Regularised Nonparametric Estimation of First-Price Private Value Auction

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Motivation

1. Alternative general direct estimation framework for games of incomplete information
2. Here: focus on fist Price IPV auction
3. Methodological contribution
4. Potentially attractive features (sample size, minimal constraints, behaviour at boundaries,...)
5. CDF and Quantile estimation

Summary

1. Empirical Game Model Model
2. Econometric Model
3. Estimation of F^{-1}: a linear ill posed inverse problem
4. Estimation of F: a non linear ill posed inverse problem
5. Simulations

Empirical game model

- $\xi \in [0,1] \subset \mathbb{R}$, unobservable (private information)
  $\xi \sim F$ (c.d.f.)
  ξ personal information available by a player of a game (type, signal...)
  F common knowledge to all players.
- X action plays by the player $X \in \mathbb{R}$

$$X = \varphi(\xi, F)$$

  φ strategy function (monotonic w.r.t. ξ)
  φ follows from an equilibrium rule (Nash,...)
- For simplicity φ (and F) are the same for each player (symmetric game).

Empirical game model

- From the econometrician viewpoint:
  - knows φ (as a function of ξ and F)
  - observes an iid sample of X
  - does not observe ξ_i
  - does not know F
- Objective : estimation of F
  and then possibly of $\varphi(., F)$ and possibly counterfactual analysis.

Empirical game model

Possible use of covariates:

- Z conditioning variables of the ξ
- W conditioning variables of the game and of the strategy

$$\xi | Z \sim F^Z \quad X = \varphi(\xi, F^Z, W)$$

- not considered here.
  $\longrightarrow$ See Gimenez and Guerre (2017)

Empirical game model

Possible economic applications:

- auctions and procurement
- non linear pricing and option pricing
- contract models
- cost of regulated firms
- duopoly or oligopoly...

A game of incomplete information: First-Price Sealed Bid auction

- First price independent private value (IPV) symmetric auction
- Nonparemetric identification and estimation: Guerre, Perrigne and Vuong (2000)
- $N+1$ bidders

  ξ : private (unobserved) value of the object

  X : bid

  F : c.d.f. of ξ

  φ : strategic function (known and increasing)

- Nash equilibrium (symmetric)

$$X = \xi - \frac{\int_0^{\xi} F^N(u)du}{F^N(\xi)} = \varphi(\xi, F).$$

- Data:

N given

iid sample of bids $X(x_1, ..., x_n)$

- **Objective**: estimate F

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- **Main feature of the model:**

$$X \sim \quad G(x) = P(X \leq x)$$

$$G(x) = P(\varphi(\xi, F) \leq x) = P(\xi \leq \varphi_F^{-1}(x))$$

where

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or

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- G, cdf of the observable, and F, the parameter of interest, are associated by this functional equation.

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- $\hat{G}_n$ estimation of the distribution of $X \Rightarrow$ Solve equation:

$$F - \hat{G}_n \ \mathbf{o} \ \varphi_F = 0$$

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- **Issue: Not well posed inverse problem**, in general

Econometric model

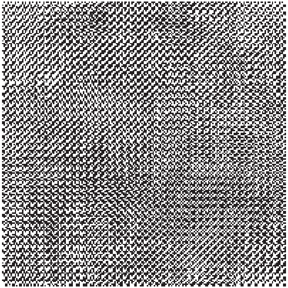
- **Well-posed or ill-posed?**
- if ill-posed, solution might not depend continuously on the data.

Econometric model

- **Well-posed or ill-posed?**
- if ill-posed, solution might not depend continuously on the data.
- Why do we care?
- Roughly speaking:
  ill-posedness amplifies noise when we invert.

