Strategy-Proof Exchange under Trichotomous Preferences

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Abstract

We study the exchange of indivisible objects without monetary transfers when each agent may be endowed with and consume more than one object. We assume that each agent has *trichotomous* preferences in the sense that she

(A) partitions objects into three: desirable, obligatory, and undesirable ones,

(B) considers a bundle acceptable if and only if it contains no undesirable objects, and

(C) ranks acceptable bundles by their numbers of desirable objects.

On this domain, we show that there is an individually rational, Pareto-efficient, and strategy-proof mechanism that is also computationally efficient.

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1 Introduction

We study the problem of reallocating indivisible objects without monetary transfers.[1] Unlike much of the earlier work on this problem, we consider situations where each agent may be endowed with and consume more than one object. We require exchange to be balanced in

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[1] See the literature following Shapley and Scarf [1974].

the sense that each agent ends up with exactly the same number of objects as she is endowed with. Agents' preferences over individual objects are coarse. An object is either desirable, obligatory, or undesirable. Each agent considers a bundle unacceptable if it contains an undesirable object. Finally, each agent ranks acceptable bundles in increasing order of the numbers of desirable objects that they contain. Our contribution is to define an individually rational, Pareto-efficient, and strategy-proof mechanism that is computationally efficient. Our positive result is in marked contrast with the impossibility results that abound in the literature on multi-object exchange without monetary transfers [Sönmez, 1999, Biró et al., 2018].

Our interest in these exchange problems stems from our search for a solution to the problem of shift-reallocation. Millions of people in many different professions, from physicians to retail workers, engage in shift work. Shift plans are often made months in advance and scenarios like the following are common: A medical practice consists of four specialist consultants Drs A, B, C, and D. This practice is responsible for ensuring that emergency medical services in their specialty are available to a given hospital at all times. That is, each week, one of the four doctors is to be designated as being *on call*. Being on call is not desirable for these doctors. However, it is necessary for their practice to maintain privileges at the hospital. Since it is a chore that they must perform, in the interest of fairness, they agree to share the weeks equally so that each member of the group is on call every fourth week.[2] To facilitate planning, the call schedule is made six months at a time, taking the doctors' preferences into consideration. For instance, the schedule from January 1 to June 30 is announced in December, and is created on the basis of the doctors' preferences as of December. Thus, on January 1, each of the four doctors is responsible for their assigned weeks until June 30. While this initial assignment may be Pareto-efficient with regards to the doctors' December preferences, six months is a long time. At some later point, say the beginning of March, there may be scope for re-optimization based on current preferences. It may well happen that Dr. A would like to attend a conference the week of April 5, Dr. D would like to help at a clinic in a remote area on June 16, and Dr. C would like to go on vacation on May 23. If each of these doctors is obliged to be on call for the respective week, a three-way trade could improve welfare in regards to current (as of March) preferences. We are interested in the design of mechanisms to identify such trades optimally and provide agents with incentives to reveal their private information about scheduling conflicts truthfully.[3,4]

[2] In the case of medical residents, such a restriction would not only be for reasons of fairness, but also for training purposes.

[3] Note that the re-optimization at the beginning of March in the above example is a static problem. We do not consider the issue of optimally timed matching as in Akbarpour et al. [2018].

[4] To the best of our knowledge, in practice these types of reassignments are typically arranged manually.

As illustrated by the previous example, we are interested in the problem of *re-allocating* shifts, or, more generally, indivisible objects, from a fixed vector of endowments rather than designing the initial schedule itself.[5] This problem is relevant for many workers: given a fixed schedule, a worker may wish to engage in other activities (e.g. work for other firms, further training or education, vacation) that are incompatible with her currently assigned responsibilities; at the same time, workers will have typically already made some commitments that limit the set of new shifts that they can take on. As long as there are possibilities of shocks to preferences or opportunities to make desirable commitments over the duration of the schedule, workers are bound to find themselves in situations where there are gains from trading pre-assigned shifts.

One important novel aspect of our model is the domain of agents' preferences that we consider. First, each agent partitions the set of objects into three (Figure 1):

1. Desirable objects. This set may include some of her endowed objects. In the shift exchange application, these correspond to shifts that she has no scheduling conflict with.

2. Obligatory objects. These correspond to shifts that she is already assigned and is therefore obliged to fulfill, but would like to trade away so that she may make other plans.

3. Undesirable objects. These are the objects that others are endowed with that she does not find desirable. If she has made other commitments and cannot take on a particular shift, it would be in this set.

Second, no agent is ever willing to accept a bundle that contains one or more undesirable objects. That is, she would rather stick to her endowment than consumes such a bundle. Finally, each agent ranks acceptable bundles in increasing order of the numbers of desirable objects they contain. We call such preferences *trichotomous* since the marginal preference on individual objects has the above mentioned three indifference classes. In the shift exchange example, a shift is undesirable to a worker if it conflicts with other activities that the worker has made or would like to make a commitment to. Otherwise, the worker is free to work during that time, so the shift is desirable. Furthermore, a worker is never willing to participate

As a result, more complex trades (three-way or larger) are often unrealized, leaving gains from trade on the table. Software solutions for managing call schedules facilitate these types of trades manually as well. The initial schedule (endowment) is similarly manually constructed. See the following for more on shift exchange in Microsoft's Staff Hub, which is used at many hospitals: https://goo.gl/FuhNtH.

[5]Without the requirement of respecting some lower bounds on welfare, the problem of designing an initial schedule is decidedly simpler: one could use a simple version of serial dictatorship where, one after another, agents select their most preferred times from what is left after those selecting before.

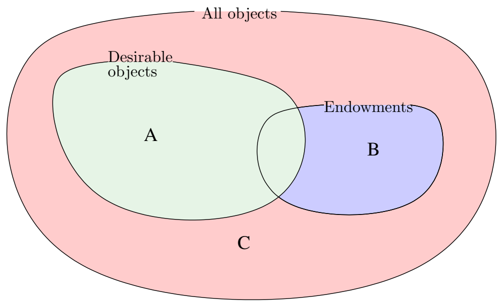

Figure 1: Under trichotomous preferences, the set of objects is partitioned into three components: desirable objects (A), obligatory objects that are not desirable but that the agent is endowed with (B), and the remaining objects, which are undesirable (C).

in an exchange that creates a scheduling conflict in the sense that she is assigned a shift that conflicts with plans that she has already made. The shifts that a worker is endowed with are necessarily her obligation and therefore she has already taken this into account in making commitments. However, she is under no such obligation for any other shift.

We focus on situations in which endowments are commonly known and exchange is balanced in the sense that each agent ends up with the same number of goods as she is endowed with. In the context of shift exchange, this mean that all participants know the initial schedule of shifts and that a worker's total workload is fixed at the number of shifts in the initial schedule.[6] Given that preferences are trichotomous, the crucial piece of information to elicit from an agent is which objects she finds desirable. An exchange mechanism asks each agent to reveal her set of desirable objects and then computes the final allocation of shifts on the basis of agents' reports as well as their commonly known endowments. We are interested in mechanisms that satisfy the following three desiderata:

1. *Individual rationality*, which says that the mechanism does not assign an unacceptable bundle to any agent. For our running example of shift exchange, this says that the mechanism ought not assign to the agent a shift that she has not indicated as being desirable.

2. *Pareto-efficiency*, which says that no further reallocation can make any agent better off without harming another.

3. *Strategy-proofness*, which says that no agent can ever profit from lying about her set of

[6] The restriction to balanced exchange is particularly meaningful for the shift exchange application as it may be imposed due to training purposes (as in the case of medical residents), fairness considerations (sharing equally the burden of taking on call), or terms of employment contracts (particularly for unionized employees who cannot work more or fewer shifts than prescribed by a collective bargaining agreement). See Dur and Ünver [forthcoming], Andersson et al. [2018] and Biró et al. [2018] for more on balanced exchange.

desirable objects, regardless of what the other agents report. In other words, it is a weakly dominant strategy for each agent to truthfully report her preferences.

Our main contribution to define a new class of exchange mechanisms satisfying these three these desiderata.[7] An *Individually Rational Priority* (IRP) mechanism is parameterized by a fixed ordering of the agents, independent of their reports, and picks a final allocation of goods in accordance with the following sequential process:

- In the first step, find the maximum number of desirable objects that the first agent in the ordering can receive in an individually rational allocation. Call this number the "promise" to the first agent.

- In the t^{th} step, find the maximum number of desirable objects that the t^{th} agent can receive in an individually rational allocation subject to the constraints imposed by the promises made to the first $t-1$ agents. Call this number the "promise" to the t^{th} agent.[8]

- After the last step, reallocate the objects in a way that the promise made to each agent is respected. First, there is necessarily at least one way to do this. Second, as we show, if there are two different re-allocations that achieve this, then every agent is indifferent between them.

An IRP mechanism selects an outcome that meets the promises made to all of the agents in this process. We show that, given an ordering, an IRP mechanism is individually rational, Pareto-efficient, and strategy-proof. While individual rationality and Pareto-efficiency are almost by definition, it is significantly harder to show that this mechanism makes truthful revelation a weakly dominant strategy for the agents. The key feature of an IRP mechanism that makes establishing strategy-proofness difficult is that the report of an agent alters the set of individually rational allocations and can thereby affect the outcomes for all agents.[9] An agent who is not first in line may therefore hope that she can influence the promises to her predecessors in such a way that the mechanism promises her more truly desirable objects. In our proof, we develop a complex combinatorial argument to show that such hopes would be misguided.

[7]For mechanisms that solve the unit endowment problem under general weak preferences, see Jaramillo and Manjunath [2012], Alcalde-Unzu and Molis [2011], Aziz and De Keijzer [2012], and Saban and Sethuraman [2013].

[8]Note that this does not pin down exactly which objects the t^{th} agent, or any agent before her, receives. It merely fixes the *number* of desirable objects that she will receive in the end.

[9]In the version of serial dictatorship described in Footnote 5, without being subjected to individual rationality constraints, an agent's report can only influence the outcomes for agents who pick after her. It is precisely this feature that makes it straightforward to show that simple priority mechanisms are strategy-proof.

Apart from their desirable allocative and incentive properties, we also show that IRP mechanisms are computationally efficient. More precisely, taking agents' desirable sets and a priority order over the agents as inputs, an IRP allocation may be computed in $O(mn^3 \log(k))$ time, where m is the total number of objects, n is the number of agents, and k is the largest number of objects an agent may be endowed with.

Since our main contribution is to show that the three main desiderata listed above are compatible under our preference restriction, we discuss how rather strong restrictions on preferences are, in a sense, necessary for this compatibility. To this end, we show an impossibility result for domains that are even slightly larger than the trichotomous domain if at least one agent is endowed with more than one object.

Related Literature

Without rather strong restrictions on preferences, individual rationality, Pareto-efficiency, and strategy-proofness are incompatible when agents are endowed with and demand multiple objects [Sönmez, 1999, Biró et al., 2018]. As we show, even for domains slightly larger than the trichotomous domain, these axioms are incompatible. Even other strategic properties such as immunity to manipulation via altering of endowments is generally incompatible with individual rationality and Pareto-efficiency [Klaus et al., 2006, Atlamaz and Klaus, 2007]. If we weaken the strategic requirement to say that successful manipulations are computationally intractable to compute, as suggested by Pini et al. [2011], then there are adaptations of the top-trading-cycles algorithm that satisfy this property while maintaining individual rationality and Pareto-efficiency [Fujita et al., 2015, Sikdar et al., 2017, 2018, Phan and Purcell, 2018].

The compatibility of individual rationality, Pareto-efficiency, and strategy-proofness in our model is driven by the restricted domain of preferences. For the two-sided one-to-one matching problem, the analog of this domain—where each agent divides the set partners into desirable and undesirable ones and is indifferent among those in the same group—results in these properties being compatible as well [Bogomolnaia and Moulin, 2004]. In the context of balanced exchange in which agents are endowed with and demand multiple objects, Andersson et al. [2018] have independently considered a model similar to ours. By making stronger assumptions on agents' preferences—that all of the objects that a given agent is endowed with are identical—they are able to design individually rational and strategy-proof mechanisms that are not only Pareto-efficient, but also maximal in the number of objects that are traded. Since there is a trade-off between generality of the model and how demanding an objective can be satisfied, our results are logically independent from theirs.

Biró et al. [2018] consider a model where, as in Andersson et al. [2018], agents are endowed with copies of objects. While they also restrict attention to balanced exchange, they study preferences that are responsive to strict, as opposed to trichotomous, orderings over the objects. They study different variations of the top-trading-cycles algorithm to analyze trade-offs, since in their model Pareto-efficiency and strategy-proofness cannot be reconciled with individual rationality.

Organization of the paper

In Section 2 we introduce the model. In Section 3 we define the IRP algorithm and characterize the outcomes that this algorithm produces. In Section 4 we show that the direct mechanism induced by the IRP algorithm is strategy-proof. In Section 5 we show how to compute IRP matchings in polynomial time. In Section 6 we show an impossibility result for a minimal extension of the preference domain beyond trichotomous preferences. All proofs are in the Appendix.

2 The Model

Let $I = \{1, \ldots, n\}$ be a set of n agents and O be a finite set of objects. Each agent $i \in I$ is endowed with a non-empty set of objects $\Omega_i \subseteq O$. We assume throughout that $\Omega_i \cap \Omega_j = \emptyset$ whenever $i \neq j$ and that $O = \cup_{i\in I}\Omega_i$.[10] For the shift exchange application, the endowment represents the pre-arranged schedule. We assume from that these endowments are known and fixed at the profile $\Omega = (\Omega_i)_{i\in I}$. In the context of shift exchange, the assumption of known endowments is reasonable since one would expect firms and workers to know who is supposed to work when (if no shift exchanges were to take place) once an initial assignment of shifts has been determined.

A *matching* is a mapping $\mu : I \to 2^O$ that satisfies the following two requirements:

1. For any pair of distinct agents $i, j \in I$, $\mu(i) \cap \mu(j) = \emptyset$.
2. For any agent $i \in I$, $|\mu(i)| = |\Omega_i|$.

We require exchange to be *balanced* in the sense that each agent ends up with as many objects as she brings to the market. For applications like shift exchange, this type of restriction is natural: contracts often specify a prefixed number of shifts per worker. Let $\mathcal{M}$ be the set of all matchings. The set of bundles that an agent $i \in I$ may consume at some matching in $\mathcal{M}$,

[10] The case where some objects are not in the endowment of any agent, i.e. $\cup_{i\in I}\Omega_i \subsetneq O$, is easy to accommodate by introducing extra (dummy) agents and objects. Details are available upon request.

is his consumption set, which we denote as

$$X_i \equiv \{B_i \subseteq O : |B_i| = |\Omega_i|\}.$$

Each agent $i \in I$ has a weak preference relation $\succsim_i$ over X_i. Given a profile of preferences $\succsim \equiv (\succsim_i)_{i \in I}$ and a matching $\mu \in \mathcal{M}$, μ is *individually rational under* $\succsim$ if each agent finds her assignment to be at least as good as her endowment—that is, for each $i \in I$, $\mu(i) \succsim_i \Omega_i$. Note that there is always at least one individually rational matching: the matching that assigns each agent her endowment. A bundle $B_i \in X_i$ such that $\Omega_i \succ_i B_i$ is *unacceptable* and a bundle $C_i \in X_i$ such that $C_i \succsim_i \Omega_i$ is *acceptable* to agent $i \in I$. Next, $\mu' \in \mathcal{M}$ *Pareto-improves* $\mu \in \mathcal{M}$ if, for each $i \in I$, $\mu'(i) \succsim_i \mu(i)$ and, for some $i \in I$, $\mu'(i) \succ_i \mu(i)$. If there is no μ' that Pareto-improves μ, then μ is *Pareto-efficient.*

We now develop assumptions on each agent's preferences. These play a key role in the remainder of this paper. The main idea is to identify a particular set of objects as being *desirable* for a given agent and define acceptability as well as the agent's preferences over acceptable bundles based on this desirable set. To this end, we fix an agent $i \in I$, i's endowment Ω_i, and a set of desirable objects $A_i \subseteq O$ for i.[11] Our first assumption is that, when we restrict attention to acceptable bundles, more desirable objects are always better.

Assumption 1 (Monotonic perfect substitution)**.** For any two sets $B_i, C_i \in X_i$ such that $B_i \succsim_i \Omega_i$ and $C_i \succsim_i \Omega_i$, $B_i \succ_i C_i$ if and only if $|B_i \cap A_i| > |C_i \cap A_i|$.

Assumption 1 says that i prefers an acceptable bundle that contains more desirable objects to an acceptable bundle that contains fewer desirable objects. It also implies that i is indifferent between any two acceptable bundles that contain the same number of desirable objects. In other words, Assumption 1 says that i's preferences are monotonic in the number of elements of A_i she receives and she views elements of A_i as perfect substitutes.

Our next assumption says that agent i exhibits an *aversion to undesirable objects* in the sense that i is not willing to accept a set of objects that contains one or more undesirable objects, no matter how many desirable objects the set contains.

Assumption 2 (Aversion to undesirable objects)**.** For any $B_i \in X_i$, $B_i \succsim_i \Omega_i$ if and only if $B_i \subseteq A_i \cup \Omega_i$.

In the context of shift exchange, Assumption 2 means that a worker is never willing to create additional scheduling conflicts by accepting a shift for which she already knows that she is not available. By contrast, any shift schedule that leaves i with only desirable shifts

[11] Note that we permit the possibility that $\Omega_i \cap A_i \neq \varnothing$.

and endowed shifts is deemed acceptable. Under this assumption, if an i reports only A_i, the constraint of individual rationality is informationally very simple: it requires that i be assigned only objects in $A_i \cup \Omega_i$. For shift exchange, this means that an individually rational matching ought not assign to her a shift that she has not indicated as desirable.

We summarize our assumptions on preferences by means of the following definition.

Definition 1. Fix an agent $i \in I$, an endowment $\Omega_i \subseteq O$, and a desirable set $A_i \subseteq O$. We say that i's preferences over sets of objects $\succsim_i$ are *trichotomous* with respect to A_i and Ω_i if $\succsim_i$ satisfies Assumptions 1 and 2.

If $\succsim_i$ is trichotomous with respect to A_i, then O is partitioned into three components: A_i, i's desirable objects; $\Omega_i \setminus A_i$, objects that i is endowed with but does not find desirable; and $O \setminus (A_i \cup \Omega_i)$, the remaining objects. It is because this partition has three components that we have chosen the name *trichotomous* for our preference restriction. Note however, that this partition is comprised of the three indifference classes of i's *marginal* preference over individual objects and not over X_i. Given A_i and Ω_i, $\succsim_i$ has $|\Omega_i \setminus A_i| + 1$ indifference classes over acceptable bundles.

Before proceeding, we discuss the relationship of our trichotomous preference domain to other approaches in the related literature. For that purpose, note first that we allow for the case where $\Omega_i \cap A_i \notin \{\emptyset, \Omega_i\}$, i.e. the case where some of i's endowed objects are desirable, while others are not. Hence, we need information about endowments *and* desirable sets of objects in order to describe agents' preferences. This is a distinct feature of our model compared with the *dichotomous* domain of Bogomolnaia and Moulin [2004].[12] Next, note that we also allow for the possibility that a non-empty strict subset of j's endowment Ω_j is desirable for $i \neq j$. This is an important way that ours is different from the the independent contribution of Andersson et al. [2018]. There, agents are assumed to either like, or find desirable, none or all of the objects another agent brings to the market. While this assumption may be sensible in the context of time banks, which is the leading application in Andersson et al. [2018], it is hard to justify in the context of shift exchange. In the latter application, we can think of objects in A_i as shifts where i has no scheduling conflicts at all, of objects in $\Omega_i \setminus A_i$ as shifts that i is responsible for but would like to trade away to engage in other activities (e.g. additional paid labor for other firms, further training/education, vacation), and of objects in $O \setminus (A_i \cup \Omega_i)$ as shifts that i is not able to take on due to existing

[12] Bogomolnaia and Moulin [2004] consider a two-sided one-to-one matching market in which each agent partitions potential match partners from the other side of the market into acceptable (better than being left unmatched) and unacceptable (worse than being left unmatched) ones. Since each agent is endowed with and cannot consume more than one "object", agents' preferences are entirely described by their sets of acceptable partners.

scheduling conflicts. Clearly, it is possible that i is able to take on some shifts initially assigned to j but not others.

Henceforth, we assume that the preferences of all agents are trichotomous. For each agent $i \in I$, the acceptable sets containing $|\Omega_i|$ objects and the ranking of these acceptable sets are then completely identified by her endowment Ω_i and her desirable set A_i.

Since we assume that preferences are trichotomous and that endowments are fixed and known, the only remaining information that we need in order to identify an agent's preference over individually rational matchings is her desirable set. We assume that this set is her private information. A *mechanism* is a mapping $\varphi : \mathcal{A}^n \to \mathcal{M}$ that associates a matching to each reported profile of desirable sets. Given a profile $A \in \mathcal{A}^n$, we denote the set of objects that i receives under mechanism φ by $\varphi_i(A)$. A mechanism is *individually rational* if it selects an individually rational matching for every profile of desirable sets. Recall that our assumptions on preferences imply that μ is individually rational at $A \in \mathcal{A}^n$ if and only if $\mu(i) \subseteq A_i \cup \Omega_i$. Similarly, a mechanism is *Pareto-efficient* if it selects a Pareto-efficient matching for every profile of desirable sets. Finally, a mechanism is *strategy-proof* if no agent can ever benefit by lying about her desirable set. That is, φ is strategy-proof if there is no $A \in \mathcal{A}^n$, $i \in I$, and $\hat{A}_i \in \mathcal{A}$ such that $\varphi_i(\hat{A}_i, A_{-i}) \succ_i^{A_i} \varphi_i(A)$, where $\succsim_i^{A_i}$ is the trichotomous preference with respect to the true desirable set A_i. Note that this is a direct mechanism since $\succsim_i^{A_i}$ is identified by Ω_i and A_i and we assume that only A_i is i's private information.

Since we have restricted attention to trichotomous preferences, an individually rational mechanism φ is strategy-proof if there are no $A \in \mathcal{A}^n$, $i \in I$, and $\hat{A}_i \in \mathcal{A}$ such that $\varphi_i(\hat{A}_i, A_{-i}) \subseteq A_i \cup \Omega_i$ and $|\varphi_i(\hat{A}_i, A_{-i}) \cap A_i| > |\varphi_i(A) \cap A_i|$.

3 The Individually Rational Priority Algorithm

In this section, we introduce an algorithm that produces individually rational and Pareto-efficient matchings. Like sequential priority, agents take turns picking their most preferred sets of objects according to some exogenous priority ranking. The crucial difference with serial dictatorship is that our algorithm constrains each agent to choose in a way that is compatible with individual rationality for all agents. Since the set of individually rational matchings varies in response to each agent's report, an agent can influence the choice set of any other agent—not just those with lower priority. This feature of our algorithm makes it much harder to show that the induced direct mechanism is strategy-proof, which we turn to in the next section, than to show that a serial dictatorship mechanism has this property.

We now describe our algorithm. For the remainder of the paper, we equate an agent's index $i \in I = \{1, \ldots, n\}$ with her priority. Note that lower indices correspond to higher priority. The counter t for the steps of the algorithm then also doubles as the reference for the successively next highest priority agent.

For the remainder of this section, fix a profile of desirable sets $A \in \mathcal{A}^n$. The *individually rational priority (IRP) algorithm* for A proceeds as follows:

Step 0: Let $\mathcal{M}^0(A)$ be the set of all individually rational matchings at A.

Step $\boldsymbol{t \in \{1, \dots, n\}}$: Let

$$K^t(A) \equiv \max_{\mu \in \mathcal{M}^{t-1}(A)} |\mu(t) \cap A_t|$$

be the *promise to agent* t and

$$\mathcal{M}^t(A) = \{\mu \in \mathcal{M}^{t-1}(A) : |\mu(t) \cap A_t| = K^t(A)\}$$

be the set of all matchings in $\mathcal{M}^{t-1}(A)$ that comply with the promise to t.[13]

At an intuitive level, the algorithm works as follows: among all individually rational matchings, let agent 1 pick those matchings that maximize the number of desirable objects she gets and collect those matchings in the set $\mathcal{M}^1(A) \subseteq \mathcal{M}^0(A)$; among all matchings in $\mathcal{M}^1(A)$, let agent 2 pick those matchings that maximize the number of desirable objects she gets and collect those matchings in the set $\mathcal{M}^2(A) \subseteq \mathcal{M}^1(A)$; ... ; among all matchings in $\mathcal{M}^{n-1}(A)$, let agent n pick those matchings that maximize the number of desirable objects she gets and collect those matchings in the set $\mathcal{M}^n(A) \subseteq \mathcal{M}^{n-1}(A)$. Note that, for any $t \leq n$, $\mathcal{M}^t(A)$ is the set of matchings that remain after agent t has made her pick(s). We now provide a formal example to illustrate the basic mechanics of the IRP algorithm.

