Why guests to Hilbert's Hotel are unwelcome: Strong anonymity and positional dominance*

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Abstract

This paper re-examines the incompatibility of Strong anonymity and Strong Pareto. We insist on Strong anonymity as an axiom of impartiality and ask how far the Paretian principle can be extended without contradicting Strong Anonymity. We show that Strong Anonymity combined with four rather innocent axioms has two consequences: (i) There is sensitivity for a person's well-being if and only if a co-finite set of people are strictly better than this person, and (ii) although Hilbert's paradox of the Grand Hotel shows that adding people to an infinite population is feasible, this addition cannot have positive social value.

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1 Introduction

Since the seminal paper of Diamond (1965), the conflict between sensitivity and impartiality in the evaluation of infinite well-being streams has been analyzed in many contribution, see e.g. Svensson (1980), Basu and Mitra (2003), Zame (2007) and Lauwers (2010), as well as Asheim (2010) for an overview. In particular, as shown by Van Liedekerke and Lauwers (1997, p. 163) by means of the following two streams,

$$\mathbf{x} = (1, 1, 1, 0, 1, 0, \dots, 1, 0, \dots) \text{ and } \mathbf{y} = (1, 0, 1, 0, 1, 0, \dots, 1, 0, \dots),$$

the axioms of **Strong Pareto**, in the sense of being sensitive to an increase of any one component of the stream if no other component is reduced, is incompatible with the axiom of **Strong anonymity**, in the sense of invariance to any permutation of a stream. In particular, $\mathbf{x} \succ \mathbf{y}$ if $\succsim$ satisfies Strong Pareto, while $\mathbf{x} \sim \mathbf{y}$ if $\succsim$ satisfies Strong Anonymity, as $\mathbf{x}$ is an infinite permutation of $\mathbf{y}$.

There are two main routes out of the dilemma that this impossibility poses:

(1) Stick with Strong Pareto and weaken Strong anonymity.

(2) Stick with Strong anonymity and weaken Strong Pareto.

Moreover, one can consider weakening both Strong Pareto and Strong anonymity.

Route (1) has been extensively explored. Strong Pareto is compatible with the axiom of **Finite anonymity**, in the sense of invariance to any *finite* permutation of a stream, if one is willing to give up completeness (Svensson 1980). In fact, there is a literature on how to extend impartiality beyond Finite anonymity. Specifically, a number of papers examine whether versions of the anonymity axiom that require invariance for specific set of permutations are compatible with Strong Pareto; see, for example, Fleurbaey and Michel (2003), Lauwers (1997, 1998), and Sakai (2010). As showed by Lauwers (1997), **Fixed-step anonymity**, which requires invariance to so-called fixed-step permutations rearranging components of a stream within each fixed range of consecutive coordinates, is compatible with strong Pareto. Mitra and Basu (2007) present a general analysis of anonymity axioms that are compatible with Strong Pareto from the view point of the requisite algebraic structure of a set of permutations. They showed that an anonymity axiom that is defined by a group of cyclic permutations is compatible with Strong Pareto (see also Adachi, Cato and Kamaga, 2014). However, Lauwers (2012) proved that a maximal group of cyclic permutation is a non-constructible object because its existence relies on the use of non-constructive mathematics like the Axiom of Choice. Therefore, it is impossible to give an explicit definition of a maximal anonymity axiom that is compatible with Strong Pareto.

In this paper we take route (2), not yet entirely explored, for avoiding the conflict between sensitivity and impartiality. Hence, we insist on Strong Anonymity as an axiom of impartiality and then ask how far the Paretian principle can be extended without contradicting Strong Anonymity. Strong Anonymity is clearly compatible with some sensitivity for the well-being of a single person, as illustrated by the *Maximin* order. Maximin is represented by the inferior of well-being taken over all people. It is invariant to any permutation of the stream, thus satisfying Strong Anonymity. If there is a sole person i that has a smaller well-being than all others, then the goodness of a stream is determined by person i, making this person a positional dictator. This also means that an increase in the well-being of person i, keeping the well-being of all others constant, makes the stream better. This is an example of *positional dominance*, as the sensitivity depends on the position of the person whose well-being is increased: Maximin is sensitive to an increase in the well-being of a single person if and only if the well-being of this person is strictly less than the well-being of all others. We will refer to this specific form of positional dominance as **Inf-restricted dominance**.

The *Leximin* order constitutes a way of extending the sensitivity of Maximin. Leximin restores Strong Pareto in the setting of finite well-being streams, while extensions of Leximin to the infinite-stream setting that insist on satisfy Strong Pareto must necessarily contradict Strong Anonymity. For example, in the infinite-stream versions of Leximin characterized by Asheim and Tungodden (2004) and Bossert, Sprumont and Suzumura (2007) Strong Pareto is maintained at the expense of weakening Strong Anonymity to Finite Anonymity. Hence, to define a version of Leximin in the infinite-stream setting that satisfies Strong Anonymity, Strong Pareto must be weakened. Asheim and Zuber (2013) do so by defining and characterizing an infinite-stream version of Leximin that has sensitivity for, not only an increase in the well-being of the worst-off, but an increase in the well-being of any person that is finitely ranked, in the sense that a co-finite set of people are strictly better than this person. This stronger form of positional dominance, which we will refer to as **Liminf-restricted dominance**, is also satisfied by the Extended rank-discounted utilitarian order defined and characterized by Zuber and Asheim (2012).

A central question posed in this paper is whether conditional dominance can be extended beyond Liminf-restricted dominance while insisting on Strong Anonymity. The answer is that Liminf-restricted dominance is as far as we can go under four additional, seemingly innocent, axioms: **Monotonicity**, **Continuity**, **Very weak Pigou-Dalton transfer** and **Critical-level consistency**. We are thereby able to characterized the extent of the Paretian principle under Strong Anonymity, provided that these four axioms are imposed: *Liminf-restricted dominance is the maximal sensitivity axiom.*

As illustrated by Hilbert's paradox of the Grand Hotel (Hilbert, 2013), one can augment an infinite population with a single person, or infinitely many people, without increasing the population's total size—which is already infinite. There is a relationship between, on the one hand, adding new people and moving the existing people to make room for the new ones and, on the other hand, increasing or decreasing the well-being of already existing people if one makes the alternative interpretation that everyone stays put. The sensitivity of increasing well-being below the limit inferior entailed by Liminf-restricted dominance means that adding people with well-being below this level has negative value as it can alternatively be interpreted as lowering the well-being of already existing people. Moreover, the insensitivity of increasing well-being at or above the limit inferior corresponds to zero value of adding people with well-being at or above this level. In both cases, adding people to a population that already is infinite has non-positive value. We show that this is a consequence of Strong Anonymity combined with the four additional axioms. So even though Hilbert's Grand Hotel can always accommodate new guests, they are invariably unwelcome in social evaluation.

We start out in Section 2 by defining four versions of conditional dominance and four different complete, reflexive and transitive binary relations that differ with respect to the conditional dominance axioms they satisfy. We introduce

Strong anonymity and the four additional axioms in Section 3. We then explore possibilities and impossibilities of positional dominance in Section 4. We state and prove the population-ethical result—that adding people cannot have positive social value—in Section 5. We finally discuss the merits of the Strong anonymity axiom in light of these results in the concluding Section 6. We strengthen our results in an appendix by weakening the Continuity axiom.

2 Positional dominance

2.1 Preliminaries

Let $\mathbb{R}$ (resp. $\mathbb{R}_+/\mathbb{R}_{++}$) be the set of all (resp. non-negative/positive) real numbers. Let $\mathbb{N}$ (resp. $\mathbb{N}_0$) denote the set of all positive (resp. non-negative) integers. An infinite stream (or allocation) of the well-being levels of infinitely many people is generically denoted by $\mathbf{x} = (x_1, x_2, \ldots, x_i, \ldots) \in \mathbb{R}^{\mathbb{N}}$, where $x_i \in \mathbb{R}$ is the well-being of person $i \in \mathbb{N}$. Throughout the paper, we restrict our attention to the set $\mathbf{X}$ of all bounded streams, which is defined by

$$\mathbf{X} = \left\{\mathbf{x} = (x_1, \ldots, x_i, \ldots) \in \mathbb{R}^{\mathbb{N}} : \sup_{i\in\mathbb{N}} |x_i| < +\infty\right\}.$$

Our notation for vector dominance between is as follows. For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \geq \mathbf{y}$ whenever $x_i \geq y_i$ for all $i \in \mathbb{N}$; $\mathbf{x} > \mathbf{y}$ if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$; and $\mathbf{x} \gg \mathbf{y}$ whenever $x_i > y_i$ for all $i \in \mathbb{N}$.

For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and for any $N \in \mathbb{N}_0$, $(\mathbf{y}_N, \mathbf{x}) \in \mathbf{X}$ is defined by

$$(\mathbf{y}_N, \mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } N = 0, \\ (y_1, \ldots, y_N, \mathbf{x}) & \text{if } 0 < N < +\infty. \end{cases}$$

Furthermore, given $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and $N = +\infty$, we write $\mathbf{z} = (\mathbf{y}_N, \mathbf{x})$ to mean $\mathbf{z} \in \mathbf{X}$ defined by

$$z_i = \begin{cases} y_n & \text{if } i = 2n - 1 \text{ and } n \in \mathbb{N}, \\ x_n & \text{if } i = 2n \text{ and } n \in \mathbb{N}, \end{cases}$$

that is, if $N = +\infty$, $(\mathbf{y}_N, \mathbf{x}) = (y_1, x_1, y_2, x_2, \ldots)$. For any $\ell \in \mathbb{R}$, let $(\ell)_{\text{con}} =$

$(\ell, \ell, \ldots) \in \mathbf{X}$. Thus, $((\ell)_{\text{con}}, \mathbf{x}) = (\ell, x_1, \ell, x_2 \ldots)$. Further, for any $\mathbf{x} \in \mathbf{X}$ and any $\ell \in \mathbb{R}$, we write $(\ell, \mathbf{x}) = (\ell, x_1, x_2, \ldots) \in \mathbf{X}$.

A *permutation* π of $\mathbb{N}$ is a bijection on $\mathbb{N}$. Let Π denote the set of all permutations of $\mathbb{N}$. For any $\mathbf{x} \in \mathbf{X}$ and any $\pi \in \Pi$, we write $\mathbf{x}_\pi = (x_{\pi(1)}, x_{\pi(2)}, \ldots) \in \mathbf{X}$.

A binary relation $\succsim$ on $\mathbf{X}$ is a subset of $\mathbf{X} \times \mathbf{X}$. For simplicity, we write $\mathbf{x} \succsim \mathbf{y}$ instead of $(\mathbf{x}, \mathbf{y}) \in \succsim$. The asymmetric and symmetric parts of $\succsim$ are denoted by $\succ$ and $\sim$, respectively. For any $k \in \mathbb{N} \setminus \{1, 2\}$ and for any $\mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathbf{X}$, if a binary relation $\succsim$ on $\mathbf{X}$ is transitive, we write, for simplicity, $\mathbf{x}^1 \succsim \mathbf{x}^2 \succsim \cdots \succsim \mathbf{x}^k$ to mean that $\mathbf{x}^\ell \succsim \mathbf{x}^{\ell+1}$ for all $\ell \in \{1, \ldots, k-1\}$, so that $\mathbf{x}^\ell \succsim \mathbf{x}^{\ell'}$ holds for all $\ell, \ell' \in \{1, \ldots, k\}$ with $\ell \geq \ell'$.

