers. Relative to the prior period, in this period a larger measure of young workers are employed with a long-term contract that prescribes a low probability of termination, and that in turn will generate a small measure of vacancies in the next period, which then is type S (② in Figure 3).

In a type S period, the job finding probability — the tightness ratio — is $(\eta - (1 - L^{(1)}))/L^{(1)}$, and in a type T period, $(\eta - (1 - L^{(2)}))/L^{(2)}$. Since $L^{(1)} < L^{(2)}$, the job finding probability is larger in the T periods than in the S periods. Note also that in these equilibria the young worker's lifetime utility/compensation is higher in the T than the S periods.

With no employment and output fluctuations, these are equilibria of the model where the idea of the paper is displayed in its purest form. In these equilibria, a large measure of vacancies in period t results in an allocation of greater market power to the unemployed workers, giving rise to an employment contract that prescribes a larger lifetime utility/compensation, together with a lower probability of termination in period $t + 1$. The lower probability of termination in $t + 1$ will then produce a small measure of vacancies in $t + 2$ which, in turn, will move the economy to greater market power for the vacant firms, larger termination probabilities for the workers, and a large measure of vacancies again.[17]

The upshot of the above discussion, therefore, is that employment and output fluctuations need not be part of the model's outcomes. Fluctuations in market tightness and compensation may stand alone as an equilibrium phenomenon, independently of any business or output dynamics. This, as discussed earlier, is in contrast with standard search or RBC models, which view labor market fluctuations as derived from productivity or aggregate demand shocks which, in turn, are taken to be the primitive of the theory.

6.3 Short labor market cycles in longer business cycles

So labor market cycles could be pure, or they could also induce job creation and destruction, and movements in aggregate employment and output. In this section, we show that the model has cycles in employment and aggregate output that differ in duration and cyclicality from those in the tightness of the market (that induces the movements in employment and output). In these equilibria, the engine for employment and output fluctuations is still the two-period tightness cycles, but the cycles in employment and output can display durations which are much longer than that of the cycles in market tightness.

Specifically, the model has a class of equilibria where the labor market oscillates in two-period cycles between S and T periods, but employment and output move over cycles that share the following cyclicality: they rise in the first period of the individual cycle, fall in the second, and then stay constant in the remaining time of the cycle.[18]

In the first and an S period of these cycles, the market condition favors firms. This induces a flow of new firms into the market to push the value of the vacant firm down to C_e.

[17]Notice that, in this equilibrium, although employment is constant in time, its composition — the division of total employment between new hires and continuing workers — fluctuates over the cycle.

[18]See Proposition 9 in Appendix B.4.

In this period, only young workers find jobs with a positive probability, and they are hired at a low expected utility but a high probability of being fired in the next period. This creates a large measure of vacancies at the beginning of the second and a type T period of the cycle. As a result, market condition favors workers in that period and vacant firms exit the market until the values of the firms that stay all become zero. In the remaining time of the cycle, the labor market continues its oscillation between S and T periods while the stock of firms in it remains constant. This can be the case because, in the remaining S periods, all firms promise a relatively high expected value to their young worker ($w_y^1 < w_y^3 = \cdots = w_y^{2n-1}$) and so their values are not large enough to attract new entrants into the market. And in the remaining T periods, all firms choose to stay in the market, with some of them hiring an old worker and anticipating the condition of the market to turn favorable to them at the beginning of the next period.[19]

So individual cycles need not have the same length. In fact, the length of each individual cycle can be any even natural number. One way to think of such an equilibrium is to view the length of each cycle as being determined ex ante by a random number generator. As such, Proposition 9 essentially depicts a scenario in which all individual cycles exhibit the same cyclicality (that is, starting with a boom which is followed by a recession) but each last for an ex ante randomly determined number of periods.

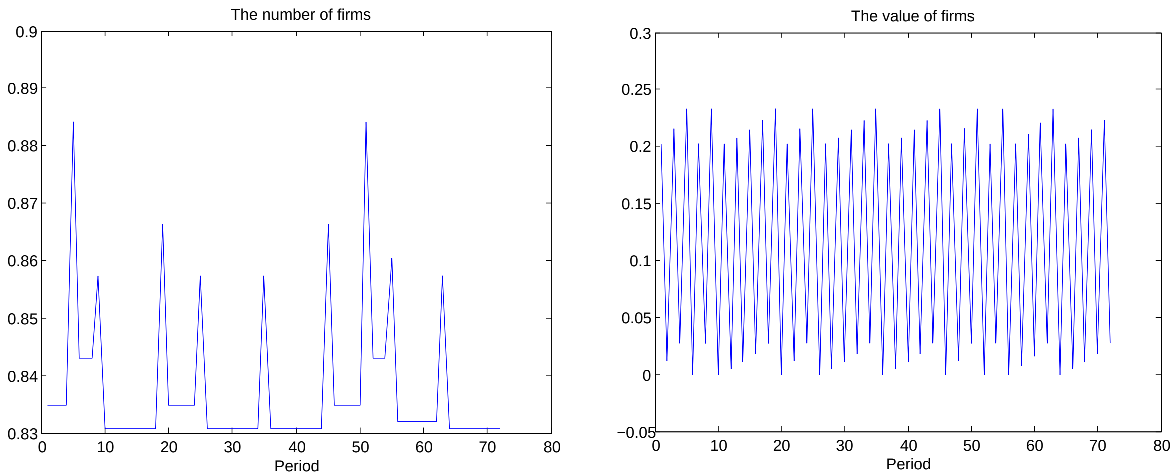
That the length of an individual cycle could be *any* even number implies that the model is able to generate both long cycles and large variability in the durations of the cycles. This, we claim, makes the outcomes of our model resemble observed business cycles which display large variability in their durations, but differ from most existing theories in the literature. In Myerson (2012) for example, all cycles share the same length which equals that of the life of the banker. In Suarez and Sussman (1997, 2007), all cycles last for two periods.

A numerical example of such an equilibrium, with the values of the model's parameters given in Table 1 in Appendix B.4.3, is depicted in Figure 5. Observe the regularity in the cyclicality of the cycles. Observe also the large variability in the durations of the recessions that the equilibrium generates.

6.4 Summary: the essence of the fluctuation

Two elements of the model are essential for fluctuations to arise in the labor market. First is the dynamical interaction between individual firm decisions on termination and the allocation of market power between the two sides of the market. More aggressive termination

[19] Here the market for old workers exists as a means for reducing competition for young workers and preventing job destruction. In a way, hiring an old worker serves as a middle state between hiring a young worker and exiting the market.



(a) Measure of firms

(b) Firms' expected values

Figure 4: Equilibrium cycles: a numerical example

policies prescribed in a slack current labor market increase the supply of vacancies in the next period, creating a tighter labor market, moving market power away from firms to workers; while competition between firms in a tight current labor market results in less aggressive termination policies that produce fewer vacancies in the next period and move market power back to the firms. The second is the entry frictions/costs that insulate, partially or completely, the labor market from outside competition and create a space in which the interaction between individual terminations and the allocation of market power can be played out. Free flows of firms into the market forces the market to give all market power to the workers at all times, breaking the mechanism for fluctuations.

The logic stated above then implies that the larger are the costs for entry or job creation, the larger fluctuations could the labor market display. This indeed is the case. In the two period cycles discussed in Section 6.1 for example, fluctuations, measured by the differences in employment or vacancies between booms and recessions, increase as the cost of job creation (or C_e) increases. To see this, observe that both employment and vacancies depend linearly on $\hat{p}_1^*$, and that $\hat{p}_1^{*(1)}$ is increasing in C_e while $\hat{p}_1^{*(2)}$ is decreasing in C_e.

The above logic also suggests that the larger is the size of the required effort ψ, the larger are the fluctuations that the market could support. First, a larger ψ plays a similar role as a larger cost of job creation does and that supports more fluctuations. Second, a larger ψ makes the incentive problem more severe — the temptation to shirk is stronger — and this forces the optimal contract to increase the utility promised to the retained old worker, making termination more effective as an incentive device, and so workers are fired more frequently in equilibrium, resulting in larger fluctuations that the market could support.

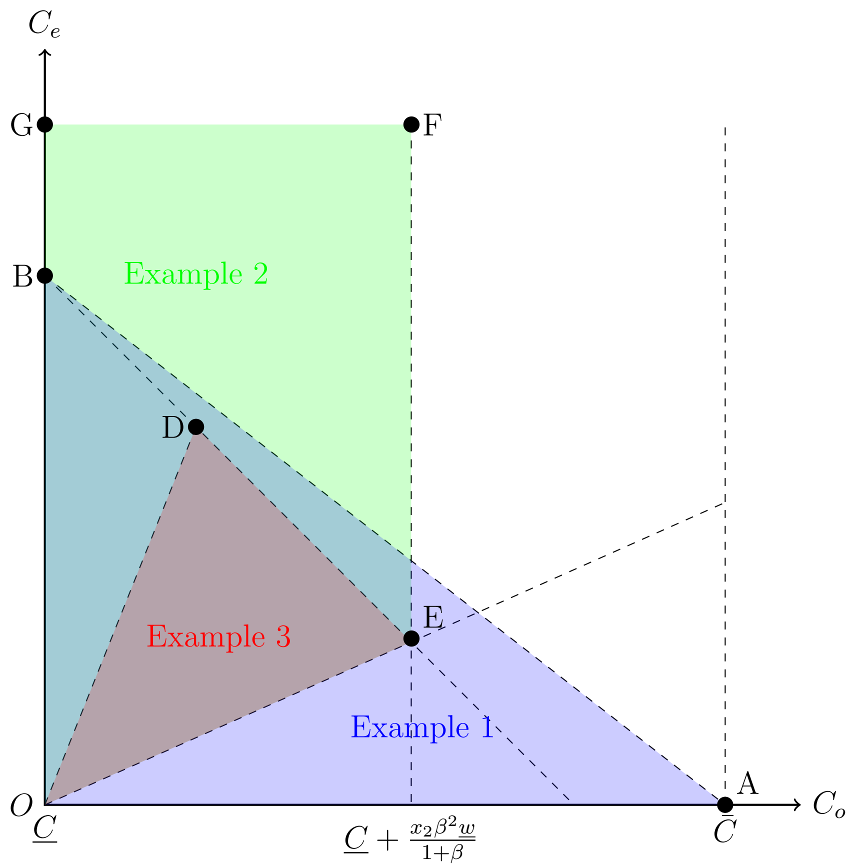


Figure 5: Summary

Lastly, observe that the overlapping generations structure of the model plays a critical role in supporting the fluctuations. What it does is to create a labor market where contracts of different durations are traded. Moral hazard gives longer-term contracts higher values, which then creates a competition for young workers (longer contracts). It is this competition that switches the market power from the firm to the worker periodically over time. Without this competition — imagine all unemployed workers are equally young, in a model where agents are perpetually young for example — all newly hired workers would be given the same monopsony contract in all periods in all equilibria with unemployment.

6.5 Co-existence

In the model, stationary and non-stationary equilibria coexist over at least a subset of the parameter space we consider. This is shown in Figure 5. Specifically, a stationary equilibrium, as described in Proposition 2, exists for the entire parameter space, i.e., for all (C_o, C_e) such that $0 \leq C_e < \overline{C} - C_o$. For any (C_o, C_e) in area OAB, the model has an equilibrium in two-period cycles in which all periods are type I, as described in Proposition

4. For any (C_o, C_e) in area OEFG, the model has an equilibrium in which type S and type T periods alternate, as described in Proposition 5. Lastly, for any (C_o, C_e) in area OED, the model displays equilibrium cycles in which each individual cycle lasts for an arbitrary even number of periods, as described in Proposition 9.

7 Output and welfare

Do stationary equilibria provide better output and welfare outcomes than non-stationary equilibria, or vice versa? Given the multiplicity in both the model's stationary and non-stationary equilibria, one way to answer this question is to compare the equilibrium which achieves the best output and welfare outcomes among all stationary equilibria with the one that attains the best output and welfare outcomes among all non-stationary equilibria. Notice that the "best" stationary equilibrium is simply the one that attains the highest measure of firms, η_s^+. This is the stationary equilibrium that offers the highest aggregate output and the maximum expected utility for the young worker. To find the "best" non-stationary equilibrium, however, is less trivial. First, in a non-stationary equilibrium, generations differ in lifetime utility and it is not obvious how welfare should be aggregated across generations. Second, the set of all non-stationary equilibria is large.

To narrow our focus, consider the "best" equilibrium among all non-stationary equilibria with the two-period cycles described in Proposition 5. In all these equilibria the economy's aggregate output is determined by the constant η. Three observations emerge.[20]

First, the maximum output attainable in a stationary equilibrium is higher than what a non-stationary equilibrium can achieve. In other words, fluctuations in the labor market may impose an output loss on the economy. An important aspect of the non-stationary equilibria is that old workers may gain or regain employment in these equilibria. This makes termination less effective as an incentive device, which reduces the value of the firm and results in a smaller measure of firms that can survive in the non-stationary equilibrium.

Second, the average lifetime utility of workers (born in both the T and S periods) in the "best" non-stationary equilibrium is lower than that of workers in the "best" stationary equilibrium. In this sense, it may be claimed that non-stationarity imposes a welfare loss on the workers as well. There are two opposite effects at work here. On the one hand, fixing the measure of firms in the economy, average lifetime utility is higher in the non-stationary equilibrium than in the stationary equilibrium — old workers may gain or regain employment in the non-stationary equilibrium. On the other hand, the maximum measure of firms that a non-stationary equilibrium can support is smaller (for their values are lower

[20]See Appendix C for proofs.

because termination is used less efficiently as an incentive mechanism), and that reduces the total surplus shared between firms and workers. In the equilibria considered, the second effect dominates and workers are worse off in a non-stationary than in a stationary equilibrium.