Example 1. There are four agents $(1, 2, 3, 4)$ and six objects $(o_1, o_2, p, q, r_1, r_2)$. Endowments and desirable sets are given by the following table:

i	Ω_i	A_i
1	$\{o_1, o_2\}$	$\{o_1, p, r_2\}$
2	$\{p\}$	$\{o_1, q\}$
3	$\{q\}$	$\{o_2\}$
4	$\{r_1, r_2\}$	$\{o_2, p\}$

In the first step of the IRP algorithm, we calculate the maximal number of desirable objects that agent 1 can obtain in an individually rational matching and the set of individually

[13] In Section 5, we show how to compute the sequences $\{K^t(A), \mathcal{M}^t(A)\}_{t=1}^n$ in polynomial time.

rational matchings that yield 1 exactly that maximal number. It is easy to see that $K^1(A) = 2$ and that $\mathcal{M}^1(A)$ consists of the following four matchings:

	1	2	3	4
μ^1	$\{o_1, p\}$	$\{q\}$	$\{o_2\}$	$\{r_1, r_2\}$
μ^2	$\{o_1, r_2\}$	$\{q\}$	$\{o_2\}$	$\{r_1, p\}$
μ^3	$\{o_1, r_2\}$	$\{p\}$	$\{q\}$	$\{r_1, o_2\}$
μ^4	$\{p, r_2\}$	$\{o_1\}$	$\{q\}$	$\{r_1, o_2\}$

Proceeding, it is easy to check that $K^2(A) = 1$, $\mathcal{M}^2(A) = \{\mu^1, \mu^2, \mu^4\}$, $K^3(A) = 1$, and $\mathcal{M}^3(A) = \{\mu^1, \mu^2\}$. Finally, note that since $1 = |\mu^2(4) \cap A_4| > |\mu^1(4) \cap A_4| = 0$, we obtain that $K^4(A) = 1$ and $\mathcal{M}^4(A) = \{\mu^2\}$.

The following lemma, whose proof follows immediately from the description above, lists an important property of the IRP algorithm that we will use repeatedly throughout our proofs.

Lemma 1. *For any pair t, t' such that $t' \leq t$ and any $\mu \in \mathcal{M}^t(A)$, we have that $|\mu(t') \cap A_{t'}| = K^{t'}(A)$.*

The preceding lemma implies in particular that all matchings in $\mathcal{M}^t(A)$ are *welfare equivalent* for all agents $t' \leq t$. Hence, all matchings in $\mu \in \mathcal{M}^n(A)$ are welfare equivalent for all agents and we refer to any matching in $\mathcal{M}^n(A)$ as an *IRP outcome* for A.[14]

We now establish that all IRP outcomes are individually rational and Pareto-efficient. First, for any $t \in \{1, \ldots, n\}$, we have $\mathcal{M}^t(A) \subseteq \mathcal{M}^{t-1}(A)$. Hence, $\mathcal{M}^n(A) \subseteq \mathcal{M}^0(A)$, which implies that IRP outcomes are individually rational. Second, let $\mu \in \mathcal{M}^n(A)$. The construction of the sequence $\{\mathcal{M}^t(A)\}_{t=1}^n$ implies that there is no matching in $\mathcal{M}^0(A)$ that Pareto-dominates μ. For any matching $\nu \in \mathcal{M} \setminus \mathcal{M}^0(A)$, there is at least one agent $j \in I$ for whom individual rationality is violated, i.e. $\mu(j) \succsim_j \Omega_j \succ_j \nu(j)$. Hence, there is no matching in $\mathcal{M} \setminus \mathcal{M}^0(A)$ that Pareto-dominates μ. Combining our last two findings, we obtain the following result.

Theorem 1. *Any $\mu \in \mathcal{M}^n(A)$ is individually rational and Pareto-efficient.*

In the remainder of this section, we develop a characterization of matchings in the sequence $\{\mathcal{M}^t(A)\}_{t=1}^n$ that plays a crucial role for our subsequent analysis. Before proceeding, note that, since we restrict attention to balanced exchange, any matching can be obtained from agents' endowments via a sequence of "trading cycles" between the agents. In particular, for any t and any $\mu \in \mathcal{M}^t(A)$, μ can be obtained from Ω in the just described manner. We will

[14]Note that, due to indifferences in agents' preferences, $\mathcal{M}^n(A)$ need not be a singleton.

show that matchings in $\mathcal{M}^t(A)$ can be characterized by the absence of trading cycles that lead to an increase in the number of desirable objects held by any agent $t' \leq t$. We now develop these ideas formally.

Given a matching $\mu \in \mathcal{M}^0(A)$, a *cycle of* μ is a sequence $C = (i^1, o^1, \ldots, i^M, o^M)$ of M distinct agents and M distinct objects such that, for each $m \in \{1, \ldots, M\}$, $o^{m-1} \in \mu(i^m)$ and $o^m \notin \mu(i^m)$, where $o^0 \equiv o^M$. For the following discussion, fix an arbitrary cycle $C = (i^1, o^1, \ldots, i^M, o^M)$ of μ. For any $m \in \{1, \ldots, M\}$, we say that i^m *points to* o^m and that o^m *points to* i^{m+1}, where $i^{M+1} \equiv i^1$. Thus, C is a sequence of agent-object pairs such that each agent i^m points to an object o^m that she does not get at μ, i.e. an object $o^m \notin \mu(i^m)$, and each object o^m points to the agent who gets it at μ, i.e. the agent i^{m+1} for whom $o^m \in \mu(i^{m+1})$. Let $I(C) \equiv \{i^1, \ldots, i^M\}$ be the set of agents involved in C and let $O(C) \equiv \{o^1, \ldots, o^M\}$ be the set of objects involved in C. Let $\mu + C$ denote the matching that results from μ by executing the trades designated by C, i.e., for each $k \in I$,

$$(\mu + C)(k) \equiv \begin{cases} \mu(k) & \text{if } k \notin I(C) \\ (\mu(k) \setminus \{o^{m-1} : i^m = k\}) \cup \{o^m : i^m = k\} & \text{if } k \in I(C). \end{cases}$$

Motivated by the definition of $\mu + C$, we say that, for any $m \in \{1, \ldots, M\}$, i^m *trades* o^{m-1} *for* o^m *in* C. We say that C

- is *individually rational* if, for each $m \in \{1, \ldots, M\}$, $o^m \in \Omega_{i^m} \cup A_{i^m}$,
- *increases* i*'s welfare* if $|(\mu + C)(i) \cap A_i| > |\mu(i) \cap A_i|$,
- *decreases* i*'s welfare* if $|(\mu + C)(i) \cap A_i| < |\mu(i) \cap A_i|$, and
- *affects* i if it either increases or decreases i's welfare.

Finally, we say that C is a *constrained improvement cycle (CIC) of* μ *for* t *at* A if C

1. individually rational,
2. increases t's welfare, and
3. does not affect any agent $t' < t$.

With these definitions, we now state the second result of this section.

Theorem 2. *For any* $t \in \{1, \ldots, n\}$, $\mu \in \mathcal{M}^t(A)$ *if and only if* $\mu \in \mathcal{M}^{t-1}(A)$ *and there does not exist a CIC of* μ *for* t *at* A.

To gain some intuition for Theorem 2, consider the t^{th} step of the IRP algorithm. Here, the number of desirable objects awarded to agent t is determined subject to the constraints imposed by the promises to agents $t' < t$. These constraints are expressed in the restriction to matchings in $\mathcal{M}^{t-1}(A)$. Now fix an arbitrary matching $\mu \in \mathcal{M}^t(A)$. If C is a CIC of μ for $t' \leq t$ at A, then $\mu + C \in \mathcal{M}^{t'-1}(A)$ given that $\mu \in \mathcal{M}^t(A) \subseteq \mathcal{M}^{t'-1}(A)$. Since C increases the welfare of t', we find that $\mu \notin \mathcal{M}^{t'}(A)$ and hence, given that $t' \leq t$, $\mu \notin \mathcal{M}^t(A)$. For the proof of part (ii), pick an arbitrary $\nu \in \mathcal{M}^{t-1}(A)$ that leaves t strictly better off than μ. Then there exists an object o that is desirable for t and that t gets in ν but not in μ. In our proof, we use o as well as the other assignments in μ and ν to construct a CIC of μ for j at A.

4 IRP Mechanisms and Strategy-proofness

Though we have fixed the priority over the agents, there are several direct mechanisms that are induced by the IRP algorithm: every selection φ from $\mathcal{M}^n$. By Lemma 1, all such direct mechanisms are welfare equivalent. We call any such mechanism a *individually rational priority (IRP) mechanism.*

For the remainder of this section, fix a true profile of desirable sets $A \in \mathcal{A}^n$, an agent $i \in I$, and a possible manipulation for i, $\hat{A}_i \in \mathcal{A}_i$. Denote the profile $(\hat{A}_i, A_{-i})$ by $\hat{A}$. Recall that $\{\mathcal{M}^t(A)\}_{t=1}^n$ and $\{\mathcal{M}^t(\hat{A})\}_{t=1}^n$ summarize the progression of the IRP algorithm for inputs A and $\hat{A}$ respectively.

We start by showing that i does not benefit by falsely claiming that an object is desirable when it is not. The first of two lemmas considers the case where she falsely claims as desirable an object o that she is not endowed with. It says that the the number of trades that she receives can only change if she receives o at every matching produced by the IRP algorithm at $\hat{A}$.

Lemma 2. *If, for some $o \notin \Omega_i \cup A_i$, $\hat{A}_i = A_i \cup \{o\}$ and $K^i(\hat{A}) \neq K^i(A)$, then, for each $\mu \in \mathcal{M}^n(\hat{A})$, $o \in \mu(i)$.*

Since preferences are trichotomous, Lemma 2 implies that falsely claiming a single undesirable object to be desirable is not beneficial. The next lemma shows that an agent cannot benefit from falsely claiming that one of her endowed objects is desirable.

Lemma 3. *If, for some $o \in \Omega_i \setminus A_i$, $\hat{A}_i = A_i \cup \{o\}$, then for each $\mu \in \mathcal{M}^i(\hat{A})$, $K^i(A) \geq |\mu(i) \cap A_i|$.*

Lemmas 2 and 3 imply that expanding the set of desirable objects can never be profitable for an agent. Next, we consider the case where i falsely declares a desirable object to be undesirable.

Lemma 4. *If, for some $o \in A_i$, $\hat{A}_i = A_i \setminus \{o\}$, then $K^i(A) \geq K^i(\hat{A})$.*

Since Lemma 4 is the main step in proving strategy-proofness of IRP mechanisms, we provide a sketch of its proof. Note first that when i shrinks her desirable set from A_i to $\hat{A}_i = A_i \setminus \{o\}$, she places an additional constraint on the choices of all agents in the IRP algorithm since she can no longer receive o. Put differently, the set of individually rational matchings shrinks from $\mathcal{M}^0(A)$ to $\mathcal{M}^0(\hat{A}) \subseteq \mathcal{M}^0(A)$. It is then immediate that a contraction can never be profitable for the highest priority agent since such a manipulation would only shrink her own choice set. Things are potentially different for $i > 1$: she might hope that shrinking the choice sets of agents $j < i$ forces these agents to leave her with a choice set $\mathcal{M}^{i-1}(\hat{A})$ that contains some matching μ that she prefers to all matchings in $\mathcal{M}^{i-1}(A)$. So assume that there exists such a matching $\mu \in \mathcal{M}^{i-1}(\hat{A})$. Then $|\mu(i) \cap A_i| > K^i(A)$. Since $\mu \notin \mathcal{M}^{i-1}(A)$ but $\mu \in \mathcal{M}^0(A)$, there exists a smallest integer $j < i$ such that $\mu \notin \mathcal{M}^j(A)$. By Theorem 2, there exists a CIC C of μ for j at A. Given that the only difference between A and $\hat{A}$ is that i ranks o as undesirable in $\hat{A}$, i points to o in C. Since $o \in A_i \setminus \Omega_i$, C cannot decrease the welfare of i at A. If $\mu + C \in \mathcal{M}^{i-1}(A)$, we obtain a contradiction to our assumption that $|\mu(i) \cap A_i| > K^i(A)$ since there is a matching that i could have picked in the IRP algorithm at A that is at least as good as μ. If $\mu + C \notin \mathcal{M}^{i-1}(A)$, Theorem 2 again implies that there has to be a CIC C' of $\mu + C$ at A for some $j' < i$. The main part of our proof of Lemma 4 shows that we can construct a third CIC C^* from C and C' such that $\mu + C^*$ (A) makes every agent $k \leq \min\{j, j'\}$ at least as well off as $(\mu + C) + C'$, and (B) makes i weakly better off than μ. Iterating this argument, we eventually obtain a contradiction to our assumption that $|\mu(i) \cap A_i| > K^i(A)$ since there exists at least one matching in $\mathcal{M}^{i-1}(A)$ that makes i weakly better off than μ.

Finally, we appeal to Lemmas 2, 3, and 4 to prove that any IRP mechanism is strategy-proof.

Theorem 3. *Any IRP mechanism is strategy-proof.*

We point out that our restriction to trichotomous preferences is important for strategy-proofness of IRP mechanisms. For example, even for the case of singleton endowments, a sequential priority over the set of individually rational matchings is not generally strategy-proof if agents' preferences have more than three indifference classes.

5 Computing IRP Matchings

In this section, we define a polynomial time algorithm (Algorithm 1) to compute matchings in $\mathcal{M}^n(A)$ based on network flows. The procedure described in Section 3 relies on an enumeration of all of the individually rational matchings ($\mathcal{M}^0(A)$), the subset of these matchings in which 1 receives the most desirable objects ($\mathcal{M}^1(A)$), and so on. The problem is that sizes of these sets are an exponential function of the number of agents and objects. The algorithm that we now develop uses the fact that each of these sets is the set of solutions to a particular network flow problem.

For the following discussion, fix an endowment Ω and a profile of desirable sets A. We work with a directed graph (V, E). The set of vertices V contains three vertices corresponding to each $i \in I$, call them i, i^A, and i^U. It also contains one vertex corresponding to each $o \in O$. Finally, it contains a source S and a sink T. That is,

$$V \equiv \left[\bigcup_{i\in I}\{i, i^A, i^U\}\right] \cup O \cup \{S, T\}$$

The set of edges E contains an edge from S to each $i \in I$, from each $i \in I$ to i^A and i^U, from i^A to each object in A_i, from i^U to each object in $\Omega_i \setminus A_i$, and from each $o \in O$ to T. That is,

$$\begin{array}{rcl} E & \equiv & \{(S, i) : i \in I\} \\ & \cup & \{(i, i^A), (i, i^U) : i \in I\} \\ & \cup & \{(i^A, o) : o \in A_i\} \\ & \cup & \{(i^U, o) : o \in \Omega_i \setminus A_i\} \\ & \cup & \{(o, T) : o \in O\}. \end{array}$$

Roughly speaking, the idea here is that copy i^A of agent i collects all desirable objects, while copy i^U collects all endowments that are not desirable. Throughout our algorithm, we only vary the capacities of each edge. The capacity of edge $e \in E$ is $q(e)$ and we initialize q as follows:

$$\begin{array}{rcll} q(S, i) & = & |\Omega_i| & i \in I \\ q(i, i^A) & = & |\Omega_i| & i \in I \\ q(i, i^U) & = & |\Omega_i \setminus A_i| & i \in I \\ q(i^A, o) & = & 1 & i \in I, o \in A_i \\ q(i^U, o) & = & 1 & i \in I, o \in \Omega_i \\ q(o, T) & = & 1 & o \in O. \end{array}$$

We say that $f : E \to \mathbb{N}$ is an (integer) flow, if $f(e) \leq q(e)$, for all $e \in E$, and $\sum_{e\in\delta^-(v)} f(e) = \sum_{e\in\delta^+(v)} f(e)$, for all $v \in V \setminus \{S, T\}$.[15] The value of a flow f through i is $f(S, i)$ and the value of a flow f is $v(f) = \sum_{i\in I} f(S, i)$. Since $q(S, i) = |\Omega_i|$, for all $i \in I$, the maximum flow through i cannot be larger than $|\Omega_i|$. Since for all $o \in O$, $i \in I$,

[15]By $\delta^+(v)$ we denote the edges that originate at node v and by $\delta^-(v)$ the edges that point to v.

and $D \in \{A, U\}$, $q(i^D, o) \in \{0, 1\}$, a flow cannot use two edges involving the same object but different agents. If no flow g has higher value than f, then f is a *maximum flow*. Let $\textsc{MaxFlow}(q)$ be the value of a maximum flow from S to T when the capacities are q. By the above observation that the flow through i cannot exceed $|\Omega_i|$, for every q that we consider, $\textsc{MaxFlow}(q) \le \sum_{i \in I} |\Omega_i|$ and, for the initial value of q, $\textsc{MaxFlow}(q) = \sum_{i \in I} |\Omega_i|$ since we can assign each agent her endowment at that capacity vector. Given $\mu \in \mathcal{M}$, we say that μ *corresponds to* f if, for each $i \in I$ and each $o \in \mu(i) \cap A_i$, $f(i^A, o) = 1$ and, for each $i \in I$ and each $o \in \mu(i) \cap \Omega_i \setminus A_i$, $f(i^u, o) = 1$.

Algorithm 1 iteratively minimizes each agent i's *capacity for endowments that are not desirable*, $q(i, i^U)$, by solving a series of network flow problems.

Algorithm 1 Procedure to compute IRP

procedure IRP(V, E, q)
    for $t = 1$ to $t = N$ **do** ▷ Loop invariant: $\textsc{MaxFlow}(q) = \sum_i |\Omega_i|$
        if $q(t, t^U) > 0$ **then** ▷ t has capacity
            $\hat{q} = q$
            $\hat{q}(t, t^U) = \hat{q}(t, t^U) - 1$ ▷ Decrement t's capacity by 1
            while $\textsc{MaxFlow}(\hat{q}) = \sum_i |\Omega_i|$ **do** ▷ As long as the loop invariant holds
                $q = \hat{q}$
                $\hat{q}(t, t^U) = \hat{q}(t, t^U) - 1$ ▷ Decrement t's capacity by 1
    return q

The following result summarizes the key properties of Algorithm 1.

Theorem 4. *Let $A \in \mathcal{A}^n$ be a profile of desirable sets and let Ω be an endowment. Suppose that V, E, and q are defined based on A and Ω as described above.*

1. *A flow f is a maximum flow in $(V, E, \mathrm{IRP}(V, E, q))$ if and only if it corresponds to an IRP outcome.*

2. *The complexity of Algorithm 1 is $O(mn^3 \log(k))$, where $m = |O|$ and $k = \max_{i \in I} |\Omega_i|$.*

6 Beyond Trichotomous Preferences

We now ask whether it is possible to weaken the assumption that agents' preferences exhibit monotonic perfect substitution. As a small departure from trichotomous preferences, suppose that agents can possibly distinguish between sets that contain no more than one desirable trade, but only if the comparison involves one particular object. Formally, consider an agent $i \in I$ who has a desirable set $A_i \subseteq O$. Designate for i a special object $a_i \in A_i$. We weaken

Assumption 1 only with regards to comparing sets containing a single desirable object, and that too only when one of them contains a_i.

Assumption 3 (Monotonic and minimally imperfect substitution)**.** For any two sets $B_i, C_i \in X_i$ such that $B_i \succsim_i \Omega_i$ and $C_i \succsim_i \Omega_i$, $B_i \succ_i C_i$ if and only if either

1. $|B_i \cap A_i| > |C_i \cap A_i|$, or
2. $|B_i \cap A_i| = |C_i \cap A_i| = 1$ and $B_i \cap A_i = \{a_i\}$.[16]

Restricted to sets that either do not contain a_i or contain more than one desirable object Assumption 3 is equivalent to Assumption 1. Note that the above permits i to favor a_i but does not require her to. Thus, if $\succsim_i$ satisfies Assumption 1 with respect to A_i, then since it favors no object, for every $a_i \in A_i$, $\succsim_i$ satisfies Assumption 3 with respect to A_i and a_i.

We now define a minimal departure from trichotomous preferences.

Definition 2. Fix an agent $i \in I$, an endowment $\Omega_i \subseteq O$, a desirable set $A_i \subseteq O$, and a special object $a_i \in A_i$. We say that i's preferences over sets of objects $\succsim_i$ are *minimally non-trichotomous* with respect to a_i, A_i and Ω_i if $\succsim_i$ satisfies Assumptions 3 and 2.

Though the departure from the trichotomous domain is very small—it is the difference between Assumptions 1 and 3—this is enough to result in an impossibility result for our three main desiderata.

Proposition 1. *Suppose at least one agent is endowed with more than one object. For the domain of minimally non-trichotomous preferences, no mechanism is individually rational, Pareto-efficient, and strategy-proof.*

Note that Proposition 1 is not a "maximal domain" result. It does not say that adding a single preference relation that is not trichotomous for a single agent $i \in I$ to the domain of trichotomous preferences leads us to an impossibility result. Rather, it tells us that, when all minimally non-trichotomous preferences are allowed, there is no mechanism that satisfies our three desiderata.

7 Conclusion

Without restrictions on the preference domain, the three central requirements of individual rationality, Pareto-efficiency, and strategy-proofness are incompatible for the exchange of indivisible goods without monetary transfers and multi-unit demand and supply. In this

[16] Since we require B_i and C_i to be acceptable to agent i, this case is possible only when $|A_i \cap \Omega_i| \leq 1$.

paper, we have focused on balanced exchange and identified a meaningful restriction on preferences where they are actually compatible when endowments are known.

Our results for this model show that, despite earlier negative results, this is an area ripe for future research.[17] We conclude with some possible directions for future research.

While we have shown an impossibility under minimally non-trichotomous preferences, there are other interesting domains where appropriate adaptations of IRP mechanisms satisfy the above mentioned desiderata. For example, this is true for the domain of "all-or-nothing" preferences where a bundle other than the endowment is acceptable if and only if the agent receives only desirable goods.[18] A systematic study of such domains would be an interesting avenue of study. Despite the desirable normative and strategic properties of IRP mechanisms for all-or-nothing preferences, it is not clear that IRP outcomes can be computed efficiently for this domain. Again, the efficient computation of individually rational and Pareto-efficient matchings beyond trichotomous preferences remains an open question. Finally, our analysis is for fixed endowments. A systematic analysis of agents' incentives to reveal their endowments remains an interesting area of research.

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[17]See also the concurrent and independent positive results of Andersson et al. [2018].

[18]A proof is available upon request.

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Appendices

A Proof of Theorem 2

We first introduce a notion of generalized CICs where we allow agents, but not objects, to reoccur. A *generalized cycle of* μ is a sequence $C \equiv (i^1, o^1, \dots, i^M, o^M)$ of M (not necessarily distinct) agents and M distinct objects such that, for each $m \in \{1, \dots, M\}$, $o^{m-1} \in \mu(i^m)$ and $o^m \notin \mu(i^m)$ (where $o^0 \equiv o^M$). For the following discussion, fix an arbitrary generalized cycle $C = (i^1, o^1, \dots, i^M, o^M)$ of μ. As for the case of cycles, we denote the matching that results by implementing the exchanges that C specifies by $\mu + C$, i.e., for each $k \in I$, we set

$$(\mu + C)(k) \equiv \begin{cases} \mu(k) & \text{if } k \notin I(C) \\ (\mu(k) \setminus \{o^{m-1} : i^m = k\}) \cup \{o^m : i^m = k\} & \text{if } k \in I(C) \end{cases}$$

We say that C is an *individually rational generalized cycle*, if, for all m, $o^m \in \Omega_{i^m} \cup A_{i^m}$. Furthermore, we refer to i^1 as the *head* of C. A *generalized constrained improvement cycle (GCIC) of* μ *for* j *at* A is an individually rational generalized cycle $C = (i^1, o^1, \dots, i^M, o^M)$ of μ at A that satisfies the following additional conditions:

1. j is the head of C: $i^1 = j$,
2. C improves its head's welfare: $o^M \in \Omega_j \setminus A_j$, $o^1 \in A_j$, and, for any $m \neq 1$ such that $i^m = j$ and $o^m \in \Omega_j \setminus A_j$, we have that $o^{m-1} \in \Omega_j \setminus A_j$
3. C does not affect agents with higher priority than its head: for any m such that $i^m < j$, either $\{o^{m-1}, o^m\} \subseteq A_{i^m}$ or $\{o^{m-1}, o^m\} \subseteq \Omega_{i^m} \setminus A_{i^m}$.

We now start our proof by verifying that if C is an individually rational generalized cycle, then $\mu + C$ is individually rational as well.