Econometric model

blurred



direct inverse

regularized inverse

- Nonparametric identification and estimation: Guerre, Perrigne and Vuong (2000)
- No direct use of regularisation techniques
- Solution based on the property:

Find "simple" known form B (e.g. F.O.C.), s.t.:

$$\xi = B(X, G, g)$$

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  $$\xi = B(X, G, g)$$

  X observed $\rightarrow$ its distribution and density can be estimated: $(\hat{G}, \hat{g}) \rightarrow \hat{\xi} \rightarrow \hat{f}$
- **Regularisation:**
  - Impose constraints ($\hat{g}$ differentiable,...)
  - Regularisation parameter: kernel bandwidth

Advantages

- No need to calculate strategic function φ
- Use well-known nonparametric kernel techniques

Possible issues

- Specific to IPV model where $\xi = B(X, G, g)$
- Well-known kernel estimates practical issues, but twice $(\hat{g}, \hat{f})$
  - need for large sample size
  - data driven kernel bandwidth selection can be a challenge
  - bias at boundaries
    $\rightarrow$ truncated estimation
- No direct treatment of ill-posedness
  - still a regularisation parameter to choose: kernel bandwidth
- Impose differentiability and shape conditions on $\hat{g}$ and $\hat{f}$
  : add more constraints to regularise the problem

- Add monotonicity constraints on bidding strategies
  Henderson, List, Millimet, Parmeter and Price (2012, JoE)
  Baysian approach:
  Kim (2015, QE)
- Boundary corrections:
  Hickman and Hubbard (2012, JAE)
- Quantile approach:
  Multiple smoothness constraints:
  Marmer and Shneyerov (2012, JoE)
  Use covariates:
  Guerre and Sabbah (2012, ET)
  Gimenes and Guerre (2016)

Econometric model

- Why do we care about being general and having less constraints?

Econometric model

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- Conceptually attractive

Econometric model

- Why do we care about being general and having less constraints?
- Conceptually attractive
- Also, what about testing the economic model?
- GPV (2000):
  "restrictions imposed by the game theoretic model (...) can constitute the basis of a test of the theory"

Our Approaches

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  Linear problem
  Asymptotic easier (asymptotic distribution)
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- One step approaches
- fewer constraints
- Quantile F^{-1} estimation:
  Linear problem
  Asymptotic easier (asymptotic distribution)
  Pb: Specific to IPV model. (same problem in GPV)
- CDF F estimation
  More general
  Nonlinear problem
  Pb: Asymptotic distribution? (same problem in GPV)

Estimation of F^{-1} in First-Price auction

Back to the main feature of the model:

$$\boxed{G = T(F)}$$

$$G(x) = F \ \mathbf{o} \ \varphi_F^{-1}(x)$$

$\Rightarrow$ **nonlinear problem in general**

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$\Rightarrow$ **nonlinear problem in general**

- Quantile form:

$$G^{-1}(\alpha) = \varphi \circ F^{-1}(\alpha)$$

$$G^{-1}(\alpha) = \frac{N}{\alpha^N} \int_0^1 \mathbf{1}(w \leq \alpha) w^{N-1} F^{-1}(w) dw$$

$\Rightarrow$ **linear in** F^{-1}:

Estimation of F^{-1} in First-Price auction

- Rewrite $G^{-1}(\alpha) = \frac{N}{\alpha^N} \int_0^1 \mathbf{1}(w \leq \alpha) w^{N-1} F^{-1}(w) dw$ as
  $$\boxed{G^{-1}(\alpha) = KF^{-1}(\alpha)}$$
- G^{-1} is identified from observed bids
- We invert the relationship to estimate F^{-1}
  $\Rightarrow$ **linear inverse problem**

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Limitation:
Not all games can be linearised through quantile form.

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- surjectivity
- and continuous K^{-1}

- continuous K^{-1}?
- Problem: not the case even in the simple First Price auction model local linearised problem
  (K is compact)
  (possible to apply Singular value Decomposition and obtain "0" eigenvalues)

- Initial linear ill-posed problem:

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- G^{-1} is identified from observed bids
  $\Rightarrow \widehat{G^{-1}}$ and $\widehat{G^{-1\prime}}$ nonparametric
  $\Rightarrow \widehat{F^{-1}}$ nonparametric (kernel)
- **Regularisation 2**: choose bandwidths
- Note: in practice monotonicity of $\widehat{F^{-1}}$ may be additionally imposed

- Practical issue: how to estimate $G^{-1\prime}$?

