Who Acquires Information in Dealer Markets?*

Jesper Rüdiger† and Adrien Vigier‡

Abstract

We study information acquisition in dealer markets. We first identify a one-sided strategic complementarity in information acquisition: the more informed traders are, the larger market makers' gain from becoming informed. We then derive the equilibrium pattern of information acquisition and examine the implications of our analysis for market liquidity and price discovery.

JEL classification: D80; D82; G10; G14; G20

Keywords: Market Microstructure; Information Acquisition; Price Discovery; Market Liquidity

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†Department of Business Administration, Universidad Carlos III de Madrid. *E-mail*: jrudiger@emp.uc3m.es.

‡Department of Economics, BI Norwegian Business School. *Email*: a.h.vigier@gmail.com.

1 Introduction

The canonical model of dealer markets (Glosten and Milgrom, 1985) assumes that traders have superior information. Yet empirical evidence shows that market makers (MMs) can be better informed than traders.[1] What does the structure of dealer markets imply about when we should expect to see one situation or the other?

Dealer markets are two-sided financial markets with MMs quoting bid and ask prices on one side of the market and traders submitting market orders on the other side (see Figure 1). The MMs buy low, and sell high, adjusting their bid-ask spreads in accordance with the adverse selection they face. Traders, on the other hand, benefit from MMs competing to offer the best quotes. The set of traders comprises *speculators* as well as *liquidity traders*; whereas the former trade for profits, the latter trade due to liquidity shocks unrelated to the asset value. MMs make profits against liquidity traders and incur losses against speculators.

$$\begin{tikzcd}
\textsc{Market Makers} \arrow[r, leftrightarrow, "\text{Trading}"] \arrow[d] & \textsc{Traders}\begin{cases}\textsc{Liquidity traders}\\ \textsc{Speculators}\end{cases} \arrow[d] \\
\text{Quote Bid and Ask Prices} & \text{Submit Market Orders}
\end{tikzcd}$$

Figure 1: Dealer Markets

Suppose now that the cost of acquiring information about an asset's value is the same for all market participants, that is, for traders and MMs: who then becomes informed? To address this question, the present paper analyses a simple two-stage model: information acquisition takes place in the first period, and trade in the second.

We first identify a one-sided strategic complementarity in information acquisition. *MMs' gain from becoming informed is increasing in the probability that traders are informed.* The logic is simple. The more informed traders are, the worse the adverse selection facing uninformed MMs. The latter thus respond with larger bid-ask spreads. This, in turn, softens price competition for informed MMs, who can now increase their own bid-ask spreads. So

[1] The empirical literature is discussed in Section 7.

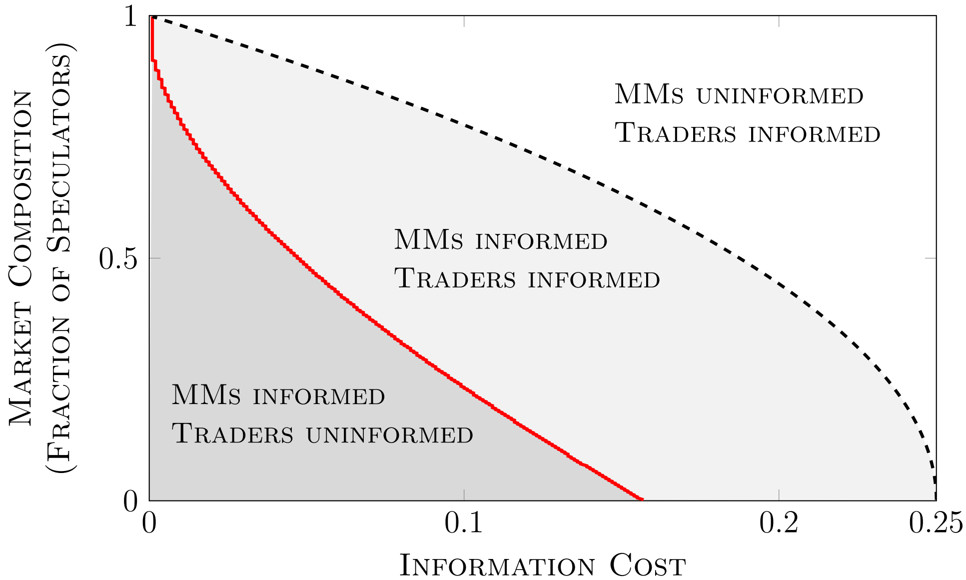


Figure 2: Who Acquires Information

MMs' incremental profit from being informed increases with the probability that traders are informed. By contrast, as traders make less profits from trading with informed MMs than with uninformed MMs, traders' gain from becoming informed is always decreasing in the probability that MMs are informed. The outcome of these observations, is a one-sided strategic complementarity in information acquisition.

We then address the question of who, in equilibrium, acquires information when the cost of information is the same for traders and MMs. We show first that the microstructure of dealer markets pins down the pattern of information acquisition, as illustrated in Figure 2. *At small information costs, MMs acquire information whereas traders choose to remain uninformed*: the market then consists only of liquidity traders, informed MMs and uninformed MMs. In sharp contrast, *at larger information costs, traders acquire information but MMs remain uninformed*. The latter cost range therefore microfounds Glosten-Milgrom types of markets. In between these two cost ranges, MMs and traders all become informed with positive probability.

The logic for who acquires information is as follows. Due to the positive fraction of liquidity traders, a MM's gain from becoming informed remains bounded away from zero as long as not all his competitors acquire information with probability one. We show that as a result, MMs' information acquisition probability is pushed towards one as the information cost tends to zero. This, in turn, implies that the probability that an informed trader finds a profitable trade *and* the profit that can be made on this trade *both* go to zero as the cost tends to zero. Since traders' gain from acquiring information is then second order in the cost of information, traders best respond by remaining uninformed at small information costs.

Consider next larger information costs. Suppose, to fix ideas, that MMs and speculators all choose to remain uninformed: both the bid price and the ask price then equal the expected asset value (since MMs face no adverse selection). Suppose now that the realized asset value is high. Both an informed MM and an informed speculator can then profit from buying the asset in order to earn the difference between the asset's high value and its expected value. Yet, whereas a speculator can *always* buy the asset at the quoted price, a MM can *only* buy when a sell order is to be executed. We show that as a result, a cost range exists in which MMs are uninformed and traders acquire information.

The effect of market composition on information acquisition is as follows. When most traders are speculators, traders acquire information whereas MMs remain uninformed; by contrast, when most traders are liquidity traders, MMs acquire information and traders remain uninformed. Intuitively, a speculator abstains from trading when she is uninformed, so MMs only recoup the cost of information from executing the orders of liquidity traders. MMs therefore stop acquiring information when the fraction of speculators comprising the market becomes sufficiently large. Information acquired by MMs in turn pins down speculators' incentives to become informed: speculators remain uninformed when the fraction of liquidity traders is large (in which case most MMs are informed), whereas speculators acquire information when liquidity traders are rare (in which case most MMs are uninformed).

We also investigate "market liquidity" (measured by the spread between ask and bid prices) and "price discovery" (how well prices reflect asset values). Market liquidity is non-monotonic in the cost of information. MMs become informed with probability tending to one as the information cost approaches zero, and no market participant acquires information when the cost is sufficiently large. The bid-ask spreads thus vanish at both ends of the cost range. In between, however, a combination of informed trading *and* informed market making induces positive spreads.

We then uncover two striking implications of the model. We show namely that *price discovery may increase (a) with the cost of information and (b) with the fraction of liquidity traders comprising the market.* Part (a) is explained by the fact that, in an intermediate cost range, raising the cost of information pushes MMs to acquire less information but induces the traders to acquire more information. The strategic complementarity in information acquisition previously highlighted, in turn, implies that the positive effect on price discovery resulting from traders' information acquisition dominates the negative effect resulting from MMs' information acquisition. Part (b), on the other hand, is a consequence of the "market collapse" occurring when liquidity trading becomes scarce.

We divide for expository purposes the analysis in two parts. We first study a model in which traders submit market orders without having observed MMs' quotes.[2] This assumption enables us to illustrate the underlying economic principles at work in the simplest possible setting. We then make quotes observable, and show that all the main results continue to hold. Yet making the quotes observable also yields novel insights. For instance, a strategic complementarity in information acquisition *amongst* MMs then arises, since information acquired by one MM then leaks to the traders through the quotes of that MM.

The related literature is discussed in the next paragraphs. Section 2 presents the baseline model of the paper, which we analyze in Sections 3 to 5. Section 3 takes the probabilities with which different market participants acquire information as given, and examines the resulting trading game. Section 4 endogenizes information acquisition. Section 5 investigates market liquidity and price discovery. Section 6 extends the baseline model by allowing traders to observe MMs' quotes before market orders are submitted. Section 7 discusses the model and results, and relates our findings to the empirical literature and to recent developments in financial markets.

Related literature. The literature on information acquisition in financial markets stretches back at least to Grossman and Stiglitz (1980) and Verrecchia (1982). The canonical model of dealer markets is due to Glosten and Milgrom (1985), who *assume* that traders are better informed than MMs. Endogenizing traders' information acquisition is relatively straightforward, if one maintains the assumption that MMs are uninformed (see, e.g., Foucault, Pagano and Röell (2013)). By contrast, the problem of information acquisition by MMs is non-trivial. If one fixes traders' information, the problem is formally equivalent to information acquisition in a standard (first-price sealed-bid common-value) auction setting, analyzed in e.g. Milgrom (1981), Lee (1984), Persico (2000), and more recently by Atakan and Ekmekci (2017). To the best of our knowledge, the present paper is the first to analyze information acquisition occurring on both sides of a dealer market, and to investigate how information acquired by one side of the market affects incentives to acquire information on the other.

Within the literature on dealer markets, Chamley (2007) allows traders to acquire costly

[2] One interpretation is that quotes are hidden limit orders, as in for instance Boulatov and George (2013).

information.[3] Leach and Madhavan (1993), Bloomfield and O'Hara (2000), and de Frutos and Manzano (2005) on the other hand take traders' information as given, and explore MMs' incentives to manipulate prices in order to learn from the order flow. Our work also relates to Biais (1993), who considers MMs with private information about inventories, to Calcagno and Lovo (2006), who explore price competition when one MM possesses superior information about fundamentals, and to Moinas (2010) who analyzes potentially informed liquidity provision with hidden limit orders. Boulatov and George (2013) analyze the choice of informed traders between supplying and demanding liquidity. However, all of these papers take MMs' information as exogenous.

Our paper is also connected to a broader literature on two-sided information acquisition. Dang (2008) analyzes a bargaining game in which the buyer can acquire information before offering a price; the seller observes the offer and can acquire information before deciding whether or not to sell. Unlike our setting, the price-setter is monopolistic and information may be acquired after the price is observed. Tirole (2009, 2015) and Bolton and Faure-Grimaud (2010) examine contracting environments in which both parties can acquire information. However, the setting they explore is quite different from ours and offers the players a great deal of commitment power, which is typically lacking in asset markets.

2 Baseline Model

We consider the market for a risky asset with random value $V \sim Ber(\frac{1}{2})$.[4] The realization of V is denoted v. There are two market makers (MMs, he), indexed by $n = 1, 2$, and one trader (she). At $t = 1$, all market participants privately decide whether to observe v for a cost $c > 0$. Trade takes place at $t = 2$: the trader decides whether to submit a market order for one unit of the asset, and MMs simultaneously choose bid and ask prices. In the baseline model, the trader does not observe the prices before placing her market order; we relax this assumption in Section 6. There is price priority on the market; whenever MMs are tied, the trading probabilities are determined as part of the equilibrium. The ask (resp. bid) price of MMn is denoted a_n (resp. b_n). Hence, the trader's profit from a buy order (resp. sell order) is $v - \hat{a}$ (resp. $\hat{b} - v$), where $\hat{a} := \min_n a_n$ and $\hat{b} := \max_n b_n$; the profit of the MM executing the order is the opposite. For expository simplicity, we assume that MMs choose prices in $[0, 1]$.

[3] Glode, Green and Lowery (2012) and Bolton, Santos and Scheinkman (2016) also examine information acquisition but in very different contexts.

[4] That is, $\mathbb{P}(V = 0) = \mathbb{P}(V = 1) = \frac{1}{2}$.

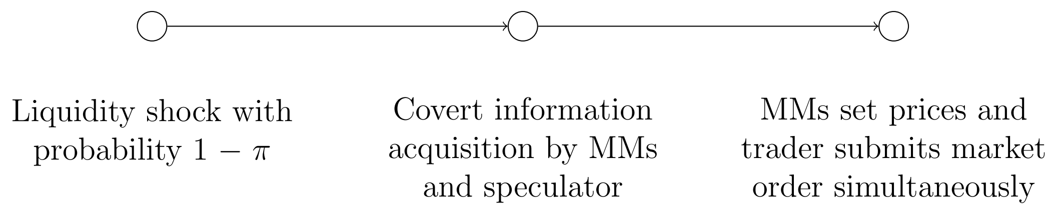


Figure 3: Timing

We say that the trader is a speculator if her objective is to maximize her expected profit. The trader is a speculator with probability π, whereas with probability $1-\pi$ the trader is privately hit by a liquidity shock before her decision in period $t=1$: she then buys or sells the asset with probability $\frac{1}{2}$ independently of all other random variables of the model. In this case we say that the trader is a liquidity trader. To make the analysis interesting we assume $\pi \in (0,1)$. Figure 3 summarizes the timing of the model.

Equilibrium. To shorten notation let MMnU denote the uninformed type of MMn, MMnH the informed type who has observed $V=1$, and MMnL the informed type who has observed $V=0$. A strategy of MMn comprises a probability p_n of acquiring information at $t=1$, and cumulative distribution functions σ_n, $\underline{\sigma}_n$ and $\overline{\sigma}_n$ specifying respectively the distribution of the bid price b_n of MMnU, MMnL and MMnH. As the bid and ask sides of the market are symmetric we assume, without loss of generality, that, conditional on MMnU, $1-a_n$ is distributed like b_n. Similarly, we assume that $1-a_n$ conditional on MMnL (resp. MMnH) is distributed like b_n conditional on MMnH (resp. MMnL). A strategy of the speculator comprises a probability q of acquiring information at $t=1$, as well as a market order as a function of her information at $t=2$. The equilibrium concept is perfect Bayesian equilibrium.

The next definition will prove useful in the following sections. Roughly, a Wilson-Engelbrecht-Milgrom-Weber-Lee (henceforth WELM) equilibrium is an equilibrium in which both MMs play the same strategy, and MMnL bids below MMnU who himself bids below MMnH.[5]

Definition 1. *A WELM equilibrium satisfies:*

(i) $p_1 = p_2 = p$;

(ii) $\sigma_1 = \sigma_2 = \sigma$, $\underline{\sigma}_1 = \underline{\sigma}_2 = \underline{\sigma}$ *and* $\overline{\sigma}_1 = \overline{\sigma}_2 = \overline{\sigma}$;

[5] After Wilson (1967), Engelbrecht-Wiggans, Milgrom and Weber (1983), and Lee (1984).

(iii) $\underline{\sigma}(0) = 1$*;*

(iv) either $p \in \{0, 1\}$ *or* σ *and* $\overline{\sigma}$ *are atomless, with supp* $(\sigma) = [0, l]$ *and supp* $(\overline{\sigma}) = [l, u]$*.*

3 A One-Sided Strategic Complementarity

In this section we fix the probabilities p_1, p_2 and q with which MM1, MM2 and the speculator are informed and study the *trading game* that results. This trading game is formally equivalent in the case $q = 0$ to a first-price sealed-bid common-value auction with (possibly) asymmetrically informed bidders.[6] In contrast to that literature however, our objective is to study and compare profits on the two sides of a dealer market, and to analyze the effect of information acquired by one side of the market on incentives to acquire information on the other side. All proofs of this section are in Appendix A.

Formal definitions, and the detailed analysis of the trading game, are relegated to Online Appendix C. We summarize the main results below. Except in knife-edge cases, in any trading equilibrium the speculator sells (respectively buys) with probability 1 when she is informed and $V = 0$ (resp. $V = 1$), and abstains when she is uninformed. The support of the bid distributions of different types of a given MM never strictly overlap: MMnL bids below MMnU, who himself bids below MMnH.

Lemma 1. *For all* p_1*,* p_2 *and* q*, a trading equilibrium exists. Each market participant's expected profit is independent of the trading equilibrium considered.*

Let Π_n (respectively $\overline{\Pi}_n$) denote the equilibrium expected profit of MMn when uninformed (resp. informed) in the trading game. Similarly, let Π_S (respectively $\overline{\Pi}_S$) denote the equilibrium expected profit of the speculator when uninformed (resp. informed) in the trading game;[7] Lemma 1 assures that these objects are well defined.

Lemma 2. *Suppose* $p_n > p_m$*. Then* $\overline{\Pi}_m = \overline{\Pi}_n > \Pi_n > \Pi_m = 0$*.*

In equilibrium, both MMs earn the same profit when informed. Intuitively, if MMnH were to make more profit than MMmH, then MMmH could increase his expected profit by bidding just above MMnH's highest bid price. Next, if $p_n > p_m$ then MMnU makes greater expected

[6] Such auctions have been analyzed, as already mentioned, by Wilson (1967), Engelbrecht-Wiggans *et al.* (1983), and Lee (1984). See also Hausch and Li (1993); Calcagno and Lovo (2006); Syrgkanis, Kempe and Tardos (2013).

[7] Hence, $\overline{\Pi}_n$ and $\overline{\Pi}_S$ are gross profits, that is, profits obtained before subtracting the cost of information.

profit than MMmU. The logic is straightforward: informed MMs pick a disproportionate share of profitable market orders; since MMn is more often informed than MMm, MMmU then faces more adverse selection than MMnU. Intuitively, an uninformed MM extracts rent from a competitor's belief that he is informed with higher probability. For our purpose the main implication is as follows: $p_n > p_m$ implies $\overline{\Pi}_n - \Pi_n < \overline{\Pi}_m - \Pi_m$, that is, MM$m$'s incremental profit from being informed is greater than that of MMn. Consequently, any equilibrium of the baseline model must be such that $p_1 = p_2$ (in fact, any equilibrium of the baseline model is a WELM equilibrium; see Theorem 2). We therefore focus in the rest of the section on profiles of information acquisition satisfying $p_1 = p_2$, and let p denote the common probability with which MMs are informed. The next theorem is the central result of this section.

Theorem 1. *There is a one-sided strategic complementarity in information acquisition.*

1. *The more information acquired by any market participant the smaller the speculator's gain from acquiring information:* $\overline{\Pi}_S - \Pi_S$ *is decreasing in* p *and* q*.*

2. *By contrast, information acquired by the speculator enhances MMs' gain from acquiring information:* $\overline{\Pi}_n - \Pi_n$ *is decreasing in* p *but increasing in* q*.*

The logic behind the effects of p and q on the speculator's gain from being informed is as follows. The speculator's only chance of making profit is against uninformed MMs. Raising p therefore reduces the speculator's chances of finding a profitable trade.[8] Higher q, on the other hand, generates greater adverse selection for uninformed MMs, which in turn induces larger bid-ask spreads. Increasing q thus reduces the profit made by the speculator on each profitable trade.

The effect of p on MMs' gain from information is straightforward, as higher p means greater competition for profitable market orders. The effect of q on MMs' gain from information is more subtle. Increasing q enhances the adverse selection problem faced by all uninformed MMs, which induces them to set wider spreads. This, in turn, softens price competition for informed MMs, who now increase their own bid-ask spreads. MMs' incremental profit from being informed therefore increases with q.

[8] Moreover, raising p increases adverse selection for uninformed MMs which, in turn, induces the latter to set wider bid-ask spreads, further reducing the informed speculator's expected profit in the trading game.

4 Information Acquisition

In this section we analyze equilibrium information acquisition in the baseline model. We show that the pattern of information acquisition is uniquely determined as a function of information cost and market composition. In particular, at small information costs MMs acquire information whereas the speculator chooses to remain uninformed. The situation is reversed at larger costs. The next theorem is this section's central result. All proofs of this section are in Appendix A.