This, however, does not mean that all workers are better offer in the stationary than in the non-stationary equilibrium. Workers are better off to be born into a T period in the "best" non-stationary equilibrium than in the "best" stationary equilibrium, but they are worse off to be born into an S period in the "best" non-stationary equilibrium than in the "best" stationary equilibrium.

8 Concluding remarks

This paper shows that moral hazard and competition in long-term contracting with optimal termination, together with costly job creation, can generate endogenous fluctuations in the labor market. We claim that these fluctuations, coming internally from the optimality in labor contracting, may contribute to the explanation of the observed "excess" volatility in unemployment which standard search models have not been able to adequately account for. For analytical tractability, the model is designed to consist of two-period-lived overlapping generations of workers so a contract is at most two-period long. An interesting extension is to consider longer worker life and hence longer contract durations. This would render the model suitable for more serious quantitative simulation and even empirical testing. We do not believe, however, that introducing longer contracts and more sophisticated employment dynamics would comprise the fundamental logic of the paper which, we hope, is better illustrated in a simple model like the one we have just studied.

References

Banerji, S., and T. Wang (2013): "Incentives, Risk Sharing and Wealth: A Model of Intrinsic Cycles," .

Boldrin, M., and M. Woodford (1990): "Equilibrium models displaying endogenous fluctuations and chaos: a survey," *Journal of Monetary Economics*, 25(2), 189–222.

Costain, J., and M. Jansen (2010): "Employment Fluctuations with Downward Wage Rigidity: The Role of Moral Hazard," *Scandinavian Journal of Economics*, 112(4), 782–811.

Eeckhout, J., and I. Lindenlaub (2018): "Unemployment cycles," Discussion paper.

Favara, G. (2012): "Agency problems and endogenous investment fluctuations," *Review of Financial Studies*, 25(7), 2301–2342.

Golosov, M., and G. Menzio (2015): "Agency Business Cycles," Discussion paper, National Bureau of Economic Research.

Gu, C., F. Mattesini, C. Monnet, and R. Wright (2013): "Endogenous Credit Cycles," *Journal of Political Economy*, 121(5), 940 – 965.

Haefke, C., M. Sonntag, and T. van Rens (2013): "Wage rigidity and job creation," *Journal of Monetary Economics*, 60(8), 887 – 899.

Hall, R. E. (2005): "Employment Fluctuations with Equilibrium Wage Stickiness," *American Economic Review*, 95(1), 50–65.

Kaplan, G., and G. Menzio (2016): "Shopping externalities and self-fulfilling unemployment fluctuations," *Journal of Political Economy*, 124(3), 771–825.

Kennan, J. (2010): "Private Information, Wage Bargaining and Employment Fluctuations," *The Review of Economic Studies*, 77(2), 633–664.

Kudlyak, M. (2014): "The cyclicality of the user cost of labor," *Journal of Monetary Economics*, 68, 53–67.

Moen, E. R., and A. Rosen (2011): "Incentives in Competitive Search Equilibrium," *Review of Economic Studies*, 78(2), 733–761.

Myerson, R. B. (2012): "A model of moral-hazard credit cycles," *Journal of Political Economy*, 120(5), 847–878.

Otrok, C., and P. M. Pourpourides (2017): "On the Cyclicality of Real Wages and Wage Differentials," *The B.E. Journal of Macroeconomics*.

Pissarides, C. A. (2009): "The Unemployment Volatility Puzzle: Is Wage Stickiness the Answer?," *Econometrica*, 77(5), 1339–1369.

Rogerson, R., and R. Shimer (2011): "Chapter 7 - Search in Macroeconomic Models of the Labor Market," vol. 4, Part A of *Handbook of Labor Economics*, pp. 619 – 700. Elsevier.

Rotemberg, J. J. (2006): "Cyclical Wages in a Search-and-Bargaining Model with Large Firms," Working Paper 12415, National Bureau of Economic Research.

Shapiro, C., and J. E. Stiglitz (1984): "Equilibrium Unemployment as a Worker Discipline Device," *American Economic Review*, 74(3), 433–44.

Shapley, L. S., and M. Shubik (1971): "The assignment game I: The core," *International Journal of Game Theory*, 1(1), 111–130.

Shimer, R. (2005): "The Cyclical Behavior of Equilibrium Unemployment and Vacancies," *American Economic Review*, 95(1), 25–49.

Sniekers, F. (2018): "Persistence and volatility of Beveridge cycles," *International Economic Review*, 59(2), 665–698.

Spear, S. E., and C. Wang (2005): "When to fire a CEO: optimal termination in dynamic contracts," *Journal of Economic Theory*, 120(2), 239–256.

Suarez, J., and O. Sussman (1997): "Endogenous Cycles in a Stiglitz-Weiss Economy," *Journal of Economic Theory*, 76(1), 47–71.

——— (2007): "Financial distress, bankruptcy law and the business cycle," *Annals of Finance*, 3(1), 5–35.

Wang, C. (2013): "Incentives and the Cost of Firing in an Equilibrium Labor Market Model with Endogenous Layoffs," *International Economic Review*, 54(2), 443–472.

Mathematical Appendix

We will use the following simplifying notation throughout the appendix.

$$\theta_i' \equiv \theta_i - C_o,\ i = 1, 2;\ \ \bar{\theta}' \equiv \bar{\theta} - C_o.$$

In particular, θ_i' is the firm's gross output net of the fixed operating cost in state i.

A Contracting

This section contains the omitted proofs in Section 3. Specifically, Section A.1 contains the proof of Proposition 1 and Section A.2 states and proves Proposition 6.

A.1 Proof of Proposition 1

Lemma 3 *For all w with which the problem that defines $V_{l,t}(w)$ has a solution, it is optimal to set $c_{ir} = 0$ for $i = 1, 2$.*

Proof. Suppose for some w, a solution that defines $V_{l,t}(w)$ has $c_{ir} = \Delta > 0$, for some i. Consider a deviation from this solution, letting $c_{ir}' = 0$ and $w_{ir}' = w_{ir} + \Delta/\beta$. Clearly, constraints (7) and (8) would continue to hold under this deviation, while the firm's value not changed. ■

Given Lemma 3, constraint (7) can be rewritten as:

$$[(1-p_2)\delta w_{2r} + p_2((1-\delta)c_{2f} + \delta w_{*t+1})] - [(1-p_1)\delta w_{1r} + p_1((1-\delta)c_{1f} + \delta w_{*t+1})] \geq \frac{(1-\delta)\psi}{x_2 - x_2'}.$$

Lemma 4 *For all w with which the problem that defines $V_{l,t}(w)$ has a solution, it is optimal to set $c_{if} = 0$ if $p_i < 1$, for $i = 1, 2$.*

Proof. Suppose for some w, $c_{if} = \Delta > 0$ and $p_i < 1$ for some i. Let $c_{if}' = 0$ and $w_{ir}' = w_{ir} + \frac{p_i \Delta}{(1-p_i)\beta}$. Then constraints (7) and (8) are still satisfied, and the firm's value is not changed. ■

Given Lemma 4, we could then focus on contracts that satisfy $p_i c_{if} = c_{if}$. It then follows that the problem of optimal long-term contracting can be restated in the following two sub-problems.

S1:

$$G(\Delta_i) = \max_{c_{if}, w_{ir}, p_i} (1-\beta)\theta_i' - p_i(1-\beta)c_{if} + p_i\beta V_{t+1} + (1-p_i)\beta V_{s,t+1}(w_{ir}),$$

s.t.

$$c_{if} \geq 0, \; w_{ir} \geq \underline{w}, \; 0 \leq p_i \leq 1,$$

$$(1-p_i)\delta w_{ir} + p_i((1-\delta)c_{if} + \delta w_{*t+1}) - (1-\delta)\psi = \Delta_i,$$

$$p_i c_{if} = c_{if}.$$

S2:

$$V_{l,t}(w) = \max_{\Delta_1, \Delta_2} x_1 G(\Delta_1) + x_2 G(\Delta_2),$$

s.t.

$$\Delta_2 - \Delta_1 \geq \frac{(1-\delta)\psi}{x_2 - x_2'},$$

$$x_1 \Delta_1 + x_2 \Delta_2 = w.$$

We first solve **S1** with c_{if} fixed at its optimal level. In other words, we solve

$$\max_{p_i, w_{ir}} p_i V_{t+1} + (1-p_i) V_{s,t+1}(w_{ir}),$$

s.t.

$$w_{ir} \geq \underline{w}, \; 0 \leq p_i \leq 1,$$

$$(1-p_i) w_{ir} + p_i w_{*t+1} = \delta_i,$$

$$p_i c_{if} = c_{if},$$

where $\delta_i \equiv (\Delta_i + (1-\delta)\psi - (1-\delta)c_{if})/\delta > 0$. To solve this problem, in turn we remove the constraint $p_i c_{if} = c_{if}$ from the constraint set, solve the resulting problem, and then show that the resulting solution does satisfy the omitted constraint $p_i c_{if} = c_{if}$.

Substituting $V_{s,t+1}(w) = (1-\beta)(\bar{\theta}' - \psi - w) + \beta V_{t+2}$ into the above problem (i.e., after removing the constraint $p_i c_{if} = c_{if}$) gives:

$$\max_{p_i, w_{ir}} p_i [V_{t+1} - (1-\beta)(\bar{\theta}' - \psi - w_{*t+1}) - \beta V_{t+2}] + (1-\beta)(\bar{\theta}' - \psi) - (1-\beta)\delta_i + \beta V_{t+2},$$

s.t.

$$w_{ir} \geq \underline{w}, \; 0 \leq p_i \leq 1,$$

$$(1-p_i) w_{ir} + p_i w_{*t+1} = \delta_i.$$

By Assumption 1, this problem is rewritten as

$$\min_{p_i, w_{ir}} p_i,$$

s.t.

$$w_{ir} \geq \underline{w}, \; 0 \leq p_i \leq 1,$$

$$(1 - p_i)w_{ir} + p_i w_{*t+1} = \delta_i.$$

Since $\delta_i \geq (1 - p_i)\underline{w} + p_i w_{*t+1} \geq w_{*t+1}$, we have that the solution to the above problem has

(i) If $\delta_i \geq \underline{w}$, then $p_i = 0$;
(ii) If $\delta_i \in [w_{*t+1}, \underline{w})$, then $w_{ir} = \underline{w}$ and $p_i = \frac{\underline{w} - \delta_i}{\underline{w} - w_{*t+1}}$.

Lemma 5 *For all w with which the problem that defines $V_{l,t}(w)$ has a solution, it is optimal to set $p_2^*(w) = 0$.*

Proof. Suppose $p_2^*(w) = 1$ for some w. Then the incentive constraint (7) can be rewritten as

$$(1 - \delta)c_{2f} + \delta w_{*t+1} - [(1 - p_1)\delta w_{1r} + p_1((1 - \delta)c_{1f} + \delta w_{*t+1})] \geq \frac{(1 - \delta)\psi}{x_2 - x_2'}.$$

This implies that $c_{2f} > 0$. Consider the following deviation from the optimal contract. Let $c_{2f}' = c_{2f} - \Delta > 0$ for some sufficiently small $\Delta > 0$. Let $w_{2r}' = \underline{w}$ and $c_{2r}' = c_{2f} + \Delta$. Let Δ and p_2' be chosen to satisfy

$$(1 - p_2')[(1 - \delta)(c_{2f} + \Delta) + \delta \underline{w}] + p_2'[(1 - \delta)(c_{2f} - \Delta) + \delta w_{*t+1}] = (1 - \delta)c_{2f} + \delta w_{*t+1}$$

and so the incentive and promise-keeping constraints (7) and (8) continue to hold with the deviation. With this deviation, the firm would obtain a net gain of

$$\begin{aligned} \epsilon =& x_2\{2p_2'(1 - \beta)\Delta + (1 - p_2')\beta[V_{s,t+1}(w_{2r}') - V_{t+1}]\} \\ \geq& x_2\{2p_2'\Delta + \beta(1 - p_2')(w_{*t+1} - \underline{w})\} \\ =& 0, \end{aligned}$$

where the inequality holds by Assumption 1. Hence it is optimal to set $p_2^*(w) < 1$. And then it is optimal to set $c_{2f} = 0$ from Lemma 4. Then it follows from the incentive constraint that

$$\delta_2 \equiv (1 - p_2)w_{2r} + p_2 w_{*t+1} > \frac{\psi}{\beta(x_2 - x_2')} > \underline{w}.$$

Thus $p_2^*(w) = 0$. ■

Lemma 6 *For all w with which the problem that defines $V_{l,t}(w)$ has a solution, it is optimal to set $c_{1f} = 0$.*

Proof. Suppose $c_{1f} > 0$ for some w. Consider a deviation from the optimal contract with $c'_{2f} = 0$ and let w'_{2r} be chosen to satisfy

$$x_2 \delta w'_{2r} = x_2 \delta w_{2r} + x_1 p_1 (1-\delta) c_{1f},$$

while holding other parts of the contract constant. With this deviation, constraints (7) and (8) continue to be satisfied, and the values for both the worker and the firm remain unchanged. ■

Lemmas 5 and 6 imply $p_i c_{if} = c_{if}$ for $i = 1, 2$. Now given Lemmas 3-6, the optimization problem that defines $V_{l,t}(w)$ can be rewritten as

$$V_{l,t}(w) = \max_{p_1, w_{1r}, w_{2r}} \left\{ (1-\beta)\bar{\theta}' + x_1 \beta [p_1 V_{t+1} + (1-p_1) V_{s,t+1}(w_{1r})] + x_2 \beta V_{s,t+1}(w_{2r}) \right\}, \quad (27)$$

s.t.

$$w_{1r},\ w_{2r} \geq \underline{w},\ \ 0 \leq p_1 \leq 1, \quad (28)$$

$$\delta w_{2r} - [(1-p_1)\delta w_{1r} + p_1 \delta w_{*t+1}] \geq \frac{(1-\delta)\psi}{x_2 - x'_2}, \quad (29)$$

$$x_1[(1-p_1)\delta w_{1r} + p_1 \delta w_{*t+1}] + x_2 \delta w_{2r} - (1-\delta)\psi = w. \quad (30)$$

Substituting the incentive constraint (29) into the promise-keeping constraint (30) yields

$$w \geq \delta w_{*t+1} + (1-\delta)\underline{w}.$$

This implies that $w_{A,t} \equiv \delta w_{*t+1} + (1-\delta)\underline{w}$ is the minimum w that an incentive compatible long-term contract can implement.