2.2 Four positional dominance axioms

Consider the following four positional dominance axioms:

Inf-restricted dominance. For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $x_i > y_i$ for some $i \in \mathbb{N}$ and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$, $\mathbf{x} \succ \mathbf{y}$ whenever $\inf_{j \in \mathbb{N}} x_j > y_i$.

Liminf-restricted dominance. For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $x_i > y_i$ for some $i \in \mathbb{N}$ and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$, $\mathbf{x} \succ \mathbf{y}$ whenever $\liminf_{j \in \mathbb{N}} x_j > y_i$.

Sup-restricted dominance. For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $x_i > y_i$ for some $i \in \mathbb{N}$ and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$, $\mathbf{x} \succ \mathbf{y}$ whenever $x_i > \sup_{j \in \mathbb{N}} y_j$.

Limsup-restricted dominance. For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $x_i > y_i$ for some $i \in \mathbb{N}$ and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$, $\mathbf{x} \succ \mathbf{y}$ whenever $x_i > \limsup_{j \in \mathbb{N}} y_j$.

Since $\liminf_{j \in \mathbb{N}} x_j \geq \inf_{j \in \mathbb{N}} x_j$, it follows that Liminf-restricted dominance implies Inf-restricted dominance, as it applies to at least as many pairs of x_i and y_i. For the same reason, as $\sup_{j \in \mathbb{N}} x_j \geq \limsup_{j \in \mathbb{N}} x_j$, we have that Limsup-restricted dominance implies Sup-restricted dominance.

Consider the following four complete, reflexive and transitive binary relations on $\mathbf{X} \times \mathbf{X}$.

Maximin: $\succsim_M^+$ represented by $W_M^+(\mathbf{x}) = \inf_{i \in \mathbb{N}} x_i = \inf_{\pi \in \Pi} x_{\pi(1)}$.

Progressive rank-discounted utilitarianism: $\succsim_R^+$ represented by

$$W_R^+(\mathbf{x}) = \inf_{\pi \in \Pi} \sum_{i=1}^{\infty} \beta^{i-1} u(x_{\pi(i)}) \,,$$

where $0 < \beta < 1$ and u is a continuous and increasing function.

Maximax: $\succsim_M^-$ represented by $W_M^-(\mathbf{x}) = \sup_{i \in \mathbb{N}} x_i = \sup_{\pi \in \Pi} x_{\pi(1)}$.

Regressive rank-discounted utilitarianism: $\succsim_R^-$ represented by

$$W_R^-(\mathbf{x}) = \sup_{\pi \in \Pi} \sum_{i=1}^{\infty} \beta^{i-1} u(x_{\pi(i)}) \,,$$

where $0 < \beta < 1$ and u is a continuous and increasing function.

We observe that Maximin satisfies Inf-restricted dominance, but not the other positional dominance axioms, and that Progressive rank-discounted utilitarianism satisfies Liminf-restricted dominance—and thus also Inf-restricted dominance—but not the two other positional dominance axioms. Likewise, Maximax satisfies Sup-restricted dominance, but not the other positional dominance axioms, while Regressive rank-discounted utilitarianism satisfies Limsup-restricted dominance—and thus also Sup-restricted dominance—but not the two other positional dominance axioms. We note that Progressive rank-discounted utilitarianism is identical to the Extended rank-discounted utilitarian order, as defined (somewhat differently) in Zuber and Asheim (2012, Definition 2)

3 Axioms

Throughout this paper we will insist on impartiality in the sense of the Strong anonymity axiom.

Strong anonymity. For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $\mathbf{x}_\pi = \mathbf{y}$ for some $\pi \in \Pi$, $\mathbf{x} \sim \mathbf{y}$.

Also, we will insist on monotonicity in the weak sense that a weak improvement of everyone's well-being does not make the well-being stream worse.

Monotonicity. For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $x_i \geq y_i$ for all $i \in \mathbb{N}$, $\mathbf{x} \succsim \mathbf{y}$.

We note that the conjunction of Liminf-restricted dominance and Monotonicity implies sensitivity to an increase in the limit inferior, and the conjunction of Limsup-restricted dominance and Monotonicty implies sensitivity to an increase in the limit supremum.

Lemma 1. *Consider* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *with* $\mathbf{x} \geq \mathbf{y}$. *Let* $\succsim$ *be a reflexive and transitive binary relation satisfying Monotonicity.*

(a) If $\succsim$ *also satisfies Liminf-restricted dominance, then* $\mathbf{x} \succ \mathbf{y}$ *whenever* $\liminf_j x_j > \liminf_j y_j$,

(b) If $\succsim$ *also satisfies Limsup-restricted dominance, then* $\mathbf{x} \succ \mathbf{y}$ *whenever* $\limsup_j x_j > \limsup_j y_j$.

Proof. Part (a). Let $0 < \varepsilon < \liminf_j x_j - \liminf_j y_j$. By the definition of the limit inferior, there are infinitely many integers k such that $y_k < \liminf_j y_j + \varepsilon$ but only finitely many integers l such that $x_l \leq \liminf_j y_j + \varepsilon < \liminf_j x_j$. So for infinitely many integers m, $x_m > \liminf_j y_j + \varepsilon > y_m$. Consider one of these integers, say i. Let $\mathbf{z}$ be derived from $\mathbf{x}$ by replacing x_i by y_i, where by the choice of ε and i, $\min\{x_i, \liminf_j x_j\} > y_i$. By Monotonicity, $\mathbf{z} \succsim \mathbf{y}$, and by Liminf-restricted dominance, $\mathbf{x} \succ \mathbf{z}$. Hence, by transitivity, $\mathbf{x} \succ \mathbf{y}$.

Part (b). The proof is similar. □

In most of this paper, we also impose a continuity requirement using the supnorm topology, based on the distance function $d\colon \mathbf{X} \times \mathbf{X} \to \mathbb{R}_+$ given by, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$d(\mathbf{x}, \mathbf{y}) = \sup_{i \in \mathbb{N}} |x_i - y_i|.$$

Continuity. For any $\mathbf{x} \in \mathbf{X}$, the sets $\{\mathbf{y} \in \mathbf{X} : \mathbf{y} \succsim \mathbf{x}\}$ and $\{\mathbf{y} \in \mathbf{X} : \mathbf{y} \precsim \mathbf{x}\}$ are closed in the supnorm topology.

The supnorm topology is a rather large topology, but it makes it possible to prove our results in a straightforward manner. In the Appendix we show how we can establish our main results also under weaker continuity properties.

All four binary relations considered in Section 2 satisfy Strong anonymity, Monotonicity and Continuity.

For any reflexive and transitive binary relation $\succsim$, let $\varepsilon^{\succsim} \in \mathbb{R}_+ \cup \{+\infty\}$ be the supremum of the set $\mathcal{E}$ defined as

$$\mathcal{E} = \{\varepsilon \in \mathbb{R}_{++} : \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbf{X} \text{ with } x_i < y_i \leq y_j < x_j \text{ for some } i, j \in \mathbb{N}, \text{ and } \\ x_k = y_k \text{ for all } k \in \mathbb{N} \setminus \{i, j\}, \mathbf{x} \precsim \mathbf{y} \text{ whenever } x_j - y_j = \varepsilon\,(y_i - x_i)\},$$

if $\mathcal{E}$ is non-empty and equal to 0 otherwise. The number $\varepsilon^{\succsim}$ measures the highest socially acceptable well-being loss for a rich person when a transfer is made from a rich person to a poor person and the poor receives a well-being gain of size 1. Notice that under the axiom of **Hammond equity** (Hammond, 1976) any loss is acceptable, so that $\mathcal{E} = \mathbb{R}_{++}$. This means that Hammond equity is equivalent to $\varepsilon^{\succsim} = +\infty$. If non-leaky transfers are acceptable, then it must be the case that $1 \in \mathcal{E}$. Hence, a reflexive and transitive binary relation $\succsim$ satisfies the **Pigou-Dalton transfer** principle (in its weak version) only if $\varepsilon^{\succsim} \geq 1$.

Here we will be concerned with a much weaker axiom than both Hammond equity and Pigou-Dalton: we simply ask that transfers from rich to poor people are acceptable if the well-beging loss for the rich person is sufficiently small.

Very weak Pigou-Dalton transfer. $\varepsilon^{\succsim} > 0$.

We have that $\varepsilon^{\succsim_M^+} = \infty$ and $\varepsilon^{\succsim_R^+} \geq 1$ (under an assumption given by Zuber and Asheim, 2012, Proposition 6), while $\varepsilon^{\succsim_M^-} = 0$ and $\varepsilon^{\succsim_R^-} = 0$ since both Maximax and Regressive rank-discounted utilitarianism have sensitivity above the supremum, but no sensitivity below the limit supremum (so that well-being gains for people below the limit supremum are worthless and cannot compensate for well-being losses for people above the supremum). Thus, of the four binary relations considered in Section 2, only Maximin and Progressive rank-discounted utilitarianism satisfy Very weak Pigou-Dalton transfer.

Our last axiom has to do with the effect of adding one person (or infinitely many people) to a population. This kind of question has been addressed in the literature on population ethics stemming from Parfit (1984) and discussed at length in Blackorby, Bossert and Donaldson (2005). In particular, Blackorby and Donaldson (1984) and Blackorby, Bossert and Donaldson (1995) have argued in favor of the existence of a critical level, such that a person's life contributes positively to the value of a population if only if the well-being of the additional person is above this critical level. In general, the critical level may depend on the distribution of utility in the population.

The next principle imposes some regularity in the level of the critical level for infinite populations. The axioms asserts that, if it is acceptable to add one person at some level of well-being to a population, then it is also acceptable to add infinitely many people at this level.

Critical-level consistency. For any $\mathbf{x} \in \mathbf{X}$ and $z \in \mathbb{R}$, $\mathbf{x} \succsim (z, \mathbf{x})$ (resp. $\mathbf{x} \precsim (z, \mathbf{x})$) if and only if $\mathbf{x} \succsim ((z)_{\text{con}}, \mathbf{x})$ (resp. $\mathbf{x} \precsim ((z)_{\text{con}}, \mathbf{x})$).