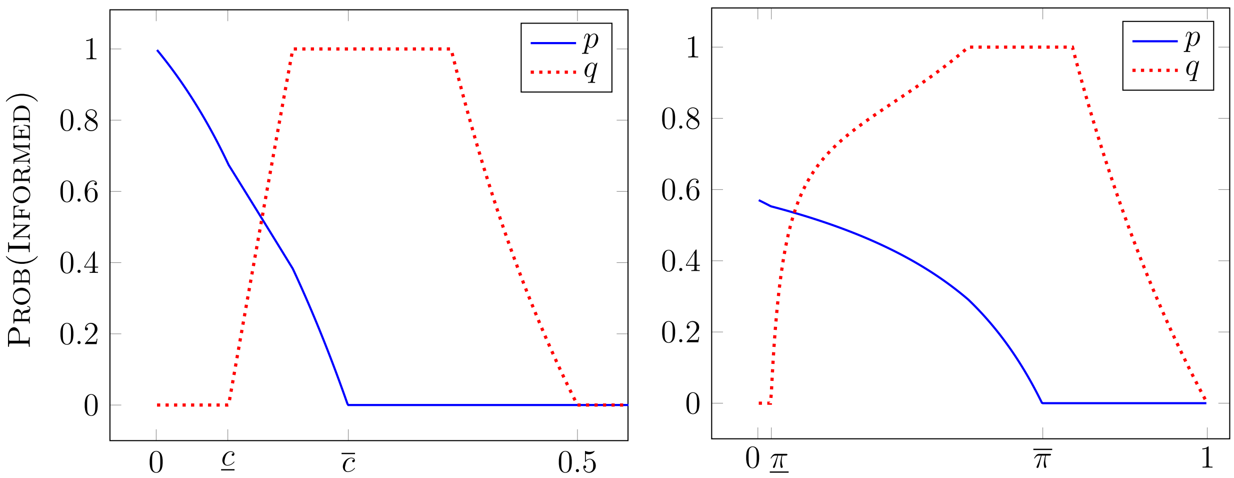
Theorem 2. *There exists an equilibrium. Moreover, any equilibrium is a WELM equilibrium. The information acquisition probabilities, p and q, are independent of the equilibrium considered; p is non-increasing in c and tends to 1 as c tends to 0. For $c > 1/2$, neither the speculator nor the MMs acquire information, and there exist $0 < \underline{c} < \overline{c} < \frac{1}{2}$ such that:*

- *if $c \in (0, \underline{c})$ then MMs acquire information with positive probability but the speculator remains uninformed;*
- *if $c \in (\underline{c}, \overline{c})$ then the MMs and the speculator all acquire information with positive probability;*
- *if $c \in (\overline{c}, \frac{1}{2})$ then the speculator acquires information with positive probability but MMs remain uninformed.*

In view of Theorem 2, standard assumptions about informational asymmetries in financial markets seem warranted for an intermediate range of information costs ($c \in (\overline{c}, \frac{1}{2})$). Yet, strikingly, at lower information costs the market consists only of liquidity traders, informed MMs and uninformed MMs. The equilibrium pattern of information acquisition is illustrated in Figure 4, panel A, for $\pi = \frac{3}{10}$.[9] The information acquisition probabilities are on the vertical axis; the information cost is on the horizontal axis. The solid curve shows the equilibrium p, and the dashed curve the equilibrium q. We also indicate the cutoff $\underline{c} > 0$ below which the speculator is uninformed, and the cutoff $\overline{c} < \frac{1}{2}$ above which MMs are uninformed.

The uniqueness of p and q as well the non-increasingness of p as a function of c all follow from Theorem 1. We summarize in the next paragraphs the logic behind who acquires information.

[9] The code used for calculating the equilibrium and simulating the prices in the following figures is available on the authors' websites.



(a) Cost of Information, c

(b) Fraction of Speculators, π

Figure 4: Equilibrium Information Acquisition

At small information costs, MMs acquire information. As uninformed MMs never set bid prices above $\frac{1}{2}$ nor set ask prices below $\frac{1}{2}$, MMn's gain from being informed in the trading game is at least as large as $\frac{(1-p)(1-\pi)}{4}$: there is probability $(1-p)$ that MMm is uninformed, in which case the informed type of MMn ensures profit $\frac{1}{2}$ whenever the trader is hit by a liquidity shock and either sells when $V = 1$ or buys when $V = 0$, which occurs with probability $\frac{1-\pi}{2}$. Thus, in equilibrium, $\frac{(1-p)(1-\pi)}{4} \leq c$, and p tends to 1 as c tends to 0.[10] This shows that, at small information costs, MMs acquire information.

At small information costs, MMs crowd out the speculator. By symmetry of the bid and ask sides of the market, the speculator's gain from being informed in the trading game may be written as

$$\overline{\Pi}_S - \Pi_S = (1-p^2)\mathbb{E}[\,\hat{b}\,|V=0,\text{ either 1 or 2 MMs are uninformed}]. \tag{1}$$

The factor $1-p^2$ represents the speculator's chances of making a positive profit. We established earlier that $\frac{(1-p)(1-\pi)}{4} \leq c$, thus,

$$1-p^2 \leq \frac{4c(1+p)}{1-\pi} \leq \frac{8c}{1-\pi}. \tag{2}$$

[10]If $\frac{(1-p)(1-\pi)}{4} > c$ then a MM's gain from acquiring information is larger than the cost of information, implying $p = 1$. But this is a contradiction, since $c > 0$.

Consider now the expected bid price $\hat{b}$ appearing on the right-hand side of (1). As p tends to 1, an uninformed MM is almost exclusively left with unprofitable market orders (when p is close to 1 an uninformed MM expects profitable market orders to be picked by his competitor). Since p tends to 1 when c tends to 0, an uninformed MM's bid-ask spread therefore tends to 1 as c tends to 0, giving[11]

$$\lim_{c \to 0} \mathbb{E}[\ \hat{b}\ |V = 0,\ \text{either 1 or 2 MMs are uninformed}] = 0. \tag{3}$$

Lastly, combining (1), (2) and (3) gives $\overline{\Pi}_S - \Pi_S < c$ for all c sufficiently small. This implies that the speculator chooses to remain uninformed at small information costs.

At intermediate information costs, the speculator acquires information whereas MMs remain uninformed. Pick $p = 0$ and $q = 1$, and consider the resulting trading game. With both MMs uninformed, Bertrand competition yields $\hat{b} = \mathbb{E}[V|\text{sell}] = \frac{1-\pi}{2} = 1 - \hat{a}$.[12] The informed speculator sells when $V = 0$ and buys when $V = 1$. Hence,

$$\overline{\Pi}_S - \Pi_S = \frac{1}{2}\hat{b} + \frac{1}{2}(1 - \hat{a}) = \frac{1-\pi}{2}. \tag{4}$$

Now, if a MM were informed he would pick all profitable market orders, that is, all sell orders of the liquidity trader when $V = 1$ and all buy orders of the liquidity trader when $V = 0$. We therefore obtain

$$\overline{\Pi}_n - \Pi_n = (1 - \pi)\left(\frac{1}{4}(1 - \hat{b}) + \frac{1}{4}\hat{a}\right) = \frac{1 - \pi^2}{4}. \tag{5}$$

Combining (4) and (5) yields $\overline{\Pi}_n - \Pi_n < \overline{\Pi}_S - \Pi_S$. The latter inequality establishes that, for $c \in [\frac{1-\pi^2}{4}, \frac{1-\pi}{2}]$, the unique equilibrium information acquisition probabilities are $q = 1$ and $p = 0$: the speculator acquires information whereas MMs remain uninformed. Intuitively, a MM's gain from acquiring information is bounded by the impossibility to turn a profit from trading with the speculator, even in cases in which the latter is uninformed. By contrast, if all MMs are uninformed the informed speculator then makes a profit with probability 1.

We end this section by investigating the effect of the composition of the market (in terms of speculation vs liquidity trading) on information acquisition.

[11] A formal proof of (3) is provided in Appendix A.

[12] As $q = 1$, the speculator sells when $V = 0$ and buys when $V = 1$: $\mathbb{E}[V|\text{sell}] = \frac{\frac{1}{2}\cdot\frac{1-\pi}{2}}{\frac{1}{2}\cdot\frac{1-\pi}{2} + \frac{1}{2}(\frac{1-\pi}{2} + \pi)} = \frac{1-\pi}{2}$.

Proposition 1. *Assume $c < \frac{1}{2}$. There exist $0 \leq \underline{\pi} \leq \overline{\pi} < 1$ such that, in any equilibrium:*

- *if $\pi \in (0, \underline{\pi})$ then MMs acquire information with positive probability but the speculator remains uninformed;*
- *if $\pi \in (\underline{\pi}, \overline{\pi})$ then the MMs and the speculator all acquire information with positive probability;*
- *if $\pi \in (\overline{\pi}, 1)$ then the speculator acquires information with positive probability but MMs remain uninformed.*

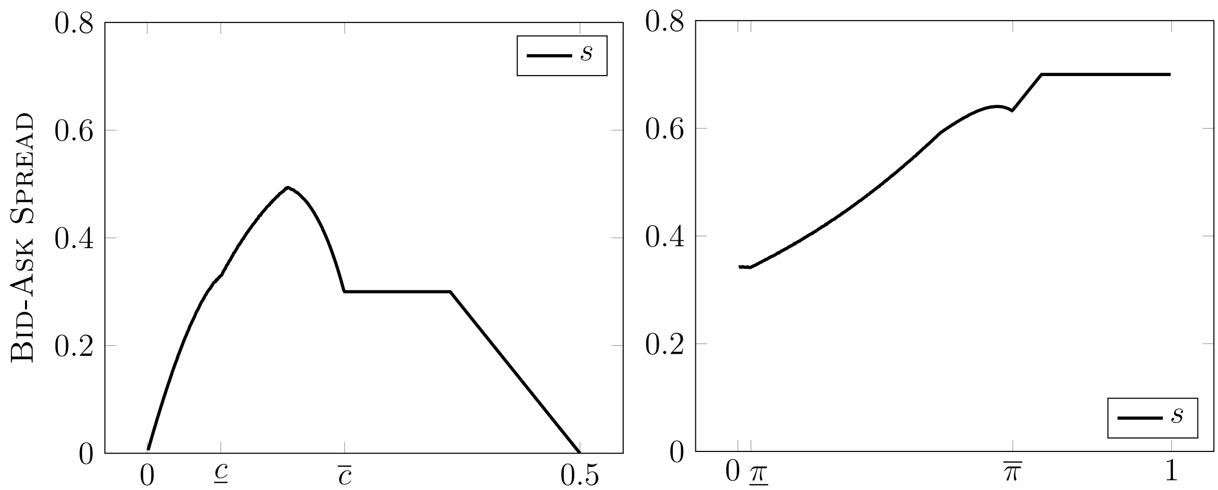
Moreover, there exists $c^* > 0$ such that $\underline{\pi} > 0$ if and only if $c < c^*$. Lastly, the equilibrium probability p with which a MM acquires information is non-increasing in π, and the equilibrium probability q with which the speculator acquires information tends to 0 as π tends to 1.

The proposition is illustrated in Figure 4, panel B, for $c = 0.15$. The vertical axis shows information acquisition, with π on the horizontal axis. The MMs exclusively recoup c by executing market orders from the liquidity trader. So MMs acquire information when liquidity shocks are frequent and remain uninformed when liquidity shocks are rare. The speculator, on the other hand, exclusively recoups c by trading with uninformed MMs. Therefore, the speculator remains uninformed when liquidity shocks are frequent, in which case most MMs are informed, and acquires information when liquidity shocks are rare, in which case most MMs are uninformed. The speculator's information acquisition probability tends to 0 as π tends to 1. If this were not the case then, conditional on $V = 0$, $\hat{b}$ would have to converge to 0 (in probability). Similarly, conditional on $V = 1$, $\hat{a}$ would have to converge to 1. However, in that case, the speculator would have no incentive at all to pay for information.

5 Market Liquidity and Price Discovery

In this section we examine the implications of the model for market liquidity and price discovery. As is usual in the literature, we measure market liquidity using the expected bid-ask spread $s := \mathbb{E}[\hat{a} - \hat{b}]$.[13] Price discovery (or inverse price discovery) is defined as the expected squared price error, $d := \mathbb{E}[(r - V)^2]$, with r denoting the realized price, i.e. $r = \hat{a}$ in case of a buy order, $r = \hat{b}$ in case of a sell order, and $r = \frac{\hat{a}+\hat{b}}{2}$ if the trader abstains.

[13] The bid-ask spread measures how much liquidity traders pay on average to trade.



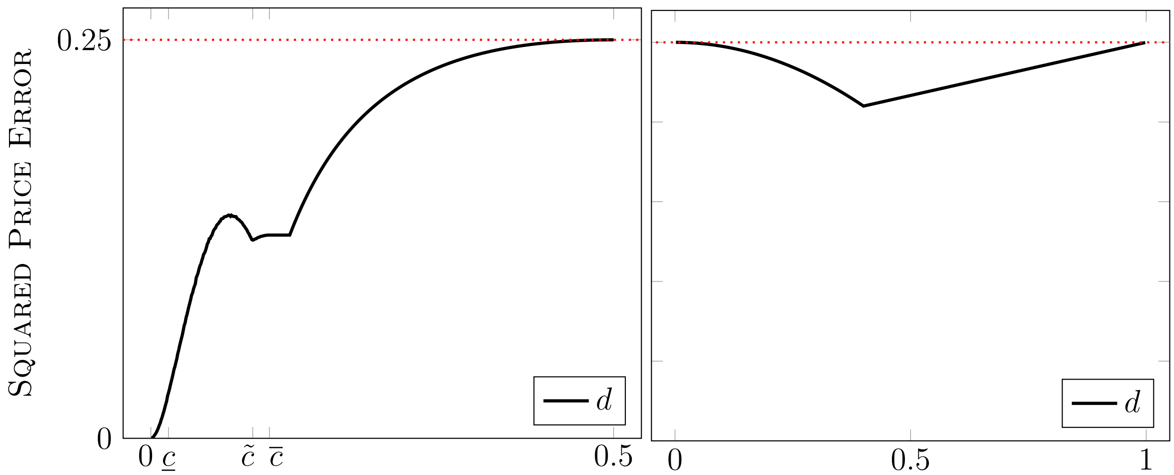
(A) Information Cost, c (B) Fraction of Speculators, π

Figure 5: Market Liquidity

Market Liquidity. The implications of the model regarding market liquidity are straightforward. For $c > \frac{1}{2}$, the MMs and the speculator all remain uninformed. Each MM, being uninformed and facing no adverse selection, sets $a_n = b_n = \frac{1}{2}$. Thus, $s = 0$ for $c > \frac{1}{2}$. On the other hand each MM acquires information with probability converging to 1 as c tends to 0 (Theorem 2). If $V = 1$, informed MMs competing to offer the best bid price then ensure that $\hat{b}$ converges to 1 (in probability) as c tends to 0. Since $V = 1$ implies $\hat{a} = 1$ whenever both MMs are informed, we conclude that s tends to 0 as c tends to 0 (the logic is the same if we condition on $V = 0$). At $c = \overline{c}$ however, the speculator acquires information with probability 1 but MMs choose to remain uninformed. This yields $\hat{b} = \frac{1-\pi}{2} = 1 - \hat{a}$, and so $s = 2\hat{a} - 1 = \pi$. The resulting non-monotonicity of the expected bid-ask spread as a function of the information cost is illustrated in Figure 5, panel A, for $\pi = \frac{3}{10}$.[14]

The effect of market composition on the expected bid-ask spread is more subtle. Figure 5, panel B, shows s as a function of π for $c = 0.15$. On the one hand increasing π reduces the probability that a market order originates from the liquidity trader; this worsens, ceteris paribus, the adverse selection problem of an uninformed MM. However, immediately below $\pi = \overline{\pi}$, the probability q with which the speculator acquires information is constant at 1, whereas

[14]This non-monotonicity holds irrespective of π, since we showed that $s = \pi$ at $c = \overline{c}$. Notice too that, moving from right to left, s goes on rising passed $c = \overline{c}$. While the speculator continues to acquire information with probability 1 immediately below $c = \overline{c}$, now MMs acquire information with some probability as well. The adverse selection problem of an uninformed MM is therefore more severe for c just below $\overline{c}$ than at $c = \overline{c}$.



(A) Information Cost, c (B) Fraction of Speculators, π

Figure 6: Price Discovery

p decreases with π; this reduces, ceteris paribus, the uninformed MMs' adverse selection problem. Immediately below $\pi = \overline{\pi}$ the latter effect dominates: increasing π indirectly benefits an uninformed MM. In Figure 5, panel B, s increases at first, but then dips around $\pi = \overline{\pi}$, before increasing again at larger values of π.

Price Discovery. Panel A of Figure 6 illustrates the expected squared price error, d, with c on the horizontal axis, and market composition $\pi = \frac{7}{10}$. The point $\tilde{c}$ marks the lowest information cost at which in equilibrium $q = 1$. First, notice that the arguments of the previous paragraph immediately establish that $d = \frac{1}{4}$ for $c > \frac{1}{2}$, whereas d tends to 0 as c tends to 0. The expected squared price error at first increases with c. Yet, d dips inside the interval $[\underline{c}, \tilde{c}]$, before increasing again at larger values of the cost. The reason is as follows. Between $\underline{c}$ and $\tilde{c}$, the probability q that the speculator acquires information goes from 0 all the way to 1. The probability p with which a MM acquires information simultaneously decreases in that interval. However, since MMs' gain from information increases with q (Theorem 1), the rate at which p falls in c *drops* within the cost interval $[\underline{c}, \tilde{c}]$. Consequently, in that cost interval, and for π sufficiently large, the positive effect from q dominates the negative effect from p.[15] Price discovery thus momentarily improves as c increases (that is, d falls).

Panel B of Figure 6 shows the expected squared price error on the vertical axis, with π on the horizontal axis, and information cost $c = \frac{3}{10}$. Price discovery is non-monotonic in π for all

[15] For π small, the positive effect from q is too weak. Price discovery is then non-decreasing in c.

$c \in (\frac{1}{4}, \frac{1}{2})$. The reason is as follows. First, notice that, for all $c \in (\frac{1}{4}, \frac{1}{2})$, the condition $c \in (\overline{c}, \frac{1}{2})$ holds irrespective of π. Hence, in equilibrium, the speculator is the only market participant acquiring information (Theorem 2) at all values of π.[16] Now, in the limit as π tends to 0 almost all market orders originate from the liquidity trader. Each MM, being uninformed and facing (almost) no adverse selection, sets a_n and b_n close to $\frac{1}{2}$. Thus $d = \mathbb{E}[(r - V)^2]$ approaches $\frac{1}{4}$. However, as π tends to 1 the probability that the speculator acquires information tends to 0 (Proposition 1). This, in turn, implies that r tends to $\frac{1}{2}$ (in probability) as π tends to 1, since the probability that the trader abstains tends to 1 as π tends to 1.[17] Thus $d = \mathbb{E}[(r - V)^2]$ approaches $\frac{1}{4}$ as π tends to 1. For all $c \in (\frac{1}{4}, \frac{1}{2})$, the expected squared price error thus attains its upper bound in the limits as $\pi \to 0$ and as $\pi \to 1$. Price discovery is therefore non-monotonic in π for all $c \in (\frac{1}{4}, \frac{1}{2})$. Intuitively, an increase of liquidity trading can improve price discovery due to the "market collapse" resulting from the disappearance of liquidity traders.

6 Observable Quotes

In this section we extend the baseline model of Section 2 by letting the MMs' quotes be observable with probability $z \in [0, 1]$ before the trader's decision in period $t = 2$; the baseline model corresponds to $z = 0$. The timing of the observable quotes model is illustrated in Figure 7. The trader is privately hit by the liquidity shock with probability $1 - \pi$, after which all market participants privately decide whether to observe v, for a cost $c > 0$. The MMs then simultaneously choose bid and ask prices. The trader observes the quotes with probability z; she then either abstains or places a market order for one unit of the asset. All proofs of this section are in Appendix B.

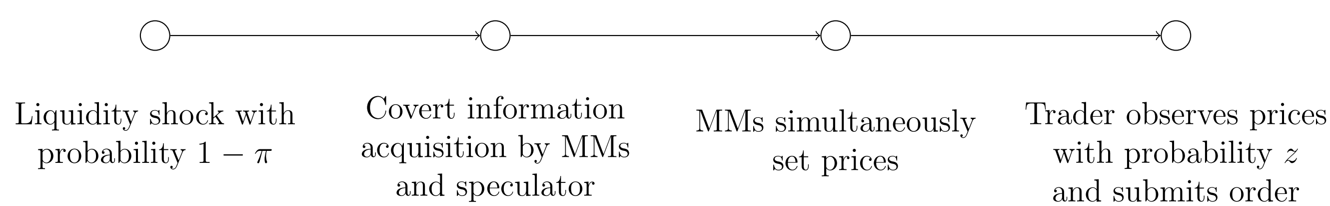


Figure 7: Timing – Observable Quotes

[16] In other words, $\underline{\pi} = \overline{\pi} = 0$.

[17] Recall, the speculator abstains when she is uninformed.

Relative to the baseline model, making the quotes observable adds multiple layers of complexity:

- Each MM's decision to acquire information now induces additional externalities on the other MM: information which the speculator learns through the prices of one MM may be used to make profit against the other MM. Two interesting consequences of this feature are that:
  (i) MM's gain from acquiring information is increasing in quote observability, z;
  (ii) MM's gain from acquiring information can increase with MMs' information, p.
- As the speculator can now learn about v without acquiring information herself, observable quotes reduce the speculator's gain from acquiring information.
- Observable quotes allow MMs to manipulate demand through prices. For instance, MMnU could masquerade as MMnH and reduce adverse selection by "jamming" the signal of MMmL.