Now given the above, and following from $V_{s,t+1}(w) = (1-\beta)(\bar{\theta}' - \psi - w) + \beta V_{t+2}$ and (30), we have

$$V_{l,t}(w) = \max_{p_1, w_{1r}, w_{2r}} \{(1-\beta^2)(\bar{\theta}' - \psi - w) + \beta^2 V_{t+2} + x_1 \beta p_1 [V_{t+1} - (1-\beta)(\bar{\theta}' - \psi - w_{*t+1}) - \beta V_{t+2}]\},$$

subject to (28) and (30). By Assumption 1, this optimization problem is equivalent to

$$\min_{p_1, w_{1r}, w_{2r}} p_1 \text{ s.t. } (28)\text{-}(30).$$

With this problem, observe then that the incentive constraint (29) must be binding. Suppose not. Then construct a deviation from the optimal contract by, while holding other parts of the contract constant, letting $w'_{2r} = w_{2r} - \Delta$ for some sufficiently small $\Delta > 0$, and choosing

Δ and p_1' be such that

$$x_1\delta[(1-p_1')w_{1r}+p_1'w_{*t+1}]+x_2\delta w_{2r}'=x_1\delta[(1-p_1)w_{1r}+p_1w_{*t+1}]+x_2\delta w_{2r}.$$

This deviation would not violate the constraints (29) and (30), but since

$$p_1-p_1'=\frac{x_2\Delta}{x_1(w_{1r}-w_{*t+1})}>0,$$

it reduces the probability of termination, weakly increasing the value of the firm. So (29) holds as an equality at the optimum.

Substituting next the binding incentive constraint (35) into constraint (30) gives

$$\delta[(1-p_1)w_{1r}+p_1w_{*t+1}]+(1-\delta)\underline{w}=w.$$

Then if $w\geq\underline{w}$, we have $p_1^*(w)=0$ and $w_{1r}=\frac{w-(1-\delta)\underline{w}}{\delta}$; if $w\in[w_{A,t+1},\underline{w})$, we have $p_1^*(w)=\frac{\underline{w}-w}{\underline{w}-w_{*t+1}}$ and $w_{1r}=\underline{w}$. This completes the proof of the proposition.

A.2 Proposition 6 and its proof

Proposition 6 *Suppose Assumption 1 is violated at t. Then for all w with which the problem that defines $V_{l,t}(w)$ has a solution, it is optimal to set $p_i(w)=1$ for $i=1,2$.*

Proof. Suppose not and suppose, with the optimal contract, $p_i(w)<1$ at some w. Then, following from Lemmas 3 and 4 in A.1, it is optimal to set $c_{ir}=c_{if}=0$. Consider a deviation from the optimal contract, letting $p_i'=1$ and $c_{if}'=\beta(1-p_i)(w_{ir}-w_{*t+1})$, while holding other parts of the contract constant. Obviously, this deviation does not violate constraints (7) and (8), but the resulting contract would make the firm weakly better off in expected utility, by a non-negative amount of

$$\epsilon=x_i\beta(1-p_i)[V_{t+1}-(1-\beta)(\bar{\theta}'-\psi-w_{*t+1})-\beta V_{t+2}]\geq 0.$$

This proves the proposition. ■

B Equilibrium

This section contains the omitted proofs in Sections 4, 5 and 6.

B.1 General anlysis

Lemma 7 *In any equilibrium, $\eta_t \leq 1$ for all t.*

Proof. Suppose to the contrary that an equilibrium exists in which $\eta_t > 1$ for some t. Then, in period t, a firms's expected value must be

$$V_t = -(1-\beta)C_o + \beta V_{t+1}.$$

This is because otherwise a vacant firm and a worker can form a new pair and make both parties strictly better off. By the free entry and exit condition, $V_t \geq 0$, which implies that $V_{t+1} > 0$. Hence, no firm exits in period $t+1$: $\eta_{t+1} \geq \eta_t > 1$. By induction, we have $\eta_\tau > 1$ for all $\tau \geq t$. But this implies that $V_t = -C_o < 0$, violating the free entry and exit condition. Hence, in equilibrium, $\eta_t \leq 1$ for all t.

Note that the proof is independent of the structure of the optimal contract as well as inequality (12). ■

Proposition 7 *In any equilibrium, Assumption 1 holds for all t.*

Proof. Fix an arbitrary t. We prove that Assumption 1 holds for $t-1$. We consider two cases in turn: (i) Assumption 1 holds for t and (ii) Assumption 1 is violated at t.

(i) Suppose that Assumption 1 holds for t. Then the optimal long-term contract in period t is described by Proposition 1, and all the equilibrium conditions for t in Definition 1 must hold. There are two subcases to consider here: Period t is type S or I, and period t is type T. If period t is type S or I, then

$$\begin{aligned} V_t =& V_{l,t}(w_{y,t}), \\ \leq& (1-\beta)(1+x_2\beta)(\bar{\theta}'-\psi) - (1-\beta)\underline{w} + x_1\beta V_{t+1} + x_2\beta^2 V_{t+2}, \\ \leq& (1-\beta)(\bar{\theta}'-\psi) - (1-\beta)(1-x_2\beta)\underline{w} + \beta V_{t+1}, \\ <& (1-\beta)(\bar{\theta}'-\psi-w_{*t}) + \beta V_{t+1}. \end{aligned}$$

The first inequality holds since $w_{y,t} \geq w_{A,t}$ in any incentive compatible long-term contract. The second inequality holds since $V_{t+1} \geq V_{s,t+1}(\underline{w})$ by (12). The last inequality holds since $w_{*t} = 0$ in a type S or I period. If period t is type T, then

$$V_t = (1-\beta)(\bar{\theta}'-\psi-\underline{w}) + \beta V_{t+1} \leq (1-\beta)(\bar{\theta}'-\psi-w_{*t}) + \beta V_{t+1},$$

where the inequality holds since $\eta_t \leq 1$.

(ii) Suppose that Assumption 1 is violated at t. Then it follows from Proposition 6 that, for all w with which the problem that defines $V_{l,t}(w)$ has a solution, it is optimal to set $p_i(w) = 1$. In other words, the optimal long-term contract is essentially the optimal short-term contract. Consider now the labor market equilibrium. Given $0 < \eta_t \leq 1$, since firms are indifferent between contracting with young and old workers, it is easy to see the prevailing expected utilities offered to young workers and old workers are

$$w_{y,t} = w_{A,t} \text{ and } w_{o,t} = \underline{w}.$$

Hence, a firm's expected value in period t is

$$V_t = (1-\beta)(\bar{\theta}' - \psi - \underline{w}) + \beta V_{t+1} \leq (1-\beta)(\bar{\theta}' - \psi - w_{*t}) + \beta V_{t+1},$$

where the inequality holds because $\eta_t \leq 1$. ■

Lemma 8 $\eta_t < 1$ *for all* t *in equilibrium if* $C_o > \underline{C}$.

Proof. Suppose to the contrary that an equilibrium exists in which $\eta_t = 1$ for some t. We claim that $\eta_{t+1} \geq \eta_t = 1$. We consider two cases in turn: (a) there is a positive measure of unemployed old workers in period t; and (b) there are no unemployed old workers in period t.

(a) Suppose that there is a positive measure of unemployed old workers in period t. In this case,

$$V_t = V_{s,t}(\underline{w}) = (1-\beta)(\underline{C} - C_o) + \beta V_{t+1}.$$

By the free-entry-and-exit condition, $V_t \geq 0$, which implies $V_{t+1} > 0$ since $C_o > \underline{C}$. Hence, no firm exits in period $t+1$: $\eta_{t+1} \geq \eta_t = 1$.

(b) Suppose that there are no unemployed old workers in period t. This is the case if and only if $1/2$ measure of firms entered long-term contracts with young workers in period $t-1$ and stay with their retained old workers in period t. This implies that Assumption 1 holds for $t-1$:

$$V_t - (1-\beta)(\underline{C} - C_o) - \beta V_{t+1} \leq 0,$$

since otherwise the optimal long-term contract in period $t-1$ specifies $p_{i,t-1}(w) = 1$ for all w and $i = 1, 2$. Here $w_{*t} = \underline{w}$ since $\eta_t = 1$. By the free-entry-and-exit condition, $V_t \geq 0$, which implies $V_{t+1} > 0$. Hence, no firm exits in period $t+1$: $\eta_{t+1} \geq \eta_t = 1$.

Hence, $\eta_{t+1} \geq \eta_t = 1$. By induction, we have $\eta_\tau = 1$ for all $\tau \geq t$ and therefore

$$V_\tau = (1-\beta)(\underline{C} - C_o) + \beta V_{\tau+1}, \ \forall \tau \geq t.$$

Hence, $V_t = \underline{C} - C_o < 0$, violating the free-entry-and-exit condition. ■

Lemma 9 *Suppose that $C_o < \underline{C}$, or $C_o = \underline{C}$ and $C_e > 0$. Then there is no exit of firms in any period in equilibrium.*

Proof. Suppose to the contrary that an equilibrium exists such that a positive measure of firms exit in some period t (i.e. $\eta_{t-1} > \eta_t$). The free-entry-and-exit condition implies that $V_t = 0$. Furthermore, by Lemma 8, $\eta_t < \eta_{t-1} \leq 1$.

Suppose that $C_o < \underline{C}$. Then a vacant firm's expected value satisfies

$$V_t \geq (1-\beta)(\underline{C} - C_o) + \beta V_{t+1} > 0,$$

a contradiction to the free-entry-and-exit condition.

Suppose that $C_o = \underline{C}$ and $C_e > 0$. We first argue that there cannot be entry of firms in any period $t' > t$. To see this, suppose, without loss of generality, that there is entry of firms in period $t+1$. Then, by the free-entry-and-exit condition, $V_{t+1} = C_e > 0$. Hence, $V_t \geq (1-\beta)(\underline{C} - C_o) + \beta V_{t+1} > 0$, a contradiction to the fact that $V_t = 0$. Hence, $\eta_\tau \leq \eta_t < \eta_{t-1} \leq 1$ for all $\tau \geq t$. Furthermore, $\eta_\tau < 1$ for all $\tau \geq t$ implies that firms can stay in the market and earn non-negative period payoff in every period $\tau \geq t$. Since $V_t = 0$, it must be the case firms earn zero payoff in every period $\tau \geq t$. This is possible only if all periods $\tau(\tau \geq t)$ are type T or type I, and $V_\tau = (1-\beta)(\underline{C} - C_o) + \beta V_{\tau+1}$ for all $\tau \geq t$. This implies that $w_{y,\tau} = \underline{w}$ for all $\tau \geq t$. It then follows from Proposition 1(iii) that $\hat{p}^*_{1\tau} = \hat{p}^*_{2\tau} = 0$ for all $\tau \geq t$. Thus, the measure of vacant firms in period $t+1$ is given by $\eta_{t+1} - 1/2 < 1/2$, a contradiction to the fact that period $t+1$ is type T or type I.

Hence, there is no exit of firms in any period in equilibrium. ■

Lemma 10 *Suppose Assumption 2 holds, and $C_0 = \underline{C}$ implies that $C_e > 0$. In any equilibrium, $\eta_t = 1$ for some t implies that $\eta_t = 1$ for all t.*

Proof. Suppose that there exists an equilibrium in which $\eta_t = 1$ for some t. By Lemma 8, it must be the case that $C_o \leq \underline{C}$. In this equilibrium, by Lemma 9, there is no exit of firms in any period. Hence, $\eta_\tau = 1$ for all $\tau > t$.

We show that $\eta_{t-1} = 1$. Clearly, this is true if $C_e \leq \underline{C} - C_o$. Assume for the rest of the proof that $C_e > \underline{C} - C_o$. Suppose to the contrary that $\eta_{t-1} < 1$. Then a positive measure of firms must enter the market in period t. By the free entry and exit condition, $V_t = C_e$.

However, since $\eta_{t+1} = 1$, we have $w_{*t+1} = \underline{w}$. Hence,

$$V_t \leq (1-\beta)(\underline{C} - C_o) + \beta V_{t+1} < C_e,$$

which is a contradiction. Hence, $\eta_{t-1} = 1$. By induction, $\eta_\tau = 1$ for all $\tau < t$. ■

Lemma 11 *Suppose Assumption 2 holds, $C_e = 0$ and $\underline{C} - C_o = 0$. In any equilibrium, $\eta_t = 1$ for all t.*

Proof. The free entry and exit condition implies that $V_t = 0$ for all t. Specifically, $V_{l,t}(w_{y,t}) = 0$ for all t. It is easy to see that the unique solution to this equality is that $w_{y,t} = \underline{w}$ for all t. By the analysis in Section 4.2, $w_{y,t} = \underline{w}$ is part of the equilibrium only if $w_{A,t} = \underline{w}$ or $w_{*t+1} = \underline{w}$. Hence, in equilibrium $\alpha_{y,t} \geq \alpha_{o,t} = 1$. By Proposition 1, $w_{y,t} = \underline{w}$ implies that employed young workers are retained with probability one after one period. Then the equilibrium evolution of η_t, L_t and $\alpha_{y,t}$ and $\alpha_{o,t}$ must satisfy

$$L_t = 1 - \frac{1}{2}\alpha_{y,t-1},$$
$$\frac{1}{2}\alpha_{y,t} + \alpha_{o,t}(L_t - \frac{1}{2}) = \eta_t - \frac{1}{2}\alpha_{y,t-1}.$$

Substituting $\alpha_{y,t} = \alpha_{o,t} = 1$ into the above equations yields $\eta_t = 1$ for all t. ■

To summarize, Lemmas 7, 8, 9, 10 and 11 prove the following proposition:

Proposition 8 *Suppose Assumption 2 holds. In any equilibrium, either $\eta_t < 1$ for all t or $\eta_t = 1$ for all t.*

B.1.1 The bounds on equilibrium firm values

Lemma 12 *The equilibrium value of the firm V_t ($t \geq 1$) satisfies*

$$\bar{\theta}' - \psi - \underline{w} \leq V_t \leq \bar{\theta}' - \psi - \frac{1}{1 + x_2\beta}\underline{w}. \tag{31}$$

Proof. It follows from (12) that, for all $s = 0, 1, 2, \ldots,$

$$\begin{aligned} V_t \geq &(1-\beta)(\bar{\theta}' - \psi - \underline{w}) + \beta V_{t+1}, \\ &\ldots \\ \geq &(1-\beta)(\bar{\theta}' - \psi - \underline{w})(1 + \beta + \cdots + \beta^s) + \beta^{s+1} V_{t+s+1}. \end{aligned}$$

Let s go to infinity and we have $V_t \geq \bar{\theta}' - \psi - \underline{w}$ for all $t \geq 1$.