Lemma 5. *Let $A \in \mathcal{A}^n$ and $\mu \in \mathcal{M}^0(A)$. If C is an individually rational generalized cycle of μ at A, then $\mu + C \in \mathcal{M}^0(A)$.*

Proof. Let $C = (i^1, o^1, \dots, i^M, o^M)$ and $\nu \equiv \mu + C$. First, for each $o \in O$, since the objects in C are distinct o cannot be in both $\nu(k)$ and $\nu(l)$ for distinct agents k and l. Next, for each $k \in I$, $\nu(k) \subseteq \Omega_k \cup A_k$ since $\mu(k) \subseteq \Omega_k \cup A_k$ and $\nu(k) \setminus \mu(k) \subseteq \Omega_k \cup A_k$ given that C is individually rational. Finally, we verify that for each $k \in I$, $|\nu(k)| = |\Omega_k|$. Since the objects in C are distinct and since $o^{m-1} \in \mu(i^m)$ as well as $o^m \notin \mu(i^m)$ for all m, we have

$$\begin{aligned} |\nu(k)| &= |\mu(k) \setminus \{o^{m-1} : i^m = k\}| + |\{o^m : i^m = k\}| \\ &= |\mu(k)| - |\{o^{m-1} : i^m = k\}| + |\{o^m : i^m = k\}| \\ &= |\Omega_k| \end{aligned}$$

Since $\nu(k) \subseteq \Omega_k \cup A_k$ and $|\nu(k)| = |\Omega_k|$, we have that $\nu(k) \succeq_k \Omega_k$. Since k was arbitrary, the last observation completes the proof. □

For the remainder of the proof, we fix $t \in \{1, \dots, n\}$. We first argue that the absence of GCICs is necessary for a matching to be compatible with the guarantee for t at A.

Lemma 6. *If $\mu \in \mathcal{M}^t(A)$, then there is no GCIC of μ for any $t' \leq t$ at A.*

Proof. Let $\mu \in \mathcal{M}^t(A)$ be arbitrary. Suppose to the contrary that $C \equiv (i^1, o^1, \dots, i^M, o^M)$ is a GCIC of μ for some $t' \leq t$ at A. Let $\nu \equiv \mu + C$. By Lemma 5, ν is individually rational, i.e. $\nu \in \mathcal{M}^0(A)$.

We first argue that $\nu \in \mathcal{M}^{t'-1}(A)$. If $t' = 1$, there is nothing left to show. So let $t' > 1$ and assume that we have already established $\nu \in \mathcal{M}^{t''}(A)$ for some $t'' < t' - 1$. Since C does not affect any agent strictly smaller than t', we have that $|\nu(t''+1) \cap A_{t''+1}| = |\mu(t''+1) \cap A_{t''+1}|$. Since $t''+1 \leq t'-1$ and $t'-1 < t$, we have that $\mathcal{M}^t(A) \subseteq \mathcal{M}^{t''+1}(A)$. Since $\mu \in \mathcal{M}^t(A)$, we obtain that $|\mu(t''+1) \cap A_{t''+1}| = K^{t''+1}(A)$. Hence, $|\nu(t''+1) \cap A_{t''+1}| = K^{t''+1}(A)$ and, given our inductive assumption that $\nu \in \mathcal{M}^{t''}(A)$, we must have $\nu \in \mathcal{M}^{t''+1}(A)$. This completes the proof that $\nu \in \mathcal{M}^{t'-1}(A)$.

To complete the proof, note that $\nu \in \mathcal{M}^{t'-1}(A)$ and the assumption that C increases the welfare of t', jointly imply $\mu \notin \mathcal{M}^{t'}(A)$. Given that $t' \leq t$, we thus obtain a contradiction to $\mu \in \mathcal{M}^t(A)$. This completes the proof. □

Since any CIC is, trivially, also a GCIC, Lemma 6 already shows the necessity part, part (i), of Theorem 2.

Showing that the absence of CICs is sufficient, i.e. part (ii) of Theorem 2, requires more work. We begin with an auxiliary lemma showing that while an agent $j < t$ may appear more than once in a GCIC of μ, it has to be the case that j either only trades desirable for desirable objects, or only trades endowed objects that are not desirable for endowed objects that are not desirable.

Lemma 7. *Let $j \in I$ be such that $j < t$, $\mu \in \mathcal{M}^{t-1}(A)$, and $C = (i^1, o^1, \dots, i^M, o^M)$ be a*

GCIC of μ for t at A. If m and m' are such that $i^m = i^{m'} = j$, then either

$$\{o^{m-1}, o^m, o^{m'-1}, o^{m'}\} \subseteq \Omega_j \setminus A_j$$
$$or$$
$$\{o^{m-1}, o^m, o^{m'-1}, o^{m'}\} \subseteq A_j.$$

Proof. Fix m and m' such that $i^m = i^{m'} = j$ and assume, without loss of generality, that $m < m' \leq M$. Since C does not affect any agent with higher priority than t, it suffices to show that either $\{o^m, o^{m'}\} \subseteq \Omega_j \setminus A_j$ or $\{o^m, o^{m'}\} \subseteq A_j$. Suppose, for the sake of contradiction, that neither is true. There are two cases. If $o^m \in \Omega_j \setminus A_j$ and $o^{m'} \in A_j$,[19] consider

$$C' \equiv (i^1, o^1, \dots i^{m-1}, o^{m-1}, i^{m'}, o^{m'}, \dots, i^M, o^M).$$

Since C is a generalized individually rational cycle of μ, C' is a generalized individually rational cycle of μ as well. Since C is a GCIC for $t > j$ and $o^m \in \Omega_j \setminus A_j$, we have that $o^{m-1} \in \Omega_j \setminus A_j$ as well. Furthermore, since C is a GCIC for $t > j$, we also obtain that, for any $\tilde{m} \in \{1, \dots, m-1, m'+1, \dots, M\}$ such that $i^{\tilde{m}} \leq t-1$, $\{o^{\tilde{m}-1}, o^{\tilde{m}}\} \subseteq A_{i^{\tilde{m}}}$ or $\{o^{\tilde{m}-1}, o^{\tilde{m}}\} \subseteq \Omega_{i^{\tilde{m}}} \setminus A_{i^{\tilde{m}}}$ (where $o^0 = o^M$). Hence, subject to relabeling the agents, C' is a GCIC of μ for j at A, which we have already argued to be impossible. On the other hand, if $o^m \in A_j$ and $o^{m'} \in \Omega_j \setminus A_j$, the following sequence yields a similar contradiction:

$$(i^m, o^m, \dots, i^{m'-1}, o^{m'-1}).$$

Thus, either $\{o^m, o^{m'}\} \subseteq A_j$ or $\{o^m, o^{m'}\} \subseteq \Omega_j \setminus A_j$. □

Building on Lemma 7, our next auxiliary lemma states that it is sufficient to consider CICs since any recurrence of an agent can be eliminated to yield a shorter cycle.

Lemma 8. *Let $\mu \in \mathcal{M}^{t-1}(A)$. If there exists a GCIC of μ for t at A, then there exists an CIC of μ for t at A.*

Proof. Let $C \equiv (i^1, o^1, \dots, i^M, o^M)$ be a GCIC of μ for t at A. We show how to construct a CIC C' of μ for t on basis of C.

We first eliminate duplicates of any $j \in I(C)$ such that $t < j$. Suppose there are two indices m and m' such that $m < m'$ and $i^m = i^{m'} = j$. Then, since $\{o^{m-1}, o^m, o^{m'-1}, o^{m'}\} \subseteq \Omega_j \cup A_j$, the sequence $(i^1, o^1, \dots, i^{m-1}, o^{m-1}, i^{m'}, o^{m'}, \dots, i^M, o^M)$ is also a GCIC of μ for t at A that has one fewer instances of j than C.

[19] Note that it may be the case that $\Omega_j \setminus \mu(j) \neq \emptyset$ and hence it may well be the case that j (re-)obtains an object from her endowment in C.

Next, we eliminate duplicates of t. Suppose there is m such that $m > 1$ and $i^m = t$. If $o^m \in \Omega_t \setminus A_t$, then since C improves its head's welfare, we have $o^{m-1} \in \Omega_t \setminus A_t$, so that $(i^1, o^1, \ldots, i^{m-1}, o^{m-1})$ is a GCIC of μ for t at A with one fewer instances of t than C. If $o^m \in A_t$, then $(i^m, o^m, \ldots, i^M, o^M)$ is a GCIC of μ for t at A with one fewer instances of t.

Finally, we eliminate duplicates of $j \in I(C)$ such that $j < t$. Suppose there are two indices m and m' such that $m < m'$ and $i^m = i^{m'} = j$. By Lemma 7, we have that either $\{o^{m-1}, o^{m'}\} \subseteq \Omega_j \setminus A_j$ or $\{o^{m-1}, o^{m'}\} \subseteq A_j$. Hence, the sequence $(i^1, o^1, \ldots, i^{m-1}, o^{m-1}, i^{m'}, o^{m'}, \ldots, i^M, o^M)$ is a GCIC of μ for t at A that has one fewer instances of j than C. $\square$

In the remainder of this proof, we establish that if $\mu \in \mathcal{M}^{t-1}(A)$ is such that $|\mu(t) \cap A_t| < K^t(A)$, then there is a GCIC of μ for t at A. By Lemma 8, we thereby establish the sufficiency part of Theorem 2.

Fix $\mu \in \mathcal{M}^{t-1}(A)$ such that $|\mu(t) \cap A_t| < K^t(A)$. We construct a GCIC of μ for t at A. We start with $i^1 = t$. Let $\nu \in \mathcal{M}^t(A)$. Since $|\mu(t) \cap A_t| < K^t(A) = |\nu(t) \cap A_t|$, there exists $o^1 \in (\nu(t) \setminus \mu(t)) \cap A_t$. For the remainder of our proof, fix such an o^1.

Suppose now that we have grown a sequence $(i^1, o^1, \ldots, i^M, o^M)$ of M agents and distinct objects in a way that satisfies the following four conditions:

$$\text{For each } m \geq 1, o^m \in \nu(i^m) \setminus \mu(i^m), \text{ and, for each } m \geq 2, o^{m-1} \in \mu(i^m) \setminus \nu(i^m). \tag{C1}$$

$$\text{For each } m \geq 1, o^m \in \Omega_{i^m} \cup A_{i^m}. \tag{C2}$$

$$\begin{gathered}\text{For each pair } m, m' \text{ such that } i^m = i^{m'} < t, \text{ either} \\ \{o^{m-1}, o^m, o^{m'-1}, o^{m'}\} \subseteq A_{i^m} \text{ or } \{o^{m-1}, o^m, o^{m'-1}, o^{m'}\} \subseteq \Omega_{i^m} \setminus A_{i^m}.\end{gathered} \tag{C3}$$

$$\text{For each } m \geq 2 \text{ such that } i^m = t, \{o^{m-1}, o^m\} \subseteq A_t. \tag{C4}$$

Note that (C1) to (C4) are satisfied for $M = 1$ by our selection of o^1. Note also that (C1) to (C4) imply that if $o^M \in \mu(i^1) \cap (\Omega_{i^1} \setminus A_{i_1})$, then $(i^1, o^1, \ldots, i^M, o^M)$ is a GCIC of μ for $i^1 = t$ at A. We now show that if $o^M \notin \mu(i^1) \cap (\Omega_{i^1} \setminus A_{i^1})$, then we can find an agent-object pair (i^{M+1}, o^{M+1}) so that $(i^1, o^1, \ldots, i^{M+1}, o^{M+1})$ satisfies (C1) to (C4). Since the sets of agents and objects are finite, we must eventually encounter an M such that $(i^1, o^1, \ldots, i^M, o^M)$ is a GCIC of μ for $i^1 = t$ at A.

Since both ν and μ are matchings, $\sum_{i \in I} |\nu(i)| = \sum_{i \in I} |\mu(i)| = \sum_{i \in I} |\Omega_i|$. Hence, $o^M \notin \mu(i^M)$ implies that there exists an agent $i^{M+1} \in I \setminus \{i^M\}$ such that $o^M \in \mu(i^{M+1})$. Since $o^M \in \nu(i^M)$, we have $o^M \in \mu(i^{M+1}) \setminus \nu(i^{M+1})$. We distinguish three cases according to the identity of i^{M+1}. In each of these cases, we identify an object o^{M+1} such that the sequence $(i^1, o^1, \ldots i^{M+1}, o^{M+1})$ satisfies (C1) to (C4).

Case 1: $\boldsymbol{i^{M+1} \notin \{i^1, \ldots, i^M\}}$ There are three subcases to consider:

Case 1.1: $i^{M+1} < t$ and $o^M \in A_{i^{M+1}}$ Since $\nu, \mu \in \mathcal{M}^{t-1}(A)$ and $i^{M+1} < t$, it follows that $|\nu(i^{M+1}) \cap A_{i^{M+1}}| = |\mu(i^{M+1}) \cap A_{i^{M+1}}|$. Since $o^M \in (\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap A_{i^{M+1}}$ there is $o^{M+1} \in (\nu(i^{M+1}) \setminus \mu(i^{M+1})) \cap A_{i^{M+1}}$. Furthermore, since $i^{M+1} \notin \{i^1, \dots, i^M\}$ and, for each m, $o^m \in \nu(i^m)$, we have that $o^{M+1} \notin \{o^1, \dots, o^M\}$.

Case 1.2: $i^{M+1} < t$ and $o^M \in \Omega_{i^{M+1}} \setminus A_{i^{M+1}}$ Since $\nu, \mu \in \mathcal{M}^{t-1}(A)$ and $i^{M+1} < t$, we have $|\nu(i^{M+1}) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})| = |\mu(i^{M+1}) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})|$. Since $o^M \in (\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})$ there is $o^{M+1} \in (\nu(i^{M+1}) \setminus \mu(i^{M+1})) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})$. Furthermore, since $i^{M+1} \notin \{i^1, \dots, i^M\}$ and, for each m, $o^m \in \nu(i^m)$, we have that $o^{M+1} \notin \{o^1, \dots, o^M\}$.

Case 1.3: $i^{M+1} > t$ Since ν and μ are individually rational matchings, $\nu(i^{M+1}) \subseteq \Omega_{i^{M+1}} \cup A_{i^{M+1}}$, $\mu(i^{M+1}) \subseteq \Omega_{i^{M+1}} \cup A_{i^{M+1}}$, and $|\nu(i^{M+1})| = |\mu(i^{M+1})| = |\Omega_{i^{M+1}}|$. Given these facts and that $o^M \in \mu(i^{M+1}) \setminus \nu(i^{M+1})$, there is $o^{M+1} \in \nu(i^{M+1}) \setminus \mu(i^{M+1})$. Furthermore, since $i^{M+1} \notin \{i^1, \dots, i^M\}$ and for each m, $o^m \in \nu(i^m)$, we have that $o^{M+1} \notin \{o^1, \dots, o^M\}$.

Case 2: $i^{M+1} \in \{i^2, \dots, i^M\} \setminus \{t\}$ Let $m \leq M$ be such that $i^m = i^{M+1}$.[20] We distinguish three subcases parallel to those of Case 1.

Case 2.1: $i^{M+1} < t$ and $o^M \in A_{i^{M+1}}$ We first show that $o^m \in A_{i^{M+1}}$. Otherwise, $o^m \in \Omega_{i^{M+1}} \setminus A_{i^{M+1}}$. Consider the following sequence:

$$C' \equiv (i^{M+1}, o^M, i^M, o^{M-1}, \dots, i^{m+1}, o^m).$$

Since we assume that $(i^1, o^1, \dots, i^M, o^M)$ satisfies (C1), we have that $o^{m'} \notin \nu(i^{m'+1})$ and $o^{m'} \in \nu(i^{m'})$ for all $m' \in \{1, \dots, M+1\}$. Hence, C' is a cycle of ν. We now show that C' is a GCIC of ν for i^{M+1} at A. By (C2) above, $o^{m'} \in \Omega_{i^{m'+1}} \cup A_{i^{m'+1}}$ for all $m' \in \{1, \dots, M\}$, so that C' is individually rational at A. Next, note that (C3) and $i^{M+1} < t$ imply that C' does not affect any agent with higher priority than i^{M+1}. Finally, (C3), $i^{M+1} < t$, and $o^M \in A_{i^{M+1}}$ imply the remaining requirements for C' to be a GCIC of ν for i^{M+1} at A. Since C' is a GCIC of ν for i^{M+1} at A, by Lemma 6, $\nu \notin \mathcal{M}^{i^{M+1}}(A)$. Since $i^{M+1} < t$, we have $\mathcal{M}^t(A) \subseteq \mathcal{M}^{i^{M+1}}(A)$ and thus $\nu \notin \mathcal{M}^t(A)$, which is a contradiction. Thus, $o^m \in A_{i^{M+1}}$.

Since m was arbitrary, (C3) and the just established fact imply that, for each $m' \leq M$ such that $i^{m'} = i^{M+1}, \{o^{m'-1}, o^{m'}\} \subseteq A_{i^{M+1}}$. Since $\nu, \mu \in \mathcal{M}^{t-1}(A) \subseteq \mathcal{M}^{i^{M+1}}(A)$, we have $|\nu(i^{M+1}) \cap A_{i^{M+1}}| = |\mu(i^{M+1}) \cap A_{i^{M+1}}|$. Since $o^M \in (\mu(i^{M+1}) \setminus$

[20] Note that i^{M+1} may appear multiple times in $\{i^2, \dots, i^M\}$. Any m such that $i^m = i^{M+1}$ works for the arguments that follow.

$\nu(i^{M+1})) \cap A_{i^{M+1}}$ and, for each $m' \in \{2, \ldots, M-1\}, o^{m'-1} \in \mu(i^{m'}) \setminus \nu(i^{m'})$ and $o^{m'} \in \nu(i^{m'}) \setminus \mu(i^{m'})$, we have that

$$|(\nu(i^{M+1}) \setminus \mu(i^{M+1})) \cap \{o^1, \ldots, o^M\} \cap A_{i^{M+1}}|$$
$$<$$
$$|(\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap \{o^1, \ldots, o^M\} \cap A_{i^{M+1}}|.$$

Thus, combining the last two observations, there exists $o^{M+1} \in [(\nu(i^{M+1}) \setminus \mu(i^{M+1})) \setminus \{o^1, \ldots, o^M\}] \cap A_{i^{M+1}}$.

Case 2.2: $\boldsymbol{i^{M+1} < t}$ and $\boldsymbol{o^M \in \Omega_{i^{M+1}} \setminus A_{i^{M+1}}}$ We first show that $o^m \in \Omega_{i^{M+1}} \setminus A_{i^{M+1}}$. Otherwise, $o^m \in A_{i^{M+1}}$. Consider the following sequence:

$$(i^m, o^m, \ldots i^M, o^M)$$

Since $o^m \in A_{i^{M+1}}$ but $o^M \in \Omega_{i^{M+1}} \setminus A_{i^{M+1}}$, by the assumptions that we have made with regards to $(i^1, o^1, \ldots, i^M, o^M)$, this sequence is a GCIC of μ for i^{M+1} at A. Hence, Lemma 6 implies that $\mu \notin \mathcal{M}^{i^{M+1}}(A)$. Since $\mu \in \mathcal{M}^t(A) \subseteq \mathcal{M}^{i^{M+1}}(A)$, we obtain a contradiction. Thus, $o^m \in \Omega_{i^{M+1}} \setminus A_{i^{M+1}}$.

Since m was arbitrary, (C3) and the just established fact imply that, for each $m' \leq M$ such that $i^{m'} = i^{M+1}, \{o^{m'-1}, o^{m'}\} \subseteq \Omega_{i^{M+1}}$. Since $\nu, \mu \in \mathcal{M}^{t-1}(A) \subseteq \mathcal{M}^{i^{M+1}}(A)$, we obtain $|\nu(i^{M+1}) \cap A_{i^{M+1}}| = |\mu(i^{M+1}) \cap A_{i^{M+1}}|$ and hence $|\nu(i^{M+1}) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})| = |\mu(i^{M+1}) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})|$. Since $o^M \in (\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})$ and, for each $m' \in \{2, \ldots, M-1\}, o^{m'-1} \in \mu(i^{m'}) \setminus \nu(i^{m'})$ as well as $o^{m'} \in \nu(i^{m'}) \setminus \mu(i^{m'})$, we have that

$$|(\nu(i^{M+1}) \setminus \mu(i^{M+1})) \cap \{o^1, \ldots, o^M\} \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})|$$
$$<$$
$$|(\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap \{o^1, \ldots, o^M\} \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})|.$$

Thus, there is $o^{M+1} \in [(\nu(i^{M+1}) \setminus \mu(i^{M+1})) \setminus \{o^1, \ldots, o^M\}] \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})$.

Case 2.3: $\boldsymbol{i^{M+1} > t}$ Since ν and μ are both feasible, $|\mu(i^{M+1}) \setminus \nu(i^{M+1})| = |\nu(i^{M+1}) \setminus \mu(i^{M+1})|$. However, since $o^M \in \mu(i^{M+1}) \setminus \nu(i^{M+1})$ and for each $m \in \{2, \ldots, M-1\}, o^{m-1} \in \mu(i^m) \setminus \nu(i^m)$ as well as $o^m \in \nu(o^m) \setminus \mu(o^m)$, we have that

$$|(\nu(i^{M+1}) \setminus \mu(i^{M+1})) \cap \{o^1, \ldots, o^M\}| < |(\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap \{o^1, \ldots, o^M\}|.$$

Thus, there is $o^{M+1} \in (\nu(i^{M+1}) \setminus \mu(i^{M+1})) \setminus \{o^1, \ldots, o^M\}$.

Case 3: $i^{M+1} = t$ and $o^M \in A_{i^{M+1}}$ By (C4), $|(\nu(t) \setminus \mu(t)) \cap \{o^1, \ldots, o^M\} \cap A_t| = |(\mu(t) \setminus \nu(t)) \cap \{o^1, \ldots, o^M\} \cap A_t|$. Since $|\nu(t) \cap A_t| > |\mu(t) \cap A_t|$, there is $o^{M+1} \in [(\nu(t) \setminus \mu(t)) \setminus \{o^1, \ldots, o^M\}] \cap A_t$.[21]

B Proofs from Section 4

Proof of Lemma 2. Let j be the highest priority agent for whom $K^j(\hat{A}) \neq K^j(A)$ and note that $K^i(A) \neq K^i(\hat{A})$ implies $j \leq i$.

We first show via induction on k that, for all $k \leq j$, $\mathcal{M}^{k-1}(A) \subseteq \mathcal{M}^{k-1}(\hat{A})$ and $o \in \mu(i)$ for each $\mu \in \mathcal{M}^{k-1}(\hat{A}) \setminus \mathcal{M}^{k-1}(A)$. For $k = 1$, both statements follow from $\hat{A}_i = A_i \cup \{o\}$ and $\hat{A}_l = A_l$ for all $l \neq i$. So consider some $k \in \{2, \ldots, j\}$ and assume that both statements have been shown for all $k' < k$. We have that

$$\begin{aligned}
\mathcal{M}^{k-1}(A) &= \{\mu \in \mathcal{M}^{k-2}(A) : |\mu(k-1) \cap A_{k-1}| = K^{k-1}(A)\} \\
&\subseteq \{\mu \in \mathcal{M}^{k-2}(\hat{A}) : |\mu(k-1) \cap A_{k-1}| = K^{k-1}(A)\} \\
&= \{\mu \in \mathcal{M}^{k-2}(\hat{A}) : |\mu(k-1) \cap \hat{A}_{k-1}| = K^{k-1}(\hat{A})\} \\
&= \mathcal{M}^{k-1}(\hat{A})
\end{aligned}$$

Here, the subset relation follows from the inductive assumption that $\mathcal{M}^{k-2}(A) \subseteq \mathcal{M}^{k-2}(\hat{A})$ and the second equality follows from the definitions of j and $\hat{A}$ since $k-1 < j \leq i$. In order to show the second part of the statement for k, fix an arbitrary $\mu \in \mathcal{M}^{k-1}(\hat{A})$ such that $o \notin \mu(i)$. By construction of $\mathcal{M}^{k-1}(\hat{A})$, we have $\mu \in \mathcal{M}^{k-2}(\hat{A})$. Since $o \notin \mu(i)$, the inductive assumption for $k-1$ implies $\mu \in \mathcal{M}^{k-2}(A)$. Since $k-1 < j \leq i$, we have that $|\mu(k-1) \cap A_{k-1}| = |\mu(k-1) \cap \hat{A}_{k-1}|$ and $K^{k-1}(A) = K^{k-1}(\hat{A})$. Combining the last two statements, we obtain $\mu \in \mathcal{M}^{k-1}(A)$.

We now use the just established statements for $k = j$ to complete the proof. Since $\mathcal{M}^{j-1}(A) \subseteq \mathcal{M}^{j-1}(\hat{A})$, $K^j(\hat{A}) \geq K^j(A)$ and the definition of j implies $K^j(\hat{A}) > K^j(A)$. Thus, $\mathcal{M}^j(\hat{A}) \subseteq \mathcal{M}^{j-1}(\hat{A}) \setminus \mathcal{M}^{j-1}(A)$. Hence, $o \in \mu(i)$ for all $\mu \in \mathcal{M}^j(\hat{A})$. Since $j \leq i$, we have that $\mathcal{M}^i(\hat{A}) \subseteq \mathcal{M}^j(\hat{A})$. Combining the last two observations we obtain that $o \in \mu(i)$ for all $\mu \in \mathcal{M}^i(\hat{A})$, which proves the desired statement. □

Proof of Lemma 3. Since $\Omega_i \cup A_i = \Omega_i \cup \hat{A}_i$, $\mathcal{M}^0(\hat{A}) = \mathcal{M}^0(A)$. Since, for each $t < i$, $\hat{A}_t = A_t$, $\mathcal{M}^t(\hat{A}) = \mathcal{M}^t(A)$. Thus, $\mathcal{M}^{i-1}(\hat{A}) = \mathcal{M}^{i-1}(A)$. By definition of $K^i(A)$ and $\mathcal{M}^i(A)$, for each $\mu \in \mathcal{M}^i(A) \subseteq \mathcal{M}^{i-1}(\hat{A})$, $K^i(A) \geq |\mu(i) \cap A_i|$. □

[21] Recall that $i^{M+1} = t$ implies $o^M \in A_{i^{M+1}}$ (as otherwise there would have been no need to extend the sequence $i^1, o^1, \ldots, i^M, o^M$) so Cases 1, 2, and 3 are exhaustive.