To make progress and keep the analysis tractable we restrict attention throughout this section to WELM equilibria.[18] While multiple WELM equilibria exist, we show in Online Appendix D that all WELM equilibria share important common properties. First, since WELM equilibria are such that different types of a MM set prices in non-overlapping intervals, the uninformed speculator learns v with probability $z\left[1-(1-p)^2\right]$, that is, as long as the speculator gets to observe the quotes and at least one MM is informed. The speculator trades if she acquires information, or if she learns v through a MM's quotes; she abstains otherwise. Second, given p and q a market participant's expected profit is the same in all WELM trading equilibria.[19] Extending previous notation, let $\Pi_n(p, q; z)$ denote MMnU's expected profit in any WELM trading equilibrium, and $\overline{\Pi}_n(p, q; z)$ denote the expected profit of MMn when informed. Similarly, let $\overline{\Pi}_S(p, q; z)$ denote the speculator's expected profit conditional on being informed, and $\Pi_S(p, q; z)$ her expected profit when she is uninformed.

We explore first the effect of the observability of the quotes on market participants' gains from being informed.

[18] We ruled out in Theorem 2 non-WELM equilibria in the case $z = 0$. Whether non-WELM equilibria can exist for $z > 0$ remains an open question, which we were unable to answer.

[19] We call WELM trading equilibrium a perfect Bayesian equilibrium of the trading game induced by the observable quotes model that satisfies: (i) $\sigma_1 = \sigma_2 = \sigma$, $\underline{\sigma}_1 = \underline{\sigma}_2 = \underline{\sigma}$ and $\overline{\sigma}_1 = \overline{\sigma}_2 = \overline{\sigma}$; (ii) $\underline{\sigma}(0) = 1$; (iii) either $p \in \{0, 1\}$ or σ and $\overline{\sigma}$ are atomless, with supp $(\sigma) = [0, l]$ and supp $(\overline{\sigma}) = [l, u]$. See Online Appendix D.

Proposition 2. *The greater the observability of the quotes the smaller the speculator's gain from acquiring information: if $p > 0$ and $q < 1$ then $\overline{\Pi}_S - \Pi_S$ is decreasing in z. By contrast, the observability of the quotes enhances MMs' gain from acquiring information: if $p > 0$ and $q < 1$ then $\overline{\Pi}_n - \Pi_n$ is increasing in z.*

The first part of the proposition is straightforward. As long as (i) MMs are informed with positive probability and (ii) the speculator is uninformed with positive probability then, increasing z: (a) increases the amount of information which the speculator can retrieve from the quotes, (b) induces MMs to set wider bid-ask spreads, by exposing uninformed MMs to greater adverse selection. Both effects in turn decrease the speculator's incentive to pay for information. The second part of the proposition is more interesting. As noted above, increasing z exposes uninformed MMs to greater adverse selection, inducing them to set wider bid-ask spreads. The larger spreads of the uninformed MMs enable in turn informed MMs to turn a greater profit from trading with the liquidity trader. The latter mechanism is naturally akin to the mechanism in Theorem 1 that induced MMs' gain from information to increase with q: increasing z indirectly increases information available to the speculator.

Our next result extends Theorem 1.

Theorem 3. *There is a one-sided strategic complementarity in information acquisition.*

1. *The more information acquired by any market participant the smaller the speculator's gain from acquiring information: $\overline{\Pi}_S - \Pi_S$ is decreasing in p and q.*

2. *By contrast, information acquired by the speculator enhances MMs' gain from acquiring information: $\overline{\Pi}_n - \Pi_n$ is increasing in q, and can be either increasing or decreasing in p.*

The logic underlying the one-side strategic complementarity in information acquisition is as in the baseline model.[20] Observable quotes do however have novel implications regarding the impact of p on MMs' gain from acquiring information. On one hand, increasing p enhances competition among MMs for profitable market orders; this effect reduces MMs' gain from acquiring information, and led $\overline{\Pi}_n - \Pi_n$ to be decreasing in p at $z = 0$. On the other hand, with observable quotes, increasing p channels additional information to the speculator, exposing uninformed MMs to greater adverse selection. The larger spreads of the uninformed

[20]Note however that with $z > 0$, the adverse effect of p on the speculator's incentive to acquire information is even larger than before since now increasing p channels information concerning v to the speculator.

MMs enable in turn informed MMs to make greater profits from trading with the liquidity trader. This feedback effect implies that MMs' gain from acquiring information can be locally increasing in their own information acquisition probability, p.

The feedback effect highlighted above opens the door to the existence of multiple WELM equilibria. However, the model's main prediction continues to hold. Who acquires information crucially depends on the cost of information: when this cost is small, MMs acquire information and crowd out the speculator; by contrast, when this cost is large, the speculator acquires information and MMs remain uninformed.

Theorem 4. *There exist* $0 < \underline{c} < \overline{c} < \frac{1}{2}$ *such that, for* $c \in (0, \underline{c}) \cup (\overline{c}, \frac{1}{2})$*, a WELM equilibrium exists and in any WELM equilibrium:*

- *if* $c \in (0, \underline{c})$ *then MMs acquire information with positive probability but the speculator remains uninformed;*
- *if* $c \in (\overline{c}, \frac{1}{2})$ *then the speculator acquires information with positive probability but MMs remain uninformed.*

Figure 8 illustrates Theorem 4, for $\pi = \frac{3}{10}$. The dashed curves correspond to the baseline model, that is, $z = 0$; the solid curves correspond to the model with perfectly observable quotes, that is, $z = 1$. The blue curves depict equilibrium p values, and the red curves equilibrium q values. First, the cutoff $\underline{c}$ below which the speculator is uninformed is higher for $z = 1$ than for $z = 0$, capturing the fact that observable quotes reduce the speculator's incentive to pay for information (Proposition 2). By contrast the cutoff $\overline{c}$ above which MMs are uninformed does not depend on z. To see that this is true, note that if $p = 0$ is an equilibrium outcome for $z = 0$ then $p = 0$ is also an equilibrium outcome for $z = 1$: intuitively, if MMs never acquire information then quote observability has no effect on any of the market participants' profits. Lastly, going from left to right in the figure, observe that whereas MMs initially acquire more information with observable quotes than without, things eventually reverse within the cost interval where q is greater for $z = 0$ than for $z = 1$. The logic is the following. We saw in Proposition 2 that, fixing p and q, MMs' gain from acquiring information increases with z. Yet we also established (Theorem 3) that MMs' gain from acquiring information increases with q. In the interval of cost where increasing z reduces q sufficiently, increasing z ultimately reduces p.

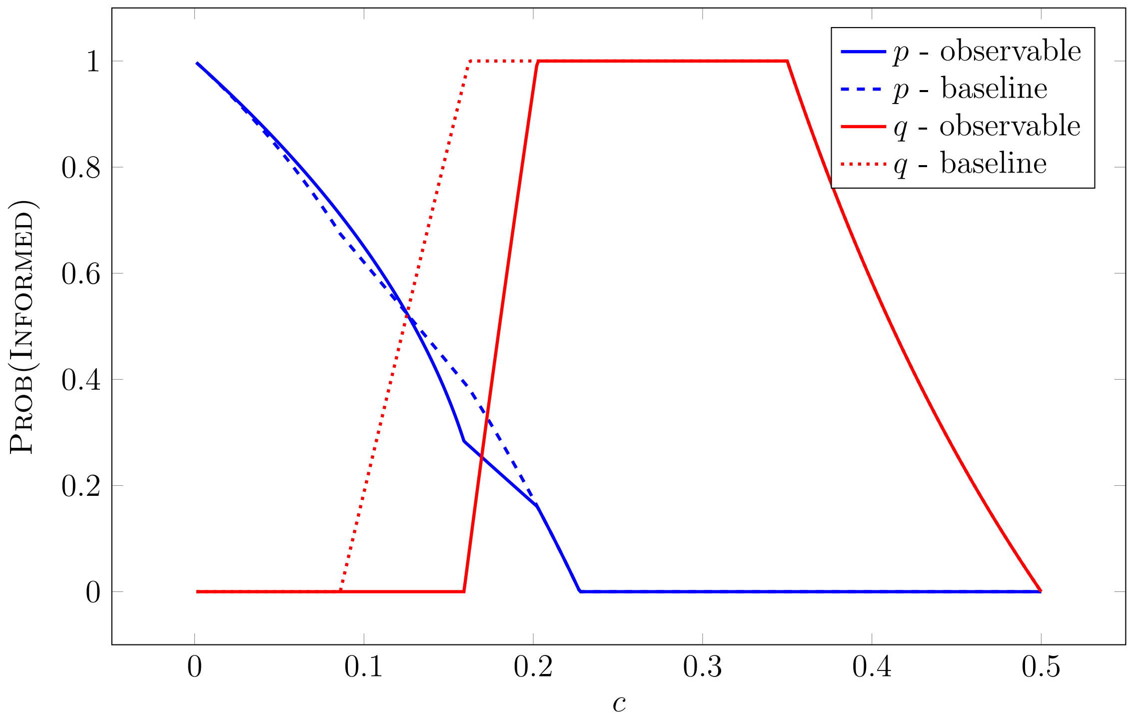


Figure 8: Equilibrium Information Acquisition: Baseline vs. Observable Quotes

7 Concluding Remarks

This paper analyzes information acquisition in dealer markets. We identify a one-sided strategic complementarity in information acquisition: the more informed traders are, the larger MMs' gain from becoming informed. When the cost of information is the same for all market participants, this strategic complementarity uniquely pins down information acquisition. We show that within a cost range information acquisition is as in the canonical model of Glosten and Milgrom (1985), that is, all speculators are informed whereas none of the MMs are. However, different configurations arise at different information costs. In particular, at small costs, information acquisition is reversed. In that case, market makers are informed, but speculators are not. Furthermore, when most traders are speculators, the speculators choose to acquire information whereas MMs remain uninformed; by contrast, when most traders are liquidity traders, MMs acquire information and traders remain uninformed. We also explore market liquidity and price discovery. Most strikingly, increasing the cost of information or the fraction of liquidity traders can improve price discovery.

Discussion of assumptions. The model we propose is stylized, yet rich enough to deliver various seemingly robust insights concerning the implications of the microstructure of dealer markets on incentives to invest in information. First, MMs' gain from acquiring information is increasing in the probability that traders are informed (one-sided strategic complementarity in information acquisition). The underlying mechanism works as follows:

- informed MMs face price competition from uninformed MMs;
- increasing the probability of informed trading increases adverse selection more for uninformed MMs than for informed MMs;
- so greater adverse selection effectively softens price competition for informed MMs.

Second, at small information costs, MMs crowd out all speculative trading. The logic is the following:

- as long as bid-ask spreads are sufficiently narrow, liquidity traders continue to trade;[21]
- combining the first point with the strategic complementarity in information acquisition implies that MMs' incentive to pay for information attains a *positive minimum* when all traders are uninformed;
- when MMs' information acquisition probability goes to 1, the probability that an informed trader finds a profitable trade *and* the profit that can be made on this trade both go to zero;
- at sufficiently small information costs, the second point implies that MMs choose to become informed with probability close to 1; the last point implies that traders remain uninformed.

The baseline model (Section 2) illustrates the previous insights in the simplest possible way. The observable quotes model (Section 6) opens a new information flow from MMs to traders, since traders benefit from MMs' information acquisition via more informative quotes. The fundamental mechanisms, however, are the same in the two models. Loosely speaking, increasing p in the observable quotes model is like increasing p and q in the baseline model.

[21] In our model liquidity traders always trade, since they have inelastic demand; under more general assumptions, they will trade if the spread is sufficiently narrow.

Some of our modelling assumptions are relatively easily relaxed, including the number of MMs, the probability that $V = 1$, or even the binary nature of the asset value, V. On the other hand, two (standard) assumptions are essential for the model's tractability:

(i) There is a single trading round. Introducing multiple trading rounds would imply that private information leaks to the market through previous quotes and trades, and that the same information can be used in several trading rounds. In this case, on the one hand MMs and traders have an incentive to act strategically and hide their private information so as to use it several times by setting less aggressive quotes or trading less aggressively, but on the other hand, they also have an incentive to act quickly before the information of other market participants is disseminated to the market. The reaction to these incentives will in turn determine how 'many times' information can be used, and how profitable it will be, thus modifying the incentives to acquire information.

(ii) The market participants acquire perfect information concerning V. Of course, in practice traders and MMs can incur varying costs in order to acquire more or less accurate information about the assets they trade. One could as a first step improve the realism of the baseline model (i.e., with unobservable quotes) by supposing that, instead of observing the realization of V, market participants can, for a cost $c > 0$, observe the realization of a binary signal correlated with V. Notice that the combination of unobservable quotes with noisy signals of V gives informed MMs an edge over the informed speculator. By contrast, with observable quotes, the situation is reversed. Alas, with noisy signals, the model loses much of its tractability once quotes are made observable. For instance, even restricting attention to binary signals, in any WELM equilibrium the speculator is one of seven possible types, whenever p_1, p_2 and q are all positive.

Relation to empirics. Our findings shed light on several well-documented empirical regularities. First, *both traders and MMs may have proprietary information.* Manaster and Mann (1996), for instance, provide evidence in connection with the market for commodity futures, Li and Heidle (2004) for stockmarkets, and Covrig and Melvin (2002) and Sapp (2002) for the foreign exchange market. Therefore, traders cannot be viewed purely as uninformed liquidity traders, and MMs cannot be viewed as only learning from their private knowledge of the order flow.[22]

[22] MMs acquiring information through this channel have been considered in, for instance, Leach and Madhavan (1993), Bloomfield and O'Hara (2000) and de Frutos and Manzano (2005). In this literature, the focus is on MMs' incentives to experiment with prices in order to learn new information.

Second, *dealer-driven price discovery can be more important than trader-driven price discovery.* For stock markets, Anand and Subrahmanyam (2008) find that "intermediaries appear to be more informed than all other institutions and individuals combined". Valseth (2013) explores government bond markets and compares the informational content of the interdealer and customer order flows: the interdealer order flow explains almost a quarter of daily yield variation, whereas the customer order flow has little explanatory power.

Third, *MMs are often asymmetrically informed.* The finding is widely documented (Albanesi and Rindi, 2000; Huang, 2002; Massa and Simonov, 2003). In our setting, ex ante identical MMs play mixed information acquisition strategies in equilibrium, and may therefore be ex-post asymmetrically informed.

Fourth, *more volatile assets exhibit larger spreads.* Stoll (1978) was first to provide evidence in the case of stocks, while Chen, Lesmond and Wei (2007) find that spreads are higher for corporate bonds with lower rating or higher maturity, which are both associated with higher price volatility.[23] We establish that the bid-ask spread is largest when there is both informed market making and informed trading. Consequently, in our setting, spreads are maximized when prices are volatile.[24]

Limit-order book markets and the 'modern market makers'. The dichotomy in our model between market makers and traders corresponds to a dealer market and mirrors Glosten and Milgrom (1985). This assumption is valid for some markets, but many markets operate via an electronic limit-order book (LOB) in which traders can choose whether to place limit orders (supply liquidity) or markets orders (demand liquidity). This naturally blurs the distinction between market makers and traders as it exists in our model.

Even if LOB markets in principle allows all traders to choose between supplying or demanding liquidity, some traders have a natural advantage in liquidity supply. In particular, high-frequency traders who specialize in algorithmic trading at high speeds operate in many LOB markets. These are not limited to market-making activities, but some high-frequency traders specialize in this. High-frequency traders have an advantage in liquidity provision in that they can cancel and update limit orders very quickly when new information arrives, leaving them less exposed to adverse selection. For instance, Menkveld (2013) analyzes a large high-frequency trader who participates in NYSE-Euronext and Chi-X and who is character-

[23]Edwards, Harris and Piwowar (2007) and Bao, Pan and Wang (2011) find similar evidence, but their measure of market liquidity is different.

[24]When MMs randomize between acquiring information and not, each MM is uncertain about the information of her competitors. The trading equilibrium thus involves mixing on the part of all MMs.

ized as a *modern market maker*, in that it mainly supplies liquidity (4 out of 5 positions are passive) and that it makes money on the spread, but loses money on its positions.

As it turns out, the description of the modern market maker fits our MMs reasonably well in the following sense. (a) The MMs we describe are not MMs in the sense that they have an obligation to post competitive quotes: we allow them to post any quotes they want. Thus, in some sense, our MMs correspond to a LOB trader who restricts himself to posting passive limit orders. (b) If we accept the hypothesis that high-frequency traders are the modern market makers, then indeed there is a dichotomy between those who can make money from market making and those who cannot: if high-frequency trading technology is required to successfully pursue market making, then this activity is limited to a small number of specialized traders, due to the large investments required to become a high-frequency trader. Therefore, our model may serve as a description of a standard dealer market which also works reasonably well as a first approximation of a LOB market with high-frequency trading market makers.

Appendix A: Proofs of Sections 3 and 4

Proof of Lemma 1: Follows from Proposition 4, in Online Appendix C.

■

Proof of Lemma 2: This is part 6 of Lemma 5, in Online Appendix C.

■

Proof of Theorem 1: Consider $p, q \in (0,1)$ (the other cases are similar). The arguments in the proof of Proposition 4 establish that a trading equilibrium exists, is unique, and satisfies: (i) $\sigma_1 = \sigma_2 = \sigma$, $\underline{\sigma}_1 = \underline{\sigma}_2 = \underline{\sigma}$ and $\overline{\sigma}_1 = \overline{\sigma}_2 = \overline{\sigma}$; (ii) $\underline{\sigma}(0) = 1$; (iii) σ and $\overline{\sigma}$ are atomless, with supp $(\sigma) = [0, l]$ and supp $(\overline{\sigma}) = [l, u]$; (iv) the speculator sells (resp. buys) with probability 1 when she is informed and $V = 0$ (resp. $V = 1$) and abstains when she is uninformed. Thus, in equilibrium,

$$\gamma := \mathbb{P}(V = 0 | \text{ sell}) = \frac{\frac{\pi q}{2} + \frac{1-\pi}{4}}{\frac{\pi q}{2} + \frac{1-\pi}{2}}. \tag{6}$$

Furthermore, (34) and $\sigma(l) = 1$ give

$$l = \frac{(1-\gamma)(1-p)}{\gamma + (1-\gamma)(1-p)}. \tag{7}$$

Next, let $\overline{\Pi}_n(l|\text{sell})$ denote MMnH's equilibrium expected profit from $b_n = l$ conditional on a sell order; (35) gives

$$\overline{\Pi}_n(l|\text{sell}) = \frac{\gamma l}{1-\gamma}. \tag{8}$$

Combining (7), (8) and the symmetry between the bid and ask sides of the market then yields[25]

$$\overline{\Pi}_n = \left(\frac{1-\pi}{2}\right)\left(\frac{\gamma(1-p)}{\gamma + (1-\gamma)(1-p)}\right). \tag{9}$$

By (6), γ increases with q. So, by (9), $\overline{\Pi}_n$ is increasing in q and decreasing in p. Since $\Pi_n = 0$, these observations establish part 2 of the theorem.

We next prove part 1 of the theorem. Since the speculator abstains when she is uninformed, $\Pi_S = 0$. So our goal is to show that $\overline{\Pi}_S$ is decreasing in p and q. Observe that, by symmetry

[25] MMnH makes expected profit given by (8) whenever the trader is hit by a liquidity shock and sells the asset. By symmetry, MMnL makes expected profit given by (8) whenever the trader is hit by a liquidity shock and buys the asset.

of the bid and ask sides of the market,

$$\overline{\Pi}_S = \int b \, dF(b), \tag{10}$$

where $F(b) := \mathbb{P}(\hat{b} \leq b | V = 0)$. As $\underline{\sigma}(0) = 1$, we can write

$$F(b) = (1-p)^2 \sigma^2(b) + 2p(1-p)\sigma(b) + p^2. \tag{11}$$

We proceed to show that an increase in either p or q induces an inverse first-order stochastic dominance shift of F. Pick an arbitrary $b \in (0, l)$. First, rearranging (34) yields

$$\sigma(b) = \frac{\gamma p b}{(1-p)[(1-\gamma)(1-b) - \gamma b]}. \tag{12}$$

So $\sigma(b)$ is increasing in p. Moreover, as $\sigma(b)$ is increasing in γ which itself is increasing in q, we conclude that $\sigma(b)$ is increasing in q as well as in p. Now, differentiating (11) with respect to p and then with respect to q gives

$$\frac{dF(b)}{dp} = 2(1-p)^2 \sigma(b) \frac{d\sigma(b)}{dp} + 2p(1-p)\frac{d\sigma(b)}{dp} + 2[(1-\sigma(b))p + (1-p)(\sigma(b) - \sigma(b)^2)],$$

$$\frac{dF(b)}{dq} = 2(1-p)^2 \sigma(b) \frac{d\sigma(b)}{dq} + 2p(1-p)\frac{d\sigma(b)}{dq}.$$

So $\frac{d\sigma(b)}{dp} > 0$ implies $\frac{dF(b)}{dp} > 0$, while $\frac{d\sigma(b)}{dq} > 0$ implies $\frac{dF(b)}{dq} > 0$. An increase in either p or q therefore induces an inverse first-order stochastic dominance shift of F. Equation (10) finishes to show that $\overline{\Pi}_S$ is decreasing in p and q.