To obtain the upper bound on V_t, we have

$$\begin{aligned} V_t \leq &V_{l,t}(w_{A,t}) \\ = &(1-\beta)(1 + x_2\beta)(\bar{\theta}' - \psi - \underline{w}) + (1-\beta)x_2\beta(\underline{w} - w_{*t+1}) + x_1\beta V_{t+1} + x_2\beta^2 V_{t+2}, \end{aligned}$$

$$\leq (1-\beta)(1+x_2\beta)(\bar{\theta}'-\psi)-(1-\beta)\underline{w}+x_1\beta V_{t+1}+x_2\beta^2 V_{t+2}. \tag{32}$$

It follows that

$$\begin{aligned} V_t + x_2\beta V_{t+1} \leq & (1-\beta)(1+x_2\beta)(\bar{\theta}'-\psi)-(1-\beta)\underline{w}+\beta(V_{t+1}+x_2\beta V_{t+2}), \\ & \dots \\ \leq & [(1-\beta)(1+x_2\beta)(\bar{\theta}'-\psi)-(1-\beta)\underline{w}](1+\beta+\cdots+\beta^s) \\ & + \beta^{s+1}(V_{t+s+1}+x_2\beta V_{t+s+2}). \end{aligned}$$

Let s go to infinity to get

$$V_t + x_2\beta V_{t+1} \leq (1+x_2\beta)(\bar{\theta}'-\psi)-\underline{w}, \ \forall t \geq 1.$$

Substitute this into (32) and we have

$$\begin{aligned} V_t \leq & (1-x_2\beta)[(1+x_2\beta)(\bar{\theta}'-\psi)-\underline{w}]+x_2^2\beta^2 V_{t+2}, \\ & \dots \\ \leq & (1-x_2\beta)[(1+x_2\beta)(\bar{\theta}'-\psi)-\underline{w}](1+x_2^2\beta^2+\cdots+x_2^{2s}\beta^{2s})+x_2^{2s+2}\beta^{2s+2}V_{t+2s+2}. \end{aligned}$$

Let s go to infinity and we have $V_t \leq \bar{\theta}'-\psi-\underline{w}/(w+x_2\beta)$ for all $t \geq 1$. ■

B.2 Stationary equilibria

Proof of Proposition 2. Define η_s^- and η_s^+ as follows:

$$\begin{aligned} \eta_s^+(C_o) &\equiv 1-\frac{x_1(1+\beta)(C_o-\underline{C})}{2x_1\beta(C_o-\underline{C})+2x_2\beta\underline{w}}, \\ \eta_s^-(C_o,C_e) &\equiv 1-\frac{x_1(1+\beta)[C_e+C_o-\underline{C}]}{2x_1\beta[C_e+C_o-\underline{C}]+2x_2\beta\underline{w}}. \end{aligned}$$

Since $C_e > 0$, we have $\eta_s^-(C_o,C_e) < \eta_s^+(C_o)$. Furthermore, $\eta_s^-(C_o,C_e) > 0$ if and only if $x_1(1-\beta)(C_e+C_o-\underline{C}) < 2x_2\beta\underline{w}$ which holds because

$$x_1(1-\beta)(C_e+C_o-\underline{C}) < x_1(1-\beta)(\bar{C}-\underline{C}) = \frac{x_1(1-\beta)x_2\beta\underline{w}}{1+x_2\beta} < x_2\beta\underline{w} < 2x_2\beta\underline{w},$$

where the first inequality holds by Assumption 2.

First, we show the model does not have a stationary equilibrium in which all periods are type T. Suppose to the contrary that such an equilibrium exists. It follows from the analysis in Section 4.3 that

$$V = (1-\beta)(\underline{C} - C_o) + \beta V.$$

Hence, $V = \underline{C} - C_o$. The free entry and exit condition requires that $V \geq 0$. This holds if and only if $\underline{C} \geq C_o$. In these equilibria, in each period, all young workers enter a long-term contract with a vacant firm and are retained with probability one after one period. Then the equilibrium values of L and α_y satisfy

$$L = 1 - [\eta - (1-L)], \tag{33}$$

$$\alpha_y = 1. \tag{34}$$

These imply that $L = 1 - \eta/2$. For a period to be a type T period, it must be the case that the measure of vacant firm $\eta - (1-L) = \eta/2 > 1/2$. Hence, $\eta > 1$. However, by Lemma 7, in equilibrium it must be $\eta \leq 1$, a contradiction. Hence, the model does not have a stationary equilibrium in which all periods are type T.

Next, consider the stationary equilibria where all periods are type S or type I. In these equilibria, in each period, all vacant firms enter a long-term contract with a young worker. Let $\hat{p}_1^*$ denote the equilibrium probability with which a young worker is fired when her first period output is low. Then the equilibrium values of L and α_y satisfy

$$L = 1 - [\eta - (1-L)](1 - x_1 \hat{p}_1^*), \tag{35}$$

$$\alpha_y = \frac{\eta - (1-L)}{\frac{1}{2}}. \tag{36}$$

Combining (35) and (36) yields

$$\alpha_y = \frac{2\eta}{2 - x_1 \hat{p}_1^*}.$$

We consider two cases in turn: (i) all periods are type S ($\alpha_y < 1$), and (ii) all periods are type I ($\alpha_y = 1$).

(i) Suppose that all periods are type S. Then $w_{y,t} = w_{A,t}$ and $\hat{p}_1^* = 1$. Hence,

$$\alpha_y = \frac{2\eta}{1 + x_2},$$

which is strictly less than 1 if and only if $\eta < (1 + x_2)/2$. In this case, the equilibrium value of a firm is given by

$$V = \bar{\theta}' - \psi - \frac{\underline{w}}{1 + x_2\beta} = \bar{C} - C_o,$$

which violates the free entry and exit condition by Assumption 2. Hence, no equilibrium exists in which all periods are type S.

(ii) Suppose all periods are type I. $\alpha_y = 1$ implies $\hat{p}_1^* = 2(1-\eta)/x_1$. For $\hat{p}_1^*$ to be a well defined probability, it must be that

$$\eta \in \left[\frac{x_2+1}{2}, 1\right].$$

In this case, the equilibrium value of a firm is given by

$$V = \bar{\theta}' - \psi - \frac{[x_1(1+\beta) - 2\beta(1-\eta)]\underline{w}}{x_1[1+\beta-2\beta(1-\eta)]}.$$

Observe that V is strictly decreasing in η, goes to $\bar{C} - C_o > 0$ as η goes to $(x_1+1)/2$, and goes to $\underline{C} - C_o$ as η goes to 1. Again the free entry and exit condition requires $0 \le V_t \le C_e$. If $\underline{C} < C_o$, there exists a unique $\eta_s^+(C_o) \in [(x_2+1)/2, 1)$ such that $V \ge 0$ if and only if $\eta \le \eta_s^+(C_o)$, where

$$\eta_s^+(C_o) \equiv 1 - \frac{x_1(1+\beta)(\bar{\theta} - \psi - \underline{w} - C_o)}{2x_1\beta(\bar{\theta} - \psi - C_o) - 2\beta\underline{w}}.$$

Notice here that η_s^+ is strictly decreasing in C_o. If $\underline{C} \ge C_o$, then $V \ge 0$ for all η and let $\eta_s^+(C_o) \equiv 1$ in this case. Since $0 < C_e < \bar{C} - C_o$, a unique $\eta_s^-(C_o, C_e) \in ((x_2+1)/2, 1)$ exists such that $V \le C_e$ if and only if $\eta \ge \eta_s^-(C_o, C_e)$, where

$$\eta_s^-(C_o, C_e) \equiv 1 - \frac{x_1(1+\beta)[C_e - (\bar{\theta} - \psi - \underline{w} - C_o)]}{2x_1\beta[C_e - (\bar{\theta} - \psi - C_o)] + 2\beta\underline{w}}.$$

Observe that η_s^- is strictly decreasing in both C_e and C_o. This completes the proof. ■

B.3 Equilibrium cycles

Proof of Lemma 1. Suppose to the contrary that an equilibrium exists in which both periods $t-1$ and t are type TI with $w_{y,t} = w_{B,t}$. This implies that in period t we have $V_t = (1-\beta)(\bar{\theta}' - \psi - \underline{w}) + \beta V_{t+1}$. Substituting this into $V_{l,t-1}(w_{B,t-1}) = V_{s,t-1}(\underline{w})$ yields $w_{y,t-1} = w_{B,t-1} = \underline{w}$. It then follows from Proposition 1(iii) that $\hat{p}_{1t-1}^* = \hat{p}_{2t-1}^* = 0$. Thus, the measure of vacant firms in period t is given by $\eta_t - \frac{1}{2}$, which is strictly less than $\frac{1}{2}$ since $\eta_t < 1$ by Proposition 8. This contradicts with our initial assumption that period t is type TI. Hence, there cannot be two successive type TI periods. ■

Proof of Lemma 2. Let period $t-1$ and t be type SI. Suppose to the contrary that period $t+1$ is type T. First, no prospective firm is willing to enter the market in period $t+1$:

$$V_{t+1} = (1-\beta)(\bar{\theta}' - \psi - \underline{w}) + \beta V_{t+2} < V_{t+2} \le C_e. \tag{37}$$

The first inequality holds because period $t+2$ is type SI with $w_{y,t+2} < w_{B,t+2}$ by Lemma 1, and that the left inequality of (31) holds strictly in this case. Hence, $\eta_{t+1} \leq \eta_t$. Then the measure of vacant firms in period $t+1$ satisfies:

$$
\begin{aligned}
&\eta_{t+1} - [\eta_t - (1 - L_t)][x_2 + x_1(1 - \hat{p}^*_{1t})] \\
=&\eta_{t+1} - [\eta_t - (\eta_{t-1} - (1 - L_{t-1}))(x_2 + x_1(1 - \hat{p}^*_{1t-1}))][x_2 + x_1(1 - \hat{p}^*_{1t})] \\
\leq& x_1\hat{p}^*_{1t}\eta_t + (1 - x_1\hat{p}^*_{1t-1})(1 - x_1\hat{p}^*_{1t})[\eta_{t-1} - (1 - L_{t-1})] \\
\leq& \frac{1}{2}x_1\hat{p}^*_{1t} + (1 - x_1\hat{p}^*_{1t-1})[\eta_{t-1} - (1 - L_{t-1})] \leq \frac{1}{2},
\end{aligned}
$$

where the first inequality holds since $\eta_{t+1} \leq \eta_t$, the second inequality holds since period t is type SI: $\eta_t - [\eta_{t-1} - (1 - L_{t-1})](1 - x_1\hat{p}^*_{1t-1}) \leq \frac{1}{2}$, and the last two inequalities hold because period $t-1$ is type SI: $\eta_{t-1} - (1 - L_{t-1}) \leq \frac{1}{2}$. That is, period $t+1$ must be a type SI period, a contradiction to our initial hypothesis that period $t+1$ is type T. Hence, period $t+1$ is type SI. ■

Proof of Proposition 4. We construct equilibria in which the measure of firms in the market rises in the first period of the cycle and falls in the second. Then the optimality of firms' entry and exit decisions implies that their expected values measured at the beginning of each period are $V^{(1)} = C_e$ and $V^{(2)} = 0$, respectively. By Proposition 1, we have

$$
V^{(1)} = (1-\beta^2)(\bar{\theta}' - \psi - \underline{w}) + \beta^2 V^{(1)} + x_1\hat{p}_1^{*(1)}\beta \left[V^{(2)} - (1-\beta)\left(\bar{\theta}' - \psi - \frac{w}{x_1}\right) - \beta V^{(1)} \right].
$$

Since $V^{(1)} = C_e$ and $V^{(2)} = 0$, we have

$$
\hat{p}_1^{*(1)} = \frac{(1-\beta^2)[C_e - (\bar{\theta}' - \psi - \underline{w})]}{-x_1\beta[(1-\beta)(\bar{\theta}' - \psi - \frac{w}{x_1}) + \beta C_e]}.
$$

Similarly,

$$
\hat{p}_1^{*(2)} = \frac{-(1-\beta^2)(\bar{\theta}' - \psi - \underline{w})}{x_1\beta[C_e - (1-\beta)(\bar{\theta}' - \psi - \frac{w}{x_1})]}.
$$

