High-frequency instruments and identification-robust inference for stochastic volatility models *

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Abstract

We introduce a novel class of generalized stochastic volatility (GSV) models that utilize high-frequency (HF) information (realized volatility (RV) measures). GSV models can accommodate nonstationary volatility process, various distributional assumptions, and exogenous regressors in the latent volatility equation. Instrumental variable methods are employed to provide a unified framework for the analysis (estimation and inference) of GSV models. We consider the parameter inference problem in GSV models with nonstationary volatility and develop identification-robust methods for joint hypotheses involving the volatility persistence parameter and the autocorrelation parameter of the composite error (or the noise ratio). For inference about the volatility persistence parameter, projection techniques can be applied. The proposed tests include the Anderson-Rubin (AR) type test, a dynamic version of the split-sample (SS) type test, and point optimal versions of these tests [AR^* and SS^*]. For distributional theory, three different sets of assumptions are considered: (1) for Gaussian errors, we provide exact tests and confidence sets; (2) for a wide class of parametric non-Gaussian errors (possibly heavy-tailed), we establish that exact Monte Carlo procedures can be applied using the statistics considered; (3) under quite general distributional assumptions, we show these tests are asymptotically valid. In simulations, the proposed tests outperform the usual asymptotic test regarding size and exhibit excellent power. We apply our inference methods to IBM's price and option data (2009-2013). We consider 175 different instruments (IV's) spanning 22 classes and analyze their ability to describe the low-frequency volatility. The IV's are compared based on the average length of confidence intervals, which produced by the proposed tests. The superior instrument set mostly consists of 5-minute HF realized measures, and these IV's produce confidence sets where the volatility persistence parameter lies roughly between 0.85 and 1.0. This outcome suggests that the volatility process is highly persistent and close to unit-root. We find RVs with higher frequency produce wider confidence intervals compared to RVs with slightly lower frequency, showing that these confidence intervals adjust to absorb the market microstructure noise or discretization error. Further, when we consider irrelevant or weak IV's [*e.g.*, jumps and signed jumps], the proposed tests produce unbounded confidence intervals. Although jumps have no or very little information content regarding the low-frequency volatility, we find evidence that there may be a nonlinear relationship between jumps and the low-frequency volatility.

Key words: Stochastic volatility, Realized variance, High frequency data, Identification robust test.

Journal of Economic Literature classification: C15, C22, C53, C58, C32.

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1. Introduction

In stochastic volatility (SV) models [proposed by Taylor (1982, 1986)], the return variation dynamics is modelled as a latent autocorrelated stochastic process. Estimation and inference are challenging in SV models due to the inherent problem of evaluating the likelihood function.[1] As a result, a variety of alternative methods have been proposed to estimate SV models.[2] For a review of the SV literature; see Ghysels, Harvey and Renault (1996), Broto and Ruiz (2004), Shephard (2005), Ahsan and Dufour (2018).

In this paper, we propose generalized stochastic volatility models (GSV), where volatility is modelled as a latent stochastic process. Instrumental-variable methods can be used to estimate GSV models, where a standard solution is to replace the unobserved volatility by a proxy. Hence, we need valid instruments (IV's) for the latent volatility. The choice of instruments plays a crucial role. As a result, we consider broad classes of IV's for the latent log volatility, *e.g.*, we use high-frequency (HF) realized measures as instruments. To the best of our knowledge, this paper is the first to propose instrumental-variable regression in the context of SV models.

GSV models can accommodate nonstationary volatility process, various distributional assumptions, and exogenous regressors in the latent volatility equation. This study allows nonstationary volatility process in GSV models and consider the hypothesis testing problem for the volatility persistence parameter, which captures the volatility clustering. Indeed, we let the autoregressive root of the latent volatility process to be close or equal to one.

The hypothesis testing problem in SV models has received much less attention, available results on hypothesis testing for SV parameters include: Harvey et al. (1994), Andersen and Sørensen (1996), Andersen et al. (1999), Gallant, Hsieh and Tauchen (1997), Durham (2006, 2007), Dufour and Valéry (2009), and Ahsan and Dufour (2019). These inference methods are based on the strong stationarity assumption, which requires time invariance of unconditional variances and autocovariances. Nonstationarity in the volatility process has been well documented for macroeconomic and financial time series data; see Pagan and Schwert (1990*a*, 1990*b*), Loretan and Phillips (1994), McConnell and Perez-Quiros (2000), Blanchard and Simon (2001), Busetti and Taylor (2003), Sensier and Dijk (2004), Cavaliere and Taylor (2007). Nonstationary volatility arises, for instance, when the variance is trending (upward or downward) or undergoes structural breaks. Several studies note that the empirical estimate of the dominant root of the SV-type process is close to unit circle; see Harvey et al. (1994), Hansen (1995), Broto and Ruiz (2004). Hansen (1995) is the only study that proposed robust regressions in the mean equation under nonstationary stochastic volatility, whereas Melino and Turnbull (1990), Harvey et al. (1994), and Ruiz (1994) estimated SV models imposing a unit root in the variance equation. However, the formal hypothesis testing problem (concerning size and power) in the latent nonstationary stochastic volatility equation is not studied in the literature.