Proof of Lemma 4. If $K^i(A) = K^i(\hat{A})$, there is nothing left to show. So for the remainder of the proof, assume that $K^i(A) \neq K^i(\hat{A})$. We argue that $K^i(A) > K^i(\hat{A})$.

Let j^* be the highest priority agent for whom $K^{j^*}(\hat{A}) \neq K^{j^*}(A)$. Note that $K^i(A) \neq K^i(\hat{A})$ implies $j^* \leq i$.

We argue first that $K^{j^*}(A) > K^{j^*}(\hat{A})$. Since $K^j(\hat{A}) = K^j(A)$ for all $j < j^*$ and $\hat{A}_k \subseteq A_k$ for all $k \in I$, we have $\mathcal{M}^{j^*-1}(\hat{A}) \subseteq \mathcal{M}^{j^*-1}(A)$. The last subset relation implies $K^{j^*}(A) \geq K^{j^*}(\hat{A})$ and thus, given that we assumed $K^{j^*}(A) \neq K^{j^*}(\hat{A})$, we have $K^{j^*}(A) > K^{j^*}(\hat{A})$.

Given that $K^{j^*}(A) > K^{j^*}(\hat{A})$, the statement of Lemma 4 follows if $j^* = i$. Henceforth, we assume that $j^* < i$. We show that, for each $\mu \in \mathcal{M}^{i-1}(\hat{A})$, there is a CIC C^* of μ for j^* at A such that $\mu + C^* \in \mathcal{M}^{i-1}(A)$ and $|(\mu + C^*)(i) \cap A_i| \geq |\mu(i) \cap A_i|$. In particular, for any $\mu \in \mathcal{M}^{i-1}(\hat{A})$ there exists a $\mu' \in \mathcal{M}^{i-1}(A)$ such that $|\mu'(i) \cap A_i| \geq |\mu(i) \cap A_i|$. Hence, $K^i(A) \geq K^i(\hat{A})$ and thus, given that we have assumed $K^i(A) \neq K^i(\hat{A})$, $K^i(A) > K^i(\hat{A})$, which yields Lemma 4.

For the remainder of the proof of Lemma 4, we fix an arbitrary $\mu \in \mathcal{M}^{i-1}(\hat{A})$ and show that a CIC of μ for j^* at A with the desired properties exists. Since the proof is quite involved, we introduce eight claims.

Our first claim is that μ *complies with the promise to agents* $j' \leq j^*-1$ in the IRP algorithm under A (i.e. satisfies $\mu \in \mathcal{M}^{j^*-1}(A)$) but not with the promise to j^* (i.e. $\mu \notin \mathcal{M}^{j^*}(A)$).

Claim 1. $\mu \in \mathcal{M}^{j^*-1}(A) \setminus \mathcal{M}^{j^*}(A)$

By Theorem 2, Claim 1 implies that there exists a CIC $C \equiv (i^1, o^1, \ldots, i^M, o^M)$ of μ for j^* at A. The CIC C remains fixed throughout the remainder of the proof of Lemma 4. Our second claim is that i receives o in C.

Claim 2. $o \in (\mu + C)(i) \setminus \mu(i)$

If there is no CIC of $\mu + C$ for any $j < i$ at A. By Claim 1 and Theorem 2, we have that $\mu + C \in \mathcal{M}^{i-1}(A)$. By Claim 2, we have that $o \in (\mu + C)(i) \setminus \mu(i)$ and thus, given that $o \in A_i$, $|(\mu + C)(i) \cap A_i| \geq |\mu(i) \cap A_i|$. Hence, C is a CIC with the desired properties and we are done.

Now, we consider the case where there is an agent $\hat{j}^* < i$ and a CIC $\hat{C} \equiv (j^1, p^1, \ldots, j^L, p^L)$ of $\mu + C$ for $\hat{j}^*$ at A. The CIC $\hat{C}$ remains fixed throughout the remainder of the proof of Lemma 4. We assume without loss of generality that there is no agent $j' < \hat{j}^*$ for whom there is a CIC of $\mu + C$ at A. Our next claim lists two important initial observations about $\hat{j}^*$, C, and $\hat{C}$.

Claim 3. $\hat{j}^* \geq j^*$ *and* $\{\mu + C, (\mu + C) + \hat{C}\} \subseteq \mathcal{M}^{\hat{j}^*-1}(A)$

In the following, we use C and $\hat{C}$ to show that there exists a third CIC $\tilde{C}$ of μ for j^* at A that satisfies the following properties:

$$\mu + \tilde{C} \in \mathcal{M}^{\hat{j}^*-1}(A) \tag{P1}$$

$$|(\mu + \tilde{C})(\hat{j}^*) \cap A_{\hat{j}^*}| \geq |((\mu + C) + \hat{C})(\hat{j}^*) \cap A_{\hat{j}^*}| \tag{P2}$$

$$|(\mu + \tilde{C})(i) \cap A_i| \geq |\mu(i) \cap A_i| \tag{P3}$$

Repeated application of this argument allows us to infer that there exists a CIC C^* of μ for j^* at A that does not decrease the welfare of i and that satisfies $\mu + C^* \in \mathcal{M}^{i-1}(A)$.[22] As mentioned above, the existence of such a CIC proves Lemma 4.

Note that since $(\mu + C) + \hat{C} \in \mathcal{M}^{\hat{j}^*-1}(A)$, a CIC $\tilde{C}$ satisfies (P1) - (P3) if and only if

1. $\mu+\tilde{C}$ makes every agent $j' \leq \hat{j}^*$ at least as well off as $(\mu+C)+\hat{C}$, i.e. $|(\mu+\tilde{C})(j') \cap A_{j'}| \geq |((\mu + C) + \hat{C})(j') \cap A_{j'}|$ for all $j' \leq \hat{j}^*$, and
2. $\tilde{C}$ does not decrease the welfare of i, i.e. $|(\mu + \tilde{C})(i) \cap A_i| \geq |\mu(i) \cap A_i|$.

In our arguments below, we always show that $\tilde{C}$ satisfies these two properties.

We start by introducing some basic notation. Given two integers $m, m' \in \{1, \ldots, M\}$, we use the notation

- $\{m, \ldots [M, 1] \ldots, m'\}$ to represent $\{m, \ldots, m'\}$ if $m \leq m'$ and $\{1, \ldots, m'\} \cup \{m, \ldots, M\}$ if $m > m'$,
- $(i^m, o^m, \ldots [o^M, i^1] \ldots, i^{m'}, o^{m'})$ to represent $(i^m, o^m, \ldots, i^{m'}, o^{m'})$ if $m \leq m'$ and $(i^m, o^m, \ldots, i^M, o^M, i^1, o^1, \ldots, i^{m'}, o^{m'})$ if $m > m'$,
- $(i^m, \ldots [i^M, i^1] \ldots, i^{m'})$ to represent $(i^m, \ldots, i^{m'})$ if $m \leq m'$ and $(i^m \ldots, i^M, i^1, \ldots, i^{m'})$ if $m > m'$, and
- $(o^m, \ldots [o^M, o^1] \ldots, o^{m'})$ to represent $(o^m, \ldots, o^{m'})$ if $m \leq m'$ and $(o^m \ldots, o^M, o^1, \ldots, o^{m'})$ if $m > m'$.

[22]To see this, note first that if $\mu + \tilde{C} \notin \mathcal{M}^{i-1}(A)$, then Theorem 2 and $\mu + \tilde{C} \in \mathcal{M}^{\hat{j}^*-1}(A)$ imply that there exists a CIC $\tilde{C}'$ of $\mu + \tilde{C}$ for some $j' \geq \hat{j}^*$. We can now apply our earlier finding to $\tilde{C}$ and $\tilde{C}'$ to get yet another CIC that satisfies the three properties listed above. Finally, note if $j' = \hat{j}^*$, then we have that $|((\mu+\tilde{C})+\tilde{C}')(\hat{j}^*) \cap A_{\hat{j}^*}| > |\mu(\hat{j}^*) \cap A_{\hat{j}^*}|$ since $|(\mu+\tilde{C})(\hat{j}^*) \cap A_{\hat{j}^*}| \geq |((\mu+C)+\hat{C})(\hat{j}^*) \cap A_{\hat{j}^*}| \geq |\mu(\hat{j}^*) \cap A_{\hat{j}^*}|$ (the second inequality follows since $\hat{C}$ is a CIC for $\hat{j}^*$) and since $\tilde{C}'$ is a CIC for $\hat{j}^*$. Since the set of agents is finite and since the maximum number of desirable objects an agent can obtain is finite, we eventually find a CIC with the desired properties.

Analogously, given two integers $l, l' \in \{1, \dots, L\}$, we use the notations $\{l, \dots [L, 1] \dots, l'\}$, $(j^n, p^l, \dots [p^L, j^1] \dots, j^{l'}, p^{l'})$, $(j^l, \dots [j^L, j^1] \dots, j^{l'})$, and $(p^l, \dots [p^L, p^1] \dots, p^{l'})$.

Next, we derive two basic properties of C and $\hat{C}$. First, we establish that C and $\hat{C}$ "intersect" in terms of objects.

Claim 4. $O(C) \cap O(\hat{C}) \neq \emptyset$

Next, we argue that C and $\hat{C}$ treat agents with higher priority than j^* "consistently".

Claim 5. *For each pair m and l such that $i^m = j^l \equiv \underline{j}$ and $\underline{j} < j^*$, either*

$$\{o^{m-1}, o^m, p^{l-1}, p^l\} \subseteq A_{\underline{j}} \text{ or } \{o^{m-1}, o^m, p^{l-1}, p^l\} \subseteq \Omega_{\underline{j}} \setminus A_{\underline{j}}.$$

We now introduce some further notation. First, let $\underline{l}^*$ be the smallest integer such that either $j^{\underline{l}^*} \in I(C) \setminus \{\hat{j}^*\}$ or $p^{\underline{l}^*} \in O(C)$. Claim 4 ensures that such $\underline{l}^*$ exists. We now use $\underline{l}^*$ to define a specific point on the cycle C. In doing so, we distinguish two cases:

1. If $j^{\underline{l}^*} \notin I(C) \setminus \{\hat{j}^*\}$ and $p^{\underline{l}^*} \in O(C)$, then let $\underline{l} = \underline{l}^*$ and let $m(\underline{l})$ be such that $p^{\underline{l}} = o^{m(\underline{l})-1}$ (where $o^0 \equiv o^M$).

2. If $j^{\underline{l}^*} \in I(C) \setminus \{\hat{j}^*\}$, then let $\underline{l} = \underline{l}^* - 1$ and let $m(\underline{l})$ be such that $j^{\underline{l}^*} = i^{m(\underline{l})}$.[23]

Note that in both two cases, our definition of $\underline{l}$ ensures that $\underline{l} \geq 1$, $j^{\underline{l}} \notin I(C) \setminus \{\hat{j}^*\}$, and $p^{\underline{l}} \in \mu(i^{m(\underline{l})})$. To understand the significance of $m(\underline{l})$, note first that the definition of $\underline{l}$ implies that $\{j^1, p^1, \dots, j^{\underline{l}}, p^{\underline{l}}\} \cap \{i^1, o^1, \dots, i^M, o^M\} \subseteq \{j^1, p^{\underline{l}}\}$. Now consider the following sequence of agent-object pairs:

$$\underline{C} \equiv (j^1, p^1, \dots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \dots, i^M, o^M).$$

In $\underline{C}$, each agent points to an object that she does not get at μ and each object, except possibly o^M, points to the agent who gets it at μ. Furthermore, except possibly j^1, all agents and objects in $\underline{C}$ are distinct from each other. These facts about $\underline{C}$ are useful for our construction of CICs in the remainder of the proof.

Second, let $\bar{l}$ be the largest integer such that either $j^{\bar{l}} \in I(C) \setminus \{\hat{j}^*\}$ or $p^{\bar{l}} \in O(C)$. Again, Claim 4 ensures that such $\bar{l}$ exists. As is the case for $\underline{l}^*$, we use $\bar{l}$ to define a specific point of the cycle C. Before doing that, we discuss the case of $p^{\bar{l}} \in O(C)$ in more detail by means of the following claim.

Claim 6. *If $p^{\bar{l}} \in O(C)$, then $\bar{l} = L$.*

[23] Note that $\underline{l}^* \geq 2$ in this case.

Now consider the case where $p^{\bar{l}} \notin O(C)$ and $j^{\bar{l}} \in I(C) \setminus \{\hat{j}^*\}$. Let $m(\bar{l})$ be such that $i^{m(\bar{l})+1} = j^{\bar{l}}$. Since C is a CIC of μ, $j^{\bar{l}} = i^{m(\bar{l})+1}$ implies that $o^{m(\bar{l})} \in \mu(j^{\bar{l}})$. By definition of $\bar{l}$, we have that $\{i^1, o^1, \ldots, i^M, o^M\} \cap \{j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L\} = \{j^{\bar{l}}\}$. Now let $m' \in \{1, \ldots, M\}$ be an arbitrary integer distinct from $m(\bar{l}) + 1$ and consider the following sequence of agent-object pairs:

$$\overline{C} \equiv (i^{m'}, o^{m'}, \ldots [o^M, i^1] \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L).$$

In $\overline{C}$, each agent points to an object that she does not get at μ and each object, except possibly p^L, points to the agent who gets it at μ. Furthermore, all agents and objects in $\overline{C}$ are distinct from each other. Again, these facts are useful for our construction of CICs in the remainder of the proof.

Before proceeding, we now complete the proof of Lemma 4 for the case of $I(C) \cap I(\hat{C}) = \{j^*\}$. Let l be such that $j^l = j^*$. Note that since $I(C) \cap I(\hat{C}) = \{j^*\}$, Claim 4 implies $O(C) \cap O(\hat{C}) = \{o^1\}$ as well as $p^{l-1} = o^1$. If $j^* = \hat{j}^*$, we arrive at a contradiction since $p^L \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$ (as $\hat{C}$ is a CIC for $\hat{j}^*$) and $o^1 \in A_{j^*}$ (as C is a CIC for j^*) imply $p^L \neq o^1$. Thus, $j^* \neq \hat{j}^*$. So, by Claim 3, $j^* < \hat{j}^*$. Since $\hat{C}$ is a CIC for $\hat{j}^*$, we obtain that $p^l \in A_{j^*}$. Given that $I(C) \cap I(\hat{C}) = \{j^*\}$ and $O(C) \cap O(\hat{C}) = \{o^1\}$,

$$\tilde{C} \equiv (j^l, p^l, \ldots [p^L, j^1] \ldots, j^{l-1}, p^{l-1}, i^2, o^2, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at A such that $\mu + \tilde{C}$ makes all agents exactly as well off as $(\mu + C) + \hat{C}$. This completes the proof of Lemma 4 in case $I(C) \cap I(\hat{C}) = \{j^*\}$.

Henceforth, we assume that $I(C) \cap I(\hat{C}) \neq \{j^*\}$. Claim 4 then implies $(I(C) \cap I(\hat{C})) \setminus \{j^*\} \neq \emptyset$.

Next, we introduce some further notation for the remainder of the proof in case $I(C) \cap I(\hat{C}) \neq \{j^*\}$. Let $\underline{m}$ be the smallest integer m such that $i^m \in I(\hat{C}) \setminus \{j^*\}$, i.e. $i^{\underline{m}} \in I(\hat{C}) \setminus \{j^*\}$ and for each $m \in \{1, \ldots, \underline{m} - 1\}$, $i^m \notin I(\hat{C}) \setminus \{j^*\}$.[24] Similarly, let $\overline{m}$ be the largest integer m' such that $i^{m'} \in I(\hat{C}) \setminus \{j^*\}$, i.e. $i^{\overline{m}} \in I(\hat{C}) \setminus \{j^*\}$ and for each $m \in \{\overline{m} + 1, \ldots, M\}$, $i^m \notin I(\hat{C}) \setminus \{j^*\}$. Note that it is possible that $\overline{m} = \underline{m}$ and that in this case, we have $(I(C) \cap I(\hat{C})) \setminus \{j^*\} = \{i^{\overline{m}}\}$, $o^{\overline{m}} \in O(\hat{C})$, and $O(C) \cap O(\hat{C}) \subseteq \{o^1, o^{\overline{m}}\}$. Finally, let m^* be such that $i^{m^*} = i$ and $o^{m^*} = o$. Note that the existence of such an m^* follows since $o \in (\mu + C)(i) \setminus \mu(i)$ by Claim 2.

Our next claim is that objects and agents that show up in both C and $\hat{C}$ either all appear "before", or all appear "after" agent i in C, starting from $i^1 (= j^*)$.

Claim 7. *$o \notin O(\hat{C})$ and if $\underline{m} \leq m^*$, then $\overline{m} \leq m^*$.*

[24]Note that the existence of such an integer follows from $(I(C) \cap I(\hat{C})) \setminus \{j^*\} \neq \emptyset$.

Our final claim lays the foundation for splicing together parts of C and $\hat{C}$ to form new CICs. As we explain below, the claim will allow us to cut and paste parts of C and $\hat{C}$ together to construct a CIC that makes agents $j' \in \{j^*, \ldots, \hat{j}^* - 1\}$ at least as well off as $(\mu + C) + \hat{C}$.

Claim 8. *Assume that $\underline{m} < \overline{m}$.*

1. *Let $\check{m} \in \{\underline{m}+1, \ldots, \overline{m}-1\}$ be such that $i^{\check{m}} < \hat{j}^*$.*
   - *(i) The cycle C does not affect $i^{\check{m}}$.*
   - *(ii) If there exists $\check{l}$ such that $j^{\check{l}} = i^{\check{m}}$, then either $\{o^{\check{m}-1}, o^{\check{m}}, p^{\check{l}-1}, p^{\check{l}}\} \subseteq A_{i^{\check{m}}}$ or $\{o^{\check{m}-1}, o^{\check{m}}, p^{\check{l}-1}, p^{\check{l}}\} \subseteq \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$.*
2. *Let $\check{m} \leq \underline{m}$ be such that $i^{\check{m}} < \hat{j}^*$.*
   - *(i) If C affects $i^{\check{m}}$, then either $j^* \notin I(\hat{C})$ or there exists l^* such that $j^{l^*} = j^*$ and $\{p^{l^*-1}, p^{l^*}\} \subseteq \Omega_{j^*} \setminus A_{j^*}$.*
   - *(ii) If $\check{m} = \underline{m}$, then $o^{\underline{m}} \in A_{i^{\underline{m}}}$ if and only if there exists a $\check{l}$ such that $j^{\check{l}} = i^{\underline{m}}$ and $\{p^{\check{l}-1}, p^{\check{l}}\} \subseteq A_{i^{\underline{m}}}$.*
3. *Let $\check{m} \geq \overline{m}$ be such that $i^{\check{m}} < \hat{j}^*$.*
   - *(i) If C affects $i^{\check{m}}$, then either $j^* \notin I(\hat{C})$ or there exists l^* such that $j^{l^*} = j^*$ and $\{p^{l^*-1}, p^{l^*}\} \subseteq A_{j^*}$.*
   - *(ii) If $\check{m} = \overline{m}$, then $o^{\overline{m}-1} \in A_{i^{\overline{m}}}$ if and only if there exists a $\check{l}$ such that $j^{\check{l}} = i^{\overline{m}}$ and $\{p^{\check{l}-1}, p^{\check{l}}\} \subseteq A_{i^{\overline{m}}}$.*
4. *If $\check{m}$ is such that $i^{\check{m}} = \hat{j}^*$, then C does not increase the welfare of $i^{\check{m}}$.*

We now use the above claims to show that either there is a CIC $\tilde{C}$ of μ for j^* at A that satisfies (P1) - (P3), or there is a CIC of μ for some agent $j' < i$ at $\hat{A}$ (contradicting $\mu \in \mathcal{M}^{i-1}(\hat{A})$), or there is a CIC of $\mu + C$ for some agent $j' < \hat{j}^*$ (contradicting our choice of $\hat{C}$). Since we only work with the reassignments implied by C and $\hat{C}$, Claim 5 implies that the CICs we construct cannot affect any agent $j' < j^*$. We use this observation repeatedly throughout the remainder of our proof. To structure our arguments, we distinguish 5 cases.

Case 1: $\underline{m} = \overline{m}$. As argued above, $I(C) \cap I(\hat{C}) \subseteq \{j^*, i^{\underline{m}}\}$ and $O(C) \cap O(\hat{C}) \subseteq \{o^1, o^{\underline{m}}\}$.

Assume first that $o^{\underline{m}} \in O(\hat{C})$. Let $\check{l}$ be such that $j^{\check{l}} = i^{\underline{m}}$ and note that $o^{\underline{m}} = p^{\check{l}-1}$ in the case we consider here. In particular, it is not possible that C and $\hat{C}$ both increase the welfare of $i^{\underline{m}}$. If $j^* \notin I(\hat{C})$, then

$$\tilde{C} \equiv (i^1, o^1, \ldots, i^{\underline{m}-1}, o^{\underline{m}-1}, j^{\check{l}}, p^{\check{l}}, \ldots [p^L, j^1] \ldots, j^{\check{l}-1}, p^{\check{l}-1}, i^{\underline{m}+1}, o^{\underline{m}+1}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3): Since $I(C) \cap I(\hat{C}) \subseteq \{j^*, i^{\underline{m}}\}$ and $O(C) \cap O(\hat{C}) \subseteq \{o^1, o^{\underline{m}}\}$ and since C and $\hat{C}$ are CICs for j^* and $\hat{j}^*$ respectively, $\tilde{C}$ is a CIC of μ for j^* at A. If $j^{\check{l}} < \hat{j}^*$, then since $o^{\underline{m}} = p^{\check{l}-1} \in A_{j^{\check{l}}}$ if and only if $p^{\check{l}} \in A_{j^{\check{l}}}$ given that $\hat{C}$ is a CIC for $\hat{j}^* > j^{\check{l}}$, we conclude that $\mu + \tilde{C}$ makes $j^{\check{l}}$ at least as well off as $(\mu + C) + \hat{C}$. If $j^{\check{l}} = \hat{j}^*$ (and thus $\check{l} = 1$), then since $o^{\underline{m}} = p^{\check{l}-1} = p^L \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$ and $p^1 \in A_{j^{\check{l}}}$, $j^{\check{l}}$ is at least well off under $\mu + \tilde{C}$ as under $(\mu + C) + \hat{C}$. Finally, $m^* \neq \underline{m}$ since otherwise $\tilde{C}$ is a CIC of μ for j^* at $\hat{A}$. Hence, $\tilde{C}$ weakly increases the welfare of i. Combining the previous arguments, we find that $\tilde{C}$ is a CIC with the desired properties.

Now suppose that $j^* \in I(\hat{C})$ so there exists an $l^* > 1$ such that $j^{l^*} = j^*$.

If $p^{l^*-1} \in \Omega_{j^*} \setminus A_{j^*}$, then $o^1 \notin O(\hat{C})$. We claim that

$$\tilde{C}' \equiv (i^1, o^1, \dots, i^{\underline{m}-1}, o^{\underline{m}-1}, j^{\check{l}}, p^{\check{l}}, \dots [p^L, j^1] \dots, j^{l^*-1}, p^{l^*-1})$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3): Since $I(C) \cap I(\hat{C}) \subseteq \{j^*, i^{\underline{m}}\}$ and $O(C) \cap O(\hat{C}) \subseteq \{o^1, o^{\underline{m}}\}$ and since C and $\hat{C}$ are CICs for j^* and $\hat{j}^*$ respectively, $\tilde{C}'$ is a CIC of μ for j^* at A. Since $\mu \in \mathcal{M}^{i-1}(\hat{A})$, we have $m^* \leq \underline{m} - 1$. As before, $\tilde{C}'$ weakly increases the welfare of $\hat{j}^*$ if $j^{\check{l}} = \hat{j}^*$. If $j^{\check{l}} \neq \hat{j}^*$, note first that a simple case distinction shows that there cannot exist a $\check{m} > \underline{m}$ such that $i^{\check{m}} < \hat{j}^*$ and such that C affects $i^{\check{m}}$.[25] Hence, given that $m^* \leq \underline{m}$, $l^* - 1 \geq \check{l} > 1$ implies that

$$(j^1, p^1, \dots, j^{\check{l}-1}, p^{\check{l}-1}, i^{\underline{m}+1}, o^{\underline{m}+1}, \dots, i^M, o^M)$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$. If $l^* - 1 < \check{l}$, $\tilde{C}'$ weakly increases the welfare of $\hat{j}^*$ compared to $(\mu + C) + \hat{C}$. Finally, as in the case of $j^* \notin I(\hat{C})$, one can show that $\tilde{C}'$ cannot make $j^{\check{l}}$ worse off than $(\mu + C) + \hat{C}$ if $j^{\check{l}} < \hat{j}^*$.