- Practical issue: how to estimate $G^{-1'}$?
- First option: follow Cheng and Sun (2006)
  $\rightarrow$ Quantile derivative kernel estimation
  $\rightarrow$ Different optimal bandwidth at every data point

Estimation of F^{-1} in First-Price auction
Method 1: "standard" nonparametric approach

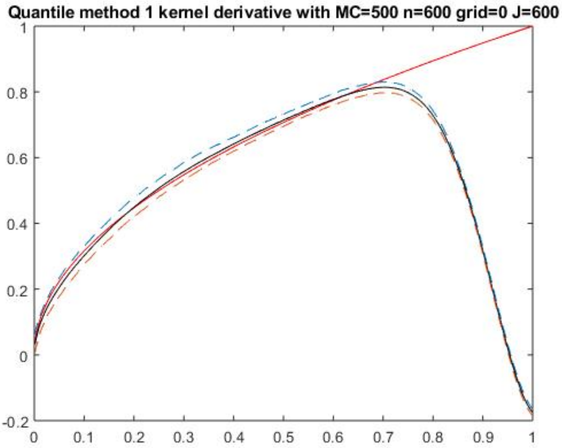

Figure: Kernel derivative

- Practical issue: how to estimate ${G^{-1}}'$?
- Second option: empirical gradient ('gradient' in Matlab)

Estimation of F^{-1} in First-Price auction
Method 1: "standard" nonparametric approach

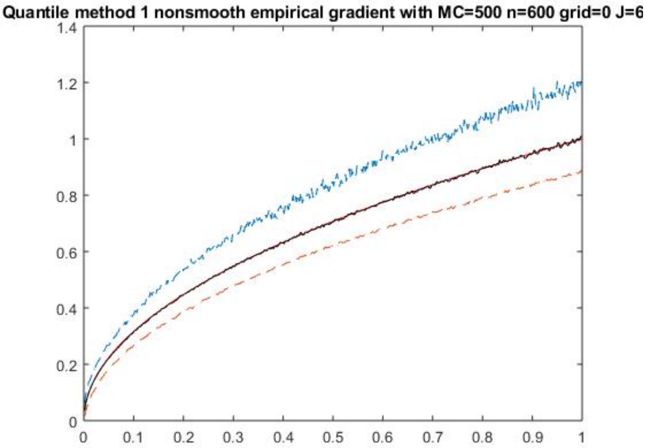

Figure: Empirical gradient

- Practical issue: how to estimate $G^{-1'}$?
- Third option: Estimate smooth spline derivative of empirical quantiles. (using 'SLM' in Matlab)

Estimation of F^{-1} in First-Price auction
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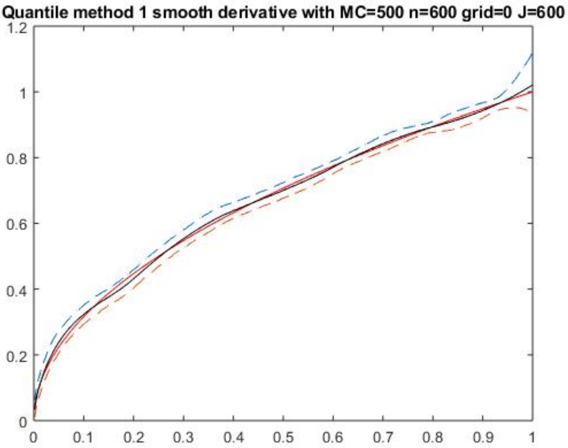

Figure: Smooth spline derivative

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- Solution:

$$\widehat{F^{-1}} = (\mu I + \widehat{K^*}\hat{K})^{-1} \widehat{K^*} \hat{r}$$

(where K^* is K adjoint operator)