■

Proof of Theorem 2:

Step 1: the equilibrium expected profit functions of the trading game (that is, $\overline{\Pi}_n$, $\overline{\Pi}_S$, Π_n, Π_S) are all continuous in p and q.

We have $\Pi_n = \Pi_S = 0$, $\overline{\Pi}_n$ given by (9), and $\overline{\Pi}_S$ given by (10), with (11) giving F and (12) giving σ. Step 1 ensues.

Step 2: there exists an equilibrium.

Define for $i = n, S$ the set-valued functions

$$\psi_i(p,q) := \begin{cases} \{0\} & \text{if} \quad \overline{\Pi}_i(p,q) - c < \Pi_i(p,q); \\ [0,1] & \text{if} \quad \overline{\Pi}_i(p,q) - c = \Pi_i(p,q); \\ \{1\} & \text{if} \quad \overline{\Pi}_i(p,q) - c > \Pi_i(p,q). \end{cases}$$

For all $(p,q) \in [0,1] \times [0,1]$, $\psi_i(p,q)$ is convex. Next, consider sequences $\{p_k\}_{k\in\mathbb{N}}$ and $\{q_k\}_{k\in\mathbb{N}}$ converging, respectively, to $p^\dagger$ and $q^\dagger$. Suppose the sequence $\{u_k\}_{k\in\mathbb{N}}$ converges to u and satisfies $u_k \in \psi_i(p_k, q_k)$ for all $k \in \mathbb{N}$. If $\psi_i(p^\dagger, q^\dagger) = [0,1]$ then $u \in \psi_i(p^\dagger, q^\dagger)$ is immediate. By Step 1, Π_i and $\overline{\Pi}_i$ are both continuous in p and q. Therefore $\psi_i(p^\dagger, q^\dagger) = \{0\}$ implies $\psi_i(p_k, q_k) = \{0\}$ for all sufficiently large k, and $\psi_i(p^\dagger, q^\dagger) = \{1\}$ implies $\psi_i(p_k, q_k) = \{1\}$ for all sufficiently large k. This shows that ψ_i has closed graph. We may therefore apply Kakutani's fixed point theorem to the correspondence $\psi_n \times \psi_S$. By construction, if $(p,q) \in \big(\psi_n(p,q), \psi_S(p,q)\big)$, an equilibrium exists in which MMs acquire information with probability p while the speculator acquires information with probability q.

Step 3: any equilibrium is a WELM equilibrium.

It follows from Lemma 2 that any equilibrium has to be such that both MMs acquire information with the same probability. The other properties are immediate from the arguments in the proof of Proposition 4, in Online Appendix C.

Step 4: the equilibrium information acquisition probabilities p and q are uniquely determined.

Assume that an equilibrium exists in which MMs acquire information with probability $p \in (0,1)$ while the speculator acquires information with probability $q \in (0,1)$ (the other cases are similar). Suppose by way of contradiction that an equilibrium exists with information acquisition probabilities p' and q', where either $p' \neq p$ or $q' \neq q$. If $p' = p$ then either $q' > q$ or $q' < q$. If $q' > q$ then, by Theorem 1, $\overline{\Pi}_S(p', q') - \Pi_S(p', q') < \overline{\Pi}_S(p,q) - \Pi_S(p,q) = c$, contradicting $q' > 0$. If $q' < q$, then $\overline{\Pi}_S(p', q') - \Pi_S(p', q') > \overline{\Pi}_S(p,q) - \Pi_S(p,q) = c$, contradicting $q' < 1$. So $p' = p$ is ruled out. Suppose next that $p' > p$. Then, $\overline{\Pi}_n(p', q) - \Pi_n(p', q) < c$. As $p' > 0$, Theorem 1 implies $q' > q$. But then, $\overline{\Pi}_S(p', q') - \Pi_S(p', q') < \overline{\Pi}_S(p,q) - \Pi_S(p,q) = c$, contradicting $q' > 0$. So $p' > p$ is ruled out as well. Lastly, suppose that $p' < p$. Then, $\overline{\Pi}_n(p',q) - \Pi_n(p',q) > c$. As $p' < 1$, we thus obtain $q' < q$. But then, $\overline{\Pi}_S(p',q') - \Pi_S(p',q') > \overline{\Pi}_S(p,q) - \Pi_S(p,q) = c$, contradicting $q' < 1$. This rules out $p' < p$ and consequently, the existence of an equilibrium with information acquisition probabilities p' and q' different from p and q.

Step 5: in equilibrium, p is non-increasing in c.

Consider $c_a < c_b$. Let p_a and q_a (resp. p_b and q_b) denote the equilibrium information acquisition probabilities given $c = c_a$ (resp. $c = c_b$). Suppose by way of contradiction that $p_b > p_a$. Then

$$\overline{\Pi}_n(p_a,q_a) - \Pi_n(p_a,q_a) \leq c_a < c_b \leq \overline{\Pi}_n(p_b,q_b) - \Pi_n(p_b,q_b).$$

Using Theorem 1, this in turn implies $q_b > q_a$. But then,

$$\overline{\Pi}_S(p_a,q_a) - \Pi_S(p_a,q_a) \geq \overline{\Pi}_S(p_b,q_b) - \pi_S(p_b,q_b) \geq c_b > c_a,$$

contradicting $q_a < 1$.

Step 6: in equilibrium, $p \to 1$ as $c \to 0$.

This step is immediate from the arguments given in the text below the statement of Theorem 2.

Step 7: in equilibrium, $q = 0$ for all sufficiently small c.

Notice first that $p = 1$ implies $\overline{\Pi}_n = 0$. Hence, in equilibrium, $p < 1$. This remark combined with Step 6 combined implies that, in equilibrium, $\overline{\Pi}_n - \Pi_n = c$ for all sufficiently small c and, as $\Pi_n = 0$, $\overline{\Pi}_n(p,q) = c$, with $\overline{\Pi}_n(p,q)$ given by (9). Now, by part 2 of Theorem 1, $\overline{\Pi}_n(p,q)$ is decreasing in p but increasing in q. So, for all sufficiently small c, in equilibrium: $p \geq p^*(c)$, where $p^*(c)$ is defined implicitly by $\overline{\Pi}_n(p^*(c),0) = c$. Using (9) yields

$$p^*(c) = \frac{1 - \pi - 4c}{1 - \pi - 2c}. \tag{13}$$

Next, the remark that $p \geq p^*(c)$ at sufficiently small c combined with part 1 of Theorem 1 shows that, in equilibrium, for all sufficiently small c:

$$\overline{\Pi}_S(p,q) - \Pi_S(p,q) \leq \overline{\Pi}_S(p^*(c),0) - \Pi_S(p^*(c),0) = \overline{\Pi}_S(p^*(c),0).$$

We then obtain, using (10), (11) and $\sigma(l) = 1$,

$$\begin{aligned}\overline{\Pi}_S(p,q) - \Pi_S(p,q) &\leq [2p(1-p^*(c)) + (1-p^*(c))^2]l(p^*(c),0) \\ &= (1-p^*(c))(2p^*(c)+1)l(p^*(c),0) \\ &\leq 3(1-p^*(c))l(p^*(c),0). \end{aligned} \tag{14}$$

with l given by (7). Substituting (13) into (7) yields

$$l(p^*(c),0) = \frac{2c}{1-\pi}. \tag{15}$$

Combining (13), (14) and (15) shows that for all sufficiently small c, in equilibrium:

$$\overline{\Pi}_S(p,q) - \Pi_S(p,q) \leq \frac{12c^2}{(1-\pi)(1-\pi-2c)}.$$

Hence, in equilibrium, $\overline{\Pi}_S(p,q) - \Pi_S(p,q) < c$ for all c sufficiently small, concluding the proof of Step 7.

<u>Step 8: there exist $0 < \underline{c} < \overline{c} < \frac{1}{2}$ such that, in equilibrium, $q = 0$ if and only if $c \leq \underline{c}$ and $p = 0$ if and only if $c \geq \overline{c}$.</u>

Follows from Lemmata 3 and 4 below.

■

Lemma 3. *Let $c \in (0, \frac{1}{2})$. Then, in equilibrium, the following are equivalent:*

(i) $p = 0$;

(ii) $p = 0 < q$;

(iii) $c \geq \overline{c}(\pi) = \frac{1-\pi^2}{4}$.

Proof: Fix $p = 0$ and $q = 1$, and consider the resulting trading game. With both MMs uninformed, Bertrand competition yields $\hat{b} = \mathbb{E}[V|\text{sell}] = \frac{1-\pi}{2} = 1 - \hat{a}$. Hence,

$$\overline{\Pi}_S(0,1) - \Pi_S(0,1) = \frac{1}{2}\hat{b} + \frac{1}{2}(1-\hat{a}) = \frac{1-\pi}{2}.$$

On the other hand,

$$\overline{\Pi}_n(0,1) - \Pi_n(0,1) = (1-\pi)\left(\frac{1}{4}(1-\hat{b}) + \frac{1}{4}\hat{a}\right) = \frac{1-\pi^2}{4}.$$

Therefore, for $c \in [\frac{1-\pi^2}{4}, \frac{1-\pi}{2}]$, the equilibrium information acquisition probabilities are $p = 0$ and $q = 1$. A similar argument establishes that for $c \in (\frac{1-\pi}{2}, \frac{1}{2})$ the equilibrium information acquisition probabilities are $p = 0$ and $q \in (0,1)$.

We next show that in equilibrium $p > 0$ for all $c < \frac{1-\pi^2}{4}$. Suppose $c < \frac{1-\pi^2}{4}$. First, notice that part 1 of Theorem 1 combined with the derivation of the previous paragraph yields $\overline{\Pi}_S(0,q) - \Pi_S(0,q) \geq \frac{1-\pi}{2}$ for all $q \in [0,1]$. So if in equilibrium $p = 0$ then $q = 1$. Yet we saw above that $\overline{\Pi}_n(0,1) - \Pi_n(0,1) = \frac{1-\pi^2}{4}$. The latter observation rules out the possibility of $p = 0$ in equilibrium.

■

Lemma 4. *Let $c \in (0, \frac{1}{2})$. Then there exists a monotone decreasing function $\underline{\pi}(\cdot)$ such that in equilibrium $q = 0$ if and only if $\pi \leq \underline{\pi}(c)$.*

Proof: Throughout the proof we restrict attention to $c < \frac{1}{2}$.

<u>Step 1: if in equilibrium $q = 0$ for a given value of π then in equilibrium $q = 0$ as well for all smaller values of π.</u>

Consider $\pi_b < \pi_a$. Let p_a and q_a (resp. p_b and q_b) denote the equilibrium information acquisition probabilities given $\pi = \pi_a$ (resp. $\pi = \pi_b$). Suppose that $q_a = 0$. We will show that $q_b = 0$ as well. First, (9) yields

$$\overline{\Pi}_n(p,0;\pi) - \Pi_n(p,0;\pi) = \left(\frac{1-\pi}{2}\right)\left(\frac{1-p}{2-p}\right). \tag{16}$$

By Lemma 3, $p_a > 0$. Since in equilibrium MMs always acquire information with probability less than 1, $p_a \in (0,1)$. Therefore, using $\overline{\Pi}_n(p_a,0;\pi_a) - \Pi_n(p_a,0;\pi_a) = c$ and solving for p_a gives

$$p_a = \frac{1-\pi_a-4c}{1-\pi_a-2c}.$$

Now define

$$p_b' := \frac{1-\pi_b-4c}{1-\pi_b-2c}.$$

Note that $p'_b > p_a$, since $\pi_b < \pi_a$. Moreover,

$$\left(\frac{1-\pi_b}{2}\right)\left(\frac{1-p'_b}{2-p'_b}\right) = c.$$

Thus, by (16),

$$\overline{\Pi}_n(p'_b, 0; \pi_b) - \Pi_n(p'_b, 0; \pi_b) = c. \tag{17}$$

On the other hand, observe that fixing $q = 0$, the speculator's expected profit functions $\overline{\Pi}_S$ and Π_S in the trading game are independent of π, since for $q = 0$ none of the price distributions depend on π. This remark, combined with part 1 of Theorem 1, yields

$$\overline{\Pi}_S(p'_b, 0; \pi_b) - \Pi_S(p'_b, 0; \pi_b) < \overline{\Pi}_S(p_a, 0; \pi_a) - \Pi_S(p_a, 0; \pi_a) \leq c. \tag{18}$$

It now follows from (17) and (18) that p'_b and 0 are the equilibrium information acquisition probabilities given $\pi = \pi_b$. This concludes the proof of Step 1.

In what follows, let

$$p^*(c, \pi) := \frac{1 - \pi - 4c}{1 - \pi - 2c}. \tag{19}$$

We also define

$$\underline{\pi}(c) := \max\big\{0, \sup\{\pi : q = 0 \text{ in equilibrium}\}\big\}.$$

Step 2: $\overline{\Pi}_n(p^*(c, \pi), 0) = c$.

Immediate from (9).

Step 3: $\underline{\pi}(c) > 0$ implies $\overline{\Pi}_S\big(p^*(c, \underline{\pi}(c)), 0\big) = c$.

Suppose $\underline{\pi}(c) > 0$. Then we can find a sequence $\{\pi_k\}_{k\in\mathbb{N}}$, with limit $\underline{\pi}(c)$, such that in equilibrium: $\pi = \pi_k$ implies $q = q_k = 0$. Hence, by Lemma 3, for all k sufficiently large, in equilibrium: $\pi = \pi_k$ implies $p = p_k > 0$. Therefore, $\overline{\Pi}_n(p_k, 0) = \overline{\Pi}_n(p_k, 0) - \Pi_n(p_k, 0) = c$ for all k sufficiently large. We conclude, using Step 2 and the monotonicity of $\overline{\Pi}_n$ with respect to p, that $p_k = p^*(c, \pi_k)$ for all k sufficiently large. As $\overline{\Pi}_S(p_k, 0) \leq c$ irrespective of k, we find that $\overline{\Pi}_S(p^*(c, \pi_k), 0) \leq c$ for all k sufficiently large. Now, we saw in Step 1 of the proof of Theorem 2 that $\overline{\Pi}_S$ is continuous in p. Moreover, (19) shows that p^* is continuous in π. Hence, $\overline{\Pi}_S\big(p^*(c, \underline{\pi}(c)), 0\big) \leq c$. A similar argument rules out $\overline{\Pi}_S\big(p^*(c, \underline{\pi}(c)), 0\big) > c$.

Step 4: $\underline{\pi}(c) > 0$ implies $\underline{\pi}(c) < 1 - 4c$.

Suppose $\underline{\pi}(c) > 0$. Reasoning as in Step 3 establishes that $p^*(c, \underline{\pi}(c)) \geq 0$. Hence, by (19), $\underline{\pi}(c) \leq 1-4c$. Next, suppose by way of contradiction that $\underline{\pi}(c) = 1-4c$. Then, $p^*(c, \underline{\pi}(c)) = 0$. Hence, by Step 3: $\overline{\Pi}_S(0,0) = c$. But that is impossible, since $\overline{\Pi}_S(0,0) = \frac{1}{2}$ whereas $c < \frac{1}{2}$.

Step 5: $\underline{\pi}(.)$ is continuous.

Step 5 follows from Step 3 and the remarks that $\overline{\Pi}_S$ is continuous in p while p^* is continuous in both of its arguments.

Step 6: $\underline{\pi}(c_a) > 0$ implies $\underline{\pi}(c_b) < \underline{\pi}(c_a)$ for all $c_b \in (c_a, \frac{1}{2})$.

Suppose by way of contradiction that there exist $0 < c_a < c_b < \frac{1}{2}$ with $0 < \underline{\pi}(c_a) \leq \underline{\pi}(c_b)$. Then, combining Steps 4 and 5 implies the existence of $0 < c_a < c_b' < \frac{1}{2}$ with $0 < \underline{\pi}(c_a) = \underline{\pi}(c_b')$. By Steps 4 and 5, the function

$$H(c) := \overline{\Pi}_S\big(p^*(c, \underline{\pi}(c_a)), 0\big) - c$$

thus crosses the horizontal axis at least twice within the open interval $\left(0, \frac{1-\underline{\pi}(c_a)}{4}\right)$, once at $c = c_a$ and once at $c = c_b'$, contradicting Lemma 6 in Online Appendix C.

■

Proof of Proposition 1: Combining Lemmata 3 and 4 establishes the main part of the proposition. We prove the remaining parts below.

Claim: the equilibrium q tends to 0 as π tends to 1.

Suppose by way of contradiction that the equilibrium probability q with which the speculator acquires information does not tend to 0 as π tends to 1. Then we can find $\varepsilon > 0$ and a sequence $\{\pi_k\}_{k\in\mathbb{N}}$ with limit 1 such that each element in the equilibrium sequence q_k is greater than ε. Therefore, by (6), γ_k converges to 1 and, by (7), l_k converges to 0. Yet, by symmetry of the bid and ask sides of the market:

$$\overline{\Pi}_S(p_k, q_k) - \Pi_S(p_k, q_k) = \overline{\Pi}_S(p_k, q_k) \leq l_k.$$

Therefore, $\overline{\Pi}_S(p_k, q_k) - \Pi_S(p_k, q_k) < c$ for all sufficiently large k, contradicting $q_k > \varepsilon$.

Claim: the equilibrium probability p with which a MM acquires information is non-increasing in π.

First, using (9) gives us

$$\frac{\partial^2 \overline{\Pi}_n}{\partial \pi^2} = -\frac{4(1-p)^2 q^2}{[(2-p)(1-\pi)+2\pi q]^3} \leq 0.$$

On the other hand,

$$\left.\frac{\partial \overline{\Pi}_n}{\partial \pi}\right|_{\pi=0} = -\frac{(1-p)[2-2q+p(2q-1)]}{2(2-p)^2} \leq 0.$$

We therefore obtain

$$\frac{\partial \overline{\Pi}_n}{\partial \pi} \leq 0. \tag{20}$$

Next, by (12), $\frac{\partial \sigma(b)}{\partial \gamma} \geq 0$. As $\frac{\partial \gamma}{\partial \pi} \geq 0$, we obtain $\frac{\partial \sigma(b)}{\partial \pi} \geq 0$. This implies, by (10) and (11), that

$$\frac{\partial \overline{\Pi}_S}{\partial \pi} \leq 0. \tag{21}$$

Now consider $\pi_b > \pi_a$. Let p_a and q_a (resp. p_b and q_b) denote the equilibrium information acquisition probabilities given $\pi = \pi_a$ (resp. $\pi = \pi_b$). Suppose by way of contradiction that $p_b > p_a$. Then

$$\overline{\Pi}_n(p_b, q_b; \pi_b) \geq c \geq \overline{\Pi}_n(p_a, q_a; \pi_a).$$

As $\pi_b > \pi_a$ and $p_b > p_a$, we conclude by (20) and part 2 of Theorem 1 that $q_b > q_a$. But then (21) and part 1 of Theorem 1 give

$$\overline{\Pi}_S(p_b, q_b; \pi_b) < \overline{\Pi}_S(p_a, q_a; \pi_a),$$

contradicting $q_b > q_a$.

■

Appendix B: Proofs of Section 6

Proof of Proposition 2: By Proposition 5, in any WELM trading equilibrium:

$$\overline{\Pi}_n(p,q;z) = \left(\frac{(1-\pi)(1-p)}{2}\right)\frac{1-\pi(1-2q)+2\pi p(1-q)z}{2-p-2\pi(1-q)+\pi p(1+2(1-q)z)}, \tag{22}$$

and

$$\sigma(b) = \frac{(1+\pi(2z-1)+2\pi q(1-z))pb}{(1-p)(1-\pi-2b(1-\pi(1-q)))}, \qquad \forall b \in [0,l]. \tag{23}$$

Differentiating (22) with respect to z gives

$$\frac{\partial\overline{\Pi}_n}{\partial z} = \frac{(1-p)^2 p(1-\pi)^2\pi(1-q)}{(2-p-2\pi(1-q)+\pi p(1+2(1-q)z))^2} > 0.$$

As $\Pi_n(p,q;z) \equiv 0$, MMn's gain from becoming informed therefore increases in z.