If $p^{l^*-1} \in A_{j^*}$, then by analogous arguments

$$(j^{l^*}, p^{l^*}, \dots [p^L, j^1] \dots, j^{\check{l}-1}, p^{\check{l}-1}, i^{\underline{m}+1}, o^{\underline{m}+1}, \dots, i^M, o^M)$$

[25] Suppose, the contrary, that such an $i^{\check{m}}$ exists. Assume without loss of generality that there is no $m' \in \{\underline{m}+1, \dots, M\}$ such that C affects $i^{m'}$ and such that $i^{m'} < i^{\check{m}}$. If C decreases the welfare of $i^{\check{m}}$, then

$$(i^1, o^M, i^M, o^{M-1}, \dots, i^{\underline{m}+1}, o^{\underline{m}}, j^{\check{l}}, p^{\check{l}}, \dots [p^L, j^1] \dots, j^{l^*-1}, p^{l^*-1})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A, which contradicts our choice of $\hat{C}$. If C increases the welfare of $i^{\check{m}}$, then

$$(i^{\underline{m}+1}, o^{\underline{m}+1}, \dots, i^M, o^M, j^{l^*}, p^{l^*}, \dots [p^L, j^1] \dots, j^{\check{l}-1}, p^{\check{l}-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$ given that $m^* \leq \underline{m}$.

is a CIC of μ for j^* at A that satisfies (P1) - (P3). We omit the details.

Finally, suppose $O(C) \cap O(\hat{C}) = \{o^1\}$. Let l^* be such that $j^{l^*} = j^*$ and note that $p^{l^*-1} = o^1$ and $p^{l^*} \in A_{j^*}$. By analogs of the above arguments

$$\tilde{C}'' \equiv (j^{l^*}, p^{l^*}, \dots [p^L, j^1] \dots, j^{\check{l}-1}, p^{\check{l}-1}, i^{\underline{m}}, o^{\underline{m}}, \dots, i^M, o^M)$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).[26]

Case 2: $\underline{m} < \overline{m}$ and C decreases the welfare of $\hat{j}^*$. Let $\check{m}$ be such that $i^{\check{m}} = \hat{j}^*$. The premise of Case 2 imply that there exist m_1, m_2, l_1, l_2 such that the following conditions are satisfied:

1. $\underline{m} \leq m_1 \leq \check{m} \leq m_2 \leq \overline{m}$
2. Either $m_1 < \check{m}$ or $\check{m} < m_2$.
3. $j^{l_1} = i^{m_1}$ and $j^{l_2} = i^{m_2}$
4. $\{j^{l_1+1}, \dots [j^L, j^1] \dots, j^{l_2-1}\} \cap I(C) \subseteq \{j^*, \hat{j}^*\}$

Assume first that $\{j^{l_1+1}, \dots [j^L, j^1] \dots, j^{l_2-1}\} \cap I(C) = \emptyset$. If $p^{l_2-1} \notin O(C)$, then

$$\tilde{C} \equiv (i^1, o^1, \dots, i^{m_1-1}, o^{m_1-1}, j^{l_1}, p^{l_1}, \dots [p^L, j^1] \dots, j^{l_2-1}, p^{l_2-1}, i^{m_2}, o^{m_2}, \dots, i^M, o^M)$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).[27] If $p^{l_2-1} \in O(C)$, then $p^{l_2-1} = o^{m_2}$, in which case

$$\tilde{C}' \equiv (i^1, o^1, \dots, i^{m_1-1}, o^{m_1-1}, j^{l_1}, p^{l_1}, \dots [p^L, j^1] \dots, j^{l_2-1}, p^{l_2-1}, i^{m_2+1}, o^{m_2+1}, \dots, i^M, o^M)$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).[28]

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[26] One difference to the case is that when $o^{\underline{m}} \notin O(\hat{C})$, it is possible for C and $\hat{C}$ to both increase the welfare of $i^{\underline{m}}$ and $i^{\underline{m}} < \hat{j}^*$. However, since $\tilde{C}''$ is a CIC of μ for j^* at A, we have $m^* \geq \underline{m}$ and

$$(j^{\check{l}}, p^{\check{l}}, \dots [p^L, j^1] \dots, j^{l^*-1}, p^{l^*-1}, i^2, o^2, \dots, i^{\underline{m}-1}, o^{\underline{m}-1})$$

is a CIC of μ for $i^{\underline{m}}$ at $\hat{A}$ given that $p^{\check{l}} \in A_{j^{\check{l}}}$ and $o^{\underline{m}-1} \in \Omega_{j^{\check{l}}} \setminus A_{j^{\check{l}}}$.

[27] To see this, note first that since $\underline{m} \leq m_1 \leq m_2 \leq \overline{m}$ the first part of Claim 8 implies that there does not exist $m' \in \{m_1 + 1, \dots, m_2 - 1\}$ such that $i^{m'} < \hat{j}^*$ and such that C affects $i^{m'}$. Next, if $m_1 = \underline{m}$ ($m_2 = \overline{m}$) the second (third) part of Claim 8 implies that $\tilde{C}$ makes i^{m_1} (i^{m_2}) at least as well off as $(\mu + C) + \hat{C}$ if $i^{m_1} < \hat{j}^*$ ($i^{m_2} < \hat{j}^*$). For agent $\hat{j}^*$, the definitions of m_1, m_2, l_1, l_2 and the assumptions of Case 2 immediately imply that $\tilde{C}$ weakly increases the welfare of $\hat{j}^*$. Finally, $\mu \in \mathcal{M}^{i-1}(\hat{A})$ implies that $m^* \in \{1, \dots, m_1 - 1, m_2, \dots, M\}$: otherwise $\tilde{C}$ is a CIC of μ for j^* at $\hat{A}$. So $\tilde{C}$ weakly increases the welfare of i. Analogous reasoning applies to all CICs that we construct in the remainder of the proof.

[28] Note that if $p^{l_2-1} = o^{m_2}$ and $i^{m_2} < \hat{j}^*$, then the third part of Claim 8 implies that C cannot increase the welfare of i^{m_2} if $m_2 = \overline{m}$ since $o^{m_2-1} \in \Omega_{i^{m_2}} \setminus A_{i^{m_2}}$ implies $\{p^{l_2-1}, p^{l_2}\} \subseteq \Omega_{i^{m_2}} \setminus A_{i^{m_2}}$ (and thus

Next, assume that there exists $l^* \in \{l_1+1, \dots [L, 1] \dots, l_2-1\}$ such that $j^{l^*} = j^*$. If $\{p^{l^*-1}, p^{l^*}\} \subseteq A_{j^*}$, then either $p^{l_2-1} \notin O(C)$ or $p^{l_2-1} = o^{m_2}$. In the former case

$$(j^{l^*}, p^{l^*}, \dots [p^L, j^1] \dots, j^{l_2-1}, p^{l_2-1}, i^{m_2}, o^{m_2}, \dots, i^M, o^M)$$

and in the latter case

$$(j^{l^*}, p^{l^*}, \dots [p^L, j^1] \dots, j^{l_2-1}, p^{l_2-1}, i^{m_2+1}, o^{m_2+1}, \dots, i^M, o^M)$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).[29] If $\{p^{l^*-1}, p^{l^*}\} \subseteq \Omega_{j^*} \setminus A_{j^*}$, then

$$(i^1, o^1, \dots, i^{m_1-1}, o^{m_1-1}, j^{l_1}, p^{l_1}, \dots [p^L, j^1] \dots, j^{l^*-1}, p^{l^*-1})$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).[30]

Finally, consider the case where $\{j^{l_1+1}, \dots [j^L, j^1] \dots, j^{l_2-1}\} \cap I(C) = \{\hat{j}^*\}$. Since all agents in $\hat{C}$ are distinct, $m_1 < \check{m} < m_2$ in this case. If $p^L \notin O(C)$, then

$$(i^1, o^1, \dots, i^{m_1-1}, o^{m_1-1}, j^{l_1}, p^{l_1}, \dots j^L, p^L, i^{\check{m}}, o^{\check{m}}, \dots, i^M, o^M)$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3). If $p^L = o^{\check{m}}$, then

$$(i^1, o^1, \dots, i^{m_1-1}, o^{m_1-1}, j^{l_1}, p^{l_1}, \dots j^L, p^L, i^{\check{m}+1}, o^{\check{m}+1}, \dots, i^M, o^M)$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).

Case 3: $\underline{m} < \overline{m}$, C does not affect $\hat{j}^*$, $p^L \notin O(C)$, and $j^{\bar{l}} \neq j^*$. By definition of $m(\bar{l})$ and the premise of this case, $m(\bar{l}) < M$ and $j^{\bar{l}} = o^{m(\bar{l})+1}$. Hence, $m(\bar{l}) < \overline{m}$. We will use this observation repeatedly throughout the proof in Case 3.

We first assume that $m(\underline{l}) \leq m(\bar{l})$.

We start by showing $m(\underline{l}) < \underline{m}$. Assume to the contrary that $m(\underline{l}) \geq \underline{m}$. Claim 7 and

$o^{m_2} \in \Omega_{i^{m_2}} \setminus A_{i^{m_2}}$). Similar reasoning applies to all CICs that we construct in which we "skip" agents $i^{\underline{m}}$ or $i^{\overline{m}}$.

[29]Since j^* exchanges two desirable objects in the case we consider here, the first and second parts of Claim 8 implies that there cannot be an $m \in \{2, \dots, \overline{m}-1\}$ such that $i^m < \hat{j}^*$ and such that C affects i^m. Analogous reasoning applies to all CICs that we construct for cases where j^* exchanges two desirable objects in $\hat{C}$.

[30]Since j^* exchanges two objects that are not desirable in the case we consider here, the first and third parts of Claim 8 implies that there cannot be an $m \in \{\underline{m}+1, \dots, M\}$ such that $i^m < \hat{j}^*$ and such that C affects i^m. Analogous reasoning applies to all CICs that we construct for cases where j^* exchanges two objects that are not desirable in $\hat{C}$.

$m(\bar{l}) < \overline{m}$ implies that $m^* \notin \{m(\underline{l}), \ldots, m(\bar{l})\}$. If $\hat{j}^* \notin \{i^{m(\underline{l})}, \ldots, i^{m(\bar{l})}\}$, then

$$\tilde{C} \equiv (j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L)$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$.[31] If there exists a $\check{m} \in \{m(\underline{l}), \ldots, m(\bar{l})\}$ such that $i^{\check{m}} = \hat{j}^*$ and $o^{\check{m}-1} \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$, then

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$. Finally, if there exists a $\check{m} \in \{m(\underline{l}), \ldots, m(\bar{l})\}$ such that $i^{\check{m}} = \hat{j}^*$ and $o^{\check{m}-1} \in A_{\hat{j}^*}$, then, given that C does not affect $\hat{j}^*$, we have $o^{\check{m}} \in A_{\hat{j}^*}$ as well. Since $m^* \notin \{m(\underline{l}), \ldots, m(\bar{l})\}$,

$$(i^{\check{m}}, o^{\check{m}}, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L)$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$. Since $m(\underline{l}) \geq \underline{m}$ necessarily leads to a contradiction, we have $m(\underline{l}) < \underline{m}$. Thus, we have established that $m(\underline{l}) < \underline{m}$.

Next, note that $m(\underline{l}) < \underline{m}$ implies that either $p^{\underline{l}} \in \mu(j^*)$ or $p^{\underline{l}} = o^1$.[32]

Assume first that $p^{\underline{l}} \in \mu(j^*) \cap (\Omega_{j^*} \setminus A_{j^*})$. If $\hat{j}^* \notin \{i^2, \ldots, i^{m(\bar{l})}\}$, then

$$(i^1, o^1, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots [p^L, j^1] \ldots, j^{\underline{l}}, p^{\underline{l}}).$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3). If there exists a $\check{m} \in \{2, \ldots, m(\bar{l})\}$ such that $i^{\check{m}} = \hat{j}^*$ and $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$, then

$$(i^1, o^1, \ldots, i^{\check{m}-1}, o^{\check{m}-1}, j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}})$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3). If there exists a $\check{m} \in \{2, \ldots, m(\bar{l})\}$ such that $i^{\check{m}} = \hat{j}^*$ and $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq A_{\hat{j}^*}$, note first that $m^* < \check{m}$ since otherwise either

$$(i^1, o^1, \ldots, i^{\check{m}-1}, o^{\check{m}-1}, j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}})$$

[31] By the first part of Claim 8, there does not exist $m \in \{\underline{m}+1, \ldots, \overline{m}-1\}$ such that $i^m < \hat{j}^*$ and such that $\tilde{C}$ affects i^m. If $m(\underline{l}) = \underline{m}$, then the second part of Claim 8 implies that $\tilde{C}$ cannot affect $i^{m(\underline{l})}$ if $i^{m(\underline{l})} < \hat{j}^*$. Analogous reasoning applies to other cases where we construct CICs for $\hat{j}^*$.

[32] Since $m(\underline{l}) < \underline{m}$, by definition of $\underline{m}$, either $m(\underline{l}) = 1$ or $i^{m(\underline{l})} \notin I(\hat{C})$. If $m(\underline{l}) \neq 1$ and $i^{m(\underline{l})} \notin I(\hat{C})$, then $p^{\underline{l}} = o^{m(\underline{l})-1}$. However, this implies $i^{m(\underline{l})-1} \in I(\hat{C})$. Since $m(\underline{l}) \neq 1$, this contradicts the definition of $\underline{m}$. Thus, $m(\underline{l}) = 1$ so either $p^{\underline{l}} = o^{m(\underline{l})-1} = o^M \in \mu(j^*)$ or $j^{\underline{l}^*} = j^*$ and $\underline{l} = \underline{l}^* - 1$ so that $p^{\underline{l}} \in (\mu + C)(j^*)$, which implies that $p^{\underline{l}} \in \mu(j^*)$ or $p^{\underline{l}} = o^1$.

(if $o^{\check{m}-1} \neq p^1$) or

$$(i^1, o^1, \ldots, i^{\check{m}-1}, o^{\check{m}-1}, j^2, p^2, \ldots, j^{\underline{l}}, p^{\underline{l}})$$

(if $o^{\check{m}-1} = p^1$) is a CIC of μ for j^* at $\hat{A}$. However, if $m^* < \check{m}$, then

$$(i^{\check{m}}, o^{\check{m}}, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L).$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$.

Next, consider the case where $p^{\underline{l}} \in A_{j^*}$. We have that either $p^{\underline{l}} = o^1$ or $p^{\underline{l}} \notin O(C)$. In both cases, $j^{\underline{l}+1} = j^*$. Since $\hat{C}$ is a CIC for $\hat{j}^* > j^*$, we obtain $p^{\underline{l}+1} \in A_{j^*}$ as well.

We argue first that $m^* < \underline{m}$. If not, then

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L)$$

is a CIC of μ for $\hat{j}^*$ given that $m(\underline{l}) \leq m(\bar{l})$, $m(\bar{l}) < \overline{m}$, Claim 8, and $\{p^{\underline{l}}, p^{\underline{l}+1}\} \subseteq A_{j^*}$ jointly imply that there is no $m' \in \{m(\underline{l}), \ldots, m(\bar{l})\}$ such that $i^{m'} < \hat{j}^*$ and such that C affects $i^{m'}$.

Now let $\tilde{l}$ be the first integer in $(\underline{l}+1, \ldots, \bar{l})$ such that either $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$ or $p^{\tilde{l}} \in O(C)$. If $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$, let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$. Since $m^* < \underline{m}$,

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\tilde{l}-1}, p^{\tilde{l}-1}, i^{\tilde{m}}, o^{\tilde{m}}, \ldots, i^M, o^M).$$

is a CIC of μ for j^* at $\hat{A}$. If $j^{\tilde{l}} \notin I(C) \setminus \{j^*\}$, we obtain a similar contradiction by letting $\tilde{m}$ be such that $o^{\tilde{m}-1} = p^{\tilde{l}}$ and considering

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\tilde{l}}, p^{\tilde{l}}, i^{\tilde{m}}, o^{\tilde{m}}, \ldots, i^M, o^M).$$

Since we have now exhausted all the possibilities for the case of $m(\underline{l}) \leq m(\bar{l})$, we now consider the case of $m(\bar{l}) < m(\underline{l})$ for the remainder of the proof in Case 3. In this case, $\underline{m} \leq m(\underline{l})$.[33]

If $\hat{j}^* \notin \{i^2, \ldots, i^{m(\bar{l})}, i^{m(\underline{l})}, \ldots, i^M\}$, then

$$(i^1, o^1, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots [p^L, j^1] \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).

[33] If $m(\bar{l}) < m(\underline{l}) < \underline{m}$, then $i^{m(\underline{l})} \notin I(\hat{C})$ and $m(\underline{l}) \neq 1$. So $p^{\underline{l}} = o^{m(\underline{l})-1}$. This implies that $i^{m(\underline{l})-1} \in I(\hat{C})$. Since $m(\underline{l}) \neq 1$, this contradicts the definition of $\underline{m}$.

If there exists a $\check{m} < m(\bar{l})$ such that $i^{\check{m}} = \hat{j}^*$ and $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$, then

$$(i^1, o^1, \ldots, i^{\check{m}-1}, o^{\check{m}-1}, j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).

If there exists a $\check{m} \geq m(\underline{l})$ such that $i^{\check{m}} = \hat{j}^*$ and $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$, then

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$ since $m^* \notin \{\underline{m}, \ldots, \check{m}-1\}$.

If there exists a $\check{m} < m(\bar{l})$ such that $i^{\check{m}} = \hat{j}^*$ and $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq A_{\hat{j}^*}$, then

$$(i^{\check{m}}, o^{\check{m}}, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L)$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$ since $m^* \notin \{\underline{m}, \ldots, \check{m}-1\}$.

Finally, if there exists a $\check{m} \geq m(\underline{l})$ such that $i^{\check{m}} = \hat{j}^*$ and $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq A_{\hat{j}^*}$, then

$$(i^1, o^1, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots j^L, p^L, i^{\check{m}}, o^{\check{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).

Case 4: $\underline{m} < \overline{m}$, C does not affect $\hat{j}^*$, $p^L \notin O(C)$, and $j^{\bar{l}} = j^*$. By definition of $m(\bar{l})$ and the premise of this case, $m(\bar{l}) = M$. Hence, $m(\bar{l}) \geq m(\underline{l})$.

Assume first that $\{p^{\bar{l}-1}, p^{\bar{l}}\} \subseteq \Omega_{j^*} \setminus A_{j^*}$.

First, we claim that $\underline{l} < \bar{l} - 1$: otherwise, $\underline{l} = \bar{l} - 1$ and, given that $p^{\bar{l}-1} \notin O(C)$ (as $p^{\bar{l}-1} \neq o^1$) we have $\{p^1, \ldots, p^{\bar{l}-1}\} \cap O(C) = \emptyset$; given that $p^L \notin O(C)$, Claim 6 and the definition of $\bar{l}$ imply $\{p^{\bar{l}}, \ldots, p^L\} \cap O(C) = \emptyset$, which contradicts Claim 4.

Second, we argue that $m(\underline{l}) \geq \underline{m}$: if $p^{\underline{l}} \notin O(C)$, we have $j^{\underline{l}+1} = i^{m(\underline{l})}$ and $\underline{l} < \bar{l} - 1$ implies $j^{\underline{l}+1} \neq j^*$ so $i^{m(\underline{l})-1} \in I(\hat{C}) \setminus \{j^*\}$; if $p^{\underline{l}} \in O(C)$, we have $j^{\underline{l}+1} = i^{m(\underline{l})-1}$ and $\underline{l} < \bar{l} - 1$ again implies $j^{\underline{l}+1} \neq j^*$ so $i^{m(\underline{l})-1} \in I(\hat{C}) \setminus \{j^*\}$.

Third, we show that $m^* < m(\underline{l})$: If not, Claim 7 and $m(\underline{l}) \geq \underline{m}$ imply $m^* \geq \overline{m}$. Let $\tilde{l}$ be the last integer in $(1, \ldots, \bar{l}-1)$ such that $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$ and let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$. Since $p^{\bar{l}-1} \notin O(C)$ in the case we consider here, we have $\{p^{\tilde{l}}, \ldots, p^{\bar{l}-1}\} \cap O(C) = \emptyset$. But then $m^* \geq \overline{m} > \tilde{m} - 1$ implies that

$$(i^1, o^1, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots, j^{\bar{l}-1}, p^{\bar{l}-1}).$$

is a CIC of μ for j^* at $\hat{A}$.

Now assume first that $\hat{j}^* \notin \{i^{m(\underline{l})}, \ldots, i^M\}$. If $o^{m(\bar{l})} \neq p^{\bar{l}}$, then Claim 8 and $m^* < m(\underline{l})$ imply that

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L)$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$. If $o^{m(\bar{l})} = p^{\bar{l}}$, we obtain a contradiction to Claim 6 since $p^{\bar{l}} \in O(C)$ implies $\bar{l} = L$ but we have $p^L \notin O(C)$ in the case we consider here.

Next, assume that there exists a $\check{m} \in \{m(\underline{l}), \ldots, M\}$ such that $i^{\check{m}} = \hat{j}^*$.

If $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$, then

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$. If $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq A_{\hat{j}^*}$, then

$$(i^{\check{m}}, o^{\check{m}}, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L)$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$.

We have now exhausted all the possibilities for the case of $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$. For the remainder of the proof in Case 4, we assume that $\{p^{\bar{l}-1}, p^{\bar{l}}\} \subseteq A_{j^*}$.

If $\hat{j}^* \notin \{i^{m(\underline{l})}, \ldots, i^M\}$, then

$$(j^{\bar{l}}, p^{\bar{l}}, \ldots [p^L, j^1] \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).

Now assume that there exists $\check{m} \in \{m(\underline{l}), \ldots, M\}$ such that $i^{\check{m}} = \hat{j}^*$.

If $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$, then, given that $\{p^{\bar{l}-1}, p^{\bar{l}}\} \subseteq A_{j^*}$, the second part of Claim 8 implies that

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$ unless $m^* \in \{m(\underline{l}), \ldots, \underline{m} - 1\}$. However, if $m^* < \underline{m}$, then

$$(j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L, i^{\check{m}}, o^{\check{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$ if $p^L \neq o^{\check{m}}$ and

$$(j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L, i^{\check{m}+1}, o^{\check{m}+1}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$ if $p^L = o^{\check{m}}$.

Finally, if $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq A_{\hat{j}^*}$, then

$$(j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L, i^{\check{m}}, o^{\check{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).

Case 5: $\underline{m} < \overline{m}$, C does not affect $\hat{j}^*$, and $p^L \in O(C)$. Since $p^L \in O(C)$, $\hat{j}^* \in I(C)$. Let $\check{m}$ be such that $i^{\check{m}} = \hat{j}^*$. Since $p^L \in O(C)$, we have $p^L = o^{\check{m}}$ and, given that C does not affect $\hat{j}^*$, $o^{\check{m}-1} \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$.

If $\check{m} < m(\underline{l})$, then

$$(i^1, o^1, \ldots, i^{\check{m}-1}, o^{\check{m}-1}, j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^M, o^M).$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).

Next, assume that $\check{m} = m(\underline{l})$. The definitions of $\underline{l}$ and $m(\underline{l})$ imply that either $p^{\underline{l}} \in \mu(\hat{j}^*) \setminus (\mu + C)(\hat{j}^*)$ or that $\underline{l} = L, j^L \notin I(C)$, and $p^L \notin O(C)$. Since $p^L \in O(C)$ in the case we consider here, $p^{\underline{l}} = o^{\check{m}-1}$. However, given that $o \notin O(\hat{C})$ by Claim 7 and, by the premise of Case 5, $o^{\check{m}-1} = p^{\underline{l}} \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$, we obtain that

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}})$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$.

For the remainder of the proof in Case 5, we assume that $\check{m} > m(\underline{l})$.

We argue first that $m(\underline{l}) < \underline{m}$: Otherwise, Claim 7 implies $m^* \notin \{m(\underline{l}), \ldots, \check{m} - 1\}$ and Claim 8 implies that

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$.

As for Case 3, $m(\underline{l}) < \underline{m}$ implies that either $p^{\underline{l}} \in \mu(j^*)$ or $p^{\underline{l}} = o^1$.

If $p^{\underline{l}} \in \Omega_{j^*} \setminus A_{j^*}$, then

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^1, o^1, \ldots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for j^* at A that satisfies (P1) - (P3).

If $p^{\underline{l}} \in A_{j^*}$, then $j^{\underline{l}+1} = j^*$.

We argue first that we must have $m^* < \underline{m}$: Otherwise, either

$$(j^1, p^1, \dots, j^{\underline{l}}, p^{\underline{l}}, i^1, o^1, \dots, i^{\tilde{m}-1}, o^{\tilde{m}-1})$$

(if $p^{\underline{l}} \neq o^1$) or

$$(j^1, p^1, \dots, j^{\underline{l}}, p^{\underline{l}}, i^2, o^2, \dots, i^{\tilde{m}-1}, o^{\tilde{m}-1})$$

(if $p^{\underline{l}} = o^1$) would be a CIC of μ for $\hat{j}^*$ at $\hat{A}$.