All four binary relations considered in Section 2 satisfy Critical-level consistency. Indeed:

- $\mathbf{x} \succ_M^+ (z, \mathbf{x})$ and $\mathbf{x} \succ_M^+ ((z)_{\text{con}}, \mathbf{x})$ if $z < \inf_{j \in \mathbb{N}} x_j$, while $\mathbf{x} \sim_M^+ (z, \mathbf{x}) \sim_M^+ ((z)_{\text{con}}, \mathbf{x})$ otherwise,
- $\mathbf{x} \succ_R^+ (z, \mathbf{x})$ and $\mathbf{x} \succ_R^+ ((z)_{\text{con}}, \mathbf{x})$ if $z < \liminf_{j \in \mathbb{N}} x_j$, while $\mathbf{x} \sim_M^+ (z, \mathbf{x}) \sim_M^+ ((z)_{\text{con}}, \mathbf{x})$ otherwise,
- $\mathbf{x} \prec_M^- (z, \mathbf{x})$ and $\mathbf{x} \prec_M^- ((z)_{\text{con}}, \mathbf{x})$ if $z > \sup_{j \in \mathbb{N}} x_j$, while $\mathbf{x} \sim_M^- (z, \mathbf{x}) \sim_M^- ((z)_{\text{con}}, \mathbf{x})$ otherwise,
- $\mathbf{x} \prec_R^- (z, \mathbf{x})$ and $\mathbf{x} \prec_R^- ((z)_{\text{con}}, \mathbf{x})$ if $z > \limsup_{j \in \mathbb{N}} x_j$, while $\mathbf{x} \sim_R^- (z, \mathbf{x}) \sim_R^- ((z)_{\text{con}}, \mathbf{x})$ otherwise.

4 Possibilities and impossibilities of positional dominance

We first introduce and prove two lemmas before turning to the main results. To introduce the first of these lemmas, recall the example of two unbounded streams used by Fleurbaey and Michel (2003, pp. 795–796) to prove that Strong anonymity is incompatible with even the Weak Pareto axiom. A small variation of their example allows us to show the conflict between these two axioms also in our setting of bounded streams: There exists $\pi \in \Pi$ such that

$$\begin{aligned} \mathbf{z} &= \left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{4}{5}, \tfrac{1}{5}, \cdots, \tfrac{k+1}{k+2}, \tfrac{1}{k+2} \cdots\right), \text{ and} \\ \mathbf{z}_\pi &= \left(\tfrac{3}{4}, \tfrac{2}{3}, \tfrac{4}{5}, \tfrac{1}{3}, \tfrac{5}{6}, \tfrac{1}{4}, \cdots, \tfrac{k+2}{k+3}, \tfrac{1}{k+1} \cdots\right), \end{aligned}$$

where by Strong anonymity $\mathbf{z}$ is indifferent to $\mathbf{z}_\pi$ even though $z_i < z_{\pi(i)}$ for all $i \in \mathbb{N}$. Notice that $\mathbf{z} = (\mathbf{z}^+, \mathbf{z}^-)$, where $\mathbf{z}^+$ is an increasing subsequence

$$\mathbf{z}^+ = \left(\tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{5}, \cdots, \tfrac{k+1}{k+2}, \cdots\right),$$

and $\mathbf{z}^-$ is an decreasing subsequence

$$\mathbf{z}^- = \left(\tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{5}, \cdots, \tfrac{1}{k+2}, \cdots\right).$$

Consider now the streams $\mathbf{x}$ and $\mathbf{y}$, where

$$\begin{aligned} \mathbf{x} = (\tfrac{2}{3}, \mathbf{z}) &= \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{4}{5}, \tfrac{1}{5}, \cdots, \tfrac{k+1}{k+2}, \tfrac{1}{k+2} \cdots\right), \text{ and} \\ \mathbf{y} = (\tfrac{1}{3}, \mathbf{z}) &= \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{4}{5}, \tfrac{1}{5}, \cdots, \tfrac{k+1}{k+2}, \tfrac{1}{k+2} \cdots\right), \end{aligned}$$

Clearly, if $\succsim$ satisfies Monotonicity, then $\mathbf{x} \succsim \mathbf{y}$, as the only difference between the two streams is that $x_1 = 2/3 > 1/3 = y_1$. However, the fact that $\mathbf{z} = (\mathbf{z}^+, \mathbf{z}^-)$ with $\mathbf{z}^+$ being an increasing sequence with $x_1 = 2/3 = z_1^+$ and $\mathbf{z}^-$ being an decreasing sequence with $y_1 = 1/3 = z_1^-$ implies that there exists $\pi' \in \Pi$ such that $\mathbf{y}_{\pi'} \geq \mathbf{x}$:

$$\mathbf{y}_{\pi'} = \left(\tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{3}, \tfrac{4}{5}, \tfrac{1}{3}, \tfrac{5}{6}, \tfrac{1}{4}, \cdots, \tfrac{k+2}{k+3}, \tfrac{1}{k+1} \cdots\right).$$

So, if $\succsim$ satisfies also Strong anonymity, then $\mathbf{y} \sim \mathbf{y}_{\pi'} \succsim \mathbf{x} \succsim \mathbf{y}$. Hence, by reflexivity and transitivity of $\succsim$, $\mathbf{x} \sim \mathbf{y}$, showing that $\succsim$ cannot be sensitive to an increase of a component from 1/3 to 2/3 when the rest of the stream equals $\mathbf{z} = (\mathbf{z}^+, \mathbf{z}^-)$. The first lemma generalizes this observation.

Lemma 2. *Consider $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $x_i > y_i = z_i$ for some $i \in \mathbb{N}$ and $x_j = y_j = z_j$ for all $j \in \mathbb{N} \setminus \{i\}$. If $\succsim$ is reflexive and transitive binary relation satisfying Strong anonymity and Monotonicity, then $\mathbf{x} \sim \mathbf{y}$ whenever $(z_j)_{j \in \mathbb{N} \setminus \{i\}}$ satisfies*

(a) there exists an increasing function $f : \mathbb{N} \to \mathbb{N} \setminus \{i\}$ such that, for all $k \in \mathbb{N}$,

$x_i \leq z_{f(k)} \leq z_{f(k+1)}$, and

(b) there exists an increasing function $g : \mathbb{N} \to \mathbb{N} \setminus \{i\}$ such that, for all $k \in \mathbb{N}$, $y_i \geq z_{g(k)} \geq z_{g(k+1)}$.

Proof. Consider the permutation function π constructed by

- $\pi(i) = f(1)$ and $\pi(f(k)) = f(k+1)$ for all $k \in \mathbb{N}$,
- $\pi(g(1)) = i$ and $\pi(g(k+1)) = g(k)$ for all $k \in \mathbb{N}$,

and $\pi(j) = j$ otherwise. Then $\mathbf{y}_\pi \geq \mathbf{x} > \mathbf{y}$, so by Strong anonymity and Monotonicity: $\mathbf{y} \sim \mathbf{y}_\pi \succsim \mathbf{x} \succsim \mathbf{y}$. Hence, by reflexivity and transitivity of $\succsim$, $\mathbf{x} \sim \mathbf{y}$. □

In fact, Lemma 2 can be used to show that $\mathbf{x}' \sim \mathbf{y}'$, where

$$\begin{aligned} \mathbf{x}' &= (x'_1, \mathbf{z}) = \left(x'_1, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{4}{5}, \tfrac{1}{5}, \cdots, \tfrac{k+1}{k+2}, \tfrac{1}{k+2} \cdots\right), \text{ and} \\ \mathbf{y}' &= (y'_1, \mathbf{z}) = \left(y'_1, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{4}{5}, \tfrac{1}{5}, \cdots, \tfrac{k+1}{k+2}, \tfrac{1}{k+2} \cdots\right), \end{aligned}$$

whenever $\liminf_{j\in\mathbb{N}} x'_j < y'_1 < x'_1 < \limsup_{j\in\mathbb{N}} y'_j$ since subsequences as specified in (a) and (b) exist. However, two such subsequences do not exist if $\liminf_{j\in\mathbb{N}} x'_j = y'_1 < x'_1 \leq \limsup_{j\in\mathbb{N}} y'_j$ or $\liminf_{j\in\mathbb{N}} x'_j \leq y'_1 < x'_1 = \limsup_{j\in\mathbb{N}} y'_j$, so that Lemma 2 does not apply under these cases. However, the non-sensitivity result of Lemma 2 can be extended to cover also such circumstances if Continuity is imposed. This extension relies on the following lemma, showing that any reflexive and transitive binary relation satisfying Strong anonymity and Continuity is invariant to adding one person (or infinitely many people) with well-being equal to the limit inferior or the limit supremum, or indeed equal to any other cluster point.

Lemma 3. *Consider $\mathbf{x} \in \mathbf{X}$ and let z be a cluster point for $\mathbf{x}$. If $\succsim$ is a reflexive and transitive binary relation satisfying Strong anonymity and Continuity, then $\mathbf{x} \sim (z, \mathbf{x}) \sim ((z)_{con}, \mathbf{x})$.*

Proof. Let $\mathbf{x} \in \mathbf{X}$ and let z be a cluster point for $\mathbf{x}$. Assume that $\succsim$ is a reflexive and transitive binary relation satisfying Strong anonymity and Continuity.

We prove that $\mathbf{x} \sim (z, \mathbf{x})$ and $\mathbf{x} \sim ((z)_{\text{con}}, \mathbf{x})$, which imply $(z, \mathbf{x}) \sim ((z)_{\text{con}}, \mathbf{x})$ by the transitivity of $\succsim$.

Since z is a cluster point of $\mathbf{x}$, there exists an increasing function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\lim_{i\to\infty} |x_{f(i)} - z| = 0$. Hence, for every $m \in \mathbb{N}$, there exists $N(m) \in \mathbb{N}$ such that $|x_{f(i)} - z| < 1/(2m)$ for all $i > N(m)$. For each $m \in \mathbb{N}$, define the increasing function $f^m\colon \mathbb{N} \to \mathbb{N}$ by $f^m(i) = f(i + N(m))$ for all $i \in \mathbb{N}$. By construction, for every $m \in \mathbb{N}$ and for all $i, j \in \mathbb{N}$,

$$|x_{f^m(i)} - x_{f^m(j)}| \leq |x_{f^m(i)} - z| + |z - x_{f^m(j)}| < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}\,. \tag{1}$$

For all $m \in \mathbb{N}$, let $\tilde{\mathbf{x}}^m = (x_{f^m(k)})_{k\in\mathbb{N}}$ and we define sequences $(\hat{\mathbf{x}}^m)_{m\in\mathbb{N}}$, $(\check{\mathbf{x}}^m)_{m\in\mathbb{N}}$ in $\mathbf{X}$ as follows. For all $m \in \mathbb{N}$,

$$\hat{x}_i^m = \check{x}_i^m = x_i \ \text{ for all } i \in \mathbb{N} \setminus \{f^m(k) : k \in \mathbb{N}\}$$

and

$$(\hat{x}^m_{f^m(k)})_{k\in\mathbb{N}} = (z, \tilde{\mathbf{x}}^m) \ \text{ and } \ (\check{x}^m_{f^m(k)})_{k\in\mathbb{N}} = ((z)_{\text{con}}, \tilde{\mathbf{x}}^m).$$

From (1) and the construction of $(\hat{\mathbf{x}}^m)_{m\in\mathbb{N}}$ and $(\check{\mathbf{x}}^m)_{m\in\mathbb{N}}$, it follows that

$$\lim_{m\to+\infty} \sup_{i\in\mathbb{N}} |x_i - \hat{x}_i^m| = \lim_{m\to+\infty} \sup_{i\in\mathbb{N}} |\tilde{x}_i^m - \hat{x}^m_{f^m(i)}| = 0\,, \tag{2a}$$

$$\lim_{m\to+\infty} \sup_{i\in\mathbb{N}} |x_i - \check{x}_i^m| = \lim_{m\to+\infty} \sup_{i\in\mathbb{N}} |\tilde{x}_i^m - \check{x}^m_{f^m(i)}| = 0\,. \tag{2b}$$

Note that for each $m \in \mathbb{N}$, there exist $\pi^m, \rho^m \in \Pi$ such that

$$(z, \mathbf{x}) = \hat{\mathbf{x}}^m_{\pi^m} \ \text{ and } \ ((z)_{\text{con}}, \mathbf{x}) = \check{\mathbf{x}}^m_{\rho^m}.$$

By Strong anonymity, we obtain $(z, \mathbf{x}) \sim \hat{\mathbf{x}}^m$ and $((z)_{\text{con}}, \mathbf{x}) \sim \check{\mathbf{x}}^m$ for all $m \in \mathbb{N}$. Since $\succsim$ satisfies Continuity, it follows from (2a) and (2b) that $(z, \mathbf{x}) \sim \mathbf{x}$ and $((z)_{\text{con}}, \mathbf{x}) \sim \mathbf{x}$. □

Lemma 3 can also be seen as a generalization of the observation that, under Strong anonymity, the streams $(1, 0, 1, 0, 1, 0, \dots)$ and $(0, 1, 0, 1, 0, 1, \dots)$ are equally good as one stream can obtained from the other through an infinite permutation. However, the second stream can also be obtained from the first by adding on person with well-being equal to the limit inferior and the first stream can be obtained from the second by adding a person with well-being equal to the limit supremum.