Further, most of the proposed inference procedures for SV models are based on large-sample approximations (asymptotic standard errors).[3] When a time series is nearly nonstationary, the asymptotic stan-

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[1] The marginal likelihood of SV models is given by a high dimensional integral, which makes the estimation by conventional maximum likelihood (ML) infeasible. This is a general feature of most nonlinear latent variable models because the latent variables must be integrated out of the joint density for the observed and latent processes, leading to an integral of high dimensionality.

[2] Major references include: the Quasi-Maximum Likelihood (QML) [Harvey, Ruiz and Shephard (1994); Ruiz (1994)], the Generalized Method of Moments (GMM) [Melino and Turnbull (1990); Andersen and Sørensen (1996)], the Efficient Method of Moments (EMM) [Gallant and Tauchen (1996); Andersen, Chung and Sørensen (1999)], the Maximum Likelihood Monte Carlo (MLMC) [Sandmann and Koopman (1998)], the Simulated Maximum Likelihood (SML) [Danielsson and Richard (1993); Danielsson (1994); Durham (2006); Liesenfeld and Jung (2000); Richard and Zhang (2007)], method base on linear-representation (LiR) [Francq and Zakoïan (2006)], the closed-form moment-based estimator (DV) [Dufour and Valéry (2006)], the ARMA-based winsorized estimator (W-ARMA-SV) [Ahsan and Dufour (2019)] and Bayesian methods based on Markov Chain Monte Carlo (MCMC) methods [Jacquier, Polson and Rossi (1994), Kim, Shephard and Chib (1998), Chib, Nardari and Shephard (2002), Flury and Shephard (2011)].

[3] Exceptions are Dufour and Valéry (2009) and Ahsan and Dufour (2019), who develop both exact and asymptotic tests of SV parameters.

dard error can be markedly different, and asymptotic approximations are very unreliable in finite samples. In the context of a standard SV model, simulation results (in this paper, see Table 2) show that tests based on asymptotic standard errors fail to control the type I errors [over-reject up to 100%] when the volatility persistence parameter approaches to the unit circle.

To be more specific, the other contributions of the paper can be summarized as follows.

First, we consider a variety of IV's for the latent log volatility, including realized volatility (RV) measures at a different sampling frequency (*e.g.*, 1-second or 5-minute sampling), sampling scheme (calendar time or tick time), and functional form (*e.g.*, jumps or kernel). We also consider subsampled versions of some of these HF IV's, realized semivariance, and realized range RV, nearest neighbor truncated RV, and HF principal component factors. Realized volatility measures (non-parametric volatility estimates from HF data) have received much attention among practitioners as an accurate measure of the true latent volatility under ideal market assumptions [see Andersen and Bollerslev (1998), Barndorff-Nielsen and Shephard (2001)]. Hence, we use RV measures as IV's for the daily latent volatility, in contrast with recent studies, where RV has been incorporated in traditional volatility models (GARCH or SV) by adding a measurement equation that connects the daily volatility measure and the realized volatility. It is worthwhile to note that several studies in SV literature such as those by Takahashi, Omori and Watanabe (2009) and Koopman and Scharth (2012) model realized volatility and daily returns simultaneously, assuming that the realized volatility includes the market microstructure noise but still contains a great deal of information regarding the latent volatility. On the other hand, daily returns contain less noise but may not have sufficient information about the latent volatility. In GARCH-type setting, examples of such models are the Multiplicative Error Model (MEM) model [Engle and Gallo (2006), the HEAVY (High-frEquency-bAsed VolatilitY) model [Shephard and Sheppard (2010), Noureldin, Shephard and Sheppard (2012)] and the Realized GARCH model [Hansen, Huang and Shek (2012)].

Second, we propose inference methods which are robust to *weak instruments* since potential HF IV's may be weak due to the *discretization error* or *market microstructure noise*.[4] The discretization error is present in the estimates of the volatility since we only observe prices at intermittent and discrete points in time and the market microstructure noise is due to bid/ask bounces, the different price impact of different types of trades, limited liquidity, or other types of market frictions. These noises may lead to a divergence between the observed price process and the true or latent "frictionless equilibrium" price process. The literature on constructing consistent volatility proxy using HF data is considerable, these include maximum likelihood estimator [Aït-Sahalia, Mykland and Zhang (2005)], quasi-maximum likelihood estimator [Xiu (2010)], Two Scales Realized Volatility (TSRV) [Zhang, Mykland and Aït-Sahalia (2005)], Multi-Scale Realized Volatility (MSRV) [Zhang (2006)], Realized Kernels (RK) [Hansen and Lunde (2006), Barndorff-Nielsen, Hansen, Lunde and Shephard (2008, 2011)], and Pre-Averaging volatility estimator (PAV) [Jacod, Li, Mykland, Podolskij and Vetter (2009)]. Other relevant references include Bandi and Russell (2006), Fan and Wang (2007), Gatheral and Oomen (2010), Kalnina and Linton (2008), Li and Mykland (2007), and Aït-Sahalia, Mykland and Zhang (2011). Thus incorporating noisy RV estimates may lead to weak identification. As a result, standard inference procedures may produce invalid confidence tests and sets.