Next, since $\hat{C}$ is a CIC for $\hat{j}^* > j^*$, we have $p^{\underline{l}+1} \in A_{j^*}$ as well. Let $\tilde{l}$ be the first integer in $(\underline{l}+1, \dots, \bar{l})$ such that either $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$ or $p^{\tilde{l}} \in O(C)$. If $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$, let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$. Since $m^* < \underline{m}$,

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \dots, j^{\tilde{l}-1}, p^{\tilde{l}-1}, i^{\tilde{m}}, o^{\tilde{m}}, \dots, i^M, o^M).$$

is a CIC of μ for j^* at $\hat{A}$. If $j^{\tilde{l}} \notin I(C) \setminus \{j^*\}$, we obtain a similar contradiction by letting $\tilde{m}$ be such that $o^{\tilde{m}-1} = p^{\tilde{l}}$ and considering

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \dots, j^{\tilde{l}}, p^{\tilde{l}}, i^{\tilde{m}}, o^{\tilde{m}}, \dots, i^M, o^M).$$

□

Proof of Theorem 3. Let A be a profile of desirable sets and, for some $i \in I$, let $\hat{A}_i$ be an alternative report for i.

We argue first that it is without loss of generality to assume that $\hat{A}_i \setminus A_i = \emptyset$. To see this, assume that $\hat{A}_i \setminus A_i \neq \emptyset$ and let $o \in \hat{A}_i \setminus A_i$. First suppose that $o \notin \Omega_i \cup A_i$. If $K^i(\hat{A}_i \setminus \{o\}) = K^i(\hat{A}_i)$, then it is immediate that we can consider the manipulation $\hat{A}_i \setminus \{o\}$ rather than $\hat{A}_i$. If $K^i(\hat{A}_i \setminus \{o\}) \neq K^i(\hat{A}_i)$, then Lemma 2 implies that $o \in \mu(i)$ for all $\mu \in \mathcal{M}^i(\hat{A})$. Since $o \in \hat{A}_i \setminus (\Omega_i \cup A_i)$, each matching in $\mathcal{M}^i(\hat{A})$ is unacceptable to i (w.r.t. her true preferences). Now suppose that $o \in \Omega_i$. Then by Lemma 3, for each $\mu \in \mathcal{M}^i(\hat{A})$ and each $\mu' \in \mathcal{M}^i(A)$, $|\mu'(i) \cap A_i| = K^i(A) \geq |\mu(i) \cap A_i|$. Hence, reporting $\hat{A}'_i \equiv \hat{A}_i \setminus \{o\}$ is at least as good for i as $\hat{A}_i$ when her true desirable set is A_i. Applying these arguments $|\hat{A}_i \setminus A_i|$ times, we end up with a manipulation $\hat{A}'_i$ that does at least as well as $\hat{A}_i$ and that satisfies $\hat{A}'_i \subseteq A_i$.

We now consider $\hat{A}_i$ such that $\hat{A}_i \subsetneq A_i$. Let $\{o^1, \dots, o^T\} = A_i \setminus \hat{A}_i$ for some $T \geq 1$. By Lemma 4 (taking $\hat{A}_i \cup \{o^1\}$ to be the true and $\hat{A}_i$ to be the false set of desirable objects), we obtain $K^i(\hat{A}_i \cup \{o^1\}) \geq K^i(\hat{A}_i)$. Proceeding inductively, T applications of Lemma 4 yield that $K^i(\hat{A}_i \cup \{o^1, \ldots, o^T\}) \geq K^i(\hat{A}_i)$. Hence, by definition of $o^1, \ldots, o^M$ we obtain that $K^i(A) \geq K^i(\hat{A}_i)$. This completes the proof.

$\square$

C Claims in Proof of Lemma 4

Proof of Claim 1. Since $\mu \in \mathcal{M}^{i-1}(\hat{A})$ and $j^* < i$, we obtain $\mu(j^*) \cap \hat{A}_{j^*} = K^{j^*}(\hat{A})$. Since $K^{j^*}(A) > K^{j^*}(\hat{A})$, we obtain $\mu \notin \mathcal{M}^{j^*}(A)$.

On the other hand, $j^* < i$ also implies that $\mu \in \mathcal{M}^{j^*-1}(\hat{A})$ given that $\mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{j^*-1}(\hat{A})$. Since $\hat{A}_j \subseteq A_j$ for all $j \in I$ and $K^j(\hat{A}) = K^j(A)$ if $j < j^*$, the definition of j^* implies that $\mathcal{M}^{j^*-1}(\hat{A}) \subseteq \mathcal{M}^{j^*-1}(A)$ and thus $\mu \in \mathcal{M}^{j^*-1}(A)$. $\square$

Proof of Claim 2. If not, then $\mu + C \in \mathcal{M}^{j^*-1}(\hat{A})$ since $\mu \in \mathcal{M}^{j^*-1}(\hat{A})$, $o \notin (\mu + C)(i)$, $\hat{A}_i = A_i \setminus \{o\}$, and $\hat{A}_j = A_j$ for all $j \neq i$. But $\mu + C \in \mathcal{M}^{j^*-1}(\hat{A})$ contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{j^*}(\hat{A})$ since $|(\mu + C)(j^*) \cap \hat{A}_{j^*}| > |\mu(j^*) \cap \hat{A}_{j^*}|$. $\square$

Proof of Claim 3. Note that the first part of Theorem 2 implies that $\hat{j}^* \geq j^*$ since $\mu + C \in \mathcal{M}^{j^*-1}(A)$ given that $\mu \in \mathcal{M}^{j^*-1}(A)$ and that C is a CIC of μ for j^* at A. Given that $\hat{j}^* \geq j^*$ and $\mu \in \mathcal{M}^{j^*-1}(A)$, our assumption that $\hat{j}^*$ is the highest priority agent for whom there exists a CIC of $\mu + C$ at A and the second part of Theorem 2 imply $\mu + C \in \mathcal{M}^{\hat{j}^*-1}(A)$. Furthermore, we also have $(\mu + C) + \hat{C} \in \mathcal{M}^{\hat{j}^*-1}(A)$ since $\mu + C \in \mathcal{M}^{\hat{j}^*-1}(A)$ and since $\hat{C}$ is a CIC of $\mu + C$ for $\hat{j}^*$ at A. $\square$

Proof of Claim 4. If $O(C) \cap O(\hat{C}) = \emptyset$, then by Claim 2, $o \notin O(\hat{C})$. Furthermore, $O(C) \cap O(\hat{C}) = \emptyset$ also implies that for all $j \in I$ and all $o' \in O(\hat{C})$, $o' \in (\mu + C)(j)$ if and only if $o' \in \mu(j)$. Combining the last two observations, we find that $\hat{C}$ is a CIC of μ for $\hat{j}^*$ at $\hat{A}$. Since $\hat{j}^* < i$, this contradicts the assumption that $\mu \in \mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{\hat{j}^*}(\hat{A})$. $\square$

Proof of Claim 5. Assume to the contrary that there is an agent $\underline{j} < j^*$ for whom the statement of the claim is violated. Without loss of generality suppose that there is no agent with higher priority than $\underline{j}$ for whom Claim 5 does not hold, that is, for all pairs m', l' such that $i^{m'} = j^{l'}$ and $i^{m'} < \underline{j}$, either

$$\{o^{m'-1}, o^{m'}, p^{l'-1}, p^{l'}\} \subseteq A_{i^{m'}} \text{ or } \{o^{m'-1}, o^{m'}, p^{l'-1}, p^{l'}\} \subseteq \Omega_{i^{m'}} \setminus A_{i^{m'}}.$$

Since C and $\hat{C}$ are CICs for agents with lower priority than $\underline{j}$, there are only two cases to consider. In either case, we construct a CIC of μ for $\underline{j}$ at A. Since $\underline{j} < j^*$, by Theorem 2, the existence of such a CIC contradicts the implication of Claim 2 that $\mu \in \mathcal{M}^{j^*-1}(A)$.

Case 1: $\{o^{m-1}, o^m\} \subseteq \Omega_{\underline{j}} \setminus A_{\underline{j}}$ and $\{p^{l-1}, p^l\} \subseteq A_{\underline{j}}$. Let $\tilde{l}$ be the first integer in $(l, \dots [L, 1] \dots, l-1)$ such that either $j^{\tilde{l}} \in I(C) \setminus \{\underline{j}\}$ or $p^{\tilde{l}} \in O(C)$. Note that the existence of an $\tilde{l}$ with the desired properties follows since $O(C) \cap O(\hat{C}) \neq \emptyset$ by Claim 4.

If $j^{\tilde{l}} \in I(C) \setminus \{\underline{j}\}$, let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$ and note that our assumption about $\underline{j}$ implies that

$$(j^l, p^l, \dots [p^L, j^1] \dots, j^{\tilde{l}}, p^{\tilde{l}}, i^{\tilde{m}}, o^{\tilde{m}}, \dots [o^M, i^1] \dots, i^{m-1}, o^{m-1})$$

is a CIC of μ for $\underline{j}$ at A.

If $j^{\tilde{l}} \notin I(C) \setminus \{\underline{j}\}$ and $p^{\tilde{l}} \notin \mu(\underline{j})$, let $\tilde{m}$ be such that $o^{\tilde{m}} = p^{\tilde{l}}$ and note that our assumption about $\underline{j}$ implies that

$$(j^l, p^l, \dots [p^L, j^1] \dots, j^{\tilde{l}}, p^{\tilde{l}}, i^{\tilde{m}+1}, o^{\tilde{m}+1}, \dots [o^M, i^1] \dots, i^{m-1}, o^{m-1})$$

is a CIC of μ for $\underline{j}$ at A.

Finally, if $j^{\tilde{l}} \notin I(C) \setminus \{\underline{j}\}$ and $p^{\tilde{l}} \in \mu(\underline{j})$, then

$$(j^l, p^l, \dots [p^L, j^1] \dots, j^{\tilde{l}}, p^{\tilde{l}})$$

is a CIC of μ for $\underline{j}$ at A since $p^{\tilde{l}} = o^{m-1} \in \Omega_{\underline{j}} \setminus A_{\underline{j}}$.

Case 2: $\{o^{m-1}, o^m\} \subseteq A_{\underline{j}}$ and $\{p^{l-1}, p^l\} \subseteq \Omega_{\underline{j}} \setminus A_{\underline{j}}$. Let $\tilde{l}$ be the last integer in $(l, \dots [L, 1] \dots, l-1)$ such that either $j^{\tilde{l}} \in I(C) \setminus \{\underline{j}\}$ or $p^{\tilde{l}} \in O(C)$. Claim 4 ensures that such $\tilde{l}$ exists.

We now establish that $p^{\tilde{l}} \notin O(C)$. If there is $\tilde{m}$ such that $p^{\tilde{l}} = o^{\tilde{m}}$, then since $p^{\tilde{l}} \in (\mu + C)(j^{\tilde{l}+1})$, we have $j^{\tilde{l}+1} = i^{\tilde{m}} \in I(C)$. Since $j^{\tilde{l}+1} \in I(C)$ is a contradiction to the definition of $\tilde{l}$ if $\tilde{l} \neq l-1$, we have $\tilde{l} = l-1$. Given that $p^{l-1} \in (\mu + C)(\underline{j})$ and C is a CIC of μ, $p^{l-1} \in O(C)$ is possible only if $p^{l-1} = o^m$. However, since $o^m \in A_{\underline{j}}$ and $p^{l-1} \in \Omega_{\underline{j}} \setminus A_{\underline{j}}$, we have $o^m \neq p^{l-1}$ and hence, by our previous arguments, $p^{l-1} \notin O(C)$.

Since $p^{\tilde{l}} \notin O(C)$, we have $j^{\tilde{l}} \in I(C)$ so that there is an $\hat{m}$ be such that $i^{\hat{m}+1} = j^{\tilde{l}}$.

By an argument analogous to that in Case 1, the following is a CIC of μ for $\underline{j}$ at A:

$$\tilde{C} \equiv (i^m, o^m, \dots [o^M, i^1] \dots, i^{\hat{m}}, o^{\hat{m}}, j^{\tilde{l}}, p^{\tilde{l}}, \dots [p^L, j^1] \dots, j^{l-1}, p^{l-1})$$

□

Proof of Claim 6. Suppose that $p^{\bar{l}} \in O(C)$. Since $\hat{C}$ is a CIC of $\mu + C$, we have $j^{\bar{l}+1} \in I(C)$. However, if $\bar{l} < L$, then $\bar{l} + 1 \neq 1$, so $j^{\bar{l}+1} \neq j^1 = \hat{j}^*$. This contradicts the definition of $\bar{l}$. $\square$

Proof of Claim 7. We argue first that if $\underline{m} \leq m^*$, then $o \notin O(\hat{C})$ and $\overline{m} \leq m^*$.

Assume to the contrary that $\underline{m} \leq m^*$ and either $o \in O(\hat{C})$ or $m^* < \overline{m}$. Let $\tilde{m}_1, \dots, \tilde{m}_T$ be such that

1. $i^{\tilde{m}_t} \in I(\hat{C}) \setminus \{j^*\}$ for all $t \in \{1, \dots, T\}$
2. $\tilde{m}_t < \tilde{m}_{t+1}$ for all $t \leq T - 1$ and $\tilde{m}_T \leq m^*$
3. For all $m' \in \{2, \dots, m^*\} \setminus \{\tilde{m}_1, \dots, \tilde{m}_T\}$, $i^{m'} \notin I(\hat{C})$

Let $\tilde{l}_1, \dots, \tilde{l}_T$ be such that $j^{\tilde{l}_t} = i^{\tilde{m}_t}$ for all $t \in \{1, \dots, T\}$. Note that since $\underline{m} < m^*$ and either $o^{m^*} = o \in O(\hat{C})$ or $m^* < \overline{m}$, there exists $t^* \leq T$ such that

$$\{p^{\tilde{l}_{t^*}}, j^{\tilde{l}_{t^*}+1}, p^{\tilde{l}_{t^*}+1}, \dots [p^L, j^1] \dots, j^{\tilde{l}_{t^*+1}-1}, p^{\tilde{l}_{t^*+1}-1}\} \cap \{o^{m^*}, i^{m^*+1}, o^{m^*+1}, \dots, i^M, o^M\} \neq \emptyset, \quad (1)$$

where $\tilde{l}_{t^*+1} = \tilde{l}_1$ if $t^* = T$. For the remainder of the proof of Claim 7, fix some t^* for which Equation (1) holds and let $\hat{l}$ be the first integer in $(\tilde{l}_{t^*}, \dots [L, 1] \dots, \tilde{l}_{t^*+1} - 1)$ such that either $p^{\hat{l}} \in \{o^{m^*}, \dots, o^M\}$ or $j^{\hat{l}} \in \{i^{m^*+1}, \dots, i^M\}$.

We distinguish two cases:

Case 1: $\hat{l} \neq \tilde{l}_{t^*}$ and $j^* \in \{j^{\tilde{l}_{t^*}+1}, \dots [j^L, j^1] \dots, j^{\hat{l}-1}\}$.

Assume first that $j^* = \hat{j}^* = j^1$. If $j^{\hat{l}} \in \{i^{m^*+1}, \dots, i^M\}$, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$ and consider the sequence

$$\tilde{C} \equiv (j^1, p^1, \dots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \dots, i^M, o^M).$$

Note that $\hat{l} > 1$ since $j^{\hat{l}} \in \{i^{m^*+1}, \dots, i^M\}$ and $j^1 = i^1 = j^*$ in the case we consider here. By the definitions of $\tilde{l}_{t^*}$ and $\tilde{l}_{t^*+1}$ as well as the fact that $\hat{l} \leq \tilde{l}_{t^*+1} - 1$, we obtain $\{p^1, \dots, p^{\hat{l}-1}\} \cap O(C) = \emptyset$. Since $p^1 \in A_{j^*}$ and $o^M \in \Omega_{j^*} \setminus A_{j^*}$, Claim 5 implies that $\tilde{C}$ is a CIC of μ for j^* at $\hat{A}$, which is a contradiction to $\mu \in \mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{j^*}(\hat{A})$. If $p^{\hat{l}} \in \{o^{m^*}, \dots, O^M\}$ and $j^{\hat{l}} \notin \{i^{m^*+1}, \dots, i^M\}$, let $\hat{m}$ be such that $o^{\hat{m}} = p^{\hat{l}}$ and consider

$$(j^1, p^1, \dots, j^{\hat{l}}, p^{\hat{l}}, i^{\hat{m}+1}, o^{\hat{m}+1}, \dots, i^M, o^M)$$

to obtain a contradiction in a similar manner as before.[34]

[34] The only difference to before is that the sequence may now contain o. However, given that $\hat{m} + 1 > m^*$, it cannot be the case that i points to o in the proposed cycle.

Henceforth, we assume that $j^* \neq \hat{j}^*$. Let $\tilde{l} \geq 2$ be such that $j^{\tilde{l}} = j^*$ and note that Claim 3 implies $j^{\tilde{l}} < \hat{j}^*$.

Assume first that $p^{\hat{l}} \in A_{j^*}$. If $j^{\hat{l}} \in \{i^{m^*+1}, \ldots, i^M\}$, then $\hat{l} \neq \tilde{l}$ since $j^{\tilde{l}} = j^* = i^1$. Analogs of the arguments above imply that

$$(j^{\tilde{l}}, p^{\tilde{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$, which contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{j^*}(\hat{A})$. If $p^{\hat{l}} \in \{o^{m^*}, \ldots, o^M\}$ and $j^{\hat{l}} \notin \{i^{m^*+1}, \ldots, i^M\}$, let $\hat{m}$ be such that $o^{\hat{m}} = p^{\hat{l}}$ and consider

$$\tilde{C}' \equiv (j^{\tilde{l}}, p^{\tilde{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}}, p^{\hat{l}}, i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^M, o^M).[35]$$

Since i does not point to o in $\tilde{C}'$, analogs of our earlier arguments imply that $\tilde{C}'$ is a CIC of μ at $\hat{A}$, which is impossible given our choice of μ.

Next, consider the case where $p^{\tilde{l}} \in \Omega_{j^*} \setminus A_{j^*}$. Since $j^* < \hat{j}^*$ and since $\hat{C}$ is a CIC of $\mu + C$ for $\hat{j}^*$, we have $p^{\tilde{l}-1} \in \Omega_{j^*} \setminus A_{j^*}$ as well. Consider the sequence

$$\tilde{C}'' \equiv (i^1, o^1, \ldots, i^{\tilde{m}_{t^*}-1}, o^{\tilde{m}_{t^*}-1}, j^{\tilde{l}_{t^*}}, p^{\tilde{l}_{t^*}}, \ldots [p^L, j^1] \ldots j^{\tilde{l}-1}, p^{\tilde{l}-1}).$$

Note that $\tilde{m}_{t^*} \geq 2$ and $\tilde{l}_{t^*} \neq \tilde{l}$ since $i^{\tilde{m}_{t^*}} \in I(C) \setminus \{j^*\}$ and $j^* = j^{\tilde{l}}$. We now show that $\{p^{\tilde{l}_{t^*}}, \ldots [p^L, p^1] \ldots, p^{\tilde{l}-1}\} \cap O(C) = \emptyset$. If $p^l \in O(C)$ for some $l \in \{\tilde{l}_{t^*}, \ldots [L, 1] \ldots, \tilde{l} - 2\}$, then, since $\hat{C}$ is a CIC of $\mu + C$, we have $j^{l+1} \in I(C)$. Since $l \in \{l_{t^*}+1, \ldots [L, 1] \ldots, l_{t^*+1} - 1\}$, the construction of the sequence $\{\tilde{l}_t\}_{t=1}^T$ implies that $j^{l+1} \in \{i^{m^*+1}, \ldots, i^M\}$. However, since $\tilde{l} \in \{\tilde{l}_{t^*} + 1, \ldots [L, 1] \ldots, \hat{l} - 1\}$ and $\{p^{\tilde{l}_{t^*}}, j^{\tilde{l}_{t^*}+1}, p^{\tilde{l}_{t^*}+1}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}\} \cap \{o^{m^*}, i^{m^*+1}, o^{m^*+1}, \ldots, i^M, o^M\} = \emptyset$, $j^{l+1} \in \{i^{m^*+1}, \ldots, i^M\}$ is impossible. If $p^{\tilde{l}-1} \in O(C)$, then, since $\hat{C}$ is a CIC of $\mu + C$, we have $p^{\tilde{l}-1} = o^1$. However, since $p^{\tilde{l}-1} \in \Omega_{j^*} \setminus A_{j^*}$ and $o^1 \in A_{j^*}$ we have $p^{\tilde{l}-1} \neq o^1$. Given that we have now established $\{p^{\tilde{l}_{t^*}+1}, \ldots [p^L, p^1] \ldots, p^{\tilde{l}-1}\} \cap O(C) = \emptyset$, similar arguments as before show that $\tilde{C}''$ satisfies all conditions of a CIC of μ for j^* at A. Furthermore, since $\tilde{m}_{t^*} - 1 < m^*$, we have that $o \notin O(\tilde{C}'')$ and $\tilde{C}''$ is also a CIC of μ for j^* at $\hat{A}$, which is a contradiction.

Case 2: $\hat{l} = \tilde{l}_{t^*}$ or $\hat{l} \neq \tilde{l}_{t^*}$ and $j^* \notin \{j^{\tilde{l}_{t^*}+1}, \ldots [j^L, j^1] \ldots, j^{\hat{l}-1}\}$.

Let $\tilde{m} \equiv \tilde{m}_{t^*}$, $\tilde{l} \equiv \tilde{l}_{t^*}$, and $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$ if $j^{\hat{l}} \in I(C)$ and $o^{\hat{m}} = p^{\hat{l}}$ if $j^{\hat{l}} \notin I(C)$. Note that the properties that define $\tilde{m}, \tilde{l}, \hat{l}, \hat{m}$ imply that if $p^{\hat{l}} \notin \{o^{m^*}, \ldots, o^M\}$, then $\hat{l} \neq \tilde{l}$ since $\tilde{m} = \tilde{m}_{t^*} \leq m^*$.

[35] Note that $\{i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^M, o^M\} = \emptyset$ if $\hat{m} = M$.

If $p^{\hat{l}} \in \{o^{m^*}, \ldots, o^M\}$ and $j^{\hat{l}} \notin I(C)$, then we claim that

$$\tilde{C} \equiv (i^1, o^1, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}}, p^{\hat{l}}, i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots i^M, o^M)$$

is a CIC of μ for j^* at A.[36] The arguments to show that $\tilde{C}$ is a cycle of μ are analogous to the corresponding arguments in the proof of Claim 5. Next, recall that, by Claim 3, $j^* \leq \hat{j}^*$. Hence, Claim 5 implies that $\tilde{C}$ does not affect any agent $k < j^* = \min\{j^*, \hat{j}^*\}$. Since $o^1 \in A_{j^*}$ as well as $o^M \in \Omega_{j^*} \setminus A_{j^*}$, and since $i^1 = j^*$ is the only instance of j^* in $\tilde{C}$, $\tilde{C}$ increases the welfare of j^*. Note that the definitions of $\tilde{m}, \tilde{l}, \hat{l}, \hat{m}$ imply that $i \in I(\tilde{C})$ only if $i = j^{\tilde{l}}$. However, since $\hat{C}$ is a CIC of $\mu + C$ and $o \in (\mu + C)(i) \setminus \mu(i)$, we have $p^{\tilde{l}} \neq o$. Hence, $o \notin (\mu + \tilde{C})(i)$ and therefore $\tilde{C}$ is a CIC of μ for j^* at $\hat{A}$ as well, contradicting the assumption that $\mu \in \mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{j^*}(\hat{A})$.

If $j^{\hat{l}} \in I(C)$, then analogous arguments establish that

$$(i^1, o^1, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$ and we again obtain a contradiction.

We have thus shown that if $\underline{m} \leq m^*$, then $o \notin O(\hat{C})$ and $\overline{m} \leq m^*$.

Finally, we now show that $o \notin O(\hat{C})$. Assume to the contrary that there exists an l such that $p^l = o$. Since $\hat{C}$ is a CIC of $\mu + C$ at A and $o^{m^*} = o$, we have that $j^{l+1} = i^{m^*} = i$. Since $i \in I(\hat{C})$, $\underline{m} \leq m^*$ so that our previous arguments imply $o \notin O(\hat{C})$, thus contradicting our hypothesis that $o \in O(\hat{C})$. □

Proof of Claim 8. We proceed by contradiction. Without loss of generality, assume that $\check{m}$ is such that $i^{\check{m}}$ is highest priority agent for whom one of the conditions in Claim 8 is violated. Whenever $i^{\check{m}} \in I(\hat{C})$, we let $\check{l}$ be such that $j^{\check{l}} = i^{\check{m}}$. Furthermore, whenever $j^* \in I(\hat{C})$, we let l^* be such that $j^{l^*} = j^*$. Since we only work with the reassignments implied by C and $\hat{C}$, Claim 5 implies that the CICs we construct cannot affect any agent $j' < j^*$. We use this observation repeatedly throughout our proof of Claim 8.