Using our two lemmas, we are able to prove three main results regarding positional dominance. The first proposition shows that any reflexive and transitive binary relation satisfying Strong anonymity, Monotonicity and Continuity is insensitive to increasing a person's well-being between the limit inferior and the limit supremum.

Proposition 1. *Consider* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *with* $x_i > y_i$ *for some* $i \in \mathbb{N}$ *and* $x_j = y_j$ *for all* $j \in \mathbb{N} \setminus \{i\}$. *The two following statements are equivalent:*

(1) For all reflexive and transitive binary relations $\succsim$ *satisfying Strong anonymity, Monotonicity and Continuity, we have that* $\mathbf{x} \sim \mathbf{y}$.

(2) $\limsup_{j\in\mathbb{N}} y_j \geq x_i$ *and* $y_i \geq \liminf_{j\in\mathbb{N}} x_j$.

Proof. Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, and assume that $x_i > y_i$ for some $i \in \mathbb{N}$ and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$.

We begin by showing that (1) implies (2). Suppose that (2) does not hold. Then there are two subcases:

Subcase (i): $x_i > \limsup_{j\in\mathbb{N}} y_j$. Consider $\succsim_R^-$. We have that $\succsim_R^-$ is a reflexive and transitive binary relation that satisfies Strong anonymity, Monotonicity and Continuity. Furthermore, $\mathbf{x} \succ \mathbf{y}$ since $\succsim_R^-$ satisfies Limsup-restricted dominance.

Subcase (ii): $\liminf_{j\in\mathbb{N}} x_j > y_i$. Consider $\succsim_R^+$. We have that $\succsim_R^+$ is a reflexive and transitive binary relation that satisfies Strong anonymity, Monotonicity and Continuity. Furthermore, $\mathbf{x} \succ \mathbf{y}$ since $\succsim_R^+$ satisfies Liminf-restricted dominance.

Next, we show that (2) implies (1). Assume that (2) holds and assume that $\succsim$ is a reflexive and transitive binary relation that satisfies Strong anonymity, Monotonicity and Continuity.

Both $u = \limsup_{j\in\mathbb{N}} x_j = \limsup_{j\in\mathbb{N}} y_j$ and $\ell = \liminf_{j\in\mathbb{N}} x_j = \liminf_{j\in\mathbb{N}} y_j$ are cluster points for $\mathbf{x}$ and $\mathbf{y}$, so by repeated use of Lemma 3 we have that

$$\mathbf{x} \sim ((\ell)_{\text{con}}, \mathbf{x}) \sim ((u)_{\text{con}}, ((\ell)_{\text{con}}, \mathbf{x})) \text{ and } \mathbf{y} \sim ((\ell)_{\text{con}}, \mathbf{y}) \sim ((u)_{\text{con}}, ((\ell)_{\text{con}}, \mathbf{y}))\,.$$

Write $\mathbf{x}' = ((u)_{\text{con}}, ((\ell)_{\text{con}}, \mathbf{x}))$ and $\mathbf{y}' = ((u)_{\text{con}}, ((\ell)_{\text{con}}, \mathbf{y}))$. By construction, $x'_{i'} > y'_{i'}$ for some $i' \in \mathbb{N}$ and $x'_j = y'_j = z_j$ for all $j \in \mathbb{N} \setminus \{i'\}$. Furthermore, $(z_j)_{j\in\mathbb{N}\setminus\{i'\}}$ satisfies

(a) there exists an increasing function $f : \mathbb{N} \to \mathbb{N} \setminus \{i'\}$ such that, for all $k \in \mathbb{N}$, $x'_{i'} \leq z_{f(k)} = u$, and

(b) there exists an increasing function $g : \mathbb{N} \to \mathbb{N} \setminus \{i'\}$ such that, for all $k \in \mathbb{N}$, $y'_{i'} \geq z_{g(k)} = \ell$.

By Lemma 2, $\mathbf{x} \sim \mathbf{x}' \sim \mathbf{y}' \sim \mathbf{x}$. Thus, $\mathbf{x} \sim \mathbf{y}$ by the transitivity of $\succsim$. □

Before stating our two last main results, we prove the following lemma which goes one step further by adding the Very weak Pigou-Dalton transfer axiom. It shows that a binary relation that satisfies also this axiom is invariant to increasing the well-being of a person that is already at the limit inferior or above, in the case where the stream under consideration has a limit inferior different from its limit supremum.

Lemma 4. *Consider* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *with* $x_i > y_i$ *for some* $i \in \mathbb{N}$ *and* $x_j = y_j$ *for all* $j \in \mathbb{N} \setminus \{i\}$. *If* $\succsim$ *is a reflexive and transitive binary relation* $\succsim$ *satisfying Strong anonymity, Monotonicity, Continuity and Very weak Pigou-Dalton transfer, then* $\mathbf{x} \sim \mathbf{y}$ *whenever* $\limsup_{j\in\mathbb{N}} y_j > \liminf_{j\in\mathbb{N}} x_j$ *and* $y_i \geq \liminf_{j\in\mathbb{N}} x_j$.

Proof. Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, and assume that $x_i > y_i$ for some $i \in \mathbb{N}$ and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$. Assume that $\succsim$ is a reflexive and transitive binary relation that satisfies Strong anonymity, Monotonicity, Continuity and Very weak Pigou-Dalton transfer.

The result follows from Proposition 1 if $\limsup_j y_j \geq x_i$.

Now, suppose that $x_i > \limsup_{j\in\mathbb{N}} y_j$. By monotonicity, $\mathbf{x} \succsim \mathbf{y}$. We need to establish that $\mathbf{y} \succsim \mathbf{x}$. Write $u = \limsup_{j\in\mathbb{N}} x_j = \limsup_{j\in\mathbb{N}} y_j$, $\ell = \liminf_{j\in\mathbb{N}} x_j = \liminf_{j\in\mathbb{N}} y_j$ and $\delta = \varepsilon(u - \ell)$, where $\varepsilon > 0$ is chosen such that $\varepsilon \in \mathcal{E}$; this is possible since $\succsim$ satisfies Very weak Pigou-Dalton transfer, implying that $\mathcal{E} \neq \emptyset$.

Assume that there exists $k \in \mathbb{N} \setminus \{i\}$ such that $u \geq x_k = y_k \geq \ell$. Define $\mathbf{y}^0$ by $y^0_i = \max\{y_i, u\}$, $y^0_k = u$ and $y^0_j = y_j$ for $j \in \mathbb{N} \setminus \{i, k\}$. Define inductively $\mathbf{x}^n$ by $x^n_i = y^{n-1}_i + \delta$ and $x^n_k = \ell$ and $x^n_j = y^{n-1}_j$ for $j \in \mathbb{N} \setminus \{i, k\}$, and $\mathbf{y}^n$ by $y^n_k = u$ and $y^n_j = x^n_j$ for $j \in \mathbb{N} \setminus \{k\}$ for $n = 1, 2, 3, \ldots, \bar{n}$ until $\mathbf{y}^{\bar{n}} \geq \mathbf{x}$. Then:

$$\mathbf{y} \sim \mathbf{y}^0 \succsim \mathbf{x}^1 \sim \mathbf{y}^1 \succsim \cdots \succsim \mathbf{x}^{\bar{n}} \sim \mathbf{y}^{\bar{n}} \succsim \mathbf{x}\,,$$

where $\mathbf{y} \sim \mathbf{y}^0$ and $\mathbf{x}^{\bar{n}} \sim \mathbf{y}^{\bar{n}}$ follow from Proposition 1, $\mathbf{y}^{n-1} \succsim \mathbf{x}^n$ for $n = 1, 2, 3, \ldots, \bar{n}$ follow from Very weak Pigou-Dalton transfer, and $\mathbf{y}^{\bar{n}} \succsim \mathbf{x}$ follows from Monotonicity.

In the case where there does not exist $k \in \mathbb{N} \setminus \{i\}$ such that $u \geq x_k = y_k \geq \ell$, it follows from Lemma 3 that $\mathbf{x} \sim \mathbf{y}$ if and only if $(u, \mathbf{x}) \sim (u, \mathbf{y})$ since $u = \limsup_{j \in \mathbb{N}} x_j = \limsup_{j \in \mathbb{N}} y_j$ is a cluster point for $\mathbf{x}$ and $\mathbf{y}$. Hence, we can use the argument above on the pair $((u, \mathbf{x}), (u, \mathbf{y}))$ instead of $(\mathbf{x}, \mathbf{y})$. $\square$

Lemma 4 implies that Limsup-restricted dominance is inconsistent with the four axioms of the lemma because such dominance cannot be imposed on streams where the limit inferior is strictly lower than the limit supremum. This almost establishes the result that any reflexive and transitive binary relation satisfying these four axioms cannot exhibit any sensitivity beyond Liminf-restricted dominance. In order to show this result we must remove the restriction made in Lemma 4 that the streams under consideration must have a limit inferior strictly lower than the limit supremum. The next proposition shows that it is actually true in general that if a reflexive and transitive binary relation satisfies Strong anonymity, Monotonicity, Continuity and Very weak Pigou-Dalton transfer, then it cannot exhibit sensitivity beyond Liminf-restricted dominance. Moreover, this result is tight since the converse follows from the fact that Progressive rank-discounted utilitarianism satisfies these axioms as well as Liminf-restricted dominance.

Proposition 2. *Consider* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *with* $x_i > y_i$ *for some* $i \in \mathbb{N}$ *and* $x_j = y_j$ *for all* $j \in \mathbb{N} \setminus \{i\}$. *The two following statements are equivalent:*

(1) For all reflexive and transitive binary relations $\succsim$ *satisfying Strong anonymity, Monotonicity, Continuity and Very weak Pigou-Dalton transfer, we have that* $\mathbf{x} \sim \mathbf{y}$.