It is easy to see that $0 \leq \hat{p}_1^{*(2)} \leq 1$ if and only if $\bar{\theta}' - \psi - \underline{w} \leq 0$ (i.e. $\underline{C} \leq C_o$). Assume for the rest of the proof that $\underline{C} \leq C_o$. Then $\hat{p}_1^{*(1)} \geq 0$ if and only if

$$
C_e < -\frac{(1-\beta)(\bar{\theta}' - \psi - \frac{w}{x_1})}{\beta},
$$

and $\hat{p}_1^{*(1)} \le 1$ if and only if

$$C_e \le \frac{(1-\beta)(1+x_2\beta)(\bar{C}-C_o)}{1-x_2\beta^2}.$$

We claim that

$$-\frac{(1-\beta)(\bar{\theta}'-\psi-\frac{w}{x_1})}{\beta} > \frac{(1-\beta)(1+x_2\beta)(\bar{C}-C_o)}{1-x_2\beta^2}.$$

To see this, note that the above inequality holds if and only if

$$-(1-\beta)(1-x_2\beta^2)(\underline{C}-C_o)+(1-\beta)(1-x_2\beta^2)\frac{x_2\underline{w}}{x_1} > \beta(1-\beta)(1+x_2\beta)(\underline{C}-C_o)+\beta(1-\beta)x_2\underline{w},$$

$$\text{or equivalently, } C_o > \underline{C} - \frac{(1-\beta)x_2\underline{w}}{x_1},$$

which holds because $C_o \ge \underline{C}$. Hence, $0 \le \hat{p}_1^{*(1)} \le 1$ if and only if

$$C_e \le \frac{(1-\beta)(1+x_2\beta)(\bar{C}-C_o)}{1-x_2\beta^2}.$$

Since $C_e > 0$, such a C_e exists if and only if $C_o < \bar{C}$. Given $\hat{p}_1^{*(1)}$ and $\hat{p}_1^{*(2)}$, it is straightforward to compute the values for $\{L^{(j)}, \alpha_y^{(j)}, \alpha_o^{(j)}, w_y^{(j)}, w_o^{(j)}, w_*^{(j)}; j=1,2\}$ and to verify that they are just as given in the proposition. ■

Lemma 13 *Suppose Assumptions 1 and 2 hold and $C_e > 0$. Suppose that $\{\eta_\tau\}_{\tau\ge1}$ constitutes the path of firm measures in an equilibrium where type SI and type T periods alternate. Then*

1. *For any two periods $t+1$ and $t+2$, where period $t+1$ is type SI and period $t+2$ is type T, the following three conditions hold:*

   (a) $\eta_{t+1} \ge \eta_{t+2}$ and

   $$\frac{1+\beta}{2(1+x_1\beta)} < \eta_{t+1} < 1.$$

   (b) If $\frac{1+\beta}{2(1+x_1\beta)} < \eta_{t+1} \le \frac{2+x_1\beta}{2(1+x_1\beta)}$, then period $t+1$ is type S and

   $$\eta_{t+2} > x_2\eta_{t+1} + \frac{x_1\beta + x_1}{2(x_1\beta+1)}.$$

   (c) If $\frac{2+x_1\beta}{2(1+x_1\beta)} < \eta_{t+1} < 1$, then when period $t+1$ is type S,

   $$\eta_{t+2} > 1 - \frac{1-\eta_{t+1}}{\beta - 2\beta(1-\eta_{t+1})},$$

and when period $t+1$ is type I,

$$\eta_{t+2} = 1 - \frac{1-\eta_{t+1}}{\beta - 2\beta(1-\eta_{t+1})}.$$

2. *The equilibrium values of the aggregate variables $\{L_\tau, \alpha_{y,t}, \alpha_{o,\tau}, w_{y,\tau}, w_{o,\tau}, w_{*\tau}, V_\tau\}_{\tau\geq 1}$ can be written as a function of $\{\eta_\tau\}_{\tau\geq 1}$. Specifically,*

   (a) *If period $t+1$ is type S, then*

$$L_{t+1} = 1 - \frac{1}{2}(1 - x_1\hat{p}^*_{1t}),\ L_{t+2} = 1 - x_2[\eta_{t+1} - \frac{1}{2}(1 - x_1\hat{p}^*_{1t})],$$
$$\alpha_{y,t+1} = 2[\eta_{t+1} - \frac{1}{2}(1 - x_1\hat{p}^*_{1t})],\ \alpha_{o,t+1} = 0,\ \alpha_{y,t+2} = 1,\ \alpha_{o,t+2} = \frac{\beta(1 - x_1\hat{p}^*_{1t}) - \hat{p}^*_{1t}}{\beta(1 - x_1\hat{p}^*_{1t})},$$
$$w_{o,t} = w_{o,t+1} = \underline{w},\ w_{y,t} = (1 - \delta\hat{p}^*_{1t})\underline{w},\ w_{y,t+1} = w_{A,t+1},$$
$$w_{*t+1} = 0,\ w_{*t+2} = \alpha_{o,t+2}\underline{w},$$
$$V_{t+1} = (1-\beta^2)(\bar{\theta}' - \psi - \underline{w}) + \beta^2 V_{t+3} + x_2\beta(1-\beta)(\underline{w} - w_{*t+2}),$$
$$V_{t+2} = (1-\beta)(\bar{\theta}' - \psi - \underline{w}) + \beta V_{t+3},$$

   where $\hat{p}^*_{1t}$ is the smaller root of

$$\frac{1}{2}x_1x_2(\hat{p}^*_{1t})^2 - [\frac{1}{2}x_1 + x_2(1-\eta_{t+1}) + x_1\beta(1-\eta_{t+2})]\hat{p}^*_{1t} + (1-\eta_{t+2})\beta = 0. \quad (38)$$

   (b) *If period $t+1$ is type I, then*

$$L_{t+1} = \frac{3}{2} - \eta_{t+1},\ L_{t+2} = \frac{1}{2} + \frac{(1-\eta_{t+1})}{\beta(2\eta_{t+1}-1)(\underline{w} - w_{*t+2})},$$
$$\alpha_{y,t+1} = 1,\ \alpha_{o,t+1} = 0,\ \alpha_{y,t+2} = 1,\ \alpha_{o,t+2} = \frac{w_{*t+2}}{\underline{w}},$$
$$w_{o,t} = w_{o,t+1} = \underline{w},\ w_{y,t} = \underline{w} - \frac{2\beta(1-\eta_{t+1})\underline{w}}{x_1(1+\beta)},\ w_{y,t+1} = \underline{w} - \frac{2(1-\eta_{t+1})\underline{w}}{x_1(1+\beta)(2\eta_{t+1}-1)},$$
$$w_{*t+1} = 0,\ 0 < w_{*t+2} < \underline{w},$$
$$V_{t+1} = (1-\beta^2)(\bar{\theta}' - \psi - \underline{w}) + \beta^2 V_{t+3} + \frac{2x_2(1-\beta)(1-\eta_{t+1})\underline{w}}{x_1(2\eta_{t+1}-1)},$$
$$V_{t+2} = (1-\beta)(\bar{\theta}' - \psi - \underline{w}) + \beta V_{t+3}.$$

In addition, suppose $\{\eta_\tau\}_{\tau\geq 1}$ satisfies the three conditions in part 1 and the expected values $\{V_\tau\}_{\tau\geq 1}$ from part 2 are such that $0 \leq V_t \leq C_e$ where $V_t = 0$ if $\eta_t < \eta_{t-1}$ and $V_t = C_e$ if $\eta_t > \eta_{t-1}$. Then an equilibrium exists such that $\{\eta_\tau\}_{\tau\geq 1}$ constitutes the equilibrium path firm *measures.*

Proof. Let period t be type T, period $t+1$ be type SI and period $t+2$ be type T. Firstly, by a similar argument to that in the proof of Lemma 2, it must hold that $\eta_{t+2} \leq \eta_{t+1}$. This proves the first claim in part 1(a).

It follows from Proposition 1 that V_{t+1} is given by

$$V_{t+1} =(1-\beta^2)(\bar{\theta}'-\psi-\underline{w})+\beta^2 V_{t+3} \\ + \hat{p}^*_{1t+1}x_1\beta\left[V_{t+2}-(1-\beta)\left(\bar{\theta}'-\psi-\frac{\underline{w}-x_2 w_{*t+2}}{x_1}\right)-\beta V_{t+3}\right]. \tag{39}$$

Since period $t+2$ is type T, we have $V_{t+2} = (1-\beta)(\bar{\theta}'-\psi-\underline{w})+\beta V_{t+3}$. Combining this and equation (39), we have

$$V_{t+1}-(1-\beta)(\bar{\theta}'-\psi-\underline{w})-\beta V_{t+2} = \hat{p}^*_{1t+1}\beta(1-\beta)x_2(\underline{w}-w_{*t+2}). \tag{40}$$

Observe also that

$$V_t =(1-\beta^2)(\bar{\theta}'-\phi-\underline{w})+\beta^2 V_{t+2} \\ + \hat{p}^*_{1t}x_1\beta\left[V_{t+1}-(1-\beta)\left(\bar{\theta}'-\psi-\frac{\underline{w}-x_2 w_{*t+1}}{x_1}\right)-\beta V_{t+2}\right]. \tag{41}$$

Since period t is type T, we have $V_t = (1-\beta)(\theta'-\bar{\psi}-\underline{w})+\beta V_{t+1}$. Combining this and equation (41) gives us

$$\hat{p}^*_{1t} = \frac{\beta[V_{t+1}-(1-\beta)(\bar{\theta}'-\psi-\underline{w})-\beta V_{t+2}]}{x_1\beta[V_{t+1}-(1-\beta)(\bar{\theta}'-\psi-\underline{w})-\beta V_{t+2}]+x_2\beta(1-\beta)(\underline{w}-w_{*t+1})} \\ = \frac{\hat{p}^*_{1t+1}\beta(\underline{w}-w_{*t+2})}{x_1\hat{p}^*_{1t+1}\beta(\underline{w}-w_{*t+2})+\underline{w}}, \tag{42}$$

where the last equality follows from (40) and the fact that $w_{*t+1}=0$ in a type SI period. We can rewrite (42) as follows

$$w_{*t+2} = \frac{\beta\hat{p}^*_{1t+1}(1-x_1\hat{p}^*_{1t})-\hat{p}^*_{1t}}{\beta\hat{p}^*_{1t+1}(1-x_1\hat{p}^*_{1t})}\underline{w}. \tag{43}$$

For $t+2$ to be a type T period, it must be that $w_{*t+2}>0$, which holds, by (43), if and only if

$$\hat{p}^*_{1t+1} > \frac{\hat{p}^*_{1t}}{\beta(1-x_1\hat{p}^*_{1t})}. \tag{44}$$

Furthermore, we know that w_{*t+2} also satisfies:

$$\frac{w_{*t+2}}{\underline{w}} = \alpha_{o,t+2} = \frac{\eta_{t+2} - (1 - L_{t+2}) - \frac{1}{2}}{\frac{1}{2} - (1 - L_{t+2})}, \tag{45}$$

where

$$L_{t+2} = 1 - [\eta_{t+1} - (1 - L_{t+1})](1 - x_1 \hat{p}^*_{1t+1}). \tag{46}$$

Substituting (43) into (45) yields

$$\frac{\beta \hat{p}^*_{1t+1}(1 - x_1 \hat{p}^*_{1t}) - \hat{p}^*_{1t}}{\beta \hat{p}^*_{1t+1}(1 - x_1 \hat{p}^*_{1t})} = \frac{\eta_{t+2} - \frac{1}{2} - [\eta_{t+1} - (1 - L_{t+1})](1 - x_1 \hat{p}^*_{1t+1})}{\frac{1}{2} - [\eta_{t+1} - (1 - L_{t+1})](1 - x_1 \hat{p}^*_{1t+1})}. \tag{47}$$

There are two cases to be discussed: (i) period $t+1$ is type I, and (ii) period $t+1$ is type S.

(i) Suppose that $t+1$ is type I, or $\eta_{t+1} - (1 - L_{t+1}) = 1/2$. Since period t is type T, $L_{t+1} = 1 - (1 - x_1 \hat{p}^*_{1t})/2$. Then we can write $\hat{p}^*_{1t}$ as a function of η_{t+1}:

$$\hat{p}^*_{1t} = \frac{2(1 - \eta_{t+1})}{x_1} > 0. \tag{48}$$

Substituting (48) and $\eta_{t+1} - (1 - L_{t+1}) = 1/2$ into (47) yields

$$\eta_{t+2} = 1 - \frac{1 - \eta_{t+1}}{\beta - 2\beta(1 - \eta_{t+1})}. \tag{49}$$

Note that if η_{t+1} and η_{t+2} satisfy (49), it must be that $\eta_{t+1} > \eta_{t+2}$. Substituting (48) into (44) yields:

$$\hat{p}^*_{1t+1} > \frac{2(1 - \eta_{t+1})}{\beta x_1 (2\eta_{t+1} - 1)}.$$

This implies that

$$\frac{2(1 - \eta_{t+1})}{\beta x_1 (2\eta_{t+1} - 1)} < 1, \text{ or } \eta_{t+1} > \frac{2 + \beta x_1}{2(1 + \beta x_1)}. \tag{50}$$

This proves that if (50) is violated then period $t+1$ must be type S. This also proves the second claim of part 1(c).