As pointed out by Dufour (1997) that statistical inference should be based on proper pivots, especially when a model involves locally almost unidentified parameters, *i.e.,* in the presence of weak IV's. The proposed inference methods include Anderson-Rubin (AR)-type tests, dynamic version of the split-sample (SS)-type tests [Dufour and Jasiak (2001)], and point-optimal versions of these tests [AR^* and SS^*]. The AR test is considered robust to weak IV's because the test has the correct size in cases where IV's are

[4] In instrumental-variable regressions, when IV's are not valid (the identification conditions are not satisfied), the standard asymptotic theory for estimators and test statistics typically collapses. Further, when IV's are weak, the limiting distributions of standard test statistics - like Student, Wald, likelihood ratio and Lagrange multiplier criteria - have non-standard distributions and often depend heavily on nuisance parameters; see Phillips (1989), Bekker (1994), Dufour (1997), Staiger and Stock (1997), and Wang and Zivot (1998). In particular, standard Wald-type procedures based on the use of asymptotic standard errors are very unreliable in the presence of weak identification.

weak, and/or strong. The *SS* test is an alternative to the *AR* test, where one can estimate the optimal IV's as well as any nuisance parameter. Further, appropriately splitting the sample into two parts, one for estimation of optimal IV's and nuisance parameters and the other for testing, also ensures exogeneity of the constructed IV's and the validity of the tests. Point-optimal versions of these tests gain power by exploiting the differences in the error covariance matrices under the null and the alternative, for a general review, the reader may consult King (1980), Dufour and King (1991), and Andrews, Moreira and Stock (2006).

Third, we consider a joint testing problem where we make an inference jointly on both the volatility persistence parameter and the autocorrelation parameter of the composite error (or the noise ratio). Hence, for inference on general (possibly nonlinear) transformations of model parameters [single parameter or a subvector], projection techniques can be applied [see Dufour (1989), Dufour (1990), Dufour and Jasiak (2001), Dufour and Taamouti (2005, 2007)].

Fourth, the proposed inference procedures are also robust to *dynamics*, *i.e.*, nonstationarity. Under the null hypothesis (even with nonstationary stochastic volatility) and appropriate assumptions on IV's, these tests can become pivotal functions with the possibility of exact inference.

Fifth, we employ three different sets of assumptions for the error distribution:

1. Assuming Gaussian errors, we provide confidence sets and tests based on standard Fisher critical values for the *AR* and *SS* test statistics. For point-optimal versions, we propose to use the Monte Carlo test (MCT) method [see Dwass (1957), Barnard (1963) and Dufour (2006)].

2. We assume that the conditional distribution of scale transformed error is completely specified up to an unknown scale factor, under which the Monte Carlo tests (MCT) method can apply for exact statistical inference. This assumption enables us to deal with non-standard error distributions. For example, even when errors have a heavy-tailed distribution, such as Cauchy distribution or more generally the family of stable distributions, which may not have moments and thus makes statistical inference complicated, our procedures provide exact solutions.

3. We show that the asymptotic validity of these procedures under quite general distributional assumptions.

Sixth, we study the statistical properties of the proposed inference procedures by simulation experiments. We find that the usual asymptotic t-tests fail to control the level, whereas the proposed tests control the level and show excellent power. These findings hold for several simulation setups: (1) the DGP is correctly specified; (2) the DGP is incorrectly specified because the error distribution is misspecified; (3) the DGP is incorrectly specified due to the violation of exogeneity assumption [IV's are not exogenous] and misspecification of error distributions; (4) the DGP is incorrectly specified due to the violation of exogeneity assumption and misspecification of error distributions as well as the instrument set includes HF IV's.

Finally, we apply the proposed procedures to IBM's price and option data (2009-2013). We consider 175 different instruments spanning 22 different classes and look at their ability to describe the low-frequency volatility. The average length of confidence intervals produced by the proposed tests is used to examine the strength of the IV's. The superior instrument set constitutes of 1-, 5- and 10-minute high-frequency realized measures and call option implied volatilities. These IV's produce confidence sets where the persistence parameter lies roughly between 0.85 and 1.0. This result shows that the latent volatility process of IBM is highly persistent and close to unit-root.

Further, we find RVs with higher frequency produce wider confidence intervals compared to RVs with slightly lower frequency, pointing out that these confidence intervals adjust to incorporate the microstructure noise or discretization error. We also find jumps and signed jumps have no or little information content regarding the low-frequency volatility, whereas their log squared versions have a strong identification strength. When we consider irrelevant or weak instruments, such as jumps and signed jumps, the proposed procedures produce unbounded confidence intervals. These confidence intervals cover the entire set of $\phi \in [0,\ 1]$. This means that under no identification, all values of ϕ are observationally equivalent, which implies that the proposed test statistics yield unbounded (valid) confidence sets with a non-zero probability.

This paper proceeds as follows: Section 2 specifies models and assumptions. Section 3 proposes finite-sample identification-robust inference procedures, whereas Section 4 extends finite-sample procedures with non-standard error distributions. Section 5 develops the asymptotic validity of the proposed tests. Section 6 presents the simulation study, and Section 7 presents the empirical applications. Section 8 offers conclusions. Proofs, Figures, and Tables are reported in the Appendix.

2. Framework

This paper presents extensions of the standard log-normal autoregressive SV model, which is described below following Taylor (1986), Shephard (1996), and Ghysels et al. (1996). $\mathbb{N}_0$ refers to the non-negative integers.