Estimation of F^{-1} in First-Price auction
Method 2: TIKHONOV regularisation approach

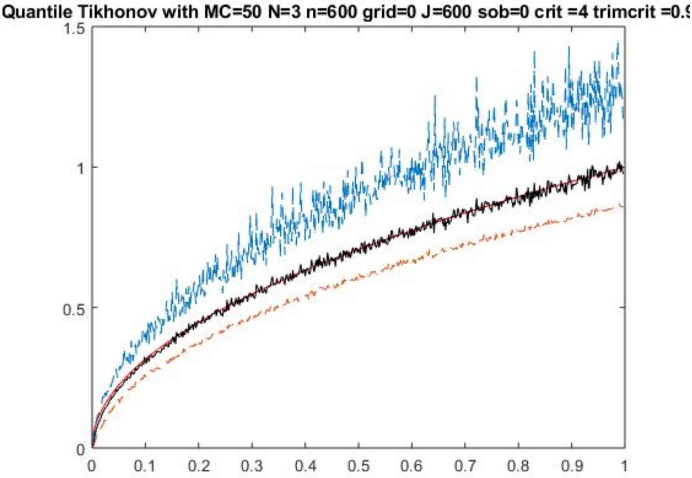

Figure: Tikhonov

- We may want a smoother estimator, hence allow it to be differentiable.
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- Solution:

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Remark We could also use a second order Sobolev penalty

- We minimise:
$$\|r - KF^{-1}\|^2 + \mu \|F^{-1\prime}\|^2$$
where μ is the regularisation parameter
- Data driven choice of μ:
Roughly, minimise:
$$\frac{1}{\mu}\|r - KF^{-1}\|^2$$

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- Instead of direct Tikhonov, we use an iterative approach ($\sim$ gradient descent)

$$\widehat{F_l^{-1}} = \widehat{F_{l-1}^{-1}} + c\widehat{K^*}(\hat{r} - \hat{K}\widehat{F_{l-1}^{-1}})$$

where $l = 1, ..., L$

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  where $l = 1, ..., L$
- Data driven choice of regularisation parameter: choose L that minimises

  $$L||\hat{r} - \hat{K}\widehat{F_{l-1}^{-1}}||^2$$

  as in Fève and Florens (2014)