Next, in any WELM trading equilibrium:

$$\overline{\Pi}_S(p,q;z) = 2p(1-p)\int_0^l b\,d\sigma(b) + (1-p)^2\int_0^l b\,d\sigma^2(b).$$

If both MMs are informed then the informed speculator makes zero profit. With probability $2(1-p)p$ one MM is informed and the other is uninformed. In this case, by symmetry of the bid and ask sides of the market, the informed speculator's expected profit equals the expected bid of the uninformed MM. With probability $(1-p)^2$ both MMs are uninformed. In this case, by symmetry of the bid and ask sides of the market, the informed speculator's expected profit equals the expected minimum bid of the uninformed MMs. On the other hand,

$$\Pi_S(p,q;z) = 2p(1-p)z\int_0^l b\,d\sigma(b).$$

If the speculator is uninformed, MMn is informed, quotes are observable and MMm is uninformed then the speculator learns v from MMn's quotes and makes profit from trading with MMm. Thus, the speculator's gain from becoming informed is

$$\overline{\Pi}_S(p,q;z) - \Pi_S(p,q;z) = 2(1-z)p(1-p)\int_0^l b\,d\sigma(b) + (1-p)^2\int_0^l b\,d\sigma^2(b). \tag{24}$$

Now, by (23), $\sigma(b)$ is increasing in z. So the right-hand side of (24) decreases in z. ■

Proposition 3. *Let $z = 1$. There exists $\hat{p} < 1$, independent of q, such that, for all $p > \hat{p}$, $\overline{\Pi}_n - \Pi_n$ is decreasing in p. There exists $\hat{q} < 1$ such that, for all $q > \hat{q}$, $\overline{\Pi}_n - \Pi_n$ is decreasing in p. If $\pi > \frac{1}{3}$ then, for p and q sufficiently small, $\overline{\Pi}_n - \Pi_n$ is increasing in p.*

Proof: Taking $z = 1$ and differentiating (22) with respect to p gives

$$\frac{\partial \overline{\Pi}_n}{\partial p} = \frac{(1-\pi)A(p,q)}{2(2\pi(p-1)q - 3\pi p + p + 2\pi - 2)^2},$$

where $A(0,0) = (1-\pi)(3\pi - 1)$, and $A(1,q) = A(p,1) = -(1+\pi)^2$.

■

Proof of Theorem 3: In any WELM trading equilibrium, $\overline{\Pi}_n$ is given by (22). Taking the derivative with respect to q:

$$\frac{\partial \overline{\Pi}_n}{\partial q} = \frac{(1-p)^2(1-\pi)^2\pi(1-zp)}{(2(1-\pi(1-q)) + 2\pi pz(1-q) - p(1-\pi))^2} > 0.$$

As $\Pi_n \equiv 0$, MMn's gain from becoming informed therefore increases in q. Next, by (23), $\sigma(b)$ is increasing in q. So the right-hand side of (24) decreases in q. The comparative statics with respect to p follow from Proposition 3.

■

Proof of Theorem 4: Suppose $z = 1$ (the proof for the case $z \in (0,1)$ is almost identical). In the rest of the proof, let p_0 (resp. q_0) denote MMs' (resp. the speculator's) unique equilibrium information acquisition probability in any equilibrium with $z = 0$. Let $\underline{c}$ and $\overline{c}$ denote the cutoffs from Theorem 2.

<u>Step 1: there exists $c^\dagger$ such that, for all $c < c^\dagger$, a WELM equilibrium exists and satisfies $p \geq p_0$ and $q = 0$.</u>

Recall: (a) p_0 tends to 1 as c tends to 0, and (b) by Proposition 7, a WELM trading equilibrium exists for all $p \geq \frac{\sqrt{2\pi}}{\sqrt{2\pi}+\sqrt{1-\pi}}$. Pick $\tilde{c} > 0$ such that $p_0 > \frac{\sqrt{2\pi}}{\sqrt{2\pi}+\sqrt{1-\pi}}$ for all $c < \tilde{c}$. Consider $c < \min\{\tilde{c}, \underline{c}\}$. Then, using Proposition 2,

$$\overline{\Pi}_n(p_0, 0; 1) - \Pi_n(p_0, 0; 1) > \overline{\Pi}_n(p_0, 0; 0) - \Pi_n(p_0, 0; 0) \geq c.$$

Next, as $\overline{\Pi}_n(1,0;1) - \Pi_n(1,0;1) = 0$, the intermediate value theorem gives $p^* > p_0$ solving

$$\overline{\Pi}_n(p^*,0;1) - \Pi_n(p^*,0;1) = c.$$

As $p^* > p_0$, Proposition 2 and Theorem 3 give

$$\overline{\Pi}_S(p^*,0;1) - \Pi_S(p^*,0;1) < \overline{\Pi}_S(p_0,0;0) - \Pi_S(p_0,0;0) \leq c.$$

A WELM equilibrium therefore exists, with $p = p^*$ and $q = 0$.

<u>Step 2: for all $\hat{p} < 1$, there exists $\hat{c} > 0$ such that, in any equilibrium, $p > \hat{p}$ whenever $c < \hat{c}$.</u>

As uninformed MMs never set bid prices above $\frac{1}{2}$ nor set ask prices below $\frac{1}{2}$, MMn's gain from being informed in the trading game is at least as large as $\frac{(1-p)(1-\pi)}{4}$. Thus, in equilibrium, $\frac{(1-p)(1-\pi)}{4} \leq c$.

<u>Step 3: there exists $\hat{c} > 0$ such that, in any WELM equilibrium, $p \geq p_0$ for all $c < \hat{c}$.</u>

Choose $\hat{p} < 1$ such that $\overline{\Pi}_n - \Pi_n$ is decreasing in p for all $p > \hat{p}$ (such a $\hat{p}$ exists, by Proposition 3). Now pick $\hat{c}$ such that, for all $c < \hat{c}$: (a) $q_0 = 0$ and (b) $p > \hat{p}$ in any WELM equilibrium (such a $\hat{c}$ exists, by virtue of Theorem 2 combined with Step 2 of the proof). Let $c < \hat{c}$ and suppose by way of contradiction that a WELM equilibrium exists satisfying $\hat{p} < p < p_0$. Then $\overline{\Pi}_n(p,q;1) - \Pi_n(p,q;1) \leq c$. We have on the other hand, $\overline{\Pi}_n(p_0,0;0) - \Pi_n(p_0,0;0) \geq c$. Yet, our choice of $\hat{p}$ combined with Proposition 2 and Theorem 3 implies

$$\overline{\Pi}_n(p,q;1) - \Pi_n(p,q;1) > \overline{\Pi}_n(p_0,0;0) - \Pi_n(p_0,0;0),$$

which clearly cannot be.

<u>Step 4: in any WELM equilibrium, $q = 0$ for all $c < \hat{c}$.</u>

Recall, we chose $\hat{c}$ such that $q_0 = 0$ for all $c < \hat{c}$. So Step 4 is immediate from Step 3 combined with the fact that, by Proposition 2 and Theorem 3, $\overline{\Pi}_S - \Pi_S$ is decreasing in p, q and z.

<u>Step 5: for $c \in \left(\frac{(1-\pi)(1+\pi)}{4}, \frac{1}{2}\right)$, a WELM equilibrium exists and satisfies $p = 0 < q$.</u>

Theorem 2 and the observation made in the text that $\overline{c} = \frac{(1-\pi)(1+\pi)}{4}$ give $p_0 = 0 < q_0$

for all $c \in \left(\frac{(1-\pi)(1+\pi)}{4}, \frac{1}{2}\right)$. Proposition 7 assures the existence of a WELM trading equilibrium whenever $p = 0$. The existence of a WELM equilibrium satisfying $p = p_0$, $q = q_0$, $\overline{\Pi}_n(p_0, q_0; 1) = \overline{\Pi}_n(p_0, q_0; 0)$, $\overline{\Pi}_S(p_0, q_0; 1) = \overline{\Pi}_S(p_0, q_0; 0)$, $\Pi_n(p_0, q_0; 1) = \Pi_n(p_0, q_0; 0)$, and $\Pi_S(p_0, q_0; 1) = \Pi_S(p_0, q_0; 0)$ is now straightforward to verify.

Step 6: for $c \in (\frac{1-\pi}{2}, \frac{1}{2})$, $p = 0 < q$ in any WELM equilibrium.

Recall, $c > \overline{c} = \frac{(1-\pi)(1+\pi)}{4}$ implies $p_0 = 0 < q_0$. Let $c \in (\frac{1-\pi}{2}, \frac{1}{2})$. In any WELM equilibrium, MMs' gain from acquiring information is bounded above by $\frac{1-\pi}{2}$. So $p = 0$ in any WELM equilibrium. Moreover, combining Theorems 2 and 3 implies that, for all $q < q_0$:

$$\overline{\Pi}_S(0, q; 1) - \Pi_S(0, q; 1) > \overline{\Pi}_S(0, q_0; 1) - \Pi_S(0, q_0; 1) = \overline{\Pi}_S(0, q_0; 0) - \Pi_S(0, q_0; 0) = c.$$

We conclude that $q \geq q_0$ in any WELM equilibrium.

■

Appendix C: Trading Game of Baseline Model (for online publication)

In this appendix we analyze the trading game induced by the baseline model. Specifically, throughout this appendix p_1, p_2 and q play the role of parameters: MMn is informed with probability p_n, and the speculator is informed with probability q. A strategy of MMn comprises cumulative distribution functions σ_n, $\underline{\sigma}_n$ and $\overline{\sigma}_n$ specifying respectively the distribution of the bid price b_n of MMnU, MMnL and MMnH. We assume in line with the baseline model that, conditional on MMnU, $1 - a_n$ is distributed like b_n. Similarly, we assume that the law of $1 - a_n$ conditional on MMnL (resp. MMnH) is the same as the law of b_n conditional on MMnH (resp. MMnL). A strategy of the speculator specifies her market order as a function of the information she possesses at that point.

The following notation will be used throughout:

- $\Pi_n(b|\text{sell})$ (respectively $\overline{\Pi}_n(b|\text{sell})$ and $\underline{\Pi}_n(b|\text{sell})$) denotes MM$n$U's (resp. MM$n$H's and MM$n$L's) expected profit conditional on a sell order, given $b_n = b$;
- $\sigma_n(b) := \mathbb{P}(b_n \leq b|\text{ MM}n\text{U})$, $\overline{\sigma}_n(b) := \mathbb{P}(b_n \leq b|\text{ MM}n\text{H})$ and $\underline{\sigma}_n(b) := \mathbb{P}(b_n \leq b|\text{ MM}n\text{L})$;
- $\Sigma_n := \text{supp}\,(\sigma_n)$ and $\overline{\Sigma}_n := \text{supp}\,(\overline{\sigma}_n)$;
- A_n (respectively $\overline{A}_n$) denotes the set of atoms in MMnU's (resp. MMnH's) strategy;
- $l_n := \sup \Sigma_n$;
- $\gamma := \mathbb{P}(V = 0|\text{sell})$.

Lemma 5. *If $p_1 = p_2 = 1$ then any trading equilibrium has $a_1 = a_2 = b_1 = b_2 = v$. If $p_1 = p_2 = 0$ then $a_1 = a_2 = \frac{1-\pi(1-2q)}{2-2\pi(1-q)}$ and $b_1 = b_2 = \frac{1-\pi}{2-2\pi(1-q)}$. Otherwise, any trading equilibrium satisfies the following properties:*

1. $\underline{\sigma}_1(0) = \underline{\sigma}_2(0) = 1$;
2. $\Sigma_1 \cup \overline{\Sigma}_1 = \Sigma_2 \cup \overline{\Sigma}_2 = [0, u]$, *where* $u \in (0, 1)$;
3. $A_1 \cup \overline{A}_1 \cup A_2 \cup \overline{A}_2 \subseteq \{0\}$;
4. *if* $p_m \in (0, 1)$ *then* $\Sigma_m \cap \overline{\Sigma}_m = \{l_m\}$ *and* $l_m < u$, *with* $l_m > 0$ *if and only if* $p_n < 1$;

5. *if* $1 > p_n \geq p_m > 0$ *then* $\mathbb{E}[V|sell] > l_m \geq l_n > 0$*, with* $l_m > l_n$ *if* $p_n > p_m$*;*

6. *if* $p_n > p_m$ *then* $0 = \underline{\Pi}_m < \underline{\Pi}_n < \overline{\Pi}_n = \overline{\Pi}_m$*.*

Proof: The cases $p_1 = p_2 = 1$ and $p_1 = p_2 = 0$ are trivial. We prove below that, in any trading equilibrium, properties 1-6 hold in the case $\min\{p_1, p_2\} \in (0, 1)$, that is, when both market MMs acquire information with positive probability but neither of them becomes informed with probability 1; the proof for the case $p_m = 0 < p_n$ is similar.

Step 1: $\underline{\sigma}_1(0) = \underline{\sigma}_2(0) = 1$. Suppose by way of contradiction that $\underline{\sigma}_1(0) < 1$. Then we can find $b' > 0$ with $b' \in \arg\max_b \underline{\Pi}_1(b|\text{sell})$ and $\mathbb{P}(b_1 \geq b'|\text{ MM1L}) > 0$. The previous remarks imply $\mathbb{P}(b_1 = b' \text{ wins}|V = 0) = 0$, for otherwise $\underline{\Pi}_1(b'|V = 0) = -\mathbb{P}(b_1 = b' \text{ wins}|V = 0)b' < 0 = \underline{\Pi}_1(0|V = 0)$. Next, $\mathbb{P}(b_1 = b' \text{ wins}|V = 0) = 0$ implies the existence of $b'' \geq b'$ with $b'' \in \arg\max_b \underline{\Pi}_2(b|\text{sell})$ and $\mathbb{P}(b_2 = b'' \text{ wins}|V = 0) > 0$. We therefore obtain $\underline{\Pi}_2(b''|\text{sell}) < 0 = \underline{\Pi}_2(0|\text{sell})$, which cannot be.

Step 2: $p_n \in (0, 1) \Rightarrow l_n \leq \inf \overline{\Sigma}_n$. Suppose by way of contradiction that $l_n > \inf \overline{\Sigma}_n$. Then we can find $b'' > b'$ with $b'' \in \arg\max_b \underline{\Pi}_n(b|\text{sell})$ and $b' \in \arg\max_b \overline{\Pi}_n(b|\text{sell})$. Next,

$$\begin{aligned}
\underline{\Pi}_n(b''|\text{sell}) &= -\gamma\mathbb{P}(b_n = b'' \text{ wins}|V = 0)b'' + (1-\gamma)\mathbb{P}(b_n = b'' \text{ wins}|V = 1)(1 - b'') \\
&= -\gamma\mathbb{P}(b_n = b'' \text{ wins}|V = 0)b'' + (1-\gamma)\overline{\Pi}_n(b''|\text{sell}) \\
&< -\gamma\mathbb{P}(b_n = b'' \text{ wins}|V = 0)b' + (1-\gamma)\overline{\Pi}_n(b''|\text{sell}) \\
&\leq -\gamma\mathbb{P}(b_n = b' \text{ wins}|V = 0)b' + (1-\gamma)\overline{\Pi}_n(b'|\text{sell}) \\
&= \underline{\Pi}_n(b'|\text{sell}).
\end{aligned}$$

The last inequality holds since $b' < b''$ and $\mathbb{P}(b_n \text{ wins}|V = 0)$ is non-decreasing in b_n. Thus $\underline{\Pi}_n(b'|\text{sell}) > \underline{\Pi}_n(b''|\text{sell}) = \max_b \underline{\Pi}_n(b|\text{sell})$, which cannot be.

Step 3: $0 < \sup \overline{\Sigma}_1 = \sup \overline{\Sigma}_2 < 1$. We start by showing that $\sup \overline{\Sigma}_1 = \sup \overline{\Sigma}_2$. Suppose by way of contradiction that this is not the case, say $u_n > u_m$, where $u_n = \sup \overline{\Sigma}_n$ and $u_m = \sup \overline{\Sigma}_m$. Then, since increasing the bid beyond $u_m + \varepsilon$ does not increase the winning probability for n, $\exists\, \varepsilon > 0$ such that

$$\overline{\Pi}_n(u_m + \epsilon|\text{sell}) > \overline{\Pi}_n(u_n - x|\text{sell}), \quad \forall x \in [0, \varepsilon],$$

contradicting $u_n \in \overline{\Sigma}_n$. Hence, $u_n = u_m$. Next, let u denote the common supremum; we claim that $u \in (0,1)$. Suppose by way of contradiction that $u = 0$. One of the two MMs does not win with probability 1 conditional on a tie at 0, say $\mathbb{P}(b_n = 0 \text{ wins}|b_n = b_m = 0) < 1$. Then bidding slightly above zero yields MMnH strictly larger expected profit than $b_n = 0$, $\overline{\Pi}_n(\epsilon|\text{sell}) > \overline{\Pi}_n(0|\text{sell})$, contradicting $u = 0$. Next, suppose by way of contradiction that $u = 1$. Then $\max_b \overline{\Pi}_n(b|\text{sell}) = 0$, for $n = 1, 2$. However, $\min\{p_1, p_2\} \in (0,1)$. Say $p_m < 1$; then $l_m \leq \mathbb{E}[V] = \frac{1}{2}$. Therefore, $\overline{\Pi}_n(\frac{3}{4}|\text{sell}) = \frac{1}{4}(1-p_m) > 0$, contradicting $\max_b \overline{\Pi}_n(b|\text{sell}) = 0$.

<u>*Step 4:* $p_n = 1 \Rightarrow l_m = 0$; $\max\{p_1, p_2\} < 1 \Rightarrow \max\{l_1, l_2\} < \mathbb{E}[V|\text{sell}]$.</u> The first part is trivial; we prove the second part. By Step 1, for both MMs and given any bid b, the probability of winning a sell order is maximized under $V = 0$. Hence, for all b,

$$\mathbb{E}[V|\text{sell}, b_n = b \text{ wins}] \leq \mathbb{E}[V|\text{sell}].$$

This implies, in turn, $l_n \leq \mathbb{E}[V|\text{sell}]$, otherwise MM$n$U could profitably deviate to $b_n = 0$. Now suppose $\max\{p_1, p_2\} < 1$ and, by way of contradiction, that $l_m = \mathbb{E}[V|\text{sell}]$. We consider two cases, $u = \mathbb{E}[V|\text{sell}]$ (Case 1) and $u > \mathbb{E}[V|\text{sell}]$ (Case 2). In Case 1, Step 2 gives $u = \mathbb{E}[V|\text{sell}] \in \overline{A}_m$. But then bidding slightly above u yields MMnH strictly larger expected profit than $b_n = u$: $\exists\, \varepsilon > 0$ such that

$$\overline{\Pi}_n(u + \epsilon|\text{sell}) > \overline{\Pi}_n(u - x|\text{sell}), \quad \forall x \in [0, \varepsilon],$$

contradicting $u = \sup \overline{\Sigma}_n$. Consider next Case 2. Note that in this case, by virtue of Steps 1 and 2, there exists $\delta > 0$ with

$$\mathbb{P}(b_m = b \text{ wins}|V = 1) < \mathbb{P}(b_m = b \text{ wins}|V = 0) - \delta, \quad \forall b \leq \mathbb{E}[V|\text{sell}].$$

Thus, $\exists\, \delta' > 0$ such that

$$\mathbb{E}[V|\text{sell}, b_m = b \text{ wins}] < \mathbb{E}[V|\text{sell}] - \delta', \quad \forall b \leq \mathbb{E}[V|\text{sell}].$$

We therefore obtain

$$\Pi_m(b|\text{sell}) = \mathbb{P}(b_m = b \text{ wins})\big(\mathbb{E}[V|\text{sell}, b_m = b \text{ wins}] - b\big) < 0, \quad \forall b \in \big[\mathbb{E}[V|\text{sell}] - \delta', \mathbb{E}[V|\text{sell}]\big],$$

giving $l_m \leq \mathbb{E}[V|\text{sell}] - \delta'$.