Assumption 2.1 STOCHASTIC VOLATILITY MODEL. *The process* $\{s_t : t \in \mathbb{N}_0\}$ *follows an SV model of the type:*

$$s_t = \sigma_t z_t, \tag{2.1}$$

$$\log \sigma_t^2 = \mu + \phi \log \sigma_{t-1}^2 + e_t, \tag{2.2}$$

where s_t *is the return observed at time* t *and* σ_t *is the corresponding volatility. The* z_t*'s and* e_t*'s, are i.i.d.* $\mathrm{N}(0,1)$ *and* $\mathrm{N}(0,\sigma_e^2)$ *random variables, respectively and* ϕ, μ, σ_e *are the fixed parameters of the model.*

Assumption 2.2 STATIONARITY. *The process* $l_t = (s_t, \log \sigma_t^2)'$ *is strictly stationary.*

The above stationarity condition entails $|\phi| < 1$ and $\log \sigma_0^2 \sim N[0, \sigma_e^2/(1-\phi^2)]$. The SV model consists of two stochastic processes, where s_t [$s_t := r_t - \mu_r$, where $r_t := 100[\log(p_t) - \log(p_{t-1})]$, μ_r is the mean of return (r_t), and p_t is the raw price of an asset] describes the dynamics of returns and $\log \sigma_t^2$ captures the dynamics of latent log volatilities. The latent process $\log \sigma_t^2$ in (2.2) can be interpreted as the random and uneven flow of new information in financial markets, while ϕ is the persistence in the volatility.

Now transforming s_t by taking logarithms of the squares, we can write the measurement equation of the model as following:

$$\begin{aligned} \log s_t^2 &= \log \sigma_t^2 + \log z_t^2 = \mathbb{E}[\log z_t^2] + \log \sigma_t^2 + \{\log z_t^2 - \mathbb{E}[\log z_t^2]\} \\ &= \mathbb{E}[\log z_t^2] + \log \sigma_t^2 + v_t \end{aligned} \tag{2.3}$$

where

$$v_t := \log z_t^2 - \mathbb{E}[\log z_t^2]. \tag{2.4}$$

This transformation entails no information loss since the distribution of z_t is symmetric [see Remark 1 of Francq and Zakoïan (2006)]. Under the standard normality assumption for z_t, the transformed error, v_t are i.i.d. according to the distribution of a centered $\log(\chi^2_{(1)})$ random variable with $\mathbb{E}[\log z_t^2] \simeq -1.2704$, $\sigma_v^2 := \mathbb{E}(v_t^2) = Var(\log z_t^2) = \pi^2/2$ and $\mathbb{E}[v_t^4] = \pi^4 + 3\sigma_v^2$ [see Abramowitz and Stegun (1970)]. Several studies approximated the distribution of v_t by a normal distribution characterized by a mean of -1.2704, and a variance of $\pi^2/2$ [see Broto and Ruiz (2004)]. Notice that the model expressed by (2.3) can be written as

$$y_t = w_t + v_t, \tag{2.5}$$

where

$$y_t := \log s_t^2 - \mathbb{E}[\log z_t^2], \quad w_t := \log \sigma_t^2. \tag{2.6}$$

Combining (2.2) and (2.5), we have the linear state space representation of SV model. Given initial condition of the variables, the SV model [in assumption 2.1] can be written as following

$$\begin{aligned} \text{State Transition Equation:} \quad w_t &= \mu + \phi w_{t-1} + e_t & (2.7)\\ \text{Measurement Equation:} \quad y_t &= w_t + v_t & (2.8) \end{aligned}$$

where w_t is a logarithm of latent daily volatility, y_t is a logarithm of daily squared returns, the matrix X_t is a set of exogenous variables that may predict the latent volatility as well as capture the leverage effect [X_t also includes the constant term in the model], while e and v are the disturbances.

It is evident from (2.7) and (2.8) that using any proxy for latent volatility (*e.g.*, replacing w_t by y_t) will induce a measurement error problem. Further, the latency of the volatility process introduces a moving average of measurement errors. We could alleviate this type of problem by using an instrumental-variable regression where we replace the unobserved variables by their proxies.

In the following assumption, we introduce generalized stochastic volatility models, where we incorporate valid IV's $\bar{Z}_{t-2}$ which are related to w_{t-1} and uncorrelated to v_{t-1}.

Assumption 2.3 GENERALIZED STOCHASTIC VOLATILITY MODEL. *The process* $\{y_t : t \in \mathbb{N}_0\}$ *follows a generalized SV model of the type:*

$$\begin{aligned} \textit{State Transition Equation:} \quad w &= \phi w_{-1} + X\beta + e & (2.9)\\ \textit{Measurement Equation:} \quad y &= w + v & (2.10)\\ \textit{Instrument Equation:} \quad w_{-1} &= \bar{Z}_{-2}\pi + u_{-1} & (2.11) \end{aligned}$$

where $w = (w_1, \ldots, w_T)'$, $w_{-1} = (w_0, \ldots, w_{T-1})'$ *with* w_0 *is given,* $y = (y_1, \ldots, y_T)'$ *are* $T \times 1$ *vector,* $X = [X_1', \vdots \cdots \vdots, X_T']'$ *is a* $T \times k$ *matrix of exogenous explanatory variables that may predicts the latent volatility as well as capture the leverage effect,* $\bar{Z}_{-2} = [\bar{Z}_{-1}', \vdots \cdots \vdots, \bar{Z}_{T-2}']'$ *is a* $T \times m$ *matrix of of variables related to* w_{-1}, *while* $e = (e_1, \ldots, e_T)'$, $v = (v_1, \ldots, v_T)'$, $u_{-1} = (u_0, \ldots, u_{T-1})'$ *are* $T \times 1$ *vector of disturbances. The matrices of unknown coefficients* ϕ, β , *and* π *have dimensions respectively* 1×1, $k \times 1$, *and* $m \times 1$.