We distinguish 12 cases to cover all the ways in which the conditions of Claim 8 may be violated for $i^{\check{m}}$. Below is a roadmap that explains which cases cover each possible violation.

- If Part (i) of the first statement is violated, then Case 1 or 2 applies.
- If Part (ii) of the first statement is violated, then Case 11 or 12 applies.
- If Part (i) of the second statement is violated, then Case 5 or 6 applies.

[36] Note that $\tilde{m} \geq 2$ and that $\{i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots i^M, o^M\} = \emptyset$ if $p^{\hat{l}} = o^M$. Note also that $o^M \in O(\tilde{C})$.

- If Part (ii) of the second statement is violated, then Case 3, 7, 11, or 12 applies.
- If Part (i) of the third statement is violated, then Case 8 or 9 applies.
- If Part (ii) of the third statement is violated, then Case 4, 10, 11, or 12 applies.
- If the fourth statement is violated, then Case 1, 3, or 4 applies.[37]

Case 1: $\check{m} \in \{\underline{m}+1, \dots, \overline{m}-1\}$, $i^{\check{m}} \leq \hat{j}^*$, $o^{\check{m}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$, and $o^{\check{m}} \in A_{i^{\check{m}}}$. By our assumption about $\check{m}$, there exist $\hat{m}_1$, $\hat{m}_2$, $\hat{l}_1$, and $\hat{l}_2$ such that

1. $\underline{m} \leq \hat{m}_1 < \check{m} < \hat{m}_2 \leq \overline{m}$
2. $i^{\hat{m}_1} = j^{\hat{l}_1}$ and $i^{\hat{m}_2} = j^{\hat{l}_2}$
3. $\{j^{\hat{l}_2+1}, \dots [j^L, j^1] \dots, j^{\hat{l}_1-1}\} \cap I(C) \subseteq \{i^{\check{m}}, j^*\}$

Assume first that $i^{\check{m}} \notin \{j^{\hat{l}_2+1}, \dots [j^L, j^1] \dots, j^{\hat{l}_1-1}\}$ and that $o^1 \notin \{p^{\hat{l}_2}, \dots [p^L, p^1] \dots, p^{\hat{l}_1-1}\}$. If $p^{\hat{l}_1-1} \notin O(C)$, then

$$(i^{\hat{m}_1}, o^{\hat{m}_1}, \dots, i^{\hat{m}_2-1}, o^{\hat{m}_2-1}, j^{\hat{l}_2}, p^{\hat{l}_2}, \dots [p^L, j^1] \dots, j^{\hat{l}_1-1}, p^{\hat{l}_1-1})$$

is a CIC of μ for $i^{\check{m}}$ at A. Since Claim 7 implies that $m^* \notin \{\hat{m}_1, \dots, \hat{m}_2 - 1\}$, the preceding sequence is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$, which contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A})$. If $p^{\hat{l}_1-1} \in O(C)$, then $p^{\hat{l}_1-1} = o^{\hat{m}_1}$ and so we obtain a similar contradiction using

$$(i^{\hat{m}_1+1}, o^{\hat{m}_1+1}, \dots, i^{\hat{m}_2-1}, o^{\hat{m}_2-1}, j^{\hat{l}_2}, p^{\hat{l}_2}, \dots [p^L, j^1] \dots, j^{\hat{l}_1-1}, p^{\hat{l}_1-1}).$$

Next, assume that $i^{\check{m}} \notin \{j^{\hat{l}_2+1}, \dots [j^L, j^1] \dots, j^{\hat{l}_1-1}\}$ and that there exists $l^* \in \{\hat{l}_2 + 1, \dots [L, 1] \dots, \hat{l}_1\}$ such that $p^{l^*-1} = o^1$. Since $o^1 \in A_{j^*}$, we must have $j^* < \hat{j}^*$ and since $\hat{C}$ is a CIC for $\hat{j}^*$, we obtain $p^{l^*} \in A_{j^*}$. If $m^* < \underline{m}$ and $p^{\hat{l}_1-1} \notin O(C)$, then

$$(j^{l^*}, p^{l^*}, \dots [p^L, j^1] \dots, j^{\hat{l}_1-1}, p^{\hat{l}_1-1}, i^{\hat{m}_1}, o^{\hat{m}_1}, \dots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$. If $m^* < \underline{m}$ and $p^{\hat{l}_1-1} = o^{\hat{m}_1}$, we obtain a similar contradiction by considering

$$(j^{l^*}, p^{l^*}, \dots [p^L, j^1] \dots, j^{\hat{l}_1-1}, p^{\hat{l}_1-1}, i^{\hat{m}_1+1}, o^{\hat{m}_1+1}, \dots, i^M, o^M).$$

[37] Note that since $\hat{C}$ is a CIC of $\mu + C$ for $\hat{j}^*$, $i^{\check{m}} = \hat{j}^*$ implies $\check{m} \in \{\underline{m}, \dots, \overline{m}\}$ and that $\hat{C}$ increases the welfare of $i^{\check{m}}$.

If $m^* \geq \underline{m}$, then Claim 7 implies $m^* \geq \overline{m}$ and

$$(i^2, o^2, \dots, i^{\hat{m}_2-1}, o^{\hat{m}_2-1}, j^{\hat{l}_2}, p^{\hat{l}_2}, \dots [p^L, j^1] \dots, j^{l^*-1}, p^{l^*-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$.

Next, assume that there exists $\check{l} \in \{\hat{l}_2 + 1, \dots [L, 1] \dots, \hat{l}_1 - 1\}$ such that $j^{\check{l}} = i^{\check{m}}$.

If $p^{\check{l}} \in A_{j^{\check{l}}}$ and $o^1 \notin \{p^{\check{l}}, \dots [p^L, p^1] \dots, p^{\hat{l}_1-1}\}$, then, given that $m^* \notin \{\underline{m}, \dots, \overline{m} - 1\}$,

$$(j^{\check{l}}, p^{\check{l}}, \dots [p^L, j^1] \dots, j^{\hat{l}_1-1}, p^{\hat{l}_1-1}, i^{\hat{m}_1}, o^{\hat{m}_1}, \dots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $j^{\check{l}}$ at $\hat{A}$ if $p^{\hat{l}_1-1} \notin O(C)$ and

$$(j^{\check{l}}, p^{\check{l}}, \dots [p^L, j^1] \dots, j^{\hat{l}_1-1}, p^{\hat{l}_1-1}, i^{\hat{m}_1+1}, o^{\hat{m}_1+1}, \dots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $j^{\check{l}}$ at $\hat{A}$ if $p^{\hat{l}_1-1} = o^{\hat{m}_1}$. Next, assume that $p^{\check{l}} \in A_{j^{\check{l}}}$ and there is $l^* \in \{\check{l} + 1, \dots [L, 1] \dots, \hat{l}_1\}$ such that $p^{l^*-1} = o^1$. If $m^* < \underline{m}$, we obtain a contradiction just as in the case where $i^{\check{m}} \notin \{j^{\hat{l}_2}, \dots [j^L, j^1] \dots, j^{\hat{l}_1}\}$ and $o^1 \in \{p^{\hat{l}_2}, \dots [p^L, p^1] \dots, p^{\hat{l}_1-1}\}$. If $m^* \geq \overline{m}$, then

$$(j^{\check{l}}, p^{\check{l}}, \dots [p^L, j^1] \dots, j^{l^*-1}, p^{l^*-1}, i^2, o^2, \dots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $j^{\check{l}}$ at $\hat{A}$.

If $p^{\check{l}} \in \Omega_{j^{\check{l}}} \setminus A_{j^{\check{l}}}$, we have $j^{\check{l}} < \hat{j}^*$ and $p^{\check{l}-1} \in \Omega_{j^{\check{l}}} \setminus A_{j^{\check{l}}}$. If $o^1 \notin \{p^{\hat{l}_2}, \dots [p^L, p^1] \dots, p^{\check{l}-1}\}$, then

$$(i^{\check{m}}, o^{\check{m}}, \dots, i^{\hat{m}_2-1}, o^{\hat{m}_2-1}, j^{\hat{l}_2}, p^{\hat{l}_2}, \dots [p^L, j^1] \dots, j^{\check{l}-1}, p^{\check{l}-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$. If there exists $l^* \in \{\hat{l}_2 + 1, \dots [L, 1] \dots, \check{l}\}$ such that $p^{l^*-1} = o^1$, we have $p^{l^*} \in A_{j^*}$. If $m^* < \underline{m}$, then

$$(j^{l^*}, p^{l^*}, \dots [p^L, j^1] \dots, j^{\check{l}-1}, p^{\check{l}-1}, i^{\check{m}}, o^{\check{m}}, \dots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$. Otherwise,

$$(i^2, o^2, \dots, i^{\hat{m}_2-1}, o^{\hat{m}_2-1}, j^{\hat{l}_2}, p^{\hat{l}_2}, \dots [p^L, j^1] \dots, j^{l^*-1}, p^{l^*-1})$$

is a CIC of μ for $j^{\check{l}}$ at $\hat{A}$.

Case 2: $\boldsymbol{\check{m} \in \{\underline{m} + 1, \dots, \overline{m} - 1\}}$, $\boldsymbol{i^{\check{m}} < \hat{j}^*}$, $\boldsymbol{o^{\check{m}-1} \in A_{i^{\check{m}}}}$, and $\boldsymbol{o^{\check{m}} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}}$. We construct a CIC of $\mu + C$ for $i^{\check m}$ at A and thus, given that $i^{\check m} < \hat j^*$, contradict the assumption that $\hat j^*$ is the highest priority agent for whom there exists a CIC of $\mu + C$ at A.

Note first that since $\check m > \underline{m}$ there exists a largest integer $\hat m \in \{\underline{m}, \dots, \check m - 1\}$ such that $i^{\hat m} \in I(\hat C)$. Let $\hat l$ be such that $j^{\hat l} = i^{\hat m}$ and note that, since C is a CIC and $\hat m \neq \check m$, $i^{\hat m} \neq i^{\check m}$. Next, let $\tilde l$ be the first integer in the sequence $(\hat l + 1, \dots [L, 1] \dots, \hat l - 1)$ such that $j^{\tilde l} \in \{i^{\check m+1}, \dots, i^{\overline{m}}\}$. Note that such an integer exists since $\check m < \overline{m}$ and $j^{\hat l} \in \{i^{\underline{m}}, \dots, i^{\check m-1}\}$. Let $\tilde m$ be such that $i^{\tilde m} = j^{\tilde l}$.

If $i^{\check m} \notin \{j^{\hat l}, \dots [j^L, j^1] \dots, j^{\tilde l}\}$, then

$$(i^{\check m}, o^{\check m-1}, \dots, i^{\hat m+1}, o^{\hat m}, j^{\hat l}, p^{\hat l}, \dots [p^L, j^1] \dots, j^{\tilde l-1}, p^{\tilde l-1}, i^{\tilde m}, o^{\tilde m-1}, \dots, i^{\check m+1}, o^{\check m}).$$

is a CIC of $\mu + C$ for $i^{\check m}$ at A.

For the remainder of the proof in Case 2 assume that $i^{\check m} \in \{j^{\hat l}, \dots [j^L, j^1] \dots, j^{\tilde l}\}$.

If $p^{\check l-1} \in \Omega_{j^{\check l}} \setminus A_{j^{\check l}}$, then analogs of our earlier arguments show

$$(i^{\check m}, o^{\check m-1}, \dots, i^{\hat m+1}, o^{\hat m}, j^{\hat l}, p^{\hat l}, \dots [p^L, j^1] \dots, j^{\check l-1}, p^{\check l-1})$$

is a CIC of $\mu + C$ for $i^{\check m}$ at A.

If $p^{\check l-1} \in A_{j^{\check l}}$, then since $i^{\check m} < \hat j^*$ and since $\hat C$ is a CIC of $\mu + C$ for $\hat j^*$ at A, we have $p^{\check l} \in A_{j^{\check l}}$. Analogs of our earlier arguments show that

$$(j^{\check l}, p^{\check l}, \dots [p^L, j^1] \dots, j^{\tilde l-1}, p^{\tilde l-1}, i^{\tilde m}, o^{\tilde m-1}, \dots, i^{\check m+1}, o^{\check m})$$

is a CIC of $\mu + C$ for $j^{\check l} = i^{\check m}$ at A.

Case 3: $\check m = \underline{m}$, $i^{\check m} \leq \hat j^*$, $o^{\check m-1} \in \Omega_{i^{\check m}} \setminus A_{i^{\check m}}$, $o^{\check m} \in A_{i^{\check m}}$, and $p^{\check l-1} \in \Omega_{i^{\check m}} \setminus A_{i^{\check m}}$. Note first that the premise of this case implies $p^{\check l-1} \notin O(C)$: Otherwise, since $\hat C$ is a CIC of $\mu + C$, we have $p^{\check l-1} = o^{\check m}$, which is incompatible with $o^{\check m} \in A_{i^{\check m}}$ and $p^{\check l-1} \in \Omega_{i^{\check m}} \setminus A_{i^{\check m}}$.

Let $\hat l$ be the last integer in $(\check l + 1, \dots [L, 1] \dots, \check l - 1)$ such that $j^{\hat l} \in I(C) \setminus \{j^*\}$ and let $\hat m$ be such that $i^{\hat m} = j^{\hat l}$. If $\{p^{\hat l}, \dots [p^L, p^1] \dots, p^{\check l-1}\} \cap O(C) = \emptyset$, then similar arguments to those employed in Case 1 show that

$$(i^{\check m}, o^{\check m}, \dots, i^{\hat m-1}, o^{\hat m-1}, j^{\hat l}, p^{\hat l}, \dots [p^L, j^1] \dots, j^{\check l-1}, p^{\check l-1})$$

is a CIC of μ for $i^{\check m}$ at $\hat A$. If $\{p^{\hat l}, \dots [p^L, p^1] \dots, p^{\check l-1}\} \cap O(C) \neq \emptyset$, the definition of $\hat l$ and our earlier finding that $p^{\check l-1} \notin O(C)$ jointly imply $\{p^{\hat l}, \dots [p^L, p^1] \dots, p^{\check l-1}\} \cap O(C) = \{o^1\}$.

Let l^* be such that $p^{l^*-1} = o^1$ and note that $j^{l^*} = j^*$. Furthermore, since $\hat{C}$ is a CIC for $\hat{j}^* > j^*$ in the case we consider here, $p^{l^*} \in A_{j^*}$ as well. Next, note first that

$$(i^2, o^2, \ldots, i^{\hat{m}-1}, o^{\hat{m}-1}, j^{\hat{l}}, p^{\hat{l}}, \ldots [p^L, j^1] \ldots, j^{l^*-1}, p^{l^*-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$ unless $m^* \leq \hat{m} - 1$. Since $\underline{m} = \check{m} < \hat{m} \leq \overline{m}$, Claim 7 implies $m^* < \check{m}$. But then,

$$(j^{l^*}, p^{l^*}, \ldots [p^L, j^1] \ldots, j^{\check{l}-1}, p^{\check{l}-1}, i^{\check{m}}, o^{\check{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$.

Case 4: $\check{m} = \overline{m}$, $i^{\check{m}} \leq \hat{j}^*$, $o^{\check{m}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$, $o^{\check{m}} \in A_{i^{\check{m}}}$, and $p^{\check{l}} \in A_{i^{\check{m}}}$. Let $\hat{l}$ be the first integer in $(\check{l}+1, \ldots [L, 1] \ldots, \check{l} - 1)$ such that $j^{\hat{l}} \in I(C) \setminus \{j^*\}$ and let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. If $\{p^{\check{l}}, \ldots [p^L, p^1] \ldots, p^{\hat{l}-1}\} \cap O(C) = \emptyset$, then, given that Claim 7 implies $m^* \notin \{\hat{m}, \ldots, \check{m} - 1\}$,

$$(j^{\check{l}}, p^{\check{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $j^{\check{l}} = i^{\check{m}}$ at $\hat{A}$. If $p^{\hat{l}-1} \in O(C)$ and $\{p^{\check{l}}, \ldots [p^L, p^1] \ldots, p^{\hat{l}-2}\} \cap O(C) = \emptyset$, we have $p^{\hat{l}-1} = o^{\hat{m}}$ and

$$(j^{\check{l}}, p^{\check{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $j^{\check{l}} = i^{\check{m}}$ at $\hat{A}$. By the definition of $\hat{l}$, the only remaining possibility is that there exists a $l^* \in \{\check{l}+1, \ldots [L, 1] \ldots, \hat{l}\}$ such that $p^{l^*-1} = o^1$. Since $\hat{C}$ is a CIC for $\hat{j}^* > j^*$, we have $p^{l^*} \in A_{j^*}$. Next, note first that

$$(i^2, o^2, \ldots, i^{\check{m}-1}, o^{\check{m}-1}, j^{\check{l}}, p^{\check{l}}, \ldots [p^L, j^1] \ldots, j^{l^*-1}, p^{l^*-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$ unless $m^* \leq \check{m} - 1$. Since $\underline{m} < \overline{m} = \check{m}$, Claim 7 implies $m^* < \underline{m}$. But then,

$$(j^{l^*}, p^{l^*}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\check{m}}, o^{\check{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$ if $p^{\hat{l}-1} \notin O(C)$ and

$$(j^{l^*}, p^{l^*}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\check{m}+1}, o^{\check{m}+1}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$ if $p^{\hat{l}-1} \in O(C)$.

Case 5: $\check{m} \leq \underline{m}$, $i^{\check{m}} < \hat{j}^*$, $o^{\check{m}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$, $o^{\check{m}} \in A_{i^{\check{m}}}$, and $\{p^{l^*-1}, p^{l^*}\} \subseteq A_{j^*}$. We argue first that $m^* \geq \overline{m}$. Assume the contrary and let $\tilde{l}$ be the first integer in $(l^*+1, \dots [L,1] \dots, l^*-1)$ such that $j^{\tilde{l}} \in I(C)$. Note that such an integer exists since $(I(C) \cap I(\hat{C})) \setminus \{j^*\} \neq \emptyset$. If $p^{\tilde{l}-1} \notin O(C)$, let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$ and consider

$$\tilde{C} \equiv (j^{l^*}, p^{l^*}, \dots [j^L, p^1] \dots, j^{\tilde{l}-1}, p^{\tilde{l}-1}, i^{\tilde{m}}, o^{\tilde{m}}, \dots, i^M, o^M).$$

Since $p^{l^*} \in A_{j^*}$, analogs of our earlier arguments show that $\tilde{C}$ is a CIC of μ for $j^* = j^{l^*}$ at A. Since we assume that $m^* \ngeq \overline{m}$, Claim 7 implies $m^* < \underline{m}$. But then, we obtain that $m^* < \tilde{m}$ and $\tilde{C}$ is a CIC of μ for j^* at $\hat{A}$, which contradicts our assumption that $\mu \in \mathcal{M}^{i-1}(\hat{A})$ given that $j^* < i$. If $p^{\tilde{l}-1} = o^{\tilde{m}-1}$ for some $\tilde{m}$, an analogous argument applies.

Now let $\hat{l}$ be the last integer in $(l^*+1, \dots [L,1] \dots, l^*-1)$ such that $j^{\hat{l}} \in I(C)$.

If $j^{\hat{l}} = i^{\check{m}}$, the premise of Case 5 implies $\check{m} = \underline{m}$. If $p^{\hat{l}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$, then we can apply our findings from Case 3 above. If $p^{\hat{l}-1} \in A_{i^{\check{m}}}$, then, since $\hat{C}$ is a CIC for $\hat{j}^*$, we have $i^{\check{m}} < \hat{j}^*$ and $p^{\hat{l}} \in A_{i^{\check{m}}}$ as well. If $p^{l^*-1} \neq o^1$, then $m^* \geq \overline{m}$ implies that

$$(j^{\hat{l}}, p^{\hat{l}}, \dots [p^L, j^1] \dots, j^{l^*-1}, p^{l^*-1}, i^1, o^1, \dots, i^{\check{m}-1}, o^{\check{m}-1}).$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$. If $p^{l^*-1} = o^1$, we obtain a similar contradiction using

$$(j^{\hat{l}}, p^{\hat{l}}, \dots [p^L, j^1] \dots, j^{l^*-1}, p^{l^*-1}, i^2, o^2, \dots, i^{\check{m}-1}, o^{\check{m}-1}).$$

For the remainder of the proof in Case 5, assume $j^{\hat{l}} \neq i^{\check{m}}$ and let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Since $\check{m} \in \{2, \dots, \underline{m}\}$, $\hat{m} > \check{m}$. If $p^{l^*-1} \neq o^1$, analogs of our earlier arguments show that

$$(i^1, o^1, \dots, i^{\hat{m}-1}, o^{\hat{m}-1}, j^{\hat{l}}, p^{\hat{l}}, \dots [p^L, j^1] \dots j^{l^*-1}, p^{l^*-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$. Again, if $p^{l^*-1} = o^1$, we obtain a similar contradiction using

$$(i^2, o^2, \dots, i^{\hat{m}-1}, o^{\hat{m}-1}, j^{\hat{l}}, p^{\hat{l}}, \dots [p^L, j^1] \dots j^{l^*-1}, p^{l^*-1}).$$

Case 6: $\check{m} \leq \underline{m}$, $i^{\check{m}} < \hat{j}^*$, $o^{\check{m}-1} \in A_{i^{\check{m}}}$, $o^{\check{m}} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$, and $\{p^{l^*-1}, p^{l^*}\} \subseteq A_{j^*}$. Let $\hat{l}$ be the first integer in $(l^*+1, \dots [L,1] \dots, l^*-1)$ such that $j^{\hat{l}} \in I(C)$.

If $j^{\hat{l}} = i^{\check{m}}$, the conditions of Case 6 imply that $\check{m} = \underline{m}$.

If $p^{\hat{l}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$, then the definition of $\hat{l}$, our assumption about $\check{m}$, and $p^{l^*} \in A_{j^*}$ imply that

$$\tilde{C} \equiv (i^{\check{m}}, o^{\check{m}-1}, \ldots, i^2, o^1, j^{l^*}, p^{l^*}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}).$$

is a CIC of $\mu{+}C$ for $i^{\check{m}}$ at A. Since $i^{\check{m}} < \hat{j}^*$, we obtain a contradiction to our assumption that $\hat{j}^*$ is the highest priority agent for whom there exists a CIC of $\mu + C$ at A.

If $p^{\hat{l}-1} \in A_{i^{\check{m}}}$, let $\tilde{l}$ be the first integer in $(\hat{l}{+}1, \ldots [L, 1] \ldots, \hat{l}{-}1)$ such that $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$ and let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$. The definitions of $\hat{l}$ and $\tilde{l}$ together with our assumption that $\underline{m} \neq \overline{m}$ imply that $j^{\tilde{l}} \in I(C) \setminus \{i^{\check{m}}\}$. Hence, $\tilde{m} > \check{m}$ and

$$(j^{\hat{l}}, p^{\hat{l}}, \ldots [p^L, j^1] \ldots, j^{\tilde{l}-1}, p^{\tilde{l}-1}, i^{\tilde{m}}, o^{\tilde{m}-1}, \ldots, i^{\check{m}+1}, o^{\check{m}})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A.

For the remainder of the proof in Case 6, assume $j^{\hat{l}} \neq i^{\check{m}}$ and let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Since $\check{m} \in \{2, \ldots, \underline{m}\}$, $\check{m} < \hat{m}$. Analogs of our earlier arguments establish that

$$\tilde{C}' \equiv (i^{\hat{m}}, o^{\hat{m}-1}, \ldots, i^2, o^1, j^{l^*}, p^{l^*}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A, which again contradicts our assumption that $\hat{j}^*$ is the highest priority agent for whom there exists a CIC of $\mu + C$ at A.

Case 7: $\boldsymbol{\check{m} = \underline{m}}$, $\boldsymbol{i^{\check{m}} < \hat{j}^*}$, $\boldsymbol{o^{\check{m}-1} \in A_{i^{\check{m}}}}$, $\boldsymbol{o^{\check{m}} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}}$, and $\boldsymbol{\{p^{\check{l}-1}, p^{\check{l}}\} \subseteq A_{i^{\check{m}}}}$. By Case 6, we can assume that, if there exists l^* such that $j^{l^*} = j^*$, then $\{p^{l^*-1}, p^{l^*}\} \subseteq \Omega_{j^*} \setminus A_{j^*}$.

Now let $\hat{l}$ be the first integer in $(\check{l} + 1, \ldots [L, 1] \ldots, \check{l} - 1)$ such that $j^{\hat{l}} \in I(C)$. If $j^{\hat{l}} \neq j^*$, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$ and note that

$$(j^{\check{l}}, p^{\check{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}-1}, \ldots, i^{\check{m}+1}, o^{\check{m}})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A. If $j^{\hat{l}} = j^*$, then

$$(j^{\check{l}}, p^{\check{l}}, \ldots [p^L, j^1] \ldots, j^{l^*-1}, p^{l^*-1}, j^{l^*}, o^M, i^M, o^{M-1}, \ldots, i^{\check{m}+1}, o^{\check{m}})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A.