Step 5: $\left(A_1 \cup \overline{A}_1\right) \cap \left(A_2 \cup \overline{A}_2\right) = \emptyset$. That $\overline{A}_n \cap \left(A_m \cup \overline{A}_m\right) = \emptyset$ is trivial. Next, suppose by way of contradiction that we can find $b \in \left(A_1 \cap A_2\right) \setminus \left(\overline{A}_1 \cup \overline{A}_2\right)$. Then $b < \mathbb{E}[V|\text{sell}]$, by virtue of Step 4. Let $\Delta = \mathbb{E}[V|\text{sell}] - b$, and consider n such that $\mathbb{P}(\text{MM}n \text{ wins}|\text{tie at } b) < 1$. Then,

$$\begin{aligned}
&\Pi_n(b+\varepsilon\Delta|\text{sell}) - \Pi_n(b|\text{sell}) \\
&= \mathbb{P}(b_m = b)\Big(\Pi_n(b+\varepsilon\Delta|\text{sell}, b_m = b) - \Pi_n(b|\text{sell}, b_m = b)\Big) \\
&+ (1-\mathbb{P}(b_m = b))\Big(\Pi_n(b+\varepsilon\Delta|\text{sell}, b_m \neq b) - \Pi_n(b|\text{sell}, b_m \neq b)\Big) \\
&= \mathbb{P}(b_m = b)\Big(\mathbb{E}[V|\text{sell}] - b - \varepsilon\Delta - \mathbb{P}(\text{MM}n \text{ wins}|\text{tie at } b)\Delta\Big) \\
&+ (1-\mathbb{P}(b_m = b))\Big(\Pi_n(b+\varepsilon\Delta|\text{sell}, b_m \neq b) - \Pi_n(b|\text{sell}, b_m \neq b)\Big) \\
&= \mathbb{P}(b_m = b)\Big((1-\varepsilon)\Delta - \mathbb{P}(\text{MM}n \text{ wins}|\text{tie at } b)\Delta\Big) \\
&+ (1-\mathbb{P}(b_m = b))\Big(\Pi_n(b+\varepsilon\Delta|\text{sell}, b_m \neq b) - \Pi_n(b|\text{sell}, b_m \neq b)\Big).
\end{aligned}$$

As $\lim_{\varepsilon\to 0}\Big(\Pi_n(b+\varepsilon\Delta|\text{sell}, b_m \neq b) - \Pi_n(b|\text{sell}, b_m \neq b)\Big) = 0$, we obtain

$$\lim_{\varepsilon\to 0}\Big(\Pi_n(b+\varepsilon\Delta|\text{sell}) - \Pi_n(b|\text{sell})\Big) = \Big(1 - \mathbb{P}(\text{MM}n \text{ wins}|\text{tie at } b)\Big)\Delta > 0,$$

contradicting $b \in A_n$.

Step 6: $\inf\left\{\Sigma_1 \cup \overline{\Sigma}_1\right\} = \inf\left\{\Sigma_2 \cup \overline{\Sigma}_2\right\}$. Assume $\max\{p_1, p_2\} < 1$ (other cases are similar), so that, by Step 2, $\inf\left\{\Sigma_1 \cup \overline{\Sigma}_1\right\} = \inf \Sigma_1$ and $\inf\left\{\Sigma_2 \cup \overline{\Sigma}_2\right\} = \inf \Sigma_2$. Suppose by way of contradiction that $b = \inf \Sigma_n > \inf \Sigma_m = b'$. Then $b \in \left(A_m \cup \overline{A}_m\right)$, otherwise MM$n$ could benefit from bidding below b and we could find $\varepsilon > 0$ such that

$$\Pi_n(b-\varepsilon|\text{sell}) > \Pi_n(b+x|\text{sell}), \quad \forall x \in [0, \varepsilon],$$

contradicting $b = \inf \Sigma_n$. Applying Step 5 thus yields $b \notin \left(A_n \cup \overline{A}_n\right)$. Next, $b = \inf \Sigma_n$ together with $b \notin A_n$ implies $\sigma_n(b) = 0$. Therefore, using Steps 1 and 2, $b \in A_m$ would imply $\Pi_m(b|\text{sell}) = -\gamma p_n b < 0$, which cannot be. Similarly, $b \in \overline{A}_m$ would imply $\overline{\Pi}_m(b|\text{sell}) = 0$, which cannot be since, by virtue of Steps 2 and 3, $\max_b \overline{\Pi}_m(b|\text{sell}) > 0$.

Step 7: $0 \in \left(\Sigma_1 \cup \overline{\Sigma}_1\right) \cap \left(\Sigma_2 \cup \overline{\Sigma}_2\right)$. Assume $\max\{p_1, p_2\} < 1$ (other cases are similar), so that, by Step 2, $\inf\left\{\Sigma_1 \cup \overline{\Sigma}_1\right\} = \inf \Sigma_1$ and $\inf\left\{\Sigma_2 \cup \overline{\Sigma}_2\right\} = \inf \Sigma_2$. Let b denote the common infinimum uncovered in Step 6, and suppose for a contradiction that $b > 0$. By Step 5, one of the MMs does not have an atom at b. Consider n such that $b \notin \left(A_n \cup \overline{A}_n\right)$. Then, by Step 1,

$$\lim_{\varepsilon \to 0} \Pi_m(b + \varepsilon|\text{sell}) = -\gamma p_n b < 0,$$

contradicting $b \in \Sigma_m$.

Step 8: $\Sigma_1 \cup \overline{\Sigma}_1 = \Sigma_2 \cup \overline{\Sigma}_2$. Suppose by way of contradiction that there exists $b' \in \left(\Sigma_n \cup \overline{\Sigma}_n\right) \setminus \left(\Sigma_m \cup \overline{\Sigma}_m\right)$, say $b' \in \Sigma_n \setminus \left(\Sigma_m \cup \overline{\Sigma}_m\right)$ (the other case is similar). Then $b' \in \arg\max_b \Pi_n(b|\text{sell})$. Moreover, by Step 7, $b' > 0$, and we can find $\delta > 0$ such that $[b' - \delta, b' + \delta] \cap \left(\Sigma_m \cup \overline{\Sigma}_m\right) = \emptyset$. Hence MM$n$ can lower his bid at b' without decreasing his winning probability, giving $\Pi_n(b' - \delta|\text{sell}) > \Pi_n(b'|\text{sell}) = \max_b \Pi_n(b|\text{sell})$, which cannot be.

Step 9: $\Sigma_1 \cup \overline{\Sigma}_1 = \Sigma_2 \cup \overline{\Sigma}_2 = [0, u]$. By Steps 2, 3, 7 and 8 all that remains to be shown is that the common support is an interval. Suppose by way of contradiction that this is not the case. Then we can find $b'' > b'$, both in the common support, and such that $(b', b'') \cap \left(\Sigma_1 \cup \overline{\Sigma}_1\right) = \emptyset$. By Step 5, there exists n such that $b'' \notin \left(A_n \cup \overline{A}_n\right)$. Hence, $\exists\, \varepsilon > 0$ such that, $\forall x \in [0, \varepsilon]$, $\Pi_m(b'' - \varepsilon|\text{sell}) > \Pi_m(b'' + x|\text{sell})$ and $\overline{\Pi}_m(b'' - \varepsilon|\text{sell}) > \overline{\Pi}_m(b'' + x|\text{sell})$, contradicting $b'' \in \left(\Sigma_m \cup \overline{\Sigma}_m\right)$.

Step 10: $p_n \in (0, 1) \Rightarrow l_n < u$. Suppose by way of contradiction that $p_n \in (0, 1)$ and $l_n = u$. Then, by Step 2, $u \in \overline{A}_n$. Assume $\mathbb{P}(\text{MM}m \text{ wins}|\text{tie at } b) < 1$ (the other case is similar). Then there exists $\varepsilon > 0$ such that

$$\overline{\Pi}_m(u + \epsilon|\text{sell}) > \overline{\Pi}_m(u - x|\text{sell}), \quad \forall x \in [0, \varepsilon],$$

contradicting $u \in \overline{\Sigma}_m$.

Step 11: $A_1 \cup \overline{A}_1 \cup A_2 \cup \overline{A}_2 \subseteq \{0\}$. Suppose by way of contradiction that there exists $b \in \left(A_m \cup \overline{A}_m\right)$, with $b > 0$. Then, by virtue of Step 4, $\exists\, \varepsilon > 0$ such that, $\forall x \in (0, \varepsilon)$, $\overline{\Pi}_n(b + x|\text{sell}) > \overline{\Pi}_n(b - x|\text{sell})$ and $\Pi_n(b + x|\text{sell}) > \Pi_n(b - x|\text{sell})$. Thus $(b - \varepsilon, b) \cap \left(\Sigma_n \cup \overline{\Sigma}_n\right) = \emptyset$, contradicting Step 9.

<u>*Step 12:* $\max_b \overline{\Pi}_1(b|\text{sell}) = \max_b \overline{\Pi}_2(b|\text{sell})$.</u> The combination of Steps 2, 3 and 11 shows that $\max_b \overline{\Pi}_1(b|\text{sell}) = \overline{\Pi}_1(u|\text{sell}) = (1-u) = \overline{\Pi}_2(u|\text{sell}) = \max_b \overline{\Pi}_2(b|\text{sell})$.

<u>*Step 13:* $0 < p_m < p_n < 1 \Rightarrow 0 < l_n < l_m$.</u> Let $0 < p_m < p_n < 1$ and suppose by way of contradiction that $l_n \geq l_m$. Note first that $l_n > 0$, for otherwise $\{0\} \in A_n \cap A_m$, which Step 5 ruled out. Hence, by Step 11, neither MM has an atom at l_n. Steps 2 and 9 therefore yield $\max_b \overline{\Pi}_n(b|\text{sell}) = \overline{\Pi}_n(l_n|\text{sell})$ and $\max_b \overline{\Pi}_m(b|\text{sell}) = \overline{\Pi}_m(l_n|\text{sell})$. On the other hand, $\overline{\Pi}_n(l_n|\text{sell}) \geq (1-p_m)(1-l_n)$ and $\overline{\Pi}_m(l_n|\text{sell}) = (1-p_n)(1-l_n)$. As $p_n > p_m$, combining the previous remarks yields

$$\max_b \overline{\Pi}_n(b|\text{sell}) > \max_b \overline{\Pi}_m(b|\text{sell}),$$

contradicting Step 12. Therefore, $l_n < l_m$. We next show that $l_n > 0$. Suppose by way of contradiction that $l_n = 0$. Then $0 \in A_n$, and, applying Step 5, $0 \notin A_m$. We therefore obtain

$$\Pi_n(0|\text{sell}) = \max_b \Pi_n(b|\text{sell}) = 0 < \max_b \Pi_m(b|\text{sell}). \tag{25}$$

Yet, as $l_m > 0$, Steps 9, 11 and 12 give

$$\begin{aligned}
\Pi_n(l_m|\text{sell}) &= -\gamma l_m + (1-\gamma)\overline{\Pi}_n(l_m|\text{sell}) \\
&= -\gamma l_m + (1-\gamma)\overline{\Pi}_m(l_m|\text{sell}) \\
&= \Pi_m(l_m|\text{sell}),
\end{aligned}$$

contradicting (25).

<u>*Step 14:* $p_n > p_m \Rightarrow 0 = \Pi_m < \Pi_n < \overline{\Pi}_n = \overline{\Pi}_m$.</u> Assume $0 < p_m < p_n < 1$ (other cases are similar). By Step 12, $\overline{\Pi}_n = \overline{\Pi}_m$. Moreover, Steps 1,2, 9, 11 and 13 give

$$\Pi_n(l_n|\text{sell}) = -\gamma\big(p_m + (1-p_m)\sigma_m(l_n)\big)l_n + (1-\gamma)\overline{\Pi}_n(l_n|\text{sell}) < \overline{\Pi}_n(l_n|\text{sell}).$$

Hence $\Pi_n < \overline{\Pi}_n$ (by symmetry of the bid and ask sides of the market). We next show that

$\Pi_n > \Pi_m$. Reasoning like we did above, and using Step 12 together with $l_n < l_m$,

$$\begin{aligned}\Pi_n(l_n|\text{sell}) &= -\gamma\big(p_m + (1-p_m)\sigma_m(l_n)\big)l_n + (1-\gamma)\overline{\Pi}_n(l_n|\text{sell})\\ &> -\gamma l_m + (1-\gamma)\overline{\Pi}_n(l_n|\text{sell})\\ &= -\gamma l_m + (1-\gamma)\overline{\Pi}_m(l_m|\text{sell})\\ &= \Pi_m(l_m|\text{sell}).\end{aligned}$$

Hence, $\Pi_n > \Pi_m$. Lastly, we show that $\Pi_m = 0$. Suppose by way of contradiction that $\Pi_m > 0$. Then Steps 2 and 7 imply $0 \in A_n$. It ensues, using Step 5, that $0 \notin \big(A_m \cup \overline{A}_m\big)$. We thus obtain $\Pi_n = 0 < \Pi_m$, contradicting $\Pi_n > \Pi_m$.

■

Proposition 4. *For all p_1, p_2 and q, a trading equilibrium exists. If either* $\max\{p_1, p_2\} < 1$ *and $q > 0$, or* $\max\{p_1, p_2\} = 1 > \min\{p_1, p_2\}$ *and $q = 0$, then there exists a unique trading equilibrium.*[26]

Proof: The cases $p_1 = p_2 = 0$ and $p_1 = p_2 = 1$ are trivial. We prove below existence and uniqueness for $0 < p_m \leq p_n < 1$ and $q > 0$ (other cases are similar).

We start by showing that in any trading equilibrium the speculator sells (resp. buys) with probability 1 when she is informed and $V = 0$ (resp. $V = 1$), and abstains when she is uninformed. First note that, by Lemma 5, $\hat{b} < 1$ with probability 1. So selling the asset is a strictly dominated strategy of the informed speculator when $V = 1$. Similarly, buying the asset is a strictly dominated strategy of the informed speculator when $V = 0$. Next, Suppose by way of contradiction that the speculator abstains with positive probability when she is informed and $V = 0$ (the other case is similar, by symmetry). Then $\mathbb{P}(\hat{b} = 0|V = 0) = 1$, otherwise the speculator would have a profitable deviation. But then $\sigma_1(0) = \sigma_2(0) = 1$, contradicting Step 5 in the proof of Lemma 5. We conclude that the speculator sells (resp. buys) with probability 1 when she is informed and $V = 0$ (resp. $V = 1$). We now prove that the speculator abstains when she is uninformed. Applying Lemma 5 gives $u < 1$ and $\max\{l_1, l_2\} < \mathbb{E}[V|\text{sell}]$. As we showed above that the speculator buys (resp. sells) with probability 1 when she is informed and $V = 1$ (resp. $V = 0$), we obtain $\mathbb{E}[V|\text{sell}] < \frac{1}{2}$. The

[26] If $\max\{p_1, p_2\} = 1$ and $q > 0$ then, given $V = 0$ (resp. given $V = 1$), the informed speculator is indifferent between selling (resp. buying) and abstaining. If $p_1 = p_2 = q = 0$, or $p_1 = p_2 = 1$ and $q = 0$, then the uninformed speculator is indifferent between trading and abstaining.

uninformed speculator's expected profit from selling the asset is therefore bounded above by

$$\mathbb{P}(\text{sell order executed by an uninformed MM})\left(\max\{l_1, l_2\} - \frac{1}{2}\right)$$
$$+\, \mathbb{P}(\text{sell order executed by an informed MM})(u-1) < 0.$$

By symmetry, the uninformed speculator's expected profit from selling the asset is negative as well.

We next derive equilibrium strategies of the MMs. Since we saw above that in any trading equilibrium the speculator trades if and only if she is informed, we obtain $\gamma = \mathbb{P}(V = 0|\text{sell}) = (\frac{\pi q}{2} + \frac{1-\pi}{4})/(\frac{\pi q}{2} + \frac{1-\pi}{2})$ in any trading equilibrium. Now, by virtue of Lemma 5, if the pricing strategies σ_m, σ_n, $\underline{\sigma}_m$, $\underline{\sigma}_n$, $\overline{\sigma}_m$ and $\overline{\sigma}_n$ are in equilibrium then $\underline{\sigma}_m(0) = \underline{\sigma}_n(0) = 1$ and there exist $0 < l_n < l_m < u < 1$ such that:

$$\big[(1-p_m) + p_m\overline{\sigma}_m(x)\big](1-x) = 1-u, \qquad \forall x \in [l_m, u]; \tag{26}$$

$$\big[(1-p_n) + p_n\overline{\sigma}_n(x)\big](1-x) = 1-u, \qquad \forall x \in [l_m, u]; \tag{27}$$

$$\overline{\sigma}_m(l_m) = 0; \tag{28}$$

$$-\gamma l_m + (1-\gamma)(1-u) = 0; \tag{29}$$

$$-\gamma x + (1-\gamma)\big[(1-p_n) + p_n\overline{\sigma}_n(x)\big](1-x) = 0, \qquad \forall x \in [l_n, l_m]; \tag{30}$$

$$(1-p_m)\sigma_m(x)(1-x) = 1-u, \qquad \forall x \in [l_n, l_m]; \tag{31}$$

$$\overline{\sigma}_n(l_n) = 0; \tag{32}$$

$$\begin{aligned} &-\gamma\big[p_m + (1-p_m)\sigma_m(x)\big]x + (1-\gamma)(1-p_m)\sigma_m(x)(1-x) \\ &\quad = -\gamma\big[p_m + (1-p_m)\sigma_m(l_n)\big]l_n + (1-\gamma)(1-p_m)\sigma_m(l_n)(1-l_n), \qquad \forall x \in [0, l_n]; \end{aligned} \tag{33}$$

$$-\gamma\big[p_n + (1-p_n)\sigma_n(x)\big]x + (1-\gamma)(1-p_n)\sigma_n(x)(1-x) = 0, \qquad \forall x \in [0, l_n]. \tag{34}$$

Equations (26) and (27) are the equiprofit conditions of, respectively, MMnH and MMmH in the bid range $[l_m, u]$; (28) is obtained by definition of l_m; equation (29) captures $\Pi_m(l_m|\text{sell}) = 0$; equations (30) and (31) are the equiprofit conditions of, respectively, MMmU and MMnH in the bid range $[l_n, l_m]$; (32) is obtained by definition of l_n; lastly, equations (33) and (34) are the equiprofit conditions of, respectively, MMnU and MMmU in the bid range $[0, l_n]$. That the system of equations (26)-(34) uniquely determines pricing strategies σ_m, σ_n, $\overline{\sigma}_m$ and $\overline{\sigma}_n$ is straightforward to check.

By construction the strategies above are in equilibrium if no MM can profitably bid outside the support of their respective strategies. Observe to begin with that no MM can profitably bid outside $[0, u]$. So we only need to check the remaining cases. To see that MMmU has no profitable deviation to $b \in (l_m, u]$ note that

$$\Pi_m(b|\text{sell}) = -\gamma b + (1-\gamma)\big[(1-p_n) + p_n\overline{\sigma}_n(b)\big](1-b), \qquad \forall b \in [l_m, u].$$

Hence, by (27),

$$\Pi_m(b|\text{sell}) = -\gamma b + (1-\gamma)(1-u), \qquad \forall b \in [l_m, u].$$

The last highlighted equation gives $\Pi_m(b|\text{sell}) < \Pi_m(l_m|\text{sell})$, for all $b \in (l_m, u]$. Similarly, to see that MMmH has no profitable deviation to $b \in [l_n, l_m)$ note that, by (30),

$$\overline{\Pi}_m(b|\text{sell}) = \big[(1-p_n) + p_n\overline{\sigma}_n(b)\big](1-b) = \frac{\gamma b}{1-\gamma}, \qquad \forall b \in [l_n, l_m].$$

Hence $\overline{\Pi}_m(b|\text{sell}) < \overline{\Pi}_m(l_m|\text{sell})$ for all $b \in [l_n, l_m)$. MMmH has no profitable deviation to $b \in [0, l_n]$ either, since, by (34),

$$\overline{\Pi}_m(b|\text{sell}) = (1-p_n)\sigma_n(b)(1-b) = \frac{\gamma\big[p_n + (1-p_n)\sigma_n(b)\big]b}{1-\gamma}, \qquad \forall b \in [0, l_n]. \tag{35}$$

Therefore, $\overline{\Pi}_m(b|\text{sell}) \leq \overline{\Pi}_m(l_n|\text{sell})$ for all $b \in [0, l_n]$, which, combined with the previous remark, gives $\overline{\Pi}_m(b|\text{sell}) < \overline{\Pi}_m(l_m|\text{sell})$ for all $b \in [0, l_n]$. This finishes to show that neither MMmU nor MMmH can profitably bid outside the support of their respective strategies. Similar arguments establish that neither MMnU nor MMnH can profitably bid outside the support of their respective strategies.

■

The following technical lemma is used in the proof of Theorem 2.

Lemma 6. *Let $p^*(c, \pi)$ be given by (19) and*

$$H(c; \pi) := \overline{\Pi}_S\big(p^*(c, \pi), 0\big) - c.$$

Then $H(c; \pi) = 0$ has exactly one solution in the interval $c \in \big(0, \frac{1-\pi}{4}\big)$.