We do not impose any stationary restriction on the latent volatility process. The assumption that the latent autoregressive volatility process is first-order is not essential to the analysis. Indeed, higher-level dynamics could be allowed, but the first-order case illustrates the main points of the paper without needless generality.

In order to handle common variables [for example the constant term] in both equations (2.9) and (2.11), we allow for the presence of common columns in the matrices $\bar{Z}_{-2}$ and X. Suppose that $\bar{Z}_{-2}$ and X have k_2 columns in common ($0 \le k_2 < m$) while the other columns of X are linearly independent of $\bar{Z}_{-2}$ such that

$$\bar{Z}_{-2} = [Z_{-2}, X_2], \quad X = [X_1, X_2], \quad \text{rank}[Z_{-2}, X_1, X_2] = l + k < T \tag{2.12}$$

where Z_{-2}, X_1, and X_2 are $T \times l$, $T \times k_1$, and $T \times k_2$ matrices, respectively with $k = k_1 + k_2$ and $m = l + k_2$.

It is important to note that no restriction is imposed on the distribution of u and it may follow any distribution (heteroskedastic or autocorrelated) since no statistical property of u has effects on the validity of the tests proposed in this paper.

To derive finite distributional theory for test statistics (proposed in Section 3), we employ the following assumptions in model (2.9)-(2.11).

Assumption 2.4 INDEPENDENCE AND RANK. *The $T\times k$ matrix X and $T\times m$ matrix $\bar{Z}_{-2}$ are independent of $T\times 1$ vectors e and v. Further,* $\mathrm{rank}(X)=k$, $1\le \mathrm{rank}(\bar{Z}_{-2})=m<T$, $1\le \mathrm{rank}[Z_{-2},X_1,X_2]=l+k<T$ *and the v_t's and e_t's are i.i.d.* $\mathrm{N}(0,\sigma_v^2)$ *and* $\mathrm{N}(0,\sigma_e^2)$ *random variables, respectively.*

Assumption 2.5 ASSUMPTIONS FOR FINITE DISTRIBUTIONAL THEORY. *The v_t's and e_t's are i.i.d.* $\mathrm{N}(0,\sigma_v^2)$ *and* $\mathrm{N}(0,\sigma_e^2)$ *random variables, respectively.*

The v_t's and e_t's are i.i.d. $\mathrm{N}(0,\sigma_v^2)$ and $\mathrm{N}(0,\sigma_e^2)$ random variables, respectively and they are orthogonal to each other with $\sigma_v^2=\pi^2/2$. We change the distributional assumption of v_t by an i.i.d. $\mathrm{N}(0,\sigma_v^2)$ distribution in line with several previous studies; see Harvey et al. (1994), Ruiz (1994), Breidt and Carriquiry (1996), Harvey and Shephard (1996), Kim et al. (1998), Chib et al. (2002), Knight, Satchell and Yu (2002), Francq and Zakoïan (2006), Omori, Chib, Shephard and Nakajima (2007). We relax the above assumptions in Section 4 and 5.

The instrumental-variable regression requires valid IV's for the latent volatility, $w_t=\log\sigma_t^2$, or the observable volatility proxy y_t which is typically the low-frequency (LF) squared return. To find valid IV's, we first look at the properties of the unobserved volatility process, w_t. If w_t's are correlated with a sufficiently long lag and the v_t's are uncorrelated, then the proxies for $w_{t-2},w_{t-3},w_{t-4},\ldots$ $(y_{t-2},y_{t-3},y_{t-4},\ldots)$ are potential clean IV's for w_{t-1} (y_{t-1}). It is important to not introduce y's with too high lags as IV's, because this requires truncating the sample in order to observe IV's for each date used in the estimation, and the good statistical properties of the instrumental-variable method begins to break down as the number of IV's ceases to be small relative to the remaining sample size. We can use realized volatility as IV's ($\bar{Z}_{t-2}$ contains past realized volatilities) since HF price data contain valuable information regarding the latent volatility. In the Section 7 below, we consider not only daily and HF IV's but also consider option implied volatility as IV's.