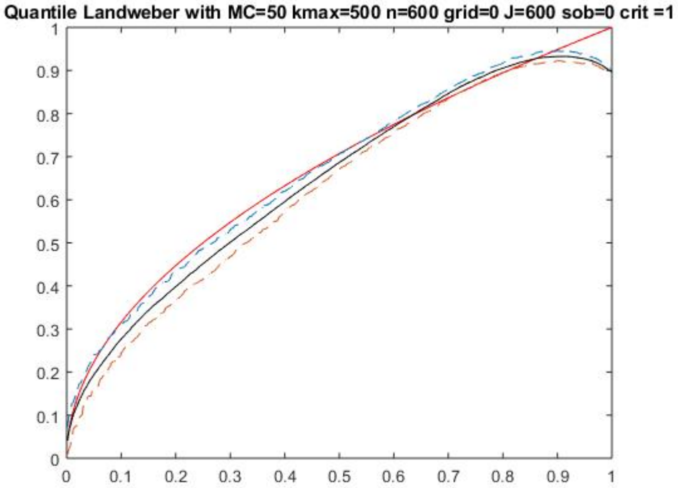

Figure: Landweber

Estimation of F^{-1} in First-Price auction

Method 3: LANDWEBER regularisation approach with Sobolev penalty

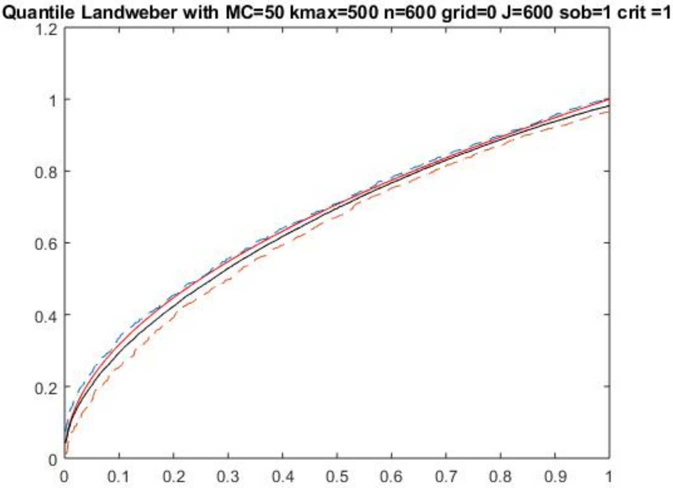

Figure: Landweber with Sobolev penalty

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Estimation of F^{-1} in First-Price auction
What do we have?

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- Data driven choice of regularisation parameters ✓

Estimation of F^{-1} in First-Price auction
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- speed of convergence ✓

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- **asymptotic distribution** ✓
- **Possibly no smoothness constraints on** $\hat{g}, \widehat{f^{-1}}, \widehat{F^{-1}}$
- **Possibly no monotonicity constraint on** $\widehat{F^{-1}}$
- **No boundaries constraints on** $\widehat{F^{-1}}$

- Why not well posed?

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  we would need:
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- and continuous T^{-1}

- Surjectivity?
  Problem: G is estimable by

$$\hat{G}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(x_i \leq x)$$

  $\hat{G}_n$ does not necessarily belong to the range of $T(F) = F \;\mathbf{o}\; \varphi_F^{-1}(x)$

- continuous T^{-1}?
  Problem: even the local linearised problem does not work
  Indeed,

$$dT_F(\tilde{F}_1)(\xi) = \frac{F(\xi)}{\int_0^{\xi} F^N} \int_0^{\xi} F^{N-1}\tilde{F}_1.$$

  is compact and then it is not invertible.

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- However, direct minimisation does not work so well
- Iterative method:
Landweber iteration

$$\hat{F}_k = \hat{F}_{k-1} + (dT_{\hat{F}_{k-1}})^*(\hat{G}_n - T(\hat{F}_{k-1}))$$

Regularisation parameter: k, number of iterations

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  if F is β times differentiable:
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- **No constraints on $\hat{G}$**
- **No monotonicity constraint on $\hat{F}$**

Monte Carlo study: Landweber Iteration

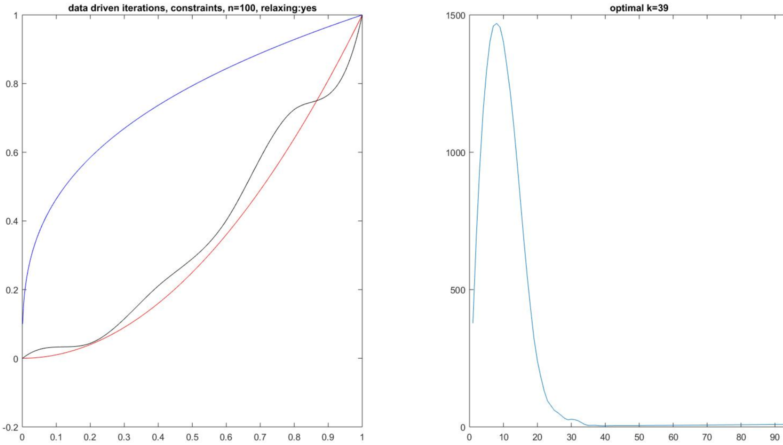

Figure: Monte Carlo study

- Possibly no constraints on $\hat{G}$
- Minimal constraints on $\hat{F}^{-1}$ or $\hat{F}$
- Possibly no other constraints
- Still solve the "cdf F estimation" non-linear inverse problem
- Solve also the quantile F^{-1} linear inverse problem
- Only one regularisation parameter (data driven)
- Offer one step estimation
- Less issues with boundaries vs GPV

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  under constraints. ('lsqlin' in Matlab)
- Why $K^* r$?
  Because $K^* r$ and $K^* K F^{-1}$ have a tractable linear form.