(2) $y_i \geq \liminf_{j \in \mathbb{N}} x_j$.

Proof. Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, and assume that $x_i > y_i$ for some $i \in \mathbb{N}$ and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$.

We begin by showing that (1) implies (2). Suppose that (2) does not hold, implying that $\liminf_{j \in \mathbb{N}} x_j > y_i$. Consider $\succsim_R^+$ with u being linear. We have that $\succsim_R^+$ is a reflexive and transitive binary relation that satisfies Strong anonymity, Monotonicity, Continuity and Very weak Pigou-Dalton transfer. Furthermore, $\mathbf{x} \succ \mathbf{y}$ since $\succsim_R^+$ satisfies Liminf-restricted dominance.

Next, we show that (2) implies (1). Assume that (2) holds and assume that $\succsim$ is a reflexive and transitive binary relation that satisfies Strong anonymity, Monotonicity, Continuity and Very weak Pigou-Dalton transfer.

There are two subcases:

Subcase (i): $\limsup_{j\in\mathbb{N}} y_j > \limsup_{i\in\mathbb{N}} x_i$. In that case, the result follows from Lemma 4.

Subcase (ii): $\limsup_{j\in\mathbb{N}} y_j = \liminf_{i\in\mathbb{N}} x_i = \ell$. In that case, ℓ is a cluster point. By Lemma 3, $((\ell)_{\text{con}}, \mathbf{x}) \sim \mathbf{x}$ and $((\ell)_{\text{con}}, \mathbf{y}) \sim \mathbf{y}$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to 0. By Lemma 4, $((\ell+\varepsilon_n)_{\text{con}}, \mathbf{x}) \sim ((\ell+\varepsilon_n)_{\text{con}}, \mathbf{y})$ for each $n \in \mathbb{N}$. Furthermore, $((\ell+\varepsilon_n)_{\text{con}}, \mathbf{x}) > ((\ell)_{\text{con}}, \mathbf{x})$ so that $((\ell+\varepsilon_n)_{\text{con}}, \mathbf{x}) \succsim ((\ell)_{\text{con}}, \mathbf{x}) \sim \mathbf{x}$ by Monotonicity. By transitivity, $((\ell+\varepsilon_n)_{\text{con}}, \mathbf{y}) \succsim \mathbf{x}$. This is true for each $n \in \mathbb{N}$ so that $((\ell)_{\text{con}}, \mathbf{y}) \succsim \mathbf{x}$ by Continuity and $\mathbf{y} \succsim \mathbf{x}$ by transitivity. On the other hand, $\mathbf{x} \succsim \mathbf{y}$ by Monotonicity. □

Proposition 2 shows that under Strong anonymity, Monotonicity, Continuity and Very weak Pigou-Dalton transfer, dominance can occur, when increasing the well-being of a single person, if and only if the well-being of this person is strictly lower than the limit inferior before the increase. Any reflexive and transitive binary relation satisfying these axioms is invariant to an increase in a person's well-being at or above the limit inferior. Notice that Regressive rank-discounted utilitarianism satisfy Limsup-restricted dominance and all the other axioms except for Very weak Pigou-Dalton transfer. Our minimal equity requirement is essential to obtain the result.

On the other hand, Inf-restricted dominance and Liminf-restricted dominance are clearly compatible with the other axioms: both Maximin and Progressive rank-discounted utilitarianism satisfy Strong anonymity, Monotonicity, Continuity and Very weak Pigou-Dalton transfer.

Our last proposition in this section shows that we can generalize the result to the case where not only one person, but infinitely many people experience an increase in well-being above the limit inferior. To obtain this result, we need the additional axiom of Critical-level consistency.

Proposition 3. *Consider* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *with* $\mathbf{x} \geq \mathbf{y}$ *and* $\mathbf{x} \neq \mathbf{y}$. *The two following statements are equivalent:*

(1) For all refexive and transitive binary relations $\succsim$ *satisfying Strong anonymity, Monotonicity, Continuity, Very weak Pigou-Dalton transfer and Critical-level consistency, we have that* $\mathbf{x} \sim \mathbf{y}$.

(2) $\ell = \liminf_{j \in \mathbb{N}} x_j = \liminf_{j \in \mathbb{N}} y_j$ *and* $y_i \geq \ell$ *for all* $i \in \mathbb{N}$ *such that* $x_i > y_i$.

Proof. Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, and assume that $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

We begin by showing that (1) implies (2). Suppose that (2) does not hold, implying that (i) $\liminf_{j \in \mathbb{N}} x_j = \liminf_{j \in \mathbb{N}} y_j$ and there exists $i \in \mathbb{N}$ such that $\min\{x_i, \liminf_{j \in \mathbb{N}} x_j\} > y_i$, or (ii) $\liminf_{j \in \mathbb{N}} x_j > \liminf_{j \in \mathbb{N}} y_j$. Consider $\succsim_R^+$ with u being linear. We have that $\succsim_R^+$ is a reflexive and transitive binary relation that satisfies Strong anonymity, Monotonicity, Continuity, Very weak Pigou-Dalton transfer and Critical-level consistency. Furthermore, $\mathbf{x} \succ \mathbf{y}$ since $\succsim_R^+$ satisfies Monotonicity and Liminf-restricted dominance (cf. Lemma 1(a) in case (ii)).

Next, we show that (2) implies (1). Assume that (2) holds and assume that $\succsim$ is a reflexive and transitive binary relation that satisfies Strong anonymity, Monotonicity Continuity, Very weak Pigou-Dalton transfer and Critical-level consistency. Denote $I_> = \{i \in \mathbb{N} : x_i > y_i\}$ the set of coordinates where utility is strictly larger in $\mathbf{x}$ than in $\mathbf{y}$.

There are two subcases:

Subcase (i): $|I_>| < +\infty$. In that case, the result follows from repeated applications of Proposition 2 and transitivity.

Subcase (ii): $|I_>| = +\infty$. Since ℓ is a cluster point, by Lemma 3 we obtain $((\ell)_{\text{con}}, \mathbf{x}) \sim \mathbf{x}$ and $((\ell)_{\text{con}}, \mathbf{y}) \sim \mathbf{y}$. Let $\mathbf{z}$ and $\hat{\mathbf{z}}$ be two streams such that:

$$z_i = \begin{cases} \ell & \text{if } i = 2n-1 \text{ and } n \in \mathbb{N}, \\ x_i & \text{if } i = 2n \text{ and } n \in \mathbb{N} \setminus I_>, \\ \sup_{j \in \mathbb{N}} x_j & \text{if } i = 2n \text{ and } n \in I_>, \end{cases}$$

and

$$\hat{z}_i = \begin{cases} \ell & \text{if } i = 2n-1 \text{ and } n \in \mathbb{N}, \\ x_i & \text{if } i = 2n \text{ and } n \in \mathbb{N} \setminus I_>, \\ \ell & \text{if } i = 2n \text{ and } n \in I_>. \end{cases}$$

By definition, $\mathbf{z} \geq ((\ell)_{\text{con}}, \mathbf{x})$ and $((\ell)_{\text{con}}, \mathbf{y}) \geq \hat{\mathbf{z}}$, so that by Monotonicity $\mathbf{z} \succsim ((\ell)_{\text{con}}, \mathbf{x})$ and $((\ell)_{\text{con}}, \mathbf{y}) \succsim \hat{\mathbf{z}}$. Now we want to show that $\mathbf{z} \sim \hat{\mathbf{z}}$ to obtain by transitivity that $((\ell)_{\text{con}}, \mathbf{y}) \succsim ((\ell)_{\text{con}}, \mathbf{x})$ and therefore $\mathbf{y} \succsim \mathbf{x}$.

Let $J = \{i \in \mathbb{N} : \exists k \in I_>, i = 2k\}$. In streams $((\ell)_{\text{con}}, \mathbf{x})$, $((\ell)_{\text{con}}, \mathbf{y})$, $\mathbf{z}$ and $\hat{\mathbf{z}}$, coordinates in J are those where the streams differ. Let f be the unique increasing bijection between $\mathbb{N}$ and $\mathbb{N} \setminus J$. Define $\tilde{\mathbf{z}}$ by $\tilde{z}_n = z_{f(n)}$ for all $n \in \mathbb{N}$: stream $\tilde{\mathbf{z}}$ collects all coordinates that are the same in $((\ell)_{\text{con}}, \mathbf{x})$, $((\ell)_{\text{con}}, \mathbf{y})$, $\mathbf{z}$ and $\hat{\mathbf{z}}$. Let $\bar{x} = \sup_{j \in \mathbb{N}} x_j$, by Proposition 2, $(\bar{x}, \tilde{\mathbf{z}}) \sim (\ell, \tilde{\mathbf{z}})$. By Critical-level consistency, $(\bar{x}, \tilde{\mathbf{z}}) \sim ((\bar{x})_{\text{con}}, \tilde{\mathbf{z}})$ and $(\ell, \tilde{\mathbf{z}}) \sim ((\ell)_{\text{con}}, \tilde{\mathbf{z}})$. Clearly, $\mathbf{z}$ can be obtained from $((\bar{x})_{\text{con}}, \tilde{\mathbf{z}})$ and $\hat{\mathbf{z}}$ can be obtained from $((\ell)_{\text{con}}, \tilde{\mathbf{z}})$ through a permutation. Therefore, by Strong anonymity and transitivity, $\mathbf{z} \sim \hat{\mathbf{z}}$.

We thus know that $\mathbf{y} \succsim \mathbf{x}$. But $\mathbf{x} \geq \mathbf{y}$ so that by Monotonicity $\mathbf{x} \succsim \mathbf{y}$. □

An implication of Proposition 3 is that even an increase in utility for an infinite number of people may not be sufficient to guarantee social dominance. For instance, Proposition 3 implies that the following two streams are equivalent:

$$(1, 0, 1, 0, \dots, 1, 0, \dots) \text{ and } (0, 0, 0, 0, \dots).$$

As mentioned in the introduction to this section Fleurbaey and Michel (2003) proved that Strong Anonymity is incompatible with the Weak Pareto axiom using an example. Proposition 3 provides a larger set of cases where the Weak Pareto axiom fails when our other axioms are satisfied: all cases where there is a strict improvement above the limit inferior changing the limit inferior. For instance, the following two streams are equivalent:

$$(1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots, \tfrac{1}{k}, \dots) \text{ and } (0, 0, 0, 0, \dots).$$

5 Infinite population ethics under strong anonymity

As we have discussed before, the literature on population ethics has discussed the effect of the addition of a person, or several people, to a population. Parfit (1984) has introduced the mere addition principle: the addition of someone with a non-negative utility level should always be acceptable. The problem is that, under mild conditions, this principle may yield the 'repugnant conclusion' by which a very large population of people with lives barely worth living may be better than a large but smaller population of people with excellent lives. To avoid this conclusion, Blackorby and Donaldson (1984) and Blackorby, Bossert and Donaldson (1995) have proposed critical-level utilitarianism, according to which adding only sufficiently good lives are socially acceptable.