Moreover, the equilibrium values of the variables $\{L_\tau, \alpha_{y,t}, \alpha_{o,\tau}, w_{y,\tau}, w_{o,\tau}, w_{*\tau}, V_\tau\}_{\tau \geq 1}$ can be written as a function of $\{\eta_\tau\}_{\tau \geq 1}$. To see this, $\eta_{t+1} - (1 - L_{t+1}) = 1/2$ implies that $L_{t+1} = 3/2 - \eta_{t+1}$. By (48), the measure of unemployed workers in period $t+2$ is

$$L_{t+2} = 1 - (1 - x_1 \hat{p}^*_{1t+1})[\eta_{t+1} - \frac{1}{2}(1 - x_1 \hat{p}^*_{1t})] = \frac{1}{2} + \frac{(1 - \eta_{t+1})}{\beta(2\eta_{t+1} - 1)(\underline{w} - w_{*t+2})}.$$

It follows from (48) and Proposition 1 that

$$w_{y,t} = \underline{w} - \frac{2\beta(1-\eta_{t+1})\underline{w}}{x_1(1+\beta)}.$$

It follows from (43), (48) and Proposition 1 that

$$w_{y,t+1} = \underline{w} - \frac{2(1-\eta_{t+1})}{x_1(1+\beta)(2\eta_{t+1}-1)}.$$

It is immediate that $\alpha_{y,t+1} = \alpha_{y,t+2} = 1$, $\alpha_{o,t+1} = 0$, $\alpha_{o,t+2} = w_{*t+2}/\underline{w}$, $w_{*t+1} = 0$ and $0 < w_{*t+2} < \underline{w}$. Notice that both w_{*t+2} and $\hat{p}^*_{1,t+1}$ are indeterminate. To see this, note that, when $\hat{p}^*_{1t}$ is given by (48), for any $\hat{p}^*_{1,t+1}$ such that w_{*t+2} given by (43) lies in $(0, \underline{w})$, equation (47) holds. Finally, $V_{t+2} = (1-\beta)(\bar{\theta}' - \psi - \underline{w}) + \beta V_{t+3}$. Substituting this, (43) and (48) into (39) gives us

$$V_{t+1} = (1-\beta^2)(\bar{\theta}' - \psi - \underline{w}) + \beta^2 V_{t+3} + \frac{2x_2(1-\beta)(1-\eta_{t+1})\underline{w}}{x_1(2\eta_{t+1}-1)}.$$

This completes the proof of part 2(b).

(ii) Suppose that $t+1$ is type S. Then $\hat{p}^*_{1t+1} = 1$ and (47) can be rewritten as

$$\frac{1}{2}x_1x_2(\hat{p}^*_{1t})^2 - [\frac{1}{2}x_1 + x_2(1-\eta_{t+1}) + x_1\beta(1-\eta_{t+2})]\hat{p}^*_{1t} + (1-\eta_{t+2})\beta = 0. \quad (51)$$

Since $\hat{p}^*_{1t+1} = 1$, inequality (44) can be rewritten as

$$\hat{p}^*_{1t} < \frac{\beta}{x_1\beta + 1}.$$

Furthermore, by (43), $w_{*t+2} < \underline{w}$ only if $\hat{p}^*_{1t} > 0$. Finally, since period $t+1$ is type S, we have $\eta_{t+1} - (1 - L_{t+1}) < \frac{1}{2}$ which, given $L_{t+1} = 1 - \frac{1}{2}(1 - x_1\hat{p}^*_{1t})$, is equivalent to

$$\hat{p}^*_{1t} < \frac{2(1-\eta_{t+1})}{x_1}.$$

Denote the left-hand side of (51) by $g(p_1)$, which is a quadratic function and $g(0) > 0$. Then η_{t+1} and η_{t+2} are consistent with an equilibrium only if there is $p_1 \in P \doteq (0, \min\{\beta/(1 + x_1\beta), 2(1-\eta_{t+1})/x_1\})$ such that $g(p_1) = 0$.

(a) Suppose $\eta_{t+1} > (2+x_1\beta)/2(1+x_1\beta)$. Then $P = (0, 2(1-\eta_{t+1})/x_1)$. It is easy to verify that the axis of symmetry of $g(\cdot)$ is larger than $2(1-\eta_{t+1})/x_1$ and therefore the relevant root of $g(p_1) = 0$ is the smaller one, which lies in P if and only if

$$g\left(\frac{2(1-\eta_{t+1})}{x_1}\right) = \beta(1-\eta_{t+2})(2\eta_{t+1}-1) - (1-\eta_{t+1}) < 0.$$

This is equivalent to

$$\eta_{t+2} > 1 - \frac{1-\eta_{t+1}}{\beta - 2\beta(1-\eta_{t+1})}.$$

For all $\eta_{t+1} < 1$, η_{t+2} exists such that $\eta_{t+2} \leq \eta_{t+1}$ and the above inequality holds. This proves the first claim of part 1(c).

(b) Suppose $\eta_{t+1} \leq (2+x_1\beta)/2(1+x_1\beta)$. Then $P = (0, \beta/(1+x_1\beta))$. Again it is easy to verify that the axis of symmetry of $g(\cdot)$ is larger than $\beta/(1+x_1\beta)$ and therefore the relevant root of $g(p_1) = 0$ is the smaller one, which lies in P if and only if

$$g\left(\frac{\beta}{1+x_1\beta}\right) = \frac{x_1x_2\beta}{2(x_1\beta+1)} - \frac{x_1}{2} + (1-\eta_{t+2}) - x_2(1-\eta_{t+1}) < 0.$$

This is equivalent to

$$\eta_{t+2} > x_2\eta_{t+1} + \frac{x_1\beta + x_1}{2(x_1\beta+1)}.$$

There exists $\eta_{t+2} \leq \eta_{t+1}$ such that the above inequality holds if and only if $\eta_{t+1} > \frac{1+\beta}{2(1+x_1\beta)}$. This completes the proof of part 1(b).

As in case (i), the equilibrium values of the variables $\{L_\tau, \alpha_{y,t}, \alpha_{o,\tau}, w_{y,\tau}, w_{o,\tau}, w_{*\tau}, V_\tau\}_{\tau\geq1}$ can be written as a function of $\{\eta_\tau\}_{\tau\geq1}$. One can easily verify that part 2(a) is true.

The analysis in cases (i) and (ii) also shows that in equilibrium it must be that $\eta_{t+1} > \frac{1+\beta}{2(1+x_1\beta)}$. This proves the second claim of part 1(a).

Lastly, it is straightforward to verify that if $\{\eta_\tau\}_{\tau\geq1}$ satisfies all the conditions in part 1 of the lemma and the expected values $\{V_\tau\}_{\tau\geq1}$ given in part 2 of the lemma are such that $0 \leq V_t \leq C_e$ where $V_t = 0$ if $\eta_t < \eta_{t-1}$ and $V_t = C_e$ if $\eta_t > \eta_{t-1}$, then an equilibrium exists in which $\{\eta_\tau\}_{\tau\geq1}$ is the path of firm measures in the market. This completes the proof the lemma. ■

Proof of Proposition 5. Proposition 5 is a direct corollary of Lemma 13. Let $\eta_t = \eta$ for all $t \geq 1$. Firstly, period $t+1$ cannot be type I, since otherwise, by Lemma 13, we have

$$\eta_{t+2} = 1 - \frac{1-\eta_{t+1}}{\beta - 2\beta(1-\eta_{t+1})} < \eta_{t+1},$$

a contradiction to η_t being constant in time. Hence, in equilibrium it must be the case that type S and type T periods alternate.

We now construct the model's equilibria in two-period cycles where the first period is type S and the second type T. Let $Z^{(j)}$ denote the value of Z in the jth period of a cycle ($j = 1, 2$). By Lemma 13, we could solve for the equilibrium values of

$$\{L^{(j)}, \alpha_y^{(j)}, \alpha_o^{(j)}, w_y^{(j)}, w_o^{(j)}, w_*^{(j)}, V^{(j)}; j = 1, 2\}$$

as a function of η, where

$$\frac{1+\beta}{2(1+x_1\beta)} < \eta < 1.$$

In particular, we have

$$V^{(1)} = \bar{\theta}' - \psi - \underline{w} + \frac{x_2\beta}{1+\beta}(\underline{w} - w_*^{(2)}),$$
$$V^{(2)} = \bar{\theta}' - \psi - \underline{w} + \frac{x_2\beta^2}{1+\beta}(\underline{w} - w_*^{(2)}),$$

where $w_*^{(2)} = \underline{w} - \frac{\hat{p}_1^{*(2)}}{\beta(1-x_1\hat{p}_1^{*(2)})}\underline{w}$ and $\hat{p}_1^{*(2)}$ is the smaller root of

$$\frac{1}{2}x_1x_2p_1^2 - \left[\frac{1}{2}x_1 + (x_2 + x_1\beta)(1-\eta)\right]p_1 + (1-\eta)\beta = 0, \tag{26}$$

which can be rewritten as

$$\frac{1}{2}x_1x_2p_1^2 - \frac{1}{2}x_1p_1 = (1-\eta)[(x_2 + x_1\beta)p_1 - \beta].$$

Observe that as η increases the right-hand side of the above equation rotates clockwise around the point $(\beta/(x_2 + x_1\beta), 0)$. Hence, $\hat{p}_1^{*(2)}$ is decreasing in η. This implies that $w_*^{(2)}$ is increasing in η and therefore both $V^{(1)}$ and $V^{(2)}$ are decreasing in η. Next, the free-entry-and exit condition requires that

$$0 \leq V^{(2)} \leq V^{(1)} \leq C_e.$$

When η goes to 1, both $V^{(1)}$ and $V^{(2)}$ go to $\underline{C} - C_o$. When η goes to $(1+\beta)/(2(1+x_1\beta))$, $w_*^{(2)}$ goes to 0, and therefore $V^{(1)}$ goes to $\bar{C} - C_o \geq 0$ and $V^{(2)}$ goes to $\underline{C} - C_o + x_2\beta^2\underline{w}/(1+\beta)$. Hence if and only if $C_o \leq \underline{C} + x_2\beta^2\underline{w}/(1+\beta)$, there exists $\eta_c^+(C_o) \in [(1+\beta)/(2(1+x_1\beta)), 1]$ such that $0 \leq V^{(2)} \leq V^{(1)}$ when $\eta \leq \eta_c^+(C_o)$, where

$$\eta_c^+(C_o) = 1 - \frac{\frac{1}{2}x_1(1+\beta)(C_o - \underline{C})[(x_2 - x_1)(1+\beta)(C_o - \underline{C}) - \underline{w}x_2\beta]}{[x_1(1+\beta)(C_o - \underline{C}) + x_2\beta\underline{w}][x_2(1+\beta)(C_o - \underline{C}) - x_2\beta^2\underline{w}]}.$$

Since $0 < C_e < \bar{C} - C_o$, there exists $\eta_c^-(C_o, C_e) \in [(1+\beta)/(2(1+x_1\beta)), 1)$ such that

$V^{(2)} \leq V^{(1)} \leq C_e$ when $\eta \geq \eta_c^-(C_o, C_e)$, where

$$\eta_c^-(C_o, C_e) = 1 - \frac{\frac{1}{2}x_1(1+\beta)(C_e + C_o - \underline{C})[(x_2 - x_1)(1+\beta)(C_e + C_o - \underline{C}) - \underline{w}x_2]}{[x_1(1+\beta)(C_e + C_o - \underline{C}) + x_2\underline{w}][x_2(1+\beta)(C_e + C_o - \underline{C}) - x_2\beta\underline{w}]}.$$

Note that $\eta_c^-(C_o, C_e) \leq \eta_c^+(C_o)$ if and only if $C_e \geq -(1-\beta)(\underline{C} - C_o)/\beta$. This completes the proof. ■

B.4 Longer Cycles

Let an individual cycle be 2n periods long, where n is any positive integer. Let $Z^{(j)}$ denote the value of a variable Z in the jth period of the cycle, $j = 1, \ldots, 2n$. Let $j = 0$ denote the period preceding the cycle and $j = 2n + 1$ the period following the cycle. The equilibria we seek to construct have $\eta^{(1)} > \eta^{(2)} = \cdots = \eta^{(2n)}$, as described in Proposition 9, which we now state and prove.

Proposition 9 *Suppose*

$$\underline{C} < C_o < \underline{C} + \frac{x_2\beta^2\underline{w}}{1+\beta}, \tag{52}$$

and

$$-\frac{1-\beta}{\beta}(\underline{C} - C_o) \leq C_e < (1-\beta)(\underline{C} - C_o) + \min\left\{1, \frac{2(1-\underline{\eta}^{(2)})}{x_1}\right\} x_2\beta(1-\beta)\underline{w}, \tag{53}$$

where

$$\underline{\eta}^{(2)}(C_0) = 1 - \frac{\frac{1}{2}x_1x_2p_1^2 - \frac{1}{2}x_1p_1}{(x_2 + x_1\beta)p_1 - \beta}, \ \textit{and}\ p_1 = \frac{-(1+\beta)(\bar{\theta} - \psi - \underline{w} - C_o)}{-x_1(1+\beta)(\bar{\theta} - \psi - \underline{w} - C_o) + \underline{w}x_2\beta}.$$

The model has an equilibrium that consists of an infinite sequence of individual cycles each of which lasting for an even number of periods. In any individual cycle with length $2n$*, n being a positive integer,* $\eta^{(1)} > \eta^{(2)} = \cdots = \eta^{(2n)}$*, type S and type T periods alternate, the optimal long-term contract is described by Proposition 1, and the values of* $\{L^{(j)}, \alpha_y^{(j)}, \alpha_o^{(j)}, w_y^{(j)}, w_o^{(j)}, w_*^{(j)}, V^{(j)}; j = 1, \cdots, 2n\}$ *are given as follow.*

1. *In any odd period (a type S period),*

$$L^{(j)} = \frac{1}{2} + \frac{1}{2}x_1\hat{p}_1^{*(j-1)},$$

$$\alpha_y^{(j)} = 2\eta^{(j)} - 1 + x_1\hat{p}_1^{*(j-1)}, \ \alpha_o^{(j)} = 0,$$

$$w_o^{(j)} = \underline{w},\ w_y^{(j)} = (1-\delta)\underline{w} + \delta w_*^{(j+1)},\ w_*^{(j)} = 0,$$

$$V^{(1)} = C^e,\ V^{(j)} = (1-\beta^2)(\bar{\theta} - \psi - \underline{w} - C_o) + (1-\beta)x_2\beta(\underline{w} - w_*^{(j+1)}) + \beta^2 V^{(j+2)},\ j \geq 3.$$