Proof: Consider any equilibrium of the trading game with $q = 0$ and a given, arbitrary, p. Define $\beta := \mathbb{E}[b_n|\text{MM}n \text{ is uninformed}]$ and $\hat{\beta} := \mathbb{E}[b_n|\text{both MMs uninformed}, b_n \geq b_m]$. By symmetry of the bid and ask sides of the market, we can write

$$\overline{\Pi}_S(p, 0) = 2p(1-p)\beta + (1-p)^2\hat{\beta} \tag{36}$$

and

$$\Pi_n(p, 0) = \left(\frac{1-\pi}{2}\right)\left[\frac{1}{2}\left(p(0-\beta) + (1-p)\frac{1}{2}(0-\hat{\beta})\right) + \frac{1}{2}(1-p)\frac{1}{2}(1-\hat{\beta})\right].$$

Rearranging the last highlighted equation gives

$$\begin{aligned}\Pi_n(p, 0) &= \frac{1}{4}(1-p) - \frac{1}{2}\left[p\beta + (1-p)\hat{\beta}\right] \\ &= \frac{1}{4}(1-p) - \frac{1}{2}\frac{\overline{\Pi}_S(p, 0)}{1-p} + \frac{1}{2}p\beta.\end{aligned}$$

Hence, as $\Pi_n(p, 0) = 0$,

$$\overline{\Pi}_S(p, 0) = \frac{1}{2}(1-p)^2 + p(1-p)\beta. \tag{37}$$

Next, using (12) gives

$$\beta = \int b\sigma'(b)db = \frac{p\left(\frac{1}{1-2b} + \ln(\frac{1}{2} - b)\right)}{4(1-p)}. \tag{38}$$

Finally, combining (19), (37) and (38) yields, for all $c \in \left(0, \frac{1-\pi}{4}\right)$,

$$H(c; \pi) = \frac{1}{(1-\pi-2c)^2}\left[\frac{(1-\pi-4c)^2}{4}\ln\left(1 - \frac{4c}{1-\pi}\right) + c(1-\pi-2c)(2c+\pi)\right].$$

Let $G(c; \pi)$ denote the expression inside the square bracket. One verifies that:

(i) $G(0; \pi) = 0$;

(ii) $G(c; \pi) \to \frac{(1-\pi)^2(1+\pi)}{16}$ as $c \to \frac{1-\pi}{4}$;

(iii) $G'(0; \pi) < 0 < G''(0; \pi)$;

(iv) $G'''(c; \pi) < 0$ for all $c \in \left(0, \frac{1-\pi}{4}\right)$.

Therefore, $G(c; \pi) = 0$ has exactly one solution in the interval $c \in \left(0, \frac{1-\pi}{4}\right)$.[27] ■

[27] On the interval $(0, \frac{1-\pi}{4})$, the function G is first convex, then concave. The function starts below the

Appendix D: Trading Game with Observable Quotes (for online publication)

In this appendix we analyze the trading game induced by the observable quotes model, with $z > 0$ denoting the probability with which the speculator gets to observe the quotes before placing her market order. Specifically, throughout this appendix p and q play the role of parameters: each MM is informed with probability p, and the speculator is informed with probability q. A strategy of MMn comprises cumulative distribution functions σ_n, $\underline{\sigma}_n$ and $\overline{\sigma}_n$ specifying respectively the distribution of the bid price b_n of MMnU, MMnL and MMnH. As the bid and ask sides of the market are symmetric we assume as usual that, conditional on MMnU, $1 - a_n$ is distributed like b_n. Similarly, we assume that the law of $1 - a_n$ conditional on MMnL (resp. MMnH) is the same as the law of b_n conditional on MMnH (resp. MMnL). A strategy of the speculator specifies her market order as a function of the information she possesses at that point. A WELM trading equilibrium is a perfect Bayesian equilibrium such that

(i) $\sigma_1 = \sigma_2 = \sigma$, $\underline{\sigma}_1 = \underline{\sigma}_2 = \underline{\sigma}$ and $\overline{\sigma}_1 = \overline{\sigma}_2 = \overline{\sigma}$;

(ii) $\underline{\sigma}(0) = 1$;

(iii) either $p \in \{0, 1\}$ or σ and $\overline{\sigma}$ are atomless, with supp $(\sigma) = [0, l]$ and supp $(\overline{\sigma}) = [l, u]$.

We focus throughout this appendix on $p \in (0, 1)$ and $q < 1$. The case $p = 0$ is almost identical. If $q = 1$, the observability of the quotes is inconsequential. The case $p = 1$ is straightforward: both MMs set prices equal to the realized asset value. Lastly, to shorten the exposition, we introduce the indicator variables I_S, I_n and Z respectively equal to 1 if and only if (a) the speculator is informed, (b) MMn is informed, (c) quotes are observable.[28]

Proposition 5. *Assume $p \in (0, 1)$ and $q < 1$. In any WELM trading equilibrium:*

$$\overline{\Pi}_n(p, q) = \left(\frac{(1-\pi)(1-p)}{2}\right) \frac{1 - \pi(1-2q) + 2\pi p(1-q)z}{2 - p - 2\pi(1-q) + \pi p(1 + 2(1-q)z)}; \tag{39}$$

$$\sigma(b) = \frac{(1 + \pi(2z-1) + 2\pi q(1-z))pb}{(1-p)(1-\pi-2b(1-\pi(1-q)))}, \qquad \forall b \in [0, l]; \tag{40}$$

horizontal axis, and ends above it. Suppose it crossed the horizontal axis twice. Then at the second crossing, the function has to be decreasing and concave. But this contradicts G ending above the horizontal axis.

[28] Note that in our terminology the speculator is informed if and only if she acquires costly information and therefore observes v "directly".

$$\overline{\sigma}(b) = \frac{2\overline{\Pi}_n(p,q) - (1-p)(1-\pi)(1-b)}{(1-b)(1-\pi)p}, \qquad \forall b \in [l,u]; \tag{41}$$

$$l = \frac{(1-\pi)(1-p)}{2 - p - 2\pi(1-q) + \pi p(1 + 2(1-q)z)}; \tag{42}$$

$$u = \frac{1 - \pi - 2\overline{\Pi}_n(p,q)}{1-\pi}. \tag{43}$$

In particular, $\overline{\Pi}_n(p,q)$*,* l *and* u *given by, respectively,* (39)*,* (42) *and* (43) *satisfy* $\overline{\Pi}_n(p,q) > 0$ *and* $0 < l < u < 1$*.*

Proof: We start with a few preliminary remarks. Observe that WELM equilibria are separating equilibria. Hence, in any WELM equilibrium, $I_S \vee \Big((I_1 \vee I_2) \wedge Z\Big) = 1$ implies that, on the equilibrium path, the speculator learns the realization of V. In this case, by sequential rationality, the speculator buys if $V = 1$ and sells if $V = 0$.[29] On the equilibrium path sell orders are thus more likely conditional on $V = 0$ than they are conditional on $V = 1$, implying $l < \frac{1}{2}$.[30] Hence, on the equilibrium path, the speculator abstains whenever $I_S \vee \Big((I_1 \vee I_2) \wedge Z\Big) = 0$.[31]

Next, by definition of a WELM trading equilibrium, MMnU's expected profit on the bid side of the market has to be zero (the same being true of course on the ask side of the market).[32] As MMnU randomizes over $[0,l]$, we obtain

$$\begin{aligned} -\frac{1}{2}\Big[\pi\Big((1-p)q\sigma(b) + p(z + (1-z)q)\Big) + \left(\frac{1-\pi}{2}\right)\big(p + (1-p)\sigma(b)\big)\Big]b \\ + \frac{1}{2}\left(\frac{1-\pi}{2}\right)(1-p)\sigma(b)(1-b) = 0, \qquad \forall b \in [0,l]. \end{aligned} \tag{44}$$

The first term in equation (44) can be decomposed as follows. With probability $\frac{1}{2}$ the asset value is $V = 0$, in which case a winning bid b induces a loss equal to b. With probability π the trader is a speculator. By the remarks made earlier in this proof the speculator sells if and only if one of the following 3 cases occurs: (i) $I_m = 0$ and $I_S = 1$, (ii) $I_m = 1$ and $Z = 1$, (iii)

[29] We suppose here, without loss of generality, that the speculators always trades when she is indifferent between trading and abstaining.

[30] MMnU is subject to greater adverse selection than in the baseline case. As $l < \frac{1}{2}$ in the baseline model, $l < \frac{1}{2}$ with observable quotes as well.

[31] Observe that on the equilibrium path, if $I_S = I_1 = I_2 = 0$ and $Z = 1$ then the speculator's expected profit from trading the asset (either buying or selling) is at most $l - \frac{1}{2} < 0$. If instead $I_S = Z = 0$ then her expected profit from trading the asset is bounded above by $\mathbb{P}(\text{trade with an uninformed MM}|I_S \vee Z = 0)(l - \frac{1}{2}) + \mathbb{P}(\text{trade with an informed MM}|I_S \vee Z = 0)(u - 1) < 0$.

[32] This must be since MMnU is indifferent between bids on the interval $[0,l]$, and the expected profit of $b_n = 0$ is zero due to the remark that, in any WELM trading equilibrium, $\mathbb{P}(b_n = 0 \text{ wins}|V = 1) = 0$.

$I_m = 1$, $Z = 0$ and $I_S = 1$. In case (i) MMnU has the winning bid with probability $\sigma(b)$; in cases (ii) and (iii) MMnU has the winning bid with probability 1. With probability $\frac{1-\pi}{2}$ the trader is hit by the liquidity shock and sells the asset: either $I_m = 1$, in which case MMnU has the winning bid with probability 1, or $I_m = 0$, in which case MMnU has the winning bid with probability $\sigma(b)$. The second term in equation (44) is decomposed as follows. With probability $\frac{1}{2}$ the asset value is $V = 1$, in which case a winning bid b induces a gain equal to $1 - b$. The probability of a sell order is the probability of a liquidity trader selling the asset, that is, $\frac{1-\pi}{2}$. Either $I_m = 1$, in which case MMnU has the losing bid, or $I_m = 0$, in which case MMnU has the winning bid with probability $\sigma(b)$.

As MMnH randomizes over $[l, u]$ we obtain in a similar way

$$\left(\frac{1-\pi}{2}\right)\left[p\overline{\sigma}(b) + (1-p)\right](1-b) = \left(\frac{1-\pi}{2}\right)(1-p)(1-l), \qquad \forall b \in [l, u]. \tag{45}$$

We can now conclude the proof of the proposition. Rearranging (44) yields (40), from which solving $\sigma(l) = 1$ gives us (42). Substituting (42) into the right-hand side of (45) and using the symmetry of the problem to write the resulting expression as $\overline{\Pi}_n(p, q)$ gives us (39). Rearranging the terms in (45) then yields (41), from which solving $\overline{\sigma}(u) = 1$ yields (43). To see that $l > 0$, substitute $b = l$ into (44). Substituting $b = u$ into (45) and using the fact that $l < \frac{1}{2}$ yields $u < 1$ and $\overline{\Pi}_n(p, q) > 0$.

■

Lemma 7. *Assume $p \in (0, 1)$ and $q < 1$. Let $\sigma(\cdot)$, $\overline{\sigma}(\cdot)$, l and u be defined by* (40)*,* (41)*,* (42) *and* (43)*, respectively. Then*

$$\arg\max_{b\in[0,1]} \left(\frac{1-\pi}{2}\right)\big[p\overline{\sigma}(b) + (1-p)\sigma(b)\big](1-b) = [l, u], \tag{46}$$

and

$$\begin{aligned}\arg\max_{b\in[0,1]} -\frac{1}{2}\Big[\pi\Big((1-p)q\sigma(b) + p(z + (1-z)q)\Big) + \left(\frac{1-\pi}{2}\right)\big(p + (1-p)\sigma(b)\big)\Big]b \\ + \frac{1}{2}\left(\frac{1-\pi}{2}\right)\big[p\overline{\sigma}(b) + (1-p)\sigma(b)\big](1-b) = [0, l].\end{aligned} \tag{47}$$

The maximum values of (46) *and* (47) *are $\overline{\Pi}_n(p, q)$, given by* (39)*, and* 0*, respectively.*

Proof: By virtue of (45),

$$\left(\frac{1-\pi}{2}\right)\big[p\overline{\sigma}(b)+(1-p)\sigma(b)\big](1-b)=\left(\frac{1-\pi}{2}\right)(1-u),\qquad \forall b\in[l,u]. \tag{48}$$

As $\overline{\sigma}(u)=\sigma(u)=1$, notice that the left-hand side of (48) is strictly decreasing in b for $b\geq u$. Next, rewriting (44) as

$$-\frac{1}{2}\Big[\pi\Big((1-p)q\sigma(b)+p(z+(1-z)q)\Big)+\left(\frac{1-\pi}{2}\right)\big(p+(1-p)\sigma(b)\big)\Big]b$$
$$+\frac{1}{2}\left(\frac{1-\pi}{2}\right)\big[p\overline{\sigma}(b)+(1-p)\sigma(b)\big](1-b)=0,\qquad \forall b\in[0,l],$$

gives

$$\left(\frac{1-\pi}{2}\right)\big[p\overline{\sigma}(b)+(1-p)\sigma(b)\big](1-b)=\Big[\pi\Big((1-p)q\sigma(b)+p(z+(1-z)q)\Big)+\frac{1-\pi}{2}\big(p+(1-p)\sigma(b)\big)\Big]b,$$

for all $b\in[0,l]$. The right-hand side of the last highlighted equation is strictly increasing in b. So combining the previous steps yields (46). Finally, (46) and the observation that the first term in the maximand of (47) is a strictly decreasing function of b together yield (47).

■

Proposition 6. *Assume $p\in(0,1)$ and $q<1$. Let $\sigma(\cdot)$, $\overline{\sigma}(\cdot)$, l and u be defined by* (40), (41), (42) *and* (43), *respectively. Define*

$$h(b):=\frac{(1-u-b)(1-\pi)-2b\pi q}{2b\pi(1-q)zp}, \tag{49}$$

and suppose that

$$1-\overline{\sigma}(b)\geq h(b),\qquad \forall b\in[l,u]. \tag{C}$$

Then a WELM trading equilibrium exists.

Proof: The following notation will be used throughout the proof. Let the cdfs σ and $\overline{\sigma}$ be defined by (40) and (41), respectively. Define also the cdf $\underline{\sigma}$ such that $\underline{\sigma}(0)=1$. Let Γ denote the set of bid-ask price pairs (b_n,a_n) consistent with the strategies σ, $\underline{\sigma}$, and $\overline{\sigma}$, that is,

$$\Gamma=\Big(\{0\}\times[1-u,1-l]\Big)\;\cup\;\Big([0,l]\times[1-l,1]\Big)\;\cup\;\Big([l,u]\times\{1\}\Big).$$

Similarly, let Γ^+ denote the set of tuples (b_1, a_1, b_2, a_2) consistent with the strategies σ, $\underline{\sigma}$, and $\overline{\sigma}$, that is,

$$\begin{aligned}\Gamma^+ = \Big\{(b_1, a_1, b_2, a_2) :\ & (b_1, a_1) \in \Gamma, (b_2, a_2) \in \Gamma, \\ & (b_n, a_n) \in [l, u] \times \{1\} \Rightarrow (b_m, a_m) \notin \{0\} \times [1-u, 1-l], \\ & (b_n, a_n) \in \{0\} \times [1-u, 1-l] \Rightarrow (b_m, a_m) \notin [l, u] \times \{1\}\Big\}.\end{aligned}$$

Let $\beta : \Gamma^+ \to \left\{0, \frac{1}{2}, 1\right\}$ represent the mapping from consistent tuples (b_1, a_1, b_2, a_2) to posterior beliefs that $V = 1$, computed through Bayes' rule. Let μ_n denote the speculator's belief that $V = 1$ based only on the quotes of MMn, with $\mu_n = \emptyset$ in case $(b_n, a_n) \notin \Gamma$.[33] Let μ denote the speculator's belief that $V = 1$ at the time she chooses her market order.

Assume the condition (C) holds with equality (the other case is similar). We aim to show that the following strategies, beliefs and tie-breaking rule comprise a trading equilibrium:

(I) $\sigma_1 = \sigma_2 = \sigma$;

(II) $\underline{\sigma}_1 = \underline{\sigma}_2 = \underline{\sigma}$;

(III) $\overline{\sigma}_1 = \overline{\sigma}_2 = \overline{\sigma}$;

(IV) if $I_S = 1$ then $\mu = v$;

(V) if $I_S \vee Z = 0$ then $\mu = \frac{1}{2}$;

(VI) if $I_S = 0$ and $Z = 1$ then:

[33] We use the terminology "speculator's belief that $V = 1$" for the probability which the speculator attaches to the event $V = 1$.

$$\mu = \begin{cases} \beta(b_1, a_1, b_2, a_2) & \text{if } (b_1, a_1, b_2, a_2) \in \Gamma^+; & \text{(50a)} \\ I_{\{1-a_m > b_n\}} & \text{if } \mu_n = 1 \text{ and } \mu_m = 0; & \text{(50b)} \\ \mu_m & \text{if } \mu_m \in \{0,1\} \text{ and } \mu_n = \emptyset; & \text{(50c)} \\ \dfrac{a_n + b_n}{2} & \text{if } \mu_m = \dfrac{1}{2}, \mu_n = \emptyset, \text{ and } b_n < a_n; & \text{(50d)} \\ 1 & \text{if } \mu_m = \dfrac{1}{2}, a_n \leq b_n, \text{ and } a_n \neq \hat{a}; & \text{(50e)} \\ 0 & \text{if } \mu_m = \dfrac{1}{2}, a_n \leq b_n, a_n = \hat{a} \text{ but } b_n \neq \hat{b}; & \text{(50f)} \\ \dfrac{a_n + b_n}{2} & \text{if } \mu_m = \dfrac{1}{2}, a_n \leq b_n, a_n = \hat{a} \text{ and } b_n = \hat{b}. & \text{(50g)} \end{cases}$$

(VII) ties are broken uniformly at random, except if $\mu_m = \frac{1}{2}, a_n \leq b_n, a_n = \hat{a}$ and $b_n = \hat{b}$, in which case any tie is broken in favor of MMn;

(VIII) the speculator's market order satisfies sequential rationality with the additional requirement that if $I_S = 0$, $Z = 1$, $\mu_m = \frac{1}{2}, a_n \leq b_n, a_n = \hat{a}$ and $b_n = \hat{b}$ (in which case, by (50g), $\mu = \frac{a_n+b_n}{2}$) then the speculator buys with probability $\frac{1}{2}$ and sells with probability $\frac{1}{2}$.

The proposed equilibrium has the following features. If the speculator is informed, her beliefs concerning V are determined by the realized value v, that is, even if the quotes suggest otherwise (see (IV)). If the speculator is uninformed and quotes are unobservable then μ is equal to the prior belief that $V = 1$, that is, $\mu = \frac{1}{2}$ (see (V)). The case in which the speculator is uninformed but gets to observe the quotes is subdivided into 7 cases. If the quotes are consistent with the proposed equilibrium strategies, then μ is derived using Bayes' rule (see (50a)). If MMn's quotes signals $V = 1$ while MMm's quotes signals $V = 0$, that is, $(b_n, a_n) \in [l, u] \times \{1\}$ and $(b_m, a_m) \in \{0\} \in [1-u, 1-l]$, then $\mu = 1$ if $1 - a_m > b_n$ and $\mu = 0$ otherwise (see (50b)). If MMn's quotes are inconsistent with the proposed equilibrium strategies but MMm's quotes signal that MMm is informed then the speculator ignores MMn and bases her beliefs exclusively on the quotes of MMm (see (50c)). The case in which MMn's quotes are inconsistent with the proposed equilibrium strategies and MMm's quotes signal that MMm is uninformed are further subdivided into 4 cases. If MMn's quotes satisfy $b_n < a_n$ then $\mu = \frac{a_n+b_n}{2}$ (see (50d)), in which case sequential rationality precludes trading between the speculator and MMn. If $a_n \leq b_n$ and MMn does not offer the best ask price then $\mu = 1$, (see (50e)), in which case sequential rationality precludes trading between the speculator and MMn.[34] If $a_n \leq b_n$, MMn offers the best ask price but not the best bid price then $\mu = 0$, (see (50f)), in which case sequential rationality precludes trading between the speculator and MMn. Lastly, if $a_n \leq b_n$ and MMn offers the best bid and ask prices then $\mu = \frac{a_n+b_n}{2}$ (see (50g)), in which case the tie breaking rule ensures that, conditional on placing a market order, the speculator trades with MMn (see (VII)).

Note that the proposed equilibrium satisfies the requirements of a WELM equilibrium; thus, repeating arguments used to prove Proposition 5, on the equilibrium path, the speculator learns the realization of V if $I_S \vee \Big((I_1 \vee I_2) \wedge Z\Big) = 1$. Moreover, since $l < \frac{1}{2}$, sequential rationality requires the speculator to abstain if $I_S \vee \Big((I_1 \vee I_2) \wedge Z\Big) = 0$.[35] It ensues that, on the equilibrium path, MMnU's expected profit on the bid side of the market can be written as the left-hand side of (44). Similarly, on the equilibrium path, MMnH's expected profit on the bid side of the market can be written as the left-hand side of (45). These remarks, Lemma 7 and the symmetry of the bid and ask sides of the market together establish that, on the equilibrium path: MMnU's expected profit is equal to 0, while MMnH's expected profit equals $\overline{\Pi}_n(p, q)$ given by (39). We establish in the rest of the proof that neither MMnU nor MMnH have a profitable deviation (which, by symmetry, implies that MMnL does not have a profitable deviation either).

Step 1: there exists no profitable deviation of MMnU to $(a_n, b_n) \notin \Gamma$, with $b_n < a_n$.