Estimation of F^{-1} in First-Price auction

Bonus: Unconstrained Least Squares

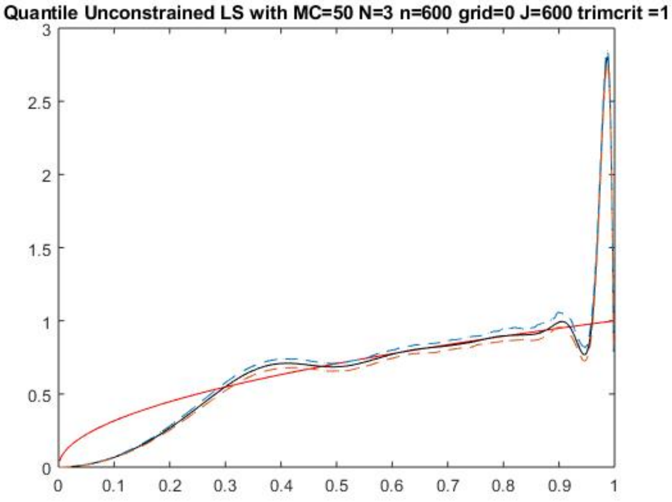

Figure: Unconstrained least squares

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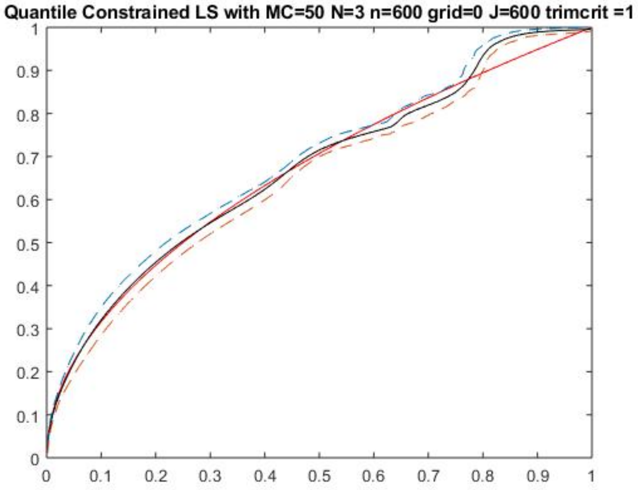


Figure: Constrained least squares

Numerical Investigation

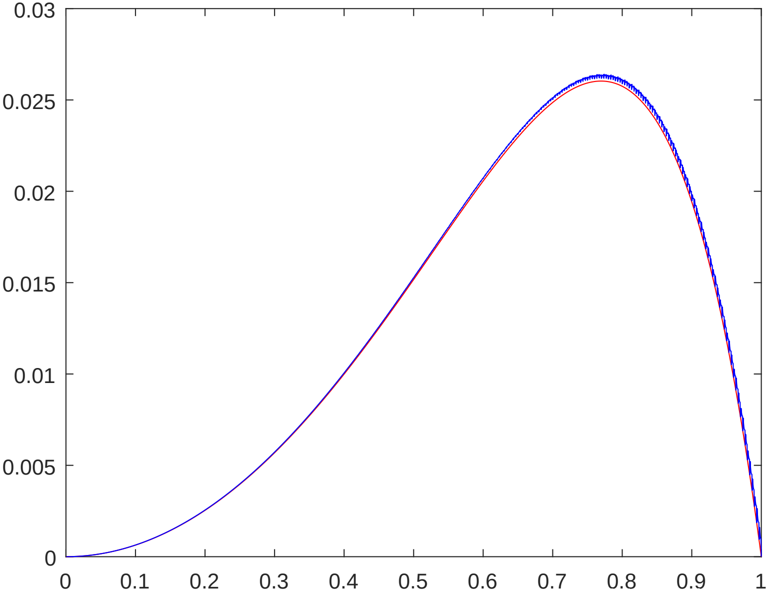
- Let us consider the quantile estimation
- Tikhonov solution:

$$\hat{F^{-1}} = (\mu I + \hat{K}^*\hat{K})^{-1}\hat{K}^*\hat{r}$$

- Lanweber solution:

$$\hat{F_l^{-1}} = \hat{F_{l-1}^{-1}} + c\hat{K}^*(\hat{r} - \hat{K}\hat{F_{l-1}^{-1}})$$