Case 8: $\boldsymbol{\check{m} \geq \overline{m}}$, $\boldsymbol{i^{\check{m}} < \hat{j}^*}$, $\boldsymbol{o^{\check{m}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}}$, $\boldsymbol{o^{\check{m}} \in A_{i^{\check{m}}}}$, and $\boldsymbol{\{p^{l^*-1}, p^{l^*}\} \subseteq \Omega_{j^*} \setminus A_{j^*}}$. We argue first that $m^* < \underline{m}$. Assume the contrary and let $\tilde{l}$ be the last integer in

$(l^*+1, \dots [L,1] \dots, l^*-1)$ such that $j^{\tilde{l}} \in I(C)$. Note that such an integer exists since $(I(C) \cap I(\hat{C})) \setminus \{j^*\} \neq \emptyset$. Let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$. If $p^{\tilde{l}-1} \notin O(C)$, consider

$$\tilde{C} \equiv (i^1, o^1, \dots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \dots [j^L, p^1] \dots, j^{l^*-1}, p^{l^*-1}).$$

Since $p^{l^*-1} \in \Omega_{j^*} \setminus A_{j^*}$, analogs of our earlier arguments show that $\tilde{C}$ is a CIC of μ for $j^* = i^1$ at A. Since we assume that $m^* \geq \underline{m}$, Claim 7 implies that $m^* \geq \overline{m}$. But then, we obtain that $m^* \geq \tilde{m}$ and $\tilde{C}$ is a CIC of μ for j^* at $\hat{A}$, which contradicts our assumption that $\mu \in \mathcal{M}^{i-1}(\hat{A})$ given that $j^* < i$. If $p^{\tilde{l}-1} \in O(C), then p^{\tilde{l}-1} = o^{\tilde{m}-1}$ and the same argument applies.

Now let $\hat{l}$ be the first integer in $(l^*+1, \dots [L,1] \dots, l^*-1)$ such that $j^{\hat{l}} \in I(C)$.

If $j^{\hat{l}} = i^{\check{m}}$, the premise of Case 8 implies $\check{m} = \overline{m}$. If $p^{\hat{l}} \in A_{i^{\check{m}}}$, then we can apply our findings from Case 4 above. If $p^{\hat{l}} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$, then, since $\hat{C}$ is a CIC for $\hat{j}^*$, we have $i^{\check{m}} < \hat{j}^*$ and $p^{\hat{l}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$ as well. If $p^{l^*} \neq o^M$in the case we consider here, then $m^* < \underline{m}$ implies that

$$(i^{\check{m}}, o^{\check{m}}, \dots, i^M, o^M, j^{l^*}, p^{l^*}, \dots, j^{\hat{l}-1}, p^{\hat{l}-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$. If $p^{l^*} = o^M$, we obtain a similar contradiction using

$$(i^{\check{m}}, o^{\check{m}}, \dots, i^M, o^M, j^{l^*+1}, p^{l^*+1}, \dots, j^{\hat{l}-1}, p^{\hat{l}-1}).$$

For the remainder of the proof in Case 8, we assume that $j^{\hat{l}} \neq i^{\check{m}}$ and let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$.

Since $\check{m} \in \{\overline{m}, \dots, M\}$, we have $\hat{m} < \check{m}$. If $p^{\hat{l}-1} \notin O(C)$ and $p^{l^*} \neq o^M$, analogs of our earlier arguments show that

$$(i^{\hat{m}}, o^{\hat{m}}, \dots, i^M, o^M, j^{l^*}, p^{l^*}, \dots, j^{\hat{l}-1}, p^{\hat{l}-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$. If $p^{\hat{l}-1} \notin O(C)$ and $p^{l^*} = o^M$, we obtain a similar contradiction using

$$(i^{\hat{m}}, o^{\hat{m}}, \dots, i^M, o^M, j^{l^*+1}, p^{l^*+1}, \dots, j^{\hat{l}-1}, p^{\hat{l}-1}).$$

If $p^{\hat{l}-1} \in O(C)$ and $p^{l^*} \neq o^M$, analogs of our earlier arguments show that

$$(i^{\hat{m}+1}, o^{\hat{m}+1}, \dots, i^M, o^M, j^{l^*}, p^{l^*}, \dots, j^{\hat{l}-1}, p^{\hat{l}-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$. Finally, if $p^{\hat{l}-1} \in O(C)$ and $p^{l^*} = o^M$, we obtain a similar contradiction using

$$(i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^M, o^M, j^{l^*+1}, p^{l^*+1}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}).$$

Case 9: $\check{m} \geq \overline{m}$, $i^{\check{m}} < \hat{j}^*$, $o^{\check{m}-1} \in A_{i^{\check{m}}}$, $o^{\check{m}} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$, and $\{p^{l^*-1}, p^{l^*}\} \subseteq \Omega_{j^*} \setminus A_{j^*}$. Let $\hat{l}$ be the last integer in $(l^*+1, \ldots [L,1] \ldots, l^*-1)$ such that $j^{\hat{l}} \in I(C)$.

If $j^{\hat{l}} = i^{\check{m}}$, the conditions of Case 9 imply that $\check{m} = \overline{m}$.

If $p^{\hat{l}} \in A_{i^{\check{m}}}$, then

$$(j^{\hat{l}}, p^{\hat{l}}, \ldots, j^{l^*-1}, p^{l^*-1}, i^1, o^M, \ldots, i^{\overline{m}+1}, o^{\overline{m}}).$$

$\tilde{C}$ is a CIC of $\mu + C$ for $i^{\check{m}}$ at A. Since $i^{\check{m}} < \hat{j}^*$, we obtain a contradiction to our assumption that $\hat{j}^*$ is the highest priority agent for whom there exists a CIC of $\mu + C$ at A.

If $p^{\hat{l}} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$ note first that $p^{\hat{l}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$ as well. Let $\tilde{l}$ be the last integer in $(\check{l}+1, \ldots [L,1] \ldots, \check{l}-1)$ such that $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$ and let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$. The definitions of $\hat{l}$ and $\tilde{l}$ imply $j^{\tilde{l}} \neq i^{\check{m}}$. Hence, we have that $\tilde{m} < \overline{m}$ and

$$(j^{\tilde{l}}, p^{\tilde{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\check{m}}, o^{\check{m}-1}, \ldots, i^{\tilde{m}+1}, o^{\tilde{m}})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A.

For the remainder of the proof in Case 9, assume that $j^{\hat{l}} \neq i^{\check{m}}$ and let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Since $\check{m} \in \{\overline{m}, \ldots, M\}$, $\hat{m} < \check{m}$. Analogs of our earlier arguments establish that

$$\tilde{C}' \equiv (j^{\hat{l}}, p^{\hat{l}}, \ldots [p^L, j^1] \ldots, j^{l^*-1}, p^{l^*-1}, i^1, o^M, \ldots, i^{\hat{m}+1}, o^{\hat{m}})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A, which contradicts our assumption that $\hat{j}^*$ is the highest priority agent for whom there exists a CIC of $\mu + C$ at A.

Case 10: $\check{m} = \overline{m}$, $i^{\check{m}} < \hat{j}^*$, $o^{\check{m}-1} \in A_{i^{\check{m}}}$, $o^{\check{m}} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$, and $\{p^{\check{l}-1}, p^{\check{l}}\} \subseteq \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$. By Case 9, we can assume that, if there exists l^* such that $j^{l^*} = j^*$, then $\{p^{l^*-1}, p^{l^*}\} \subseteq A_{j^*}$.

Now let $\hat{l}$ be the last integer in $(\check{l}+1, \ldots [L,1] \ldots, \check{l}-1)$ such that $j^{\hat{l}} \in I(C)$. If $j^{\hat{l}} \neq j^*$, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$ and note that

$$(i^{\check{m}}, o^{\check{m}-1}, \ldots, i^{\hat{m}+1}, o^{\hat{m}}, j^{\hat{l}}, p^{\hat{l}}, \ldots [p^L, j^1] \ldots, j^{\check{l}-1}, p^{\check{l}-1})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A. If $j^{\hat{l}} = j^*$, then

$$(i^{\check{m}}, o^{\check{m}-1}, \ldots, i^2, o^1, j^{l^*}, p^{l^*}, \ldots [p^L, j^1] \ldots, j^{\check{l}-1}, p^{\check{l}-1})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A.

Case 11: $\check{m} \in \{\underline{m}, \ldots, \overline{m}\}$, $i^{\check{m}} < \hat{j}^*$, $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq A_{i^{\check{m}}}$, and $\{p^{\check{l}-1}, p^{\check{l}}\} \subseteq \Omega_{j^{\check{l}}} \setminus A_{j^{\check{l}}}$. Let $\hat{l}$ be the last integer in $(\check{l}+1, \ldots [L, 1] \ldots, \check{l}-1)$ such that $j^{\hat{l}} \in I(C) \setminus \{j^*\}$ and let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$.

If $\hat{m} < \check{m}$, then

$$(i^{\check{m}}, o^{\check{m}-1}, \ldots, i^{\hat{m}+1}, o^{\hat{m}}, j^{\hat{l}}, p^{\hat{l}}, \ldots [p^L, j^1] \ldots, j^{\check{l}-1}, p^{\check{l}-1})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A.

If $\hat{m} > \check{m}$ and $\{p^{\hat{l}}, \ldots [p^L, p^1] \ldots, p^{\check{l}-1}\} \cap O(C) = \emptyset$, then Claim 7 implies that

$$(i^{\check{m}}, o^{\check{m}}, \ldots, i^{\hat{m}-1}, o^{\hat{m}-1}, j^{\hat{l}}, p^{\hat{l}}, \ldots [p^L, j^1] \ldots, j^{\check{l}-1}, p^{\check{l}-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$. By the premise of this case and the definition of $\hat{l}$, the only remaining possibility is that there exists $l^* \in (\hat{l}+1, \ldots [L, 1] \ldots, \check{l}-1)$ such that $p^{l^*-1} = o^1$. But then,

$$(i^{\check{m}}, o^{\check{m}-1}, \ldots, i^2, o^1, j^{l^*}, p^{l^*}, \ldots [p^L, j^1] \ldots, j^{\check{l}-1}, p^{\check{l}-1})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A.

Case 12: $\check{m} \in \{\underline{m}, \ldots, \overline{m}\}$, $i^{\check{m}} < \hat{j}^*$, $\{o^{\check{m}-1}, o^{\check{m}}\} \subseteq \Omega_{j^*} \setminus A_{i^{\check{m}}}$, and $\{p^{\check{l}-1}, p^{\check{l}}\} \subseteq A_{\check{l}}$. Let $\hat{l}$ be the first integer in $(\check{l}+1, \ldots [L, 1] \ldots, \check{l}-1)$ such that $j^{\hat{l}} \in I(C) \setminus \{j^*\}$ and let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$.

If $\hat{m} < \check{m}$ and $\{p^{\check{l}}, \ldots [p^L, p^1] \ldots, p^{\hat{l}-1}\} \cap O(C) = \emptyset$, then Claim 7

$$(j^{\check{l}}, p^{\check{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $j^{\check{l}}$ at $\hat{A}$. If $p^{\hat{l}-1} \in O(C)$ and $\{p^{\check{l}}, \ldots [p^L, p^1] \ldots, p^{\hat{l}-2}\} \cap O(C) = \emptyset$, we have $p^{\hat{l}-1} = o^{\hat{m}}$ and

$$(j^{\check{l}}, p^{\check{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $j^{\check{l}} = i^{\check{m}}$ at $\hat{A}$. By the definition of $\hat{l}$, the only remaining possibility is that there exists a $l^* \in (\check{l}+1, \ldots [L,1] \ldots, \hat{l}-1)$ such that $p^{l^*-1} = o^1$. If $m^* \geq \check{m}$, then

$$(j^{\check{l}}, p^{\check{l}}, \ldots [p^L, j^1] \ldots, j^{l^*-1}, p^{l^*-1}, i^2, o^2, \ldots, i^{\check{m}-1}, o^{\check{m}-1})$$

is a CIC of μ for $j^{\check{l}}$ at $\hat{A}$. Otherwise, Claim 7 implies $m^* < \underline{m}$. Note that $p^{l^*-1} = o^1$ implies $p^{l^*} \in A_{j^*}$. If $p^{\hat{l}-1} \notin O(C)$, then

$$(j^{l^*}, p^{l^*}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$. If $p^{\hat{l}-1} \in O(C)$, the same is true for

$$(j^{l^*}, p^{l^*}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^M, o^M).$$

If $\hat{m} > \check{m}$, then

$$(j^{\check{l}}, p^{\check{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}-1}, \ldots, i^{\check{m}+1}, o^{\check{m}})$$

is a CIC of $\mu + C$ for $j^{\check{l}}$ at A.

□

D Proof of Theorem 4

To prove part 1 of the Theorem, we show by induction on t that the maximum flows of (V, E, q) correspond to $\mathcal{M}^t(A)$ after the t^{th} iteration of the for-loop on line 2 of Algorithm 1.

As a base case, we show that for (V, E, q), where q is the initial capacity vector defined above, f is a maximum flow if and only if it corresponds to some matching $\mu \in \mathcal{M}^0(A)$. Take any matching $\mu \in \mathcal{M}^0(A)$. The definition of a matching and trichotomous preferences imply $|\mu(i)| = |\Omega_i|$ and $\mu(i) \subseteq \Omega_i \cup A_i$. Define a flow f by setting, for all $i \in I$, $f(S, i) = |\Omega_i|$, $f(i, i^A) = |\mu(i) \cap A_i|$, $f(i, i^U) = |\mu(i) \cap (\Omega_i \setminus A_i)|$, $f(i^A, o) = 1$ for all $o \in \mu(i) \cap A_i$, $f(i^U, o) = 1$ for all $o \in \mu(i) \cap (\Omega_i \setminus A_i)$, and, for all $o \in \cup_{i \in I} \mu(i)$, $f(o, T) = 1$. Clearly, f is a maximum flow in (V, E, q). Conversely, take a maximum flow f in (V, E, q) and let μ be the matching that corresponds to f. Since all maximum flows have value $\sum_i |\Omega_i|$, we have $|\mu(i)| = |\Omega_i|$ for all $i \in I$. Since (V, E, q) contains no paths between i and an object in $O \setminus (\Omega_i \cup A_i)$, we have $\mu(i) \subseteq \Omega_i \cup A_i$. Hence, $\mu \in \mathcal{M}^0(A)$.

As an induction hypothesis, suppose that at the start of the t^{th} iteration of the loop, the maximum flows of (V, E, q) correspond to $\mathcal{M}^{t-1}(A)$. Let $\hat{q}$ be the capacity at the end of the

t^{th} iteration of the for-loop, f be a maximum flow in $(V, E, \hat{q})$, and μ be the matching that corresponds to f. We show first that $\mu \in \mathcal{M}^t(A)$. By construction, $|\mu(t) \cap A_t| = |\Omega_t| - \hat{q}(i, i^U)$. Assume that there is a matching $\mu' \in \mathcal{M}^t(A)$ such that $|\mu'(t) \cap A_t| > |\mu(t) \cap A_t|$. Since the set of maximum flows of (V, E, q) corresponds to $\mathcal{M}^{t-1(A)}$ and since $\mu' \in \mathcal{M}^t(A)$, just as argued in the base case, μ' induces a maximum flow f' in (V, E, q). Given that $|\mu'(t) \cap A_t| > |\mu(t) \cap A_t|$, f' is a maximum flow in $(V, E, \tilde{q})$, where $\tilde{q}(t, t^U) = \hat{q}(t, t^U) - 1$ and $\tilde{q}(e) = \hat{q}(e)$ for all $e \in E \setminus \{(t, t^U)\}$. But then, the t^{th} iteration of the loop would have concluded with the capacity of (i, i^U) no higher than $\hat{q}(t, t^U) - 1$. This contradiction shows that $\mu \in \mathcal{M}^t(A)$. Conversely, take any matching $\hat{\mu} \in \mathcal{M}^t$ and let $\hat{f}$ be the induced flow, defined as in the base case. By our previous arguments, we have $|\hat{\mu}(t) \cap A_t| = |\mu(t) \cap A_t|$. Since $|\hat{\mu}(t)| = |\mu(t)| = |\Omega_t|$, we obtain that $|\hat{\mu}(t) \setminus A_t| = |\mu(t) \setminus A_t| = \hat{q}(t, t^U)$. Given that $\mu \in \mathcal{M}^t \subseteq \mathcal{M}^{t-1}$ and that $\hat{q}(e) = q(e)$ for all $e \in E \setminus \{(t, t^U)\}$, the induction hypothesis implies that $\hat{f}$ is a maximum flow in $(V, E, \hat{q})$.

We now turn to part 2 of the Theorem. Let $m = |O|$ and $n = |I|$. Then there are at most m edges out of i^A for each $i \in I$, so the Ford-Fulkerson algorithm computes MaxFlow in $O(mn^2)$ time. We invoke MaxFlow while minimizing the flow along (i, i^U), so it suffices to call it $O(log(k))$ times for each agent, where $k = \max_{i \in I} |\Omega_i|$. Thus, the complexity of IRP is $O(mn^3 log(k))$. □

E Proof of Proposition 1

We first show incompatibility for the case of three agents, two of whom are endowed with a single object while the third of whom is endowed with two objects. So, let $I \equiv \{1, 2, 3\}$ and $O \equiv \{\omega_1, \omega_2, \omega_{31}, \omega_{32}\}$. Let $\Omega_1 \equiv \{\omega_1\}, \Omega_2 \equiv \{\omega_2\}$, and $\Omega_3 \equiv \{\omega_{31}, \omega_{32}\}$.

Suppose that $\succsim$ is a minimally non-trichotomous preference profile such that $A_1 = A_2 = \{\omega_{31}, \omega_{32}\}, a_1 = a_2 = \omega_{31}, A_3 = \{\omega_1, \omega_2\}$, and $a_3 = \omega_2$.

A matching $\mu \in \mathcal{M}$ is individually rational if and only if $\omega_2 \notin \mu(1)$ and $\omega_1 \notin \mu(2)$. Pareto-efficiency requires that μ can be individually rational and Pareto-efficient only if $\mu(3) = \{\omega_1, \omega_2\}$. Hence, there are two Pareto-efficient and individually rational matchings: μ^1 such that $\mu^1(1) = \{\omega_{31}\}, \mu^1(2) = \{\omega_{32}\}$, and $\mu^1(3) = \{\omega_1, \omega_2\}$ and μ^2 such that $\mu^2(1) = \{\omega_{32}\}, \mu^2(2) = \{\omega_{31}\}$, and $\mu^2(3) = \{\omega_1, \omega_2\}$.

Suppose now that φ is strategy-proof, Pareto-efficient, and individually rational.

Case 1: $\boldsymbol{\varphi(\succsim) = \mu^1}$. Consider the trichotomous preference relation $\succsim_2'$ for agent 2 such that $A_2' = \{\omega_{31}\}$. At $(\succsim_1, \succsim_2', \succsim_3)$, a matching $\mu \in \mathcal{M}$ is individually rational if and only if $\omega_2 \notin \mu(1)$, $\omega_1 \notin \mu(2)$, and $\omega_{32} \notin \mu(2)$. There are two Pareto-efficient and individually rational matchings at $(\succsim_1, \succsim_2', \succsim_3)$: ν' such that $\nu'(1) = \{\omega_{31}\}, \nu'(2) = \{\omega_2\}$, and $\nu'(3) = \{\omega_1, \omega_{32}\}$ and μ^2 defined above.

If $\varphi(\succsim_1, \succsim_2', \succsim_3) = \mu^2$, then agent 2 has an incentive to report $\succsim_2'$ when the true preference profile is $\succsim$. Hence, strategy-proofness implies that $\varphi(\succsim_1, \succsim_2', \succsim_3) = \nu'$.

Now, consider the trichotomous preference relation $\succsim_3'$ for agent 3 such that $A_3 = \{\omega_2\}$. At $(\succsim_1, \succsim_2', \succsim_3')$ a matching $\mu \in \mathcal{M}$ is individually rational if and only if $\omega_2 \notin \mu(1), \omega_1 \notin \mu(2), \omega_{32} \notin \mu(2)$, and $\omega_1 \notin \mu(3)$. Thus, $\mu(1) = \{\omega_1\}$. So the unique individually rational and Pareto-efficient matching is λ' such that $\lambda'(1) = \{\omega_1\}, \lambda'(2) = \{\omega_{31}\}$, and $\lambda'(3) = \{\omega_2, \omega_{32}\}$. Therefore, $\varphi(\succsim_1, \succsim_2', \succsim_3') = \lambda'$. However, this contradicts the strategy-proofness of φ since agent 3 has an incentive to report $\succsim_3'$ when the true preference profile is $(\succsim_1, \succsim_2', \succsim_3)$.

Case 2: $\boldsymbol{\varphi(\succsim) = \mu^2}$. Consider the trichotomous preference relation $\hat{\succsim}_1$ for agent 1 such that $\hat{A}_1 = \{\omega_{31}\}$. At $(\hat{\succsim}_1, \succsim_2, \succsim_3)$, a matching $\mu \in \mathcal{M}$ is individually rational if and only if $\omega_2 \notin \mu(1), \omega_{32} \notin \mu(1)$, and $\omega_1 \notin \mu(2)$. Thus, there are two individually rational and Pareto-efficient matchings at $(\hat{\succsim}_1, \succsim_2, \succsim_3)$: $\hat{\nu}$ such that $\hat{\nu}(1) = \{\omega_1\}, \hat{\nu}(2) = \{\omega_{31}\}$, and $\hat{\nu}(3) = \{\omega_2, \omega_{32}\}$ and μ^1 defined above.

If $\varphi(\hat{\succsim}_1, \succsim_2, \succsim_3) = \mu^1$, then agent 1 has an incentive to report $\hat{\succsim}_1$ when the true preference profile is $\succsim$. Hence, strategy-proofness implies $\varphi(\hat{\succsim}_1, \succsim_2, \succsim_3) = \hat{\nu}$.

Consider the minimally non-trichotomous preference relation $\hat{\succsim}_3$ for agent 3 such that $A_3 = \{\omega_1, \omega_2\}$ and $a_3 = \{\omega_1\}$. The individually rational and Pareto-efficient matchings at $(\hat{\succsim}_1, \succsim_2, \hat{\succsim}_3)$ are, again, μ^1 and $\hat{\nu}$. By strategy-proofness $\varphi(\hat{\succsim}_1, \succsim_2, \hat{\succsim}_3) = \hat{\nu}$. Otherwise, $\varphi(\hat{\succsim}_1, \succsim_2, \hat{\succsim}_3) = \mu^1$ and agent 3 has an incentive to report $\hat{\succsim}_3$ when the true preference profile is $(\hat{\succsim}_1, \succsim_2, \succsim_3)$.

Finally, consider the trichotomous preference relation $\bar{\succsim}_3$ for agent 3 such that $\bar{A}_3 = \{\omega_1\}$. At $(\hat{\succsim}_1, \succsim_2, \bar{\succsim}_3)$, a matching $\mu \in \mathcal{M}$ is individually rational if and only if $\omega_2 \notin \mu(1), \omega_{32} \notin \mu(1), \omega_1 \notin \mu(2)$, and $\omega_2 \notin \mu(3)$. Thus, $\mu(2) = \{\omega_2\}$. So the unique individually rational and Pareto-efficient matching is $\hat{\lambda}$ such that $\hat{\lambda}(1) = \{\omega_{31}\}, \hat{\lambda}(2) = \{\omega_2\}$, and $\hat{\lambda}(3) = \{\omega_1, \omega_{32}\}$.

This contradicts the strategy-proofness of φ since agent 3 has an incentive to report $\bar{\succsim}_3$ when the true preference profile is $(\hat{\succsim}_1, \succsim_2, \hat{\succsim}_3)$.

Now we consider the case where the three agents are endowed with more objects. Suppose $\Omega_1 = \{\omega_{11}, \omega_{12}, \ldots, \omega_{1k_1}\}$, $\Omega_2 = \{\omega_{21}, \omega_{22}, \ldots, \omega_{2k_2}\}$, and $\Omega_2 = \{\omega_{31}, \omega_{32}, \ldots, \omega_{3k_3}\}$ such that $k_3 \geq 2$. We now require that for each $i \in I$, and each "extra" objects $o \in \{\omega_{12}, \ldots, \omega_{1k_1}\} \cup \{\omega_{22}, \ldots, \omega_{2k_2}\} \cup \{\omega_{33}, \ldots, \omega_{3k_3}\}$, i does not consider o to be desirable, i.e. $o \notin A_i$. Individual rationality requires that 1 retains the objects $\omega_{12}, \ldots, \omega_{1k_1}$, 2 retains the objects $\omega_{22}, \ldots, \omega_{2k_2}$, and 3 retains the objects $\omega_{33}, \ldots, \omega_{3k_3}$. The remainder of the proof is exactly as above.

Finally, we consider the case where there are more than three agents. Let $1, 2$, and 3 be three of the agents in I such that $|\Omega_3| \geq 2$. For each $i \in I \setminus \{1, 2, 3\}$, we consider only preferences such that $A_i = \Omega_i$ and for each $o \in \Omega_i$, there is no $j \in I$ such that $o \in A_j$. That is, agents other than 1, 2, and 3 are degenerate in the sense that individual rationality requires that they consume their endowments. Thus, the proof proceeds as above. $\square$