3. Finite-sample procedures

In this section, we consider the problem of testing the volatility persistence parameter in a GSV model, *i.e.*, testing restriction about volatility clustering. We propose four finite-sample procedures, which are valid under assumptions 2.4 - 2.5. Let us now consider the null hypothesis:

$$H_0:\phi=\phi_0 \tag{3.1}$$

The instrument substitution method is based on replacing unobserved variables with a set of IV's. First, we substitute (2.10) into (2.9):

$$y=\phi y_{-1}+X\beta+e+v-\phi v_{-1} \tag{3.2}$$

Then subtracting $\phi_0 y_{-1}$ on both sides of (3.2), we get:

$$y-\phi_0 y_{-1}=(\phi-\phi_0)y_{-1}+X\beta+e+v-\phi v_{-1} \tag{3.3}$$

Since $\mathbb{E}[y_{t-1}v_{t-1}]\neq 0$, we need to find IV's for w_{-1} to solve this endogeneity problem. Suppose we have IV's $\bar{Z}_{-2}$ for w_{-1}, thus equation (2.10) and (2.11) yields

$$y_{-1}=\bar{Z}_{-2}\pi+\eta_{-1}, \tag{3.4}$$

where $\eta_{-1}:=v_{-1}+u_{-1}$ and from assumption 2.5, $\bar{Z}_{-2}$ is independent of v_{-1}. Now again substituting the $y_{-1}=\bar{Z}_{-2}\pi+\eta_{-1}$, to get:

$$\begin{aligned} y-\phi_0 y_{-1} &= \bar{Z}_{-2}\pi(\phi-\phi_0)+X\beta+(\phi-\phi_0)\eta_{-1}+e+v-\phi v_{-1}\\ &= \bar{Z}_{-2}\pi(\phi-\phi_0)+X\beta+(\phi-\phi_0)[v_{-1}+u_{-1}]+e+v-\phi v_{-1}\end{aligned}$$

$$= \bar{Z}_{-2}\pi(\phi-\phi_0) + X\beta + (\phi-\phi_0)u_{-1} + e + v - \phi_0 v_{-1}$$

or equivalently,

$$y - \phi_0 y_{-1} = \bar{Z}_{-2}\pi(\phi-\phi_0) + X\beta + \xi\,, \tag{3.5}$$

where

$$\xi := (\phi-\phi_0)u_{-1} + e + v - \phi_0 v_{-1}\,. \tag{3.6}$$

Using (2.12), we can write (3.5) as

$$y - \phi_0 y_{-1} = Z_{-2}\delta + X\beta_* + \xi\,, \tag{3.7}$$

where

$$\delta := \pi_1(\phi-\phi_0)\,, \quad \beta_* := (\beta_1', \beta_{2*}')'\,, \quad \beta_{2*} := \beta_2 + \pi_2(\phi-\phi_0)\,, \quad \pi := (\pi_1', \pi_2')'\,, \tag{3.8}$$

where π_i is a $k_i \times 1$ vector.

3.1. Anderson-Rubin-type procedure (AR)

Since $v_t - \phi_0 v_{t-1}$ is an MA(1) process, thus ξ's are serially correlated. However, when $\phi = \phi_0 = 0$, under the H_0, ξ is distributed $\mathrm{N}(0, \sigma_\xi^2 I_T)$ where $\sigma_\xi^2 = \sigma_e^2 + \sigma_v^2$. As a result, the model in equation (3.7) satisfies all the assumptions of the classical linear model with probability 1. Furthermore, since $\delta = 0$ when $\phi = \phi_0$, we can test H_0 by a standard F-test of the following null hypothesis.

$$H_0^* : \delta = 0 \tag{3.9}$$

This F-statistic has the form

$$AR(\phi_0) = \frac{(y-\phi_0 y_{-1})'(M[X] - M[X, Z_{-2}])(y - \phi_0 y_{-1})/l}{(y-\phi_0 y_{-1})' M[X, Z_{-2}](y-\phi_0 y_{-1})/(T-l-k)}, \tag{3.10}$$

where $M(A) = I - A(A'A)^{-1}A'$.

The above F-statistic is commonly referred to as the Anderson-Rubin (AR) statistic which yields the confidence set $C_\phi(\alpha) = \{\phi_0 : AR(\phi_0) \le F_\alpha(l, T-l-k)\}$ for ϕ when the disturbances are i.i.d. normal. When the normality assumption holds, *i.e.*, $\xi \sim \mathrm{N}(0, \sigma_\xi^2 I_T)$ and X and Z_{-2} is exogenous, we have $AR(\phi_0) \sim F(l, T-l-k)$, and the $H_0(\phi_0)$ can be assessed by using a critical region of the form $\{AR(\phi_0) > f(\alpha)\}$, where $f(\alpha) = F_\alpha(l, T-l-k)$ is the $(1-\alpha)$-quantile of the $F(l, T-l-k)$ distribution. A confidence set with level $1-\alpha$ for ϕ is then given by

$$C_\phi(\alpha) = \{\phi_0 : AR(\phi_0) \le F_\alpha(l, T-l-k)\} = \{\phi : Q(\phi) \le 0\} \tag{3.11}$$

where $Q(\phi) = \phi' A\phi + b'\phi + c$, $A = y_{-1}' H y_{-1}$, $b = -2y_{-1}' H y$, $c = y' H y$, $H = M[X] - [1 + f(\alpha)(l/T-l-k)]M[X, Z_{-2}]$, and $f(\alpha) = F_\alpha(l, T-l-k)$; see Dufour and Taamouti (2005).[5]