In any case, most approaches assume that there exist levels of well-being such that adding someone with a well-being at these levels is a strict social improvement. However, this is true when one considers a finite population. In contrast, the following axiom is concerned with the addition of finitely or infinitely many people to a population that already has infinitely many people. It asserts that the social value of adding finitely or infinitely many lives is non-positive regardless of their well-being levels.

Non-positive value of additional lives. For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if there exists an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $x_i = y_{f(i)}$ for all $i \in \mathbb{N}$, then $\mathbf{x} \succsim \mathbf{y}$.

We obtain the surprising result this population-ethical axiom is a result of the axioms that we have already imposed.

Proposition 4. *Assume that $\succsim$ is a reflexive and transitive binary relation satisfying Strong anonymity, Monotonicity, Continuity, Very weak Pigou-Dalton transfer and Critical-level consistency. Then $\succsim$ satisfies Non-positive value of additional lives.*

Proof. Consider $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ such that there exists an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $x_i = y_{f(i)}$ for all $i \in \mathbb{N}$. Assume that $\succsim$ is a reflexive and transitive binary relation satisfying Strong Anonymity, Monotonicity, Continuity, Very weak Pigou-Dalton, and Critical-level consistency.

Write $\ell(\mathbf{x}) = \liminf_{j\in\mathbb{N}} x_j$ and $\ell(\mathbf{y}) = \liminf_{j\in\mathbb{N}} y_j$, and let $z = \sup_{j\in\mathbb{N}} y_j$. By the relationship between $\mathbf{x}$ and $\mathbf{y}$, $z \geq \ell(\mathbf{x}) \geq \ell(\mathbf{y})$. Write $\tilde{\mathbf{x}} = ((\ell(\mathbf{x}))_{\text{con}}, \mathbf{x})$ and $\tilde{\mathbf{y}} = ((\ell(\mathbf{y}))_{\text{con}}, \mathbf{y})$. By Strong anonymity, adding an additional person with well-being z to $\tilde{\mathbf{x}}$ is equivalent to increasing the well-being of an existing person from $\ell(\mathbf{x})$ to z. So by Proposition 2, $(z, \tilde{\mathbf{x}}) \sim \tilde{\mathbf{x}}$. Thus, by Critical-level consistency, $((z)_{\text{con}}, \tilde{\mathbf{x}}) \sim \tilde{\mathbf{x}}$. By the definition of z, there exists $\pi \in \Pi$ such that $((z)_{\text{con}}, \tilde{\mathbf{x}}) \geq \tilde{\mathbf{y}}_\pi$, so by Monotonicity, $((z)_{\text{con}}, \tilde{\mathbf{x}}) \succsim \tilde{\mathbf{y}}_\pi$. By Strong anonymity, $\tilde{\mathbf{y}}_\pi \sim \tilde{\mathbf{y}}$, while by Lemma 3, $\mathbf{x} \sim \tilde{\mathbf{x}}$ and $\mathbf{y} \sim \tilde{\mathbf{y}}$. Hence,

$$\mathbf{x} \sim \tilde{\mathbf{x}} \sim ((z)_{\text{con}}, \tilde{\mathbf{x}}) \succsim \tilde{\mathbf{y}}_\pi \sim \tilde{\mathbf{y}} \sim \mathbf{y}\,,$$

which by the transitivity of $\succsim$ implies that $\succsim$ satisfies Non-positive value of additional lives. □

One can even be more specific about cases where the addition of a person is a matter of social indifference or has negative social value. By Lemma 3 and Proposition 3, adding one person—or finitely or infinitely many people—at the limit inferior or above is a matter of social indifference. Indeed, adding any number of people at the limit inferior is socially indifferent by Lemma 3. Then increasing their well-being level is also socially indifferent by Proposition 3.

On the other hand, if a reflexive and transitive binary relation $\succsim$ satisfies Liminf-restricted dominance, then adding at least one person below the limit inferior has negative social value. Indeed, like before, adding any number of people at the limit inferior is socially indifferent. Then decreasing the level of well-being of one person from the limit inferior to a strictly lower value has negative social value by Liminf-restricted dominance.

6 Concluding remarks

In the view of Van Liedekerke and Lauwers (1997, p. 164) there are good reasons why the route taken in this paper, where we insist on Strong anonymity and weaken Strong Pareto, has been left unexplored. In their opinion, Strong anonymity is an unreasonable impartiality requirement. This view seems compelling if there exists a natural isomorphism between people in different alternatives, like in the case where the identity of people remains the same independently of the well-being they experience. For example, if the number of people in all generations is fixed, then it is not logically impossible to assume that people are the same independently of their well-beings. So if, following an example provided by Van Liedekerke and Lauwers (1997, p. 164), there are 100 people in each generation, with one alternative giving 99 of them a well-being of 1 and the last one a well-being of 0, while in the other alternative 99 gets a well-being of 0 and only one a well-being of 1, it might be hard to argue that the streams are equally good, even though one stream is an infinite permutation of the other. If one agrees with this position, then the fact that Strong anonymity together with four rather innocent axioms implies that adding people to an infinite population never has positive value can be taken as a further indication that Strong anonymity is indeed too strong, giving more weight to the kind of arguments put forward by opponents of this axiom (see e.g., Jonsson, 2019).

However, in comparisons where the number of people in any one generation varies between two alternatives, there exists no such natural isomorphism. In such settings the arguments against the axiom of Strong Anonymity are weakened. We have shown that it is a consequence of Strong anonymity in combination with four other axioms that only people finitely ranked from the bottom matter. The asymmetry between the sensitivity for people finitely ranked from the bottom and the insensitivity for people finitely ranked from the top is a consequence of an innocent equity axiom, Very weak Pigou-Dalton transfer, stating that when making a transfer between a richer and a poorer person, the poorer person's change in well-being must have positive relative weight. Furthermore, we have shown that—when sensitivity is limited in this way—it follows that adding people to an infinite population cannot have positive social value. These conclusions might, at first, appear to be a bitter pill to swallow for those that find Strong anonymity attractive. Further reflection might, however, provide insights into why it is reasonable to require insensitivity at or above the limit inferior and to conclude that an infinite population cannot be improved by adding additional people.

Appendix

In the appendix we show that our main results, i.e., Propositions 3 and 4, can be strengthened by employing weaker continuity axioms. To this end, we prove stronger variants of Lemma 3 and Proposition 2 with weaker continuity axioms.

Let $d_1 \colon \mathbf{X} \times \mathbf{X} \to \mathbb{R}_+$ be the distance function given by, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$d_1(\mathbf{x}, \mathbf{y}) = \min \left\{ 1, \sum_{i=1}^{\infty} |x_i - y_i| \right\}.$$

Using the distance function d_1, we define weak continuity as follows; see Svensson (1980).

Weak continuity. For any $\mathbf{x} \in \mathbf{X}$, the sets $\{\mathbf{y} \in \mathbf{X} : \mathbf{y} \succsim \mathbf{x}\}$ and $\{\mathbf{y} \in \mathbf{X} : \mathbf{x} \succsim \mathbf{y}\}$ are closed in $(\mathbf{X}, d_1)$.

The following lemma shows that Lemma 3 can be strengthened by using Weak continuity instead of Continuity.

Lemma 5. *Let $\mathbf{x} \in \mathbf{X}$ and z be a cluster point of $\mathbf{x}$. If $\succsim$ is a reflexive and transitive binary relation satisfying Strong anonymity and Weak continuity, then $\mathbf{x} \sim (z, \mathbf{x}) \sim ((z)_{con}, \mathbf{x})$.*

Proof. We show that $\mathbf{x} \sim (z, \mathbf{x})$ and $\mathbf{x} \sim ((z)_{\text{con}}, \mathbf{x})$, which imply $(z, \mathbf{x}) \sim ((z)_{\text{con}}, \mathbf{x})$ by the transitivity of $\succsim$.

Since z is a cluster point of $\mathbf{x}$, there exists a subsequence of $\mathbf{x}$ that converges to z. Furthermore, it is well-known that any sequence of real numbers has a monotone subsequence and that every subsequence of a convergent sequence has the same limit. Therefore, there exists a monotone subsequence of $\mathbf{x}$ that converges to z, that is, there exists an increasing function $f \colon \mathbb{N} \to \mathbb{N}$ such that $(x_{f(k)})_{k \in \mathbb{N}} \in \mathbf{X}$ is monotone and converges to z. Without loss of generality, we assume that $(x_{f(k)})_{k \in \mathbb{N}}$ is non-decreasing.

Let $\tilde{\mathbf{x}} = (x_{f(k)})_{k \in \mathbb{N}}$ and, for any $n \in \mathbb{N}$, define $\tilde{\mathbf{x}}(n)$ by

$$\tilde{\mathbf{x}}(n) = (\tilde{x}_1, \ldots, \tilde{x}_{n-1}, z, \tilde{x}_n, \tilde{x}_{n+1}, \ldots).$$

Then, we obtain that

$$\begin{aligned}
\lim_{n\to+\infty}\sum_{i=1}^{\infty}|\tilde{x}_i-\tilde{x}(n)_i| &= \lim_{n\to+\infty}\left(z-\tilde{x}_n+\lim_{N\to+\infty}\sum_{i=n+1}^{\infty}|\tilde{x}_i-\tilde{x}(n)_i|\right)\\
&= \lim_{n\to+\infty}\left(z-\tilde{x}_n+\lim_{N\to+\infty}(\tilde{x}_N-\tilde{x}_n)\right)\\
&= \lim_{n\to+\infty}2(z-\tilde{x}_n)\\
&= 0.
\end{aligned}$$

Thus, for any $\varepsilon\in(0,1)$, there exists an increasing sequence $\{n_t\}_{t\in\mathbb{N}}$ in $\mathbb{N}$ satisfying

$$n_t+t<n_{t+1}\ \text{ for each } t\in\mathbb{N}$$

and we can define the sequence $\{\tilde{\mathbf{x}}^n\}_{n\in\mathbb{N}\cup\{0\}}$ in $\mathbf{X}$ that satisfies

$$\sum_{i=1}^{\infty}|\tilde{x}_i^n-\tilde{x}_i^{n-1}|<\frac{\varepsilon}{2^n}\ \text{ for each } n\in\mathbb{N}$$

as follows:

$$\begin{aligned}
\tilde{\mathbf{x}}^0 &= (\tilde{x}_1,\ \ldots,\ \tilde{x}_{n_1-1},\ \tilde{x}_{n_1},\ \tilde{x}_{n_1+1},\ \ldots,\ \tilde{x}_{n_2-1},\ \tilde{x}_{n_2},\ \tilde{x}_{n_2+1},\ \ldots),\\
\tilde{\mathbf{x}}^1 &= (\tilde{x}_1,\ \ldots,\ \tilde{x}_{n_1-1},\ z,\ \tilde{x}_{n_1},\ \ldots,\ \tilde{x}_{n_2-2},\ \tilde{x}_{n_2-1},\ \tilde{x}_{n_2},\ \ldots),\\
\tilde{\mathbf{x}}^2 &= (\tilde{x}_1,\ \ldots,\ \tilde{x}_{n_1-1},\ z,\ \tilde{x}_{n_1},\ \ldots,\ \tilde{x}_{n_2-2},\ z,\ \tilde{x}_{n_2-1},\ \ldots),
\end{aligned}$$

and so forth. Formally, $\tilde{\mathbf{x}}^0=\tilde{\mathbf{x}}$ and for each $n\in\mathbb{N}$, $\tilde{\mathbf{x}}^n$ is defined as follows. For each $i\in\{n_t : t\in\{1,\ldots,n\}\}$,

$$\tilde{x}_i^n=z$$

and the subsequence of $\tilde{\mathbf{x}}^n$ composed of all the other components coincides with $\tilde{\mathbf{x}}$. Analogously, we define $\tilde{\mathbf{x}}^\infty$ by, for each $i\in\{n_t : t\in\mathbb{N}\}$,

$$\tilde{x}_i^\infty=z$$

and the subsequence of $\tilde{\mathbf{x}}^\infty$ composed of all the other components coincides with