Suppose MMnU deviates to $(\tilde{a}_n, \tilde{b}_n) \notin \Gamma$, with $\tilde{b}_n < \tilde{a}_n$. Observe first that in this case, applying (IV), (V), (50c) and (50d), the speculator trades with MMnU if and only if $I_S \vee (I_m \wedge Z) = 1$ (notice that if $I_S = I_m = 0$ while $Z = 1$ then (50d) yields $b_n < \mu < a_n$). So the "demand" facing MMnU is the same as it is on the equilibrium path. In consequence, MMnU's expected profit on the bid side of the market can be written like the maximand of (47), with $b = \tilde{b}_n$. Yet, by virtue of Lemma 7, the maximand of (47) is maximized when MMnU sticks to the proposed equilibrium strategy. The symmetry between the bid and ask sides of the market finishes to establish that $(\tilde{a}_n, \tilde{b}_n)$ is not a profitable deviation of MMnU.

[34] Observe that $a_n \neq \hat{a}$ implies $\hat{a} < 1$. So, for $\mu = 1$, the speculator's expected profit from buying the asset is strictly positive. On the other hand, the speculator's expected profit from selling is at most 0. Sequential rationality therefore requires the speculator to buy.

[35] See the third footnote in the proof of Proposition 5.

Step 2: there exists no profitable deviation of MMnU to $(a_n, b_n) \notin \Gamma$, with $a_n \leq b_n$.

Suppose MMnU deviates to $(\tilde{a}_n, \tilde{b}_n) \notin \Gamma$, with $\tilde{a}_n \leq \tilde{b}_n$. Now in this case, applying (IV), (V), (50c), (50e), (50f), (50g), (VII) and (VIII) the speculator trades with MMn if and only if either (a) $I_S \vee (I_m \wedge Z) = 1$ or (b) $I_S \vee I_m = 0$, $Z = 1$, $\tilde{a}_n = \hat{a}$ and $\tilde{b}_n = \hat{b}$. Moreover, in the latter event, (VIII) assures that the speculator buys with probability $\frac{1}{2}$ and sells with probability $\frac{1}{2}$. These remarks enable us to write the expected profit of MMnU as

$$\begin{aligned}
&\Big\{ -\frac{1}{2}\Big[\pi\Big((1-p)q\sigma(\tilde{b}_n) + p(z+(1-z)q)\Big) + \frac{1-\pi}{2}\big(p + (1-p)\sigma(\tilde{b}_n)\big)\Big]\tilde{b}_n \\
&\qquad\qquad + \frac{1}{2}\left(\frac{1-\pi}{2}\right)(1-p)\sigma(\tilde{b}_n)(1-\tilde{b}_n)\Big\} \\
&+ \Big\{ -\frac{1}{2}\Big[\pi\Big((1-p)q\sigma(1-\tilde{a}_n) + p(z+(1-z)q)\Big) + \frac{1-\pi}{2}\big(p + (1-p)\sigma(1-\tilde{a}_n)\big)\Big](1-\tilde{a}_n) \\
&\qquad\qquad + \frac{1}{2}\left(\frac{1-\pi}{2}\right)(1-p)\sigma(1-\tilde{a}_n)\tilde{a}_n\Big\} \\
&+ \left\{\pi(1-q)(1-p)z\sigma(\tilde{b}_n)\sigma(1-\tilde{a}_n)\left[\frac{1}{2}\left(-\frac{\tilde{b}_n}{2} + \frac{\tilde{a}_n}{2}\right) + \frac{1}{2}\left(\frac{1-\tilde{b}_n}{2} + \frac{\tilde{a}_n - 1}{2}\right)\right]\right\},
\end{aligned}$$

where the first two curly brackets capture case (a) in the previous paragraph, and the last curly bracket captures case (b). Now, using Lemma 7, the term inside the first curly bracket is at most 0. By symmetry, the same remark applies to the second curly bracket. Finally, the third curly bracket is equal to $\pi(1-q)(1-p)z\sigma(\tilde{b}_n)\sigma(1-\tilde{a}_n)\left(\frac{\tilde{a}_n - \tilde{b}_n}{2}\right)$, which, since $\tilde{a}_n \leq \tilde{b}_n$, is at most 0. So $(\tilde{a}_n, \tilde{b}_n)$ is not a profitable deviation of MMnU.

Step 3: there exists no profitable deviation of MMnU to $(a_n, b_n) \in \Gamma$.

Suppose MMnU deviates to $(\tilde{a}_n, \tilde{b}_n) \in \Gamma$, say $\tilde{a}_n = 1$ and $\tilde{b}_n \in [l, u]$ (the other case is analogous, by symmetry). Consider first the ask side of the market: either $V = 1$ or $\tilde{a}_n \neq \hat{a}$ with probability 1. So the expected profit of MMnU on the ask side of the market is at most 0. Next, consider the bid side of the market. By virtue of (IV), (V), (50a) and (50b) the speculator sells and trades with MMnU if and only if $V = 0$ and:

- either $I_S = 1$;
- or $I_m \wedge Z = 1$ and $\tilde{b}_n \leq 1 - a_m$.

Thus MMnU's expected profit on the bid side of the market may be written as

$$-\frac{1}{2}\left[\pi\Big(q+(1-q)zp\mathbb{P}(\tilde{b}_n \le 1-a_m|\text{MM}m\text{L})\Big)+\frac{1-\pi}{2}\right]\tilde{b}_n$$
$$+\frac{1}{2}\left(\frac{1-\pi}{2}\right)[(1-p)+p\overline{\sigma}(\tilde{b}_n)](1-\tilde{b}_n).$$

Conditional on MMmL, $1-a_m$ is distributed according to the cdf $\overline{\sigma}$. Hence, using condition (C), $\mathbb{P}(\tilde{b}_n \le 1-a_m|\text{MM}m\text{L}) = 1-\overline{\sigma}(\tilde{b}_n) \ge h(\tilde{b}_n)$. Substituting this inequality into the last highlighted expression shows that MMnU's expected profit on the bid side of the market is bounded above by

$$-\frac{1}{2}\left[\pi\big(q+(1-q)zph(\tilde{b}_n)\big)+\frac{1-\pi}{2}\right]\tilde{b}_n+\frac{1}{2}\left(\frac{1-\pi}{2}\right)[(1-p)+p\overline{\sigma}(\tilde{b}_n)](1-\tilde{b}_n). \tag{51}$$

By (46), we can rewrite (51) as

$$-\frac{1}{2}\left[\pi\big(q+(1-q)zph(\tilde{b}_n)\big)+\frac{1-\pi}{2}\right]\tilde{b}_n+\frac{(1-\pi)(1-u)}{4},$$

which, by definition of $h(\tilde{b}_n)$, is equal to 0. So $(\tilde{a}_n, \tilde{b}_n)$ is not a profitable deviation of MMnU.

Step 4: there exists no profitable deviation of MMnH to $(a_n, b_n) \notin \Gamma$, with $b_n < a_n$.

Suppose MMnH deviates to $(\tilde{a}_n, \tilde{b}_n) \notin \Gamma$, with $\tilde{b}_n < \tilde{a}_n$. Note to start with that MMnH's expected profit on the ask side of the market has to be non-positive. Consider next the bid side of the market. Observe that by (IV), (V), (50c) and (50d), the speculator never sells to MMnH. Hence, the "demand" facing MMnU is the same as it is on the equilibrium path. In consequence, MMnH's expected profit on the bid side of the market can be written like the maximand of (46), with $b = \tilde{b}_n$. Yet, by virtue of Lemma 7, the maximand of (46) is maximized when MMnH sticks to the proposed equilibrium strategy. So $(\tilde{a}_n, \tilde{b}_n)$ is not a profitable deviation of MMnH.

Step 5: there exists no profitable deviation of MMnH to $(a_n, b_n) \notin \Gamma$, with $a_n \le b_n$.

Suppose MMnH deviates to $(\tilde{a}_n, \tilde{b}_n) \notin \Gamma$, with $\tilde{a}_n \le \tilde{b}_n$. We start by showing that MMnH cannot make positive expected profit against the speculator. First, by virtue of (IV) and (50c), if $I_S \vee (I_m \wedge Z) = 1$ then the speculator never sells. Furthermore, it is impossible to make profit against the speculator if she buys, since $V = 1$ and $\tilde{a}_n \le 1$. Hence, conditional on

$I_S \vee (I_m \wedge Z) = 1$, MM$n$H makes at most zero profit against the speculator. Next, by virtue of (V), (50e), (50f) and (50g), if $I_S \vee I_m = 0$ then the only case in which MMnH trades with the speculator is if $Z = 1$, $\tilde{a}_n = \hat{a}$ and $\tilde{b}_n = \hat{a}$. Furthermore, in that case, by (VIII) the speculator buys and sells the asset with probabilities $\frac{1}{2}$ each. Applying (VII), the expected profit made by MMnH against the speculator is then

$$\frac{1}{2}(1 - \tilde{b}_n) + \frac{1}{2}(\tilde{a}_n - 1) = \frac{\tilde{a}_n - \tilde{b}_n}{2}.$$

Yet $\tilde{a}_n \leq \tilde{b}_n$. Thus MM$n$H makes at most zero expected profit against the speculator. The expected profit of MMnH is then bounded above by the expected profit made against the liquidity trader, which we can write as $\frac{1-\pi}{2}\left[p\overline{\sigma}(\tilde{b}_n) + (1-p)\sigma(\tilde{b}_n)\right](1 - \tilde{b}_n) + \frac{1-\pi}{2}\mathbb{P}(\tilde{a}_n = \hat{a})(\tilde{a}_n - 1)$. Since the second term is non-positive, the former expression is at most equal to $\frac{1-\pi}{2}\left[p\overline{\sigma}(\tilde{b}_n) + (1-p)\sigma(\tilde{b}_n)\right](1 - \tilde{b}_n)$, which by (46) is at most equal to MMnH's expected profit in the proposed equilibrium. So $(\tilde{a}_n, \tilde{b}_n)$ is not a profitable deviation of MMnH.

Step 6: there exists no profitable deviation of MMnH to $(a_n, b_n) \in \Gamma$.

There are two possible cases. MMnH could deviate to masquerade as MMnL or MMnH could deviate to masquerade as MMnU. Suppose MMnH deviates to masquerade as MMnL. Then $b_n = 0 < \hat{b}$ with probability 1. So the expected profit of MMnH on the bid side of the market is 0. On the other hand, since $V = 1$, the profit of MMnH on the ask side of the market is bounded above by 0. Since sticking to his proposed equilibrium strategy yields MMnH an expected profit of $\overline{\Pi}(p, q) > 0$, deviating to masquerade as MMnL is therefore not a profitable deviation. Next, suppose MMnH deviates to masquerade as MMnU. Reasoning as above, the expected profit of MMnH on the ask side of the market is bounded above by 0. Consider now the bid side of the market, with $b_n = \tilde{b}_n \in [0, l]$. Since $V = 1$, we deduce from (IV), (V) and (50a) that the speculator never sells. MMnH's expected profit on the bid side of the market can thus be written as $\left(\frac{1-\pi}{2}\right)\left[p\overline{\sigma}(\tilde{b}_n) + (1-p)\sigma(\tilde{b}_n)\right](1 - \tilde{b}_n)$, which, applying Lemma 7, is bounded above by MMnH's expected profit on the bid side of the market in the proposed equilibrium. So deviating to masquerade as MMnU is not a profitable deviation either.

■

Lemma 8. *Assume* $p \in (0,1)$ *and* $q < 1$. *Let* $\overline{\Pi}_n(p,q)$, l, u *and* $h(\cdot)$ *be defined by* (39), (42), (43), *and* (49) *respectively. Then:*

(i) for all $\varepsilon > 0$, $p > 1 - \frac{2\varepsilon}{1-\pi}$ *implies* $\overline{\Pi}_n(p,q) < \varepsilon$;

(ii) for all $\delta > 0$, $p > 1 - \delta$ *implies* $l < \delta$ *and* $h(b) < 0$ *for all* $b \in [\delta, u]$;

(iv) $1 - u > \frac{1-p}{2}$.

Proof: By Lemma 7,

$$\overline{\Pi}_n(p,q) = \left(\frac{1-\pi}{2}\right)(1-u) = \left(\frac{1-\pi}{2}\right)(1-p)(1-l).$$

Hence, $1 - p < \frac{2\varepsilon}{1-\pi}$ implies $\overline{\Pi}_n(p,q) < \varepsilon$, giving part (i) of the lemma. Part (iii) follows from the remark that $l < \frac{1}{2}$.

We now show part (ii) of the lemma. The denominator on the right-hand side of (42) is minimized at $q = 0$ and $z = 0$, with minimum value $(2-p)(1-\pi) > 1 - \pi$. Hence,

$$l \leq \frac{(1-\pi)(1-p)}{1-\pi} = 1 - p.$$

Pick a $\delta > 0$. Then, $p > 1 - \delta$ implies $l < \delta$. We next show that choosing $p > 1 - \delta$ also implies $h(b) < 0$ for all $b \in [\delta, u]$. First, rearranging (49) gives

$$-\frac{1}{2}\left[\pi\big(q + (1-q)zph(b)\big) + \frac{1-\pi}{2}\right] b + \frac{(1-\pi)(1-u)}{4} = 0,$$

which, by Lemma 7, we can rewrite as

$$\left[\pi\big(q + (1-q)zph(b)\big) + \frac{1-\pi}{2}\right] b = \overline{\Pi}_n(p,q).$$

Solving for $h(b)$ gives

$$h(b) = \frac{2\overline{\Pi}_n(p,q) - b(1-\pi) - 2b\pi q}{2bpz\pi(1-q)}.$$

In particular,

$$h(b) \leq \frac{1}{2bpz\pi(1-q)}\left[2\overline{\Pi}_n(p,q) - b\,(1-\pi)\right], \qquad \forall b \in [l,u]. \tag{52}$$

Now let $\varepsilon := \frac{\delta(1-\pi)}{2}$. By part (i) of the lemma, $p > 1 - \frac{2\varepsilon}{1-\pi}$ implies $\overline{\Pi}_n(p,q) < \varepsilon$, so, $p > 1-\delta$ implies $\overline{\Pi}_n(p,q) < \varepsilon$. Finally, using (52), $p > 1-\delta$ implies

$$h(b) < \frac{1}{2bpz\pi(1-q)}\left[2\varepsilon - \delta\left(1-\pi\right)\right] = 0, \qquad \forall b \in [\delta, u].$$

■

Proposition 7. *There exists a function $\overline{z}(.) > 0$, independent of q, such that a WELM trading equilibrium exists whenever $z \leq \overline{z}(p)$. Moreover, $\overline{z}(p) = 1$ for $p = 0$ and all $p \geq \frac{\sqrt{2\pi}}{\sqrt{2\pi}+\sqrt{1-\pi}}$. If $q = 1$, a WELM trading equilibrium exists for all values of p and z.*

Proof: We remarked at the beginning of this appendix that if $q = 1$ or $p = 1$ (or both) the existence of a WELM trading equilibrium then follows from the existence of a trading equilibrium in the baseline model. That $\overline{z}(p) = 1$ for $p = 0$ is easy to show. We assume in the rest of the proof that $p \in (0,1)$ and $q < 1$.

<u>Step 1: there exists $\overline{z}(p) > 0$, independent of q, such that $z \leq \overline{z}(p)$ implies that a WELM trading equilibrium exists.</u>

Define, for all $b \in [l,u]$, $D(b) := 1 - \overline{\sigma}(b) - h(b)$, where $\overline{\sigma}(\cdot)$, l, u and $h(\cdot)$ are defined respectively by (41), (42), (43) and (49). Thus,

$$D(b) = \frac{(1-\pi)(1-b) - 2\overline{\Pi}_n(p,q)}{(1-\pi)(1-b)p} - \frac{2\overline{\Pi}_n(p,q) - b(1-\pi) - 2b\pi q}{2bpz\pi(1-q)}, \qquad \forall b \in [l,u], \tag{53}$$

with $\overline{\Pi}_n(p,q)$ given by (39). By Proposition 6, it suffices for our purpose to show the existence of $\overline{z}(p) > 0$, independent of q, such that $z \leq \overline{z}(p)$ implies $D(b) \geq 0$ for all $b \in [l,u]$. First, straightforward algebra establishes that $h(l) = 1$ and $\overline{\sigma}(l) = 0$. Hence,

$$D(l) = 0. \tag{54}$$

Next, differentiating (53) gives

$$D'(b) = \frac{\overline{\Pi}_n(p,q)}{p}\left(\frac{1}{b^2\pi(1-q)z} - \frac{2}{(1-b)^2(1-\pi)}\right), \qquad \forall b \in [l,u]. \tag{55}$$

The bracketed expression on the right-hand side of (55) is decreasing in b and increasing in

q, so

$$D'(b) \geq \frac{\overline{\Pi}_n(p,q)}{p}\left[\frac{1}{u^2\pi z} - \frac{2}{(1-u)^2(1-\pi)}\right], \qquad \forall b \in [l,u].$$

We showed in Lemma 8 that $1-u > \frac{1-p}{2}$, so the last inequality implies

$$D'(b) \geq \frac{\overline{\Pi}_n(p,q)}{p}\left[\frac{1}{\pi z} - \frac{8}{(1-p)^2(1-\pi)}\right], \qquad \forall b \in [l,u]. \tag{56}$$

The expression inside the square bracket is independent of q, and tends to $+\infty$ as z tends to 0. Hence, there exists $\overline{z}(p) > 0$, independent of q, such that $z \leq \overline{z}(p)$ implies $D'(b) \geq 0$ for all $b \in [l,u]$. Since $D(l) = 0$, we obtain $D(b) \geq 0$ for all $b \in [l,u]$ whenever $z \leq \overline{z}(p)$.

Step 2: $\overline{z}(p) = 1$ for all $p \geq \frac{\sqrt{2}}{\sqrt{2}+\sqrt{1-\pi}}$

By virtue of (55),

$$D'(b) \geq \frac{\overline{\Pi}_n(p,q)}{p}\left[\frac{1}{b^2\pi} - \frac{2}{(1-b)^2(1-\pi)}\right], \qquad \forall b \in [l,u]. \tag{57}$$

Define

$$\delta := \frac{\sqrt{1-\pi}}{\sqrt{2\pi}+\sqrt{1-\pi}}.$$

Thus,

$$\frac{1}{\delta^2\pi} = \frac{2}{(1-\delta)^2(1-\pi)},$$

and, using (57), $D'(b) \geq 0$ for all $b \leq \delta$. By Lemma 8, $p > 1-\delta$ implies $l < \delta$. So $p > 1-\delta$ implies $D'(b) \geq 0$ for all $b \in [l,\delta]$. By (54), $p > 1-\delta$ therefore implies $D(b) \geq 0$ for all $b \in [l,\delta]$. Yet, by Lemma 8, $p > 1-\delta$ also implies $h(b) < 0$ for all $b \in [\delta,u]$. So $p > 1-\delta$ implies $D(b) \geq 0$ for all $b \in [l,u]$.

■

Proposition 8. *A WELM trading equilibrium exists for all values of p and q if $z \leq \frac{(1-\pi)^2}{8\pi(\sqrt{2\pi}+\sqrt{1-\pi})^2}$. In particular, for $(1-\pi)^2 \geq 8\pi\left(\sqrt{2\pi}+\sqrt{1-\pi}\right)^2$, a WELM trading equilibrium exists for all values of p, q and z.*

Proof: Recall: if $q = 1$ or $p \in \{0,1\}$ (or both) the existence of a WELM trading equilibrium then follows from the existence of a trading equilibrium in the baseline model. We therefore assume in the rest of the proof that $p \in (0,1)$ and $q < 1$.

Next, assume $z \leq \frac{(1-\pi)^2}{8\pi(\sqrt{2\pi}+\sqrt{1-\pi})^2}$. By Proposition 7, a WELM trading equilibrium exists whenever $p \geq \tilde{p} := \frac{\sqrt{2\pi}}{\sqrt{2\pi}+\sqrt{1-\pi}}$. Next, let, as in the proof of Proposition 7, $D(b) := 1 - \overline{\sigma}(b) - h(b)$. By (56), $p < \tilde{p}$ implies

$$D'(b) \geq \frac{\overline{\Pi}_n(p,q)}{p} \left[\frac{1}{\pi z} - \frac{8}{(1-\tilde{p})^2(1-\pi)} \right], \quad \forall b \in [l,u].$$

Yet,

$$\frac{1}{\pi z} - \frac{8}{(1-\tilde{p})^2(1-\pi)} \geq 0 \iff z \leq \frac{(1-\pi)^2}{8\pi(\sqrt{2\pi}+\sqrt{1-\pi})^2}.$$

Thus, $p < \tilde{p}$ implies $D'(b) \geq 0$ for all $b \in [l,u]$. Since $D(l) = 0$, we obtain $D(b) \geq 0$ for all $b \in [l,u]$ whenever $p < \tilde{p}$. By Proposition 6, a WELM trading equilibrium therefore exists for all $p < \tilde{p}$.

■

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