Unfortunately, this property does not extend to a more general $AR(\phi_0)$ statistic where $\phi_0 \ne 0$ because in this case under the H_0, the covariance matrix of ξ is not spherical. When $\phi_0 \ne 0$, it is easy to see that the model (3.7) under H_0 does not satisfy all the assumptions of the classical linear model. In this case, under the null hypothesis, $\xi = e + v - \phi_0 v_{-1}$ is an MA(1) process which makes the standard t-tests and F-tests are invalid because the standard errors are wrong. We could correct the standard errors by a Generalized Least Squares (GLS) type transformation. The model defined by equation (3.7) can be transformed under

[5]When the disturbances are i.i.d with finite fourth-order moments, the AR-statistic converges under H_0 to a χ^2 distributed random variable when the sample size gets large. This large sample distribution of the AR-statistic does not depend on the value of π which makes it a more reliable statistic for practical purposes than the Wald statistic.

the H_0 to a model such that the AR-type tests will be valid, and the distribution of the test statistic will follow the F-distribution. Now under the H_0 model,

$$\xi = e + v - \phi_0 v_{-1}$$

is an MA(1) process. Under the assumption 2.5, $\xi_t \sim \mathrm{N}(0, \sigma_\xi^2 \Sigma(\rho))$ where

$$\Sigma(\rho) := \begin{pmatrix} 1 & -\rho & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -\rho & 1 & -\rho & 0 & & & & \vdots \\ 0 & -\rho & 1 & -\rho & \ddots & & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & -\rho & 1 & -\rho & 0 \\ \vdots & & & & 0 & -\rho & 1 & -\rho \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\rho & 1 \end{pmatrix}, \tag{3.12}$$

$$\sigma_\xi^2 := (1 + \phi_0^2)\sigma_v^2 + \sigma_e^2, \tag{3.13}$$

$$\rho := \frac{-Cov(\xi_t \xi_{t-1})}{Var(\xi_t)} = \frac{\phi_0 \sigma_v^2}{(1 + \phi_0^2)\sigma_v^2 + \sigma_e^2}. \tag{3.14}$$

Clearly, ρ is a function of ϕ_0, σ_e^2, and σ_v^2. $\Sigma(\rho)$ is a *Toeplitz* $([1, -\rho, 0, \ldots, 0])$ matrix with dimension $T \times T$. Because $\Sigma(\rho)$ is a real symmetric positive-definite matrix, there exists a $T \times T$ matrix C, such that $C\Sigma(\rho)C' = I_T$. If the $\Sigma(\rho)$ matrix is known, then we can propose the following transformation. Multiply equation (3.7) by C to make the error covariance matrix spherical. However, the Σ matrix is not known in reality since ρ is not known, thus instead of $\Sigma(\rho)$, we can use an estimator of it denoting $\Sigma(\rho)$. On setting the noise ratio

$$\lambda := \sigma_v^2 / \sigma_e^2 \ \in [0, \infty), \tag{3.15}$$

we can write ρ as

$$\rho(\phi_0, \lambda) = \frac{\phi_0 \lambda}{(1 + \phi_0^2)\lambda + 1}. \tag{3.16}$$

We can do a joint test such that under the null ρ is known. In any economic model, the disturbances contain important information, and particularly in the context of serially correlated models, researchers may be interested in joint inference. Consider the following null hypothesis:

$$H_0 : \phi = \phi_0, \quad \lambda = \lambda_0. \tag{3.17}$$

Now under the null, we can write ρ as

$$\rho_0 := \rho(\phi_0, \lambda_0) = \frac{\phi_0 \lambda_0}{(1 + \phi_0^2)\lambda_0 + 1} \in [-1/2, 1/2], \tag{3.18}$$

and the joint null hypothesis becomes

$$H_0 : \phi = \phi_0, \quad \rho = \rho_0. \tag{3.19}$$

Note that, the joint null values (ϕ_0, ρ_0) must satisfy the following equation:

$$\lambda_0 = \frac{\rho_0}{\phi_0 - \rho_0(1+\phi_0)^2} \in [0,\ \infty). \tag{3.20}$$

See Table 1 for the corresponding values of λ_0.

Since ρ_0 is known under the joint null, we can consider the following transformed model:

$$C_0(y - \phi_0 y_{-1}) = C_0 Z_{-2}\delta + C_0 X\beta_* + C_0\xi\,, \tag{3.21}$$

where $C_0 = C(\rho_0)$ is a $T \times T$ matrix such that $C_0\Sigma(\rho_0)C_0' = I_T$. The variance-covariance matrix of $\xi^* := C_0(\xi)$ is now an i.i.d. $\mathrm{N}(0, \sigma_\xi^2 I_T)$ distribution. The F-statistic now has the form

$$AR(\phi_0, \rho_0) = \frac{y(\phi_0,\rho_0)'(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\,y(\phi_0,\rho_0)/l}{y(\phi_0,\rho_0)' M_{C_0}[X, Z_{-2}]\,y(\phi_0,\rho_0)/(T-l-k)}\,, \tag{3.22}$$