2. In any even period (a type T period),

$$L^{(j)} = 1 - x_2(\eta^{(j-1)} - \frac{1}{2} + \frac{1}{2}x_1\hat{p}_1^{*(j-2)}),$$

$$\alpha_y^{(j)} = 1,\ \alpha_o^{(j)} = \frac{w_*^{(j)}}{\underline{w}},$$

$$w_o^{(j)} = \underline{w},\ w_y^{(j)} = (1-\delta\hat{p}_1^{*(j)})\underline{w},\ w_*^{(j)} = \underline{w} - \frac{\hat{p}_1^{*(j-2)}}{\beta(1-x_1\hat{p}_1^{*(j-2)})}\underline{w},$$

$$V^{(2)} = 0,\ V^{(j)} = (1-\beta)(\bar{\theta} - \psi - \underline{w} - C_o) + \beta V^{j+1},\ j \geq 4,$$

where $\hat{p}_1^{*(0)}$ is the smaller root of

$$\frac{1}{2}x_1x_2p^2 - \left[\frac{1}{2}x_1 + x_2(1-\eta^{(1)}) + x_1\beta(1-\eta^{(2)})\right]p + (1-\eta^{(2)})\beta = 0, \tag{54}$$

and $\hat{p}_1^{*(j)}$ $(j = 2, 4, \cdots, 2n-2)$ is the smaller root of

$$\frac{1}{2}x_1x_2p^2 - \left[\frac{1}{2}x_1 + x_2(1-\eta^{(2)}) + x_1\beta(1-\eta^{(2)})\right]p + (1-\eta^{(2)})\beta = 0. \tag{55}$$

Proof. After some algebra, one can verify that $\{\eta^{(j)}\}_{j=1}^{2n}$ satisfies the conditions given in part 1 of Lemma 13 if and only if

$$\eta^{(1)} > \eta^{(2)} > \frac{1+\beta}{2(1+x_1\beta)}, \tag{56}$$

and

$$\begin{cases} \eta^{(1)} < \frac{\eta^{(2)}}{x_2} - \frac{x_1+x_1\beta}{2x_2(1+x_1\beta)} & \text{if } \frac{1+\beta}{2(1+x_1\beta)} < \eta^{(2)} \leq \frac{1+x_2}{2}, \\ \eta^{(1)} < 1 - \frac{\beta(1-\eta^{(2)})}{1+2\beta(1-\eta^{(2)})} & \text{if } \eta^{(2)} > \frac{1+x_2}{2}. \end{cases} \tag{57}$$

It follows from part 2 of Lemma 13 that the equilibrium values of $V^{(j)}$ can be written as a function of $\{\eta_t\}_{t\geq 1}$:

$$V^{(1)} = (1-\beta^2)(\bar{\theta}' - \psi - \underline{w}) + \beta^2 V^{(3)} + (1-\beta)x_2\beta(\underline{w} - w_*^{(2)}), \tag{58}$$

$$V^{(2j)} = (1-\beta)(\bar{\theta}' - \psi - \underline{w}) + \beta V^{(2j+1)},\ j = 1, \ldots, n, \tag{59}$$

$$V^{(2j+1)} = (1-\beta^{2n-2j})(\bar{\theta}' - \psi - \underline{w}) + \beta^{2n-2j} V^{(2n+1)} \\ + \frac{x_2 \beta}{1+\beta}(1-\beta^{2n-2j})(\underline{w} - w_*^{(4)}),\ j = 1, \ldots, n-1, \tag{60}$$

where

$$w_*^{(2)} = \underline{w} - \hat{p}_1^{*(0)} \underline{w} / [\beta(1 - x_1 \hat{p}_1^{*(0)})], \tag{61}$$

$$w_*^{(4)} = \underline{w} - \hat{p}_1^{*(2)} \underline{w} / [\beta(1 - x_1 \hat{p}_1^{*(2)})], \tag{62}$$

where $\hat{p}_1^{*(0)}$ is the smaller root of (38) with $\eta_{t+1} = \eta^{(1)}$ and $\eta_{t+2} = \eta^{(2)}$, and $\hat{p}_1^{*(2)}$ is the smaller root of (38) with $\eta_{t+1} = \eta_{t+2} = \eta^{(2)}$. That is, $\hat{p}_1^{*(0)}$ and $\hat{p}_1^{*(2)}$ are the smaller roots of equations (54) and (55), respectively. Now the optimality of the firm's entry and exit decisions implies

$$V^{(1)} = V^{(2n+1)} = C_e, \quad V^{(2)} = 0.$$

Substituting these into (58)-(60) gives us

$$\bar{\theta}' - \psi - \underline{w} + \frac{x_2 \beta (1-\beta)}{1-\beta^{2n}}(\underline{w} - w_*^{(2)}) + \frac{x_2 \beta^3 (1-\beta^{2n-2})}{(1+\beta)(1-\beta^{2n})}(\underline{w} - w_*^{(4)}) = C_e, \tag{63}$$

$$\bar{\theta}' - \psi - \underline{w} + \frac{x_2 \beta^{2n} (1-\beta)}{1-\beta^{2n}}(\underline{w} - w_*^{(2)}) + \frac{x_2 \beta^2 (1-\beta^{2n-2})}{(1+\beta)(1-\beta^{2n})}(\underline{w} - w_*^{(4)}) = 0, \tag{64}$$

both equations are a function of $\{n, \eta^{(1)}, \eta^{(2)}\}$, following from (61), (62), (54) and (55). ■

In the remainder of this appendix, we show in B.4.1 and B.4.2 that for a positive measure set of the pairs (C_o, C_e) (namely the pairs (C_o, C_e) that satisfy (52) and (53)), there do exist $n \geq 1$, $\eta^{(1)}$ and $\eta^{(2)}$ that solve (63) and (64) subject to (56) and (57). Note that with three variables and two equality constraints, there are likely multiple solutions for $(n, \eta^{(1)}, \eta^{(2)})$ to equations (63) and (64). In other words, our model permits equilibria where individual cycles share the same cyclicality (a boom followed by a number of recession periods), but not the same length, that is, the n does not need to be constant across all individual cycles. Moreover, when a solution exists, we have $0 < w_*^{(2)} < w_*^{(4)} < \underline{w}$ since $\eta^{(1)} > \eta^{(2)}$, and therefore

$$0 = V^{(2)} < V^{(3)} < V^{(5)} < \cdots < V^{(2n-1)} < V^{(1)} = C_e,$$

$$0 = V^{(2)} < V^{(4)} < \cdots < V^{(2n)} < V^{(2n+1)} = C_e.$^{[21]}$$

That is, the firm's entry and exit decisions are optimal.

To summarize, any $\{n \geq 1, \eta^{(1)}, \eta^{(2)}\}$ that solve equations (63) and (64) subject to (56) and (57) would describe a complete cycle in an equilibrium that has the characteristics we

[21]When $n = 1$, it is obvious that $0 = V^{(2)} < V^{(1)} = C_e$.

imposed in Proposition 9. If in addition the $\eta^{(1)}$ in one cycle is greater than or equal to the $\eta^{(2)}$ in the preceding cycle, then the sequence $\{\eta_t\}_{t\geq 1}$ would constitute an equilibrium path of firm measures by Lemma 13. In particular, this is the case when n is constant for all individual cycles.

B.4.1 Two-Period Cycles

Lemma 14 *Suppose (C_o, C_e) satisfies (52) and (53) where the first inequality in (53) holds with equality. There exist $\eta^{(1)}$ and $\eta^{(2)}$ that solve (63) and (64) subject to (56) and (57) at $n = 1$.*

Proof. When $n = 1$, (63) and (64) read

$$V^{(1)} = \bar{\theta}' - \psi - \underline{w} + \frac{x_2\beta}{1+\beta}(\underline{w} - w_*^{(2)}) = C_e, \tag{65}$$

$$V^{(2)} = \bar{\theta}' - \psi - \underline{w} + \frac{x_2\beta^2}{1+\beta}(\underline{w} - w_*^{(2)}) = 0. \tag{66}$$

It is easy to see that if there exist $\eta^{(1)}$ and $\eta^{(2)}$ that solve (65) and (66) subject to (56) and (57), then (C_o, C_e) satisfies the following conditions:

$$\underline{C} < C_o < \underline{C} + \frac{x_2\beta^2\underline{w}}{1+\beta}, \tag{67}$$

$$C_e = -\frac{1-\beta}{\beta}(\underline{C} - C_o). \tag{68}$$

Next we show that for any pair of (C_o, C_e) satisfying conditions (67) and (68) there exist $\eta^{(1)}$ and $\eta^{(2)}$ that solve (65) and (66) subject to (56) and (57). Note that if (68) holds, (65) is equivalent to (66). Hence it suffices to show that for any C_o satisfying condition (67) there exists $\eta^{(1)}$ and $\eta^{(2)}$ that solve (66) subject to (56) and (57). This is equivalent to showing that for all $p_1 \in (0, \beta/(1+x_1\beta))$ there exist $\eta^{(1)}$ and $\eta^{(2)}$ such that (56) and (57) hold and p_1 is the smaller root of (54).

Fix $p_1 \in (0, \beta/(1+x_1\beta))$. For any $\eta^{(2)} \in ((1+\beta)/(2(1+x_1\beta)), 1)$, let

$$\eta^{(1)} = 1 - \frac{\frac{1}{2}x_1x_2p_1^2 - [\frac{1}{2}x_1 + x_1\beta(1-\eta^{(2)})]p_1 + (1-\eta^{(2)})\beta}{x_2p_1} \in (0,1).$$

Clearly, if (56) and (57) hold, p_1 is the smaller root of (54). We want to show that there exists $\eta^{(2)}$ such that $(\eta^{(1)}, \eta^{(2)})$ satisfies conditions (56) and (57). Note first that $\eta^{(1)} > \eta^{(2)}$

if and only if

$$\eta^{(2)} > \underline{\eta}^{(2)} \equiv 1 - \frac{\frac{1}{2}x_1x_2p_1^2 - \frac{1}{2}x_1p_1}{(x_2 + x_1\beta)p_1 - \beta},$$

where, as we will show later, $\underline{\eta}^{(2)} \in ((1+\beta)/(2(1+x_1\beta)), 1)$ for all $p_1 \in (0, \beta/(1+x_1\beta))$. Note also that when $\eta^{(1)} = \eta^{(2)}$ (i.e., $\eta^{(2)} = \underline{\eta}^{(2)}$), condition (57) holds. By continuity, there exists $\eta^{(2)} \in (\underline{\eta}^{(2)}, 1)$ such that $(\eta^{(1)}, \eta^{(2)})$ satisfies conditions (56) and (57).

We complete the proof by showing $\underline{\eta}^{(2)} > (1+\beta)/(2(1+x_1\beta))$. This inequality, after some tedious algebra, can be rewritten as

$$x_1x_2p_1^2 - \left[(1+x_1\beta) - \frac{x_2\beta(x_1\beta + x_2)}{1+x_1\beta}\right]p_1 + \beta - \frac{x_2\beta^2}{1+x_1\beta} > 0,$$

where the left-hand side is a a quadratic function of p_1, denoted by $g(p_1)$. We claim that $g(\cdot)$ is strictly decreasing in $(0, \beta/(1+x_1\beta))$. Since $x_1x_2 > 0$, it suffices to show that the symmetric axis of $g(\cdot)$ is greater than $\beta/(1+x_1\beta)$, i.e.,

$$\frac{(1+x_1\beta) - \frac{x_2\beta(x_1\beta+x_2)}{1+x_1\beta}}{2x_1x_2} > \frac{\beta}{1+x_1\beta},$$

or equivalently

$$x_1(2x_1 - 1)\beta^2 + (x_1^2 + 2x_2 - 1)\beta + 1 > 0.$$

Clearly, the above inequality holds when $x_1 \geq \frac{1}{2}$. When $0 < x_1 < \frac{1}{2}$, the left-hand side of the above inequality is a concave function of β, denoted by $h(\beta)$. Since $h(0) > 0$ and $h(1) > 0$, $h(\beta) > 0$ for all $\beta \in (0,1)$. Finally, since $g(\cdot)$ is strictly decreasing in $(0, \beta/(1+x_1\beta))$ and $g(\beta/(1+x_1\beta)) = 0$, we have $g(p_1) > 0$ for all $p_1 \in (0, \beta/(1+x_1\beta))$. ■

B.4.2 Longer Cycles

Lemma 15 *Suppose (C_o, C_e) satisfies (52) and (53) where the first inequality in (53) holds strictly. There exist $n > 1$, $\eta^{(1)}$ and $\eta^{(2)}$ that solve (63) and (64) subject to (56) and (57).*

Proof. To simplify notations, let $p_1^{(2j)}$ denote $\hat{p}_1^{*(2j)}$, $j = 0, 1$. For a given pair of $(w_*^{(2)}, w_*^{(4)})$, we can solve $\eta^{(1)}$ and $\eta^{(2)}$ from (63) and (64):

$$\eta^{(2)} = 1 - \frac{\frac{1}{2}x_1x_2(p_1^{(2)})^2 - \frac{1}{2}x_1p_1^{(2)}}{(x_2 + x_1\beta)p_1^{(2)} - \beta}, \tag{69}$$

and

$$\eta^{(1)} = 1 - \frac{\frac{1}{2}x_1x_2(p_1^{(0)})^2 - [\frac{1}{2}x_1 + x_1\beta(1-\eta^{(2)})]p_1^{(0)} + (1-\eta^{(2)})\beta}{x_2p_1^{(0)}}, \tag{70}$$

where $p_1^{(2j-2)} = \beta(\underline{w} - w_*^{(2j)})/[x_1\beta(\underline{w} - w_*^{(2j)}) + \underline{w}]$, $j = 1, 2$. It is easy to see that $\eta^{(2)}$ is decreasing in $p_1^{(2)}$. Since $p_1^{(2)}$ is decreasing in $w_*^{(4)}$, $\eta^{(2)}$ defined in (69) is increasing in $w_*^{(4)}$.