Numerical Investigation



- More developed K^*r and K^*K:

$$K^*r(w) = w^{N-1}\int_0^1 \mathbf{1}_{w\leq\alpha} r(\alpha)d\alpha$$

$$K^*K\lambda(y) = \int_0^1 \left[y^{N-1}w^{N-1}(1-max(w,y))\right]\lambda(w)dw$$

- As N increases, it is even more difficult numerically close to deal with values close to 0.

- More developed K^*r and K^*K:

$$K^*r(w) = w^{N-1} \int_0^1 \mathbf{1}_{w \leq \alpha} r(\alpha) d\alpha$$

$$K^*K\lambda(y) = \int_0^1 \left[y^{N-1} w^{N-1} (1 - max(w, y)) \right] \lambda(w) dw$$

- Numerical precision for integrals and ability to deal with values close to 0 seem to matter

Introduction of Bids with Noise

- Assume now that X is not observable, but that we observe $Y = X + \epsilon$,
  where ϵ is a random noise with density $e(\epsilon)$.
- noise: measurement error or deviation from Nash equilibrium.
- $H(y)$: c.d.f of Y.
  We have:

$$H(y) = \int G(y - \epsilon)e(\epsilon)d\epsilon = KG(y) \tag{1}$$

  where K is the convolution by $e(\epsilon)$ operator.
- Problem becomes $H = KT(F) = M(F)$ with $M = KT$.
- As a first step, let us assume that $e(\epsilon)$ is known and then M is also known.

Introduction of Bids with Noise

We can solve the problem in two different ways.

1. We can implement a deconvolution of H in order to estimate G.
   For example we can use the method developed by Delaigue and Hall (2015)
   Then we can use the "corrected" $\hat{G}$ in our procedures introduced previously.

2 We can compute directly

$$\hat{F}_{k+1} = \hat{F}_k + dM^*_{\hat{F}_k}(\hat{H} - M(\hat{F}_k)), \qquad k = 1, ..., K$$

If ϵ is symmetric, we can simplify and the Landweber iteration is:

$$\hat{F}_{k+1} = \hat{F}_k + dT^*_{\hat{F}_k}K(\hat{H} - KT_{\hat{F}_k}(\hat{F}_k))$$

and we only have to choose the regularisation parameter. (here, the number of iterations)

- Note that methods à la GPV are not straightforward to apply here:
    - based on the reconstruction of ξ from $g(x)$ and x.
    - but here we cannot observe x