where $y(\phi_0,\rho_0) = C_0(y - \phi_0 y_{-1})$, $M_{C_0}[A] = I - A[A'\Sigma(\rho_0)^{-1}A]^{-1}A'\Sigma(\rho_0)^{-1}$. A central feature of most situations where instrumental-variable methods are required come from the fact that IV's may be used to solve an endogeneity or an errors-in-variables problem. It is very rare that one can or should use all the possible valid IV's. A drawback of the AR method is that it loses power when too many IV's are used. However, the AR procedure is *robust to missing IV's* (or *instrument exclusion*) [see Dufour and Taamouti (2007)]. Alternative methods of inference aimed at being robust to weak identification [Wang and Zivot (1998), Kleibergen (2002), Moreira (2003)] do not enjoy this type of robustness. In the case of feasible GLS-type transformation, when we use an estimate of ρ, the test statistic is no longer F-distributed, it converges under H_0 to a χ^2 distributed random variable in a large sample. The tests and confidence sets obtained by the instrument substitution method can be interpreted as likelihood ratio (LR) procedures (based on appropriately chosen reduced form alternatives), or equivalently as profile likelihood techniques [for further discussion of such techniques, see Bates and Watts (1988), Meeker and Escobar (1995) and Chen and Jennrich (1996).

3.2. Anderson-Rubin-type point optimal procedure (AR^*)

In this section, we propose a point optimal version of AR-type tests. Point optimal (PO) tests provide simple and effective methods for creating exact small sample tests with excellent power properties in a wide variety of problems in linear regression. The empirical evidence in the literature indicates that in general, PO tests often outperform other testing methods in terms of power. Besides, exact small-sample critical values for PO tests can be computed in most cases. Thus one does not have to rely on the asymptotic properties of the test statistic to make inferences. For a general review of PO tests, the reader may consult King (1980), King (1987) and Dufour and King (1991).

A PO test for $\rho = \rho_0$ against $\rho = \rho_1$, which follows from King (1980) and Dufour and King (1991) and given as

$$S(\rho_0, \rho_1) = \frac{\hat{\xi}'\Sigma(\rho_0)^{-1}\hat{\xi}}{\tilde{\xi}'\Sigma(\rho_1)^{-1}\tilde{\xi}}, \tag{3.23}$$

where $|\rho_0| \le 1/2$, $|\rho_1| \le 1/2$, and $\hat{\xi}$ and $\tilde{\xi}$ are the GLS residual vectors corresponding to covariance matrices $\Sigma(\rho_0)$ and $\Sigma(\rho_1)$, respectively. The test rejects the null for large values of $S(\rho_0, \rho_1)$. The test based on (3.23) is point optimal, and it gains power by exploiting the differences in the error covariance matrices under the null and the alternative.

However the choice of ρ_1 is important. To obtain a test of $\rho = \rho_0$ against $\rho > \rho_0$, we select a value of ρ_1, such that $\rho_0 < \rho_1 \le 1/2$ and apply the test based on $S(\rho_0, \rho_1)$. Similarly, testing $\rho = \rho_0$ against $\rho < \rho_0$, we select ρ_1, such that $-1/2 \leq \rho_1 < \rho_0$. For example, we may choose ρ_1 such that $\rho_1 = \rho_0 - \bar{\Delta}$ where $0 < \bar{\Delta} < 1$.

As pointed out by King (1987), the PO test can be viewed as a partition of the sample space into two regions, a rejection region and a non-rejection region. If the observed sample falls in the rejection region, the null is rejected. Otherwise, null is not rejected. An invariant test is one for which each pair of points in the sample space that can be related by a transformation (*i.e.*, one can be obtained as a transformation of the other) either both fall into the rejection region or both fall into the non-rejection region. Consider an AR-type statistic similar to (3.23) for $\rho = \rho_0$ against $\rho = \rho_1$:

$$\overline{AR}(\phi_0, \rho_0, \rho_1) = \frac{y(\phi_0, \rho_0)' M_{C_0}[X] y(\phi_0, \rho_0)}{y(\phi_0, \rho_1)' M_{C_1}[X, Z_{-2}] y(\phi_0, \rho_1)}, \tag{3.24}$$

where

$$y(\phi_0, \rho_0) = C_0(y - \phi_0 y_{-1}), \quad y(\phi_0, \rho_1) = C_1(y - \phi_0 y_{-1}),$$
$$M_{C_i}[A] = I - A[A'\Sigma(\rho_i)^{-1}A]^{-1}A'\Sigma(\rho_i)^{-1}, i = 0, 1.$$

Note that it is difficult to derive the analytical null distribution of (3.24) even under the Gaussian assumption, while the MCT method described in Section 4 can be implemented and confidence set for ϕ and ρ with level $(1-\alpha)$ is obtained by inverting the tests.

It is worth noting that $\overline{AR}(\phi_0, \rho_0, \rho_1)$ can become degenerate in the limit. Thus we consider a monotonic transformation of $\overline{AR}(\phi_0, \rho_0, \rho_1)$, which is given as:

$$AR^*(\phi_0, \rho_0, \rho_1) = T[\overline{AR}(\phi_0, \rho_0, \rho_1) - 1]. \tag{3.25}$$

For finite-sample inference, both $\overline{AR}(\phi_0, \rho_0, \rho_1)$ and $AR^*(\phi_0, \rho_0, \rho_1)$ lead to identical results since a monotonic transformation does not change the rank of the statistic in the MCT method. On the other hand, $AR^*(\phi_0, \rho_0, \rho_1)$ is more appropriate for proving the asymptotic validity.

3.3. Split-sample-type procedure (SS)

Finite-sample inferences similar to the previous section may alternatively be obtained by applying a split-sample technique. If λ (or