$\tilde{\mathbf{x}}$. By the definitions of $\{\tilde{\mathbf{x}}^n\}_{n\in\mathbb{N}\cup\{0\}}$ and $\tilde{\mathbf{x}}^\infty$, we obtain that

$$\sum_{i=1}^{\infty} |\tilde{x}_i^\infty - \tilde{x}_i^0| \le \lim_{N\to+\infty} \sum_{n=1}^{N} \sum_{i=1}^{\infty} |\tilde{x}_i^n - \tilde{x}_i^{n-1}| < \varepsilon. \tag{3}$$

We now define the sequence $\{\mathbf{x}^n\}_{n\in\mathbb{N}\cup\{0\}}$ using the sequence $\{\tilde{\mathbf{x}}^n\}_{n\in\mathbb{N}\cup\{0\}}$ as follows. For each $n \in \mathbb{N}\cup\{0\}$,

$$x_i^n = x_i \ \text{ for all } i \in \mathbb{N} \setminus \{f(k) : k \in \mathbb{N}\}$$

and

$$(x_{f(k)}^n)_{k\in\mathbb{N}} = \tilde{\mathbf{x}}^n.$$

Analogously, we define $\mathbf{x}^\infty$ by

$$x_i^\infty = x_i \ \text{ for all } i \in \mathbb{N} \setminus \{f(k) : k \in \mathbb{N}\}$$

and

$$(x_{f(k)}^\infty)_{k\in\mathbb{N}} = \tilde{\mathbf{x}}^\infty.$$

Note that $\mathbf{x}^0 = \mathbf{x}$. From (3) and the definitions of $\{\mathbf{x}^n\}_{n\in\mathbb{N}\cup\{0\}}$ and $\mathbf{x}^\infty$, it follows that

$$\sum_{i=1}^{\infty} |x_i^1 - x_i^0| = \sum_{i=1}^{\infty} |\tilde{x}_i^1 - \tilde{x}_i^0| \le \sum_{i=1}^{\infty} |\tilde{x}_i^\infty - \tilde{x}_i^0| = \sum_{i=1}^{\infty} |x_i^\infty - x_i^0| < \varepsilon. \tag{4}$$

Note that there exist permutations $\pi, \rho \in \Pi$ such that $\mathbf{x}^1 = (z, \mathbf{x})_\pi$ and $\mathbf{x}^\infty = ((z)_{\text{con}}, \mathbf{x})_\rho$. Thus, it follows from (4) that for any $\varepsilon \in (0,1)$, there exist $\pi, \rho \in \Pi$ such that

$$d_1((z, \mathbf{x})_\pi, \mathbf{x}) \le d_1(((z)_{\text{con}}, \mathbf{x})_\rho, \mathbf{x}) < \varepsilon. \tag{5}$$

Let $m \in \mathbb{N}$. By (5), there exist $\pi^m, \rho^m \in \Pi$ such that

$$d_1((z, \mathbf{x})_{\pi^m}, \mathbf{x}) \le d_1(((z)_{\text{con}}, \mathbf{x})_{\rho^m}, \mathbf{x}) < \frac{1}{m}.$$

Consider the sequences $((z, \mathbf{x})_{\pi^m})_{m\in\mathbb{N}}$ and $(((z)_{\text{con}}, \mathbf{x})_{\rho^m})_{m\in\mathbb{N}}$ in $\mathbf{X}$. By Strong anonymity, we obtain that $(z, \mathbf{x})_{\pi^m} \sim (z, \mathbf{x})$ and $((z)_{\text{con}}, \mathbf{x})_{\rho^m} \sim ((z)_{\text{con}}, \mathbf{x})$ for each $m \in \mathbb{N}$. Since $\succsim$ satisfies Weak continuity and

$$\lim_{m\to+\infty} d_1((z, \mathbf{x})_{\pi^m}, \mathbf{x}) = \lim_{m\to+\infty} d_1(((z)_{\text{con}}, \mathbf{x})_{\rho^m}, \mathbf{x}) = 0,$$

we obtain $(z, \mathbf{x}) \sim \mathbf{x}$ and $((z)_{\text{con}}, \mathbf{x}) \sim \mathbf{x}$. □

Note that the variant of Lemma 4 that is stated using Weak continuity instead of Continuity can be proved by using Lemma 5 instead of Lemma 3. Thus, in what follows, we will use Lemma 4 to establish a variant of Proposition 2.

To state a variant of Proposition 2, we define another weakening of Continuity. Weak upper semi-continuity requires an evaluation be upper semi-continuous with respect to streams that have a constant subsequence.

Weak upper semi-continuity. For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and for any $N \in \mathbb{N}_0 \cup \{\infty\}$, the set $\{z \in \mathbb{R} : (\mathbf{y}_N, (z)_{\text{con}}) \succsim (\mathbf{y}_N, \mathbf{x})\}$ is closed.

The following proposition is a variant of Proposition 2 using Critical-level consistency and the two weaker continuity axioms instead of Continuity.

Proposition 5. *Consider* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *with* $x_i > y_i$ *for some* $i \in \mathbb{N}$ *and* $x_j = y_j$ *for all* $j \in \mathbb{N} \setminus \{i\}$. *The two following statements are equivalent:*

(1) For all reflexive and transitive binary relations $\succsim$ *satisfying Strong anonymity, Monotonicity, Weak continuity, Very weak Pigou-Dalton transfer, Critical-level consistency and Weak upper semi-continuity, we have that* $\mathbf{x} \sim \mathbf{y}$.

(2) $y_i \geq \liminf_{j\in\mathbb{N}} x_j$.

Proof. Since Continuity implies Weak continuity and Weak upper-semi continuity, the proof that (1) implies (2) is analogous to that of Proposition 2.

Next, we prove that (2) implies (1). Let $\succsim$ be a reflexive and transitive binary relation on $\mathbf{X}$ that satisfies Strong anonymity, Monotonicity, Weak continuity, Very weak Pigou-Dalton transfer, Critical-level consistency and Weak upper semi-continuity. Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, and suppose that there exists $i \in \mathbb{N}$ such that $x_i > y_i$

and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$. By Monotonicity,

$$\mathbf{x} \succsim \mathbf{y}.$$

Furthermore, if $\limsup_{j\in\mathbb{N}} y_j > \liminf_{j\in\mathbb{N}} x_j$, it follows from Proposition 4 that $\mathbf{x} \sim \mathbf{y}$. Hence, we assume that $\limsup_{j\in\mathbb{N}} y_j = \liminf_{j\in\mathbb{N}} x_j$. Let

$$\ell = \lim_{j\to+\infty} x_j = \lim_{j\to+\infty} y_j.$$

For any convergent stream $\mathbf{z} \in \mathbf{X}$, we define $H(\mathbf{z}) \subseteq \mathbb{N}$ by

$$H(\mathbf{z}) = \left\{ j \in \mathbb{N} : z_j \geq \lim_{k\to+\infty} z_k \right\}.$$

Note that $H(\mathbf{x}) = H(\mathbf{y})$.

To show that $\mathbf{y} \succsim \mathbf{x}$, we distinguish two cases. First, we suppose that $|H(\mathbf{x})| = +\infty$. We define $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \mathbf{X}$ by

$$\bar{\mathbf{x}} = (\mathbf{x}, (\ell)_{\text{con}}) \;\text{ and }\; \bar{\mathbf{y}} = (\mathbf{y}, (\ell)_{\text{con}}).$$

By Strong anonymity, we obtain $\bar{\mathbf{x}} \sim ((\ell)_{\text{con}}, \mathbf{x})$ and $\bar{\mathbf{y}} \sim ((\ell)_{\text{con}}, \mathbf{y})$. Since ℓ is a cluster point of $\mathbf{x}$ and $\mathbf{y}$, it follows from Lemma 5 and the transitivity of $\succsim$ that

$$\mathbf{x} \sim \bar{\mathbf{x}} \;\text{ and }\; \mathbf{y} \sim \bar{\mathbf{y}}.$$

Using $\bar{\mathbf{y}}$, we define $\mathbf{y}^* \in \mathbf{X}$ by, for all $j \in \mathbb{N}$,

$$y_j^* = \begin{cases} \min\{\bar{y}_j, \ell\} & \text{if } j \text{ is odd}, \\ \ell & \text{if } j \text{ is even}, \end{cases}$$

Note that $y_j^* = \min\{y_{(j+1)/2}, \ell\}$ if j is odd. By Monotonicity, we obtain

$$\bar{\mathbf{y}} \succsim \mathbf{y}^*.$$

Since $\succsim$ is transitive, it follows that

$$\mathbf{y} \succsim \mathbf{y}^*.$$

Now, let $s = \sup_{j\in\mathbb{N}} x_j$ and $\delta = s - \ell$. Note that $\delta > 0$ since $s \geq x_i > y_i \geq \ell$. We define the sequence $\{\mathbf{x}^n\}_{n\in\mathbb{N}_0}$ in $\mathbf{X}$ as follows. For each $n \in \mathbb{N}_0$ and each $j \in \mathbb{N}$,

$$x_j^n = \begin{cases} \bar{x}_j = x_{(j+1)/2} & \text{if } j \text{ is odd and } (j+1)/2 \notin H(\mathbf{x}), \\ \ell + \dfrac{\delta}{2^n} & \text{if } j \text{ is odd and } (j+1)/2 \in H(\mathbf{x}), \\ \ell & \text{if } j \text{ is even.} \end{cases}$$

By Monotonicity, we obtain

$$\mathbf{x}^0 \succsim \bar{\mathbf{x}}.$$

Let $n \in \mathbb{N}_0$. We show that

$$\mathbf{x}^{n+1} \succsim \mathbf{x}^n.$$

We define $\tilde{\mathbf{x}}^n \in \mathbf{X}$ by $\tilde{x}_2^n = \ell + \frac{\delta}{2^{n+1}}$ and $\tilde{x}_j^n = x_j^n$ for all $j \in \mathbb{N} \setminus \{2\}$. By Monotonicity, we obtain

$$\tilde{\mathbf{x}}^n \succsim \mathbf{x}^n.$$

Let $\pi \in \Pi$ be defined by, for each $j \in \mathbb{N}$,

$$\pi(j) = \begin{cases} j+1 & \text{if } j \text{ is odd,} \\ j-1 & \text{if } j \text{ is even.} \end{cases}$$