We now proceed to prove that for any given (C_o, C_e) satisfying conditions (52) and (53) where the first inequality in (53) holds strictly, there exist n, $w_*^{(2)}$ and $w_*^{(4)}$, with $n > 1$ and $w_*^{(2)} < w_*^{(4)}$, that solve equations (63) and (64), and the corresponding $\eta^{(1)}$ and $\eta^{(2)}$ given by (69) and (70) satisfy conditions (56) and (57):

$$\eta^{(1)} > \eta^{(2)} > \frac{1+\beta}{2(1+x_1\beta)}, \tag{56}$$

and

$$\begin{cases} \eta^{(1)} < \frac{\eta^{(2)}}{x_2} - \frac{1+2x_1\beta}{2(1+x_1\beta)} & \text{if } \frac{1+\beta}{2(1+x_1\beta)} < \eta^{(2)} \leq \frac{1+x_2}{2}, \\ \eta^{(1)} < 1 - \frac{\beta(1-\eta^{(2)})}{1+2\beta(1-\eta^{(2)})} & \text{if } \eta^{(2)} > \frac{1+x_2}{2}. \end{cases} \tag{57}$$

Now denote the left-hand side of equations (64) by $V^{(2)}(n, \eta^{(1)}, \eta^{(2)})$. Since $w_*^{(2)} < w_*^{(4)}$, we have

$$\bar{\theta}' - \psi - \underline{w} + \frac{x_2\beta^2}{1+\beta}(\underline{w} - w_*^{(4)}) < V^{(2)}(n, \eta^{(1)}, \eta^{(2)}) < \bar{\theta}' - \psi - \underline{w} + \frac{x_2\beta^2}{1+\beta}(\underline{w} - w_*^{(2)}).$$

Let $\underline{w}_*^{(4)}(C_o) \equiv \underline{w} + (1+\beta)(\bar{\theta}' - \psi - \underline{w})/x_2\beta^2 \in (0, \underline{w})$. Then $\eta^{(2)}(\underline{w}_*^{(4)}) = \underline{\eta}^{(2)}(C_o)$, where the latter is defined in Proposition 9. Since $w_*^{(4)} > \underline{w}_*^{(4)}$ by equation (64) and $\eta^{(2)}$ is increasing in $w_*^{(4)}$, we have $\eta^{(2)} > \underline{\eta}^{(2)}$ (i.e., $\eta^{(1)} > \eta^{(2)}$). Since $\underline{\eta}^{(2)}$ is a decreasing function of C_o over $(\underline{C}, \underline{C}+x_2\beta^2\underline{w}/(1+\beta))$ and approaches $(1+\beta)/[2(1+x_1\beta)]$ as C_o approaches $\underline{C}+x_2\beta^2\underline{w}/(1+\beta)$, inequality (56) holds.

Furthermore, let $w_*^{(4)} = (1+\varepsilon)\underline{w}_*^{(4)}$, where $\varepsilon > 0$ is small enough such that $w_*^{(4)} \in (0, \underline{w})$. Substituting this into (64) gives

$$w_*^{(2)} = \frac{1+\beta}{x_2\beta^2}(\underline{C} - C_o) + \underline{w} - H(\varepsilon, n) < w_*^{(4)}, \tag{71}$$

where

$$H(\varepsilon, n) = \frac{\varepsilon(1-\beta^{2n-2})}{x_2(1-\beta^2)\beta^{2n-2}}\left[\frac{1+\beta}{x_2\beta^2}(\underline{C} - C_o) + \underline{w}\right].$$

Observe that for any fixed $n > 1$, $H(\varepsilon, n) \to 0$ as $\varepsilon \to 0$. Let $\varepsilon \propto \beta^n$, then $H(\varepsilon, n) \to +\infty$ and $\varepsilon \to 0$ as $n \to \infty$. Thus for any $\bar{\varepsilon} > 0$ and $H_0 \in (0, +\infty)$, there is $n > 1$ and $0 < \varepsilon < \bar{\varepsilon} < 0$ such that $H(\varepsilon, n) = H_0$.

Using (71) and (64), equation (63) can be rewritten as

$$-\frac{1-\beta}{\beta}(\underline{C} - C_o) + x_2\beta(1-\beta)H(\varepsilon, n) = C_e.$$

Let $\hat{C}$ be the C_o such that $\underline{\eta}^{(2)}(\hat{C}) = (1+x_2)/2$. Then:

(i) Suppose $\hat{C} < C_o < \underline{C} + x_2\beta^2\underline{w}/(1+\beta)$, i.e., $\underline{\eta}^{(2)} < (1+x_2)/2$. Then inequality (53) can be simplified to read

$$-\frac{1-\beta}{\beta}(\underline{C} - C_o) < C_e < (1-\beta)(\underline{C} - C_o) + x_2\beta(1-\beta)\underline{w},$$

which is equivalent to

$$0 < H(\varepsilon, n) < \frac{1+\beta}{x_2\beta^2}(\underline{C} - C_o) + \underline{w}.$$

Furthermore, since $\underline{\eta}^{(2)} < (1+x_2)/2$, there is $\bar{\varepsilon} > 0$ such that $\eta^{(2)}(w_*^{(4)}(\varepsilon)) \leq (1+x_2)/2$ for all $0 < \varepsilon < \bar{\varepsilon}$. Then inequality (57) can be simplified as follows:

$$\begin{array}{rc} & \eta^{(2)} < \eta^{(1)} < \frac{\eta^{(2)}}{x_2} - \frac{1+2x_1\beta}{2(1+x_1\beta)}, \\ \iff & 0 < w_*^{(2)} < w_*^{(4)}, \\ \iff & 0 < H(\varepsilon, n) < \frac{1+\beta}{x_2\beta^2}(\underline{C} - C_o) + \underline{w}. \end{array}$$

Recall that for any $\bar{\varepsilon} > 0$ and $H_0 \in (0, +\infty)$, there is n and ϵ with $n > 1$ and $0 < \varepsilon < \bar{\varepsilon} < 0$ such that $H(\varepsilon, n) = H_0$. Thus, when $\hat{C} < C_o < \underline{C} + x_2\beta^2\underline{w}/(1+\beta)$ and C_e satisfies inequality (53), there is n and ϵ with $n > 1$ and $0 < \varepsilon < \bar{\varepsilon}$ such that the corresponding $(n, w_*^{(2)}, w_*^{(4)})$ solves equations (64) and (63) and the corresponding $(\eta^{(1)}, \eta^{(2)})$ satisfies conditions (56) and (57).

(ii) Suppose $\underline{C} < C_o \leq \hat{C}$, i.e., $\underline{\eta}^{(2)} \geq (1+x_2)/2$. Then inequality (53) can be simplified as

$$-\frac{1-\beta}{\beta}(\underline{C} - C_o) \leq C_e < (1-\beta)(\underline{C} - C_o) + \frac{2x_2\beta(1-\beta)(1-\underline{\eta}^{(2)})\underline{w}}{x_1},$$

which is equivalent to

$$0 < H(\varepsilon, n) < \frac{1+\beta}{x_2\beta^2}(\underline{C} - C_o) + \frac{2(1-\underline{\eta}^{(2)})\underline{w}}{x_1}.$$

Furthermore, since $\underline{\eta}^{(2)} \geq (1+x_2)/2$, $\eta^{(2)}(w_*^{(4)}(\varepsilon)) > (1+x_2)/2$ for all $\varepsilon > 0$. Then inequality (57) can be simplified as

$$\begin{array}{rc} & \eta^{(2)} < \eta^{(1)} \leq 1 - \frac{\beta(1-\eta^{(2)})}{1+2\beta(1-\eta^{(2)})}, \\ \iff & \underline{w} - \frac{2(1-\eta^{(2)})\underline{w}}{x_1} \leq w_*^{(2)} < w_*^{(4)}, \\ \iff & 0 < H(\varepsilon, n) \leq \frac{1+\beta}{x_2\beta^2}(\underline{C} - C_o) + \frac{2(1-\eta^{(2)})\underline{w}}{x_1}. \end{array}$$

Note that $\eta^{(2)}(w_*^{(4)}(\varepsilon)) \to \underline{\eta}^{(2)}$ as $\varepsilon \to 0$. Thus, when $0 < C_o \leq \hat{C}$ and C_e satisfies inequality

(53), there is $n > 1$ and $\varepsilon > 0$ such that the so defined $(n, w_*^{(2)}, w_*^{(4)})$ solves equation (64) and (63) and the corresponding $(\eta^{(1)}, \eta^{(2)})$ satisfies conditions (56) and (57).

This completes the proof of the lemma. ■

B.4.3 Numerical Example

Table 1: A numerical example

Preferences			
$\beta = 0.9$			
Moral hazard			
$\theta_1 = 0$	$\theta_2 = 100$	$\psi = 20$	
$x_1 = 0.4$	$x_2 = 0.6$	$x_1' = 0.8$	$x_2' = 0.2$
Costs			
$C_e = 0.2328$	$C_o = 31.8182$		

C Welfare comparison

This appendix presents the calculations behind the output and welfare comparisons in Section 7. Remember we are comparing the stationary equilibria in Proposition 2 and the non-stationary equilibria in Proposition 5. Our goal is to compare the aggregate output and worker welfare achieved in the two types of equilibria. Notice that given that (C_o, C_e) satisfies equation (25), the economy has a constant stock of firms in any given stationary or non-stationary equilibrium. Let $\eta_s^+ (\eta_c^+)$ denote the maximum stock of firms that a stationary (non-stationary) equilibrium can support.

We first show that the maximum measure of firms (and hence the maximum aggregate output) that a stationary equilibrium can support is larger than that a non-stationary equilibrium can (i.e., $\eta_s^+ > \eta_c^+$). After some algebra, we can show that $\eta_s^+ > \eta_c^+$ if and only if

$$x_1(1+\beta)(x_2 + x_1\beta)(C_o - \underline{C})^2 + x_1 x_2 \beta \underline{w}(1 + \beta - \beta^2)(C_o - \underline{C}) + (1-\beta)(x_2\beta\underline{w})^2 > 0,$$

which holds for all $C_o > \underline{C}$.

Next, we compare workers' welfare across the stationary and non-stationary equilibria. Specifically, compare workers' life-time expected utility, w_s, in a stationary equilibrium when $\eta = \eta_s^+$ with those, $w_c^{(j)}$ $(j = 1, 2)$, in a two-period cycles where $\eta = \eta_c^+$. We prove below that $w_c^{(2)} > w_s > w_c^{(1)}$.

Note first that the lifetime expected utility of a worker born in period t is measured by

$$\alpha_{y,t}w_{y,t} + (1-\alpha_{y,t})\delta w_{*,t+1}.$$

In a stationary equilibrium where $\eta = \eta_s^+$, a worker's expected utility is given by

$$w_s = \left[1 - \frac{2\delta(1-\eta_s^+)}{x_1}\right]\underline{w}.$$

In the two-period cycles where $\eta = \eta_c^+$ the expected utility of a worker born in a type T period is given by

$$w_c^{(2)} = (1-\delta\hat{p}_1^{*(2)})\underline{w}, \text{ with } \hat{p}_1^{*(2)} = \frac{(1+\beta)(C_o - \underline{C})}{x_2\beta\underline{w} + x_1(1+\beta)(C_o-\underline{C})}. \tag{72}$$

It is easy to verify that $2(1-\eta_s^+)/x_1 > \hat{p}_1^{*(2)}$ and therefore $w_s < w_c^{(2)}$. The expected utility of a worker born in a type S period is given by

$$w_c^{(1)} = (2\eta_c^+ - 1 + x_1\hat{p}_1^{*(2)})(1-\delta)\underline{w} + \delta w_*^{(2)},$$

where $\hat{p}_1^{*(2)}$ is given by (72) and $w_*^{(2)} = \underline{w} - (1+\beta)(C_o - \underline{C})/x_2\beta^2$. We then have

$$\begin{aligned} w_c^{(1)} - w_s = &-\frac{x_1(C_o-\underline{C})}{x_2\beta\underline{w} + x_1(1+\beta)(C_o-\underline{C})}\frac{x_1(1+\beta)(C_o-\underline{C}) + x_2\beta(1-\beta)\underline{w}}{-x_2(1+\beta)(C_o-\underline{C}) + x_2\beta^2\underline{w}}\underline{w} \\ &- \frac{C_o-\underline{C}}{x_2\beta} + \frac{(C_o-\underline{C})\underline{w}}{x_2\underline{w} + x_1(C_o-\underline{C})} \\ < 0, \end{aligned}$$

where the last inequality holds for all $C_o > \underline{C}$.

Finally, we compare the average expected utility of young workers in the non-stationary equilibrium, $(w_c^{(1)} + w_c^{(2)})/2$, with that in the stationary equilibrium, w_s:

$$\begin{aligned} &w_c^{(1)} + w_c^{(2)} - 2w_s, \\ &= -\frac{x_1(C_o-\underline{C})}{x_2\beta\underline{w} + x_1(1+\beta)(C_o-\underline{C})}\frac{x_1(1+\beta)(C_o-\underline{C}) + x_2\beta(1-\beta)\underline{w}}{-x_2(1+\beta)(C_o-\underline{C}) + x_2\beta^2\underline{w}}\underline{w} \\ &\quad + \frac{(C_o-\underline{C})\underline{w}}{x_2\underline{w} + x_1(C_o-\underline{C})}\left[2 - \frac{x_2\beta\underline{w} + x_1\beta(C_o-\underline{C})}{x_2\beta\underline{w} + x_1(1+\beta)(C_o-\underline{C})} - \frac{x_2\underline{w} + x_1(C_o-\underline{C})}{x_2\beta\underline{w}}\right] \\ &< 0, \end{aligned}$$

where the last inequality holds whenever $C_o > \underline{C}$.