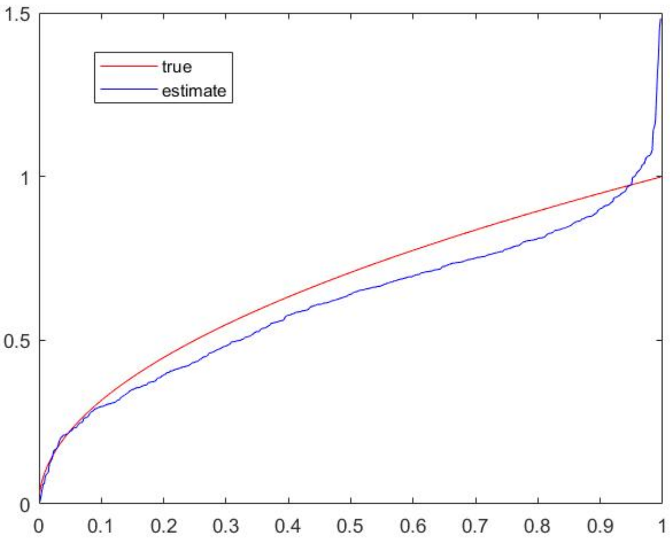


Figure: Tikhonov no noise

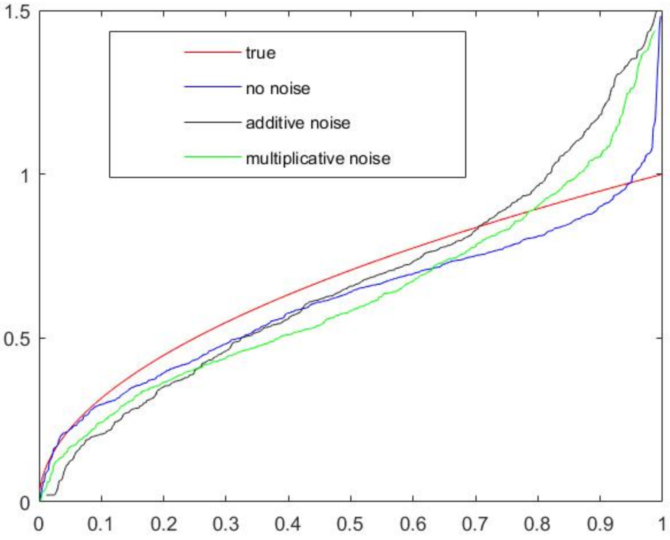


Figure: Tikhonov with noise

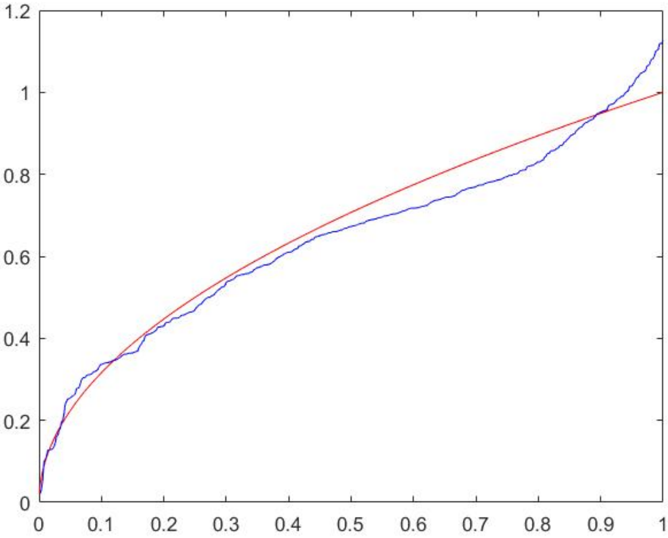


Figure: Landweber no noise

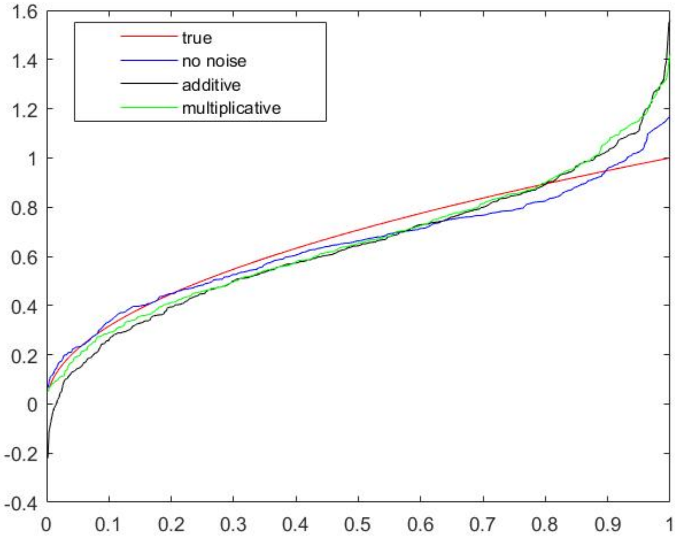


Figure: Landweber no noise