Note that

$$\tilde{\mathbf{x}}_\pi^n = \left(\ell + \frac{\delta}{2^{n+1}}, \mathbf{x}^n\right).$$

Thus, it follows from Strong anonymity that

$$\tilde{\mathbf{x}}^n \sim \left(\ell + \frac{\delta}{2^{n+1}}, \mathbf{x}^n\right).$$

Since $\succsim$ is transitive, we obtain

$$\left(\ell + \frac{\delta}{2^{n+1}}, \mathbf{x}^n\right) \succsim \mathbf{x}^n.$$

By Critical-level consistency, we obtain

$$\left(\left(\ell + \frac{\delta}{2^{n+1}}\right)_{\text{con}}, \mathbf{x}^n\right) \succsim \mathbf{x}^n.$$

Now, let $f\colon \mathbb{N} \to H(\mathbf{x})$ be the increasing bijection, that is, $f(n)$ is the n-th smallest element of $H(\mathbf{x})$. Using f, we define $\rho \in \Pi$ by, for each $j \in \mathbb{N}$,

$$\rho(j) = \begin{cases} 2(2f(k)-1) & \text{if } j \text{ is the } k\text{-th odd number,} \\ 2k-1 & \text{if there exists } k \in \mathbb{N} \text{ such that } j = 2(2f(k)-1), \\ j & \text{otherwise.} \end{cases}$$

Note that in $\left(\left(\ell + \frac{\delta}{2^{n+1}}\right)_{\text{con}}, \mathbf{x}^n\right)$, the $2(2f(k)-1)$-th component is $x^n_{f(k)} = \ell + \frac{\delta}{2^n}$ and each odd-numbered (i.e. each $(2k-1)$-th) component is $\ell + \frac{\delta}{2^{n+1}}$. Thus,

$$\left(\left(\ell + \frac{\delta}{2^{n+1}}\right)_{\text{con}}, \mathbf{x}^n\right)_\rho = \left(\left(\ell + \frac{\delta}{2^n}\right)_{\text{con}}, \mathbf{x}^{n+1}\right).$$

By Strong anonymity and the transitivity of $\succsim$, we obtain

$$\left(\left(\ell + \frac{\delta}{2^n}\right)_{\text{con}}, \mathbf{x}^{n+1}\right) \succsim \mathbf{x}^n.$$

Since $\ell + \frac{\delta}{2^n}$ is a cluster point of $((\ell + \frac{\delta}{2^n})_{\text{con}}, \mathbf{x}^{n+1})$, it follows from Lemma 5 that

$$\left(\left(\ell + \frac{\delta}{2^n}\right)_{\text{con}}, \mathbf{x}^{n+1}\right) \sim \mathbf{x}^{n+1}.$$

Thus, by the transitivity of $\succsim$, we obtain $\mathbf{x}^{n+1} \succsim \mathbf{x}^n$.

Using the sequence $\{\mathbf{x}^n\}_{n\in\mathbb{N}_0}$, we show that $\mathbf{y}^* \succsim \bar{\mathbf{x}}$, which implies $\mathbf{y} \succsim \mathbf{x}$ by the transitivity of $\succsim$. Using the increasing bijection f, we define $\sigma \in \Pi$ by, for all $j \in \mathbb{N}$,

$$\sigma(j) = \begin{cases} 2f(k) - 1 & \text{if } j \text{ is the } k\text{-th even number,} \\ 2k & \text{if there exists } k \in \mathbb{N} \text{ such that } j = 2f(k) - 1, \\ j & \text{otherwise.} \end{cases}$$

By Strong anonymity, we obtain that

$$\bar{\mathbf{x}} \sim \bar{\mathbf{x}}_\sigma \text{ and } \mathbf{y}^* \sim \mathbf{y}^*_\sigma$$

and for all $n \in \mathbb{N}_0$,

$$\mathbf{x}^n \sim \mathbf{x}^n_\sigma.$$

Since $\mathbf{x}^0 \succsim \bar{\mathbf{x}}$ and $\mathbf{x}^{n+1} \succsim \mathbf{x}^n \succsim \bar{\mathbf{x}}$ for all $n \in \mathbb{N}_0$, we obtain by the transitivity of $\succsim$ that for all $n \in \mathbb{N}_0$,

$$\mathbf{x}^n_\sigma \succsim \bar{\mathbf{x}}_\sigma.$$

Note that, for all $m \in \mathbb{N}$ and for all $n \in \mathbb{N}_0$,

$$\bar{x}_{\sigma(2m+1)} = y^*_{\sigma(2m+1)} = x^n_{\sigma(2m+1)}$$

and

$$x^n_{\sigma(2m)} = \ell + \frac{\delta}{2^n} \text{ and } y^*_{\sigma(2m)} = \ell.$$

Since $\lim_{n\to+\infty} \ell + \frac{\delta}{2^n} = \ell$, it follows from Weak upper semi-continuity that

$$\mathbf{y}^*_\sigma \succsim \bar{\mathbf{x}}_\sigma.$$

By the transitivity of $\succsim$, we obtain $\mathbf{y}^* \succsim \bar{\mathbf{x}}$.

Finally, we consider the case where $|H(\mathbf{x})| < +\infty$. Let $\hat{\mathbf{x}} = ((\ell)_{\text{con}}, \mathbf{x})$ and $\hat{\mathbf{y}} = ((\ell)_{\text{con}}, \mathbf{y})$. Note that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ satisfy that $\hat{x}_i > \hat{y}_i \geq \liminf_{j\in\mathbb{N}} \hat{x}_j$ for some $i \in \mathbb{N}$, $\hat{x}_j = \hat{y}_j$ for all $j \in \mathbb{N} \setminus \{i\}$, and $|H(\hat{\mathbf{x}})| = |H(\hat{\mathbf{y}})| = +\infty$. Thus, as we showed above, $\hat{\mathbf{x}} \succsim \hat{\mathbf{y}}$ holds. Since ℓ is a cluster point of $\mathbf{x}$ and $\mathbf{y}$, it follows from Lemma 5 that $\mathbf{x} \sim \hat{\mathbf{x}}$ and $\mathbf{y} \sim \hat{\mathbf{y}}$. By the transitivity of $\succsim$, we obtain $\mathbf{y} \succsim \mathbf{x}$. $\square$

Using Lemma 5 and Proposition 5, we obtain the following stronger variants of Propositions 3 and 4 with the weaker continuity axioms. Since their proofs are analogous to those of Propositions 3 and 4, we state them without proof.

Proposition 6. *Consider* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *with* $\mathbf{x} \geq \mathbf{y}$ *and* $\mathbf{x} \neq \mathbf{y}$. *The two following statements are equivalent:*

(1) For all refexive and transitive binary relations $\succsim$ *satisfying Strong anonymity, Monotonicity, Weak continuity, Weak upper semi-continuity, Very weak Pigou-Dalton transfer and Critical-level consistency, we have that* $\mathbf{x} \sim \mathbf{y}$.

(2) $\ell = \liminf_{j \in \mathbb{N}} x_j = \liminf_{j \in \mathbb{N}} y_j$ *and* $y_i \geq \ell$ *for all* $i \in \mathbb{N}$ *such that* $x_i > y_i$.

Proposition 7. *Assume that* $\succsim$ *is a reflexive and transitive binary relation satisfying Strong anonymity, Monotonicity, Weak continuity, Weak upper semi-continuity, Very weak Pigou-Dalton transfer and Critical-level consistency. Then* $\succsim$ *satisfies Non-positive value of additional lives.*

References

Tsuyoshi Adachi, Susumu Cato, Kohei Kamaga, Extended anonymity and Paretian relations on infinite utility streams, Mathematical Social Sciences 72 (2014) 24–32

Geir B. Asheim, Intergenerational Equity, Annual Review of Economics 2 (2010) 197–222

Geir B. Asheim, Bertil Tungodden, Resolving distributional conflicts between generations, Economic Theory 24 (2004) 221–230

Geir B. Asheim, Stéphane Zuber, A complete and strongly anonymous leximin relation on infinite streams, Social Choice and Welfare 41 (2013) 819–834

Kaushik Basu, Tapan Mitra, Aggregating infinite utility streams with intergenerational equity: the impossibility of being Paretian, Econometrica 71 (2003) 1557–1563

Charles Blackorby, Walter Bossert, David Donaldson, Intertemporal population ethics: critical-level utilitarian principles, Econometrica 63 (1995) 1303–1320

Charles Blackorby, Walter Bossert, David Donaldson, Population Issues in Social Choice Theory, Welfare Economics, and Ethics. Cambridge University Press, Cambridge, UK, 2005

Charles Blackorby, David Donaldson, Social criteria for evaluating population change, Journal of Public Economics 25 (1984) 13–33

Walter Bossert, Yves Sprumont, Kotaro Suzumura, Ordering infinite utility streams, Journal of Economic Theory 135 (2007) 579–589

Peter A. Diamond, The evaluation of infinite utility streams, Econometrica 33 (1965) 170–177

Marc Fleurbaey, Philippe Michel, Intertemporal equity and the extension of the Ramsey criterion, Journal of Mathematical Economics 39 (2003) 777–802

Peter J. Hammond, Equity, Arrow's conditions, and Rawls' difference principle. Econometrica 44 (1976) 793–804

David Hilbert, David Hilbert's Lectures on the Foundations of Arithmetics and Logic 1917–1933. William Ewald, Wilfried Sieg (eds.), Springer Verlag, Heidelberg, 2013

Adam Jonsson, On permutations and equity among generations, mimeo, Luleå University of Technology, 2019

Luc Lauwers, Infinite utility: insisting on strong monotonicity, Australasian Journal of Philosophy 75 (1997) 222–233

Luc Lauwers, Intertemporal objective functions: Strong Pareto versus anonymity, Mathematical Social Sciences 35 (1998) 37–55

Luc Lauwers, Ordering infinite utility streams comes at the cost of a non-Ramsey set, Journal of Mathematical Economics 46 (2010) 32–37

Luc Lauwers, Intergenerational equity, efficiency, and constructibility, Economic Theory 49 (2012) 227–242

Tapan Mitra, Kaushik Basu, On the existence of Paretian social welfare quasi-orderings for infinite utility streams with extended anonymity. In: John Roemer, Kotaro Suzumura (eds.) Intergenerational Equity and Sustainability. Palgrave, London, 2007

Derek Parfit, Reasons and Persons. Oxford University Press, Oxford, 1984

Toyotaka Sakai, A characterization and an impossibility of finite length anonymity for infinite generations, Journal of Mathematical Economics 46 (2010) 877–883

Lars-Gunnar Svensson, Equity among generations, Econometrica 48 (1980) 1251–1256

Luc Van Liedekerke, Luc Lauwers, Sacrificing the patrol: Utilitarianism, future generations and infinity, Economics and Philosophy 13 (1997) 159–174

William R. Zame, Can intergenerational equity be operationalized? Theoretical Economics 2 (2007) 187–202

Stéphane Zuber, Geir B. Asheim, Justifying social discounting: The rank-discounted utilitarian approach, Journal of Economic Theory 147 (2012) 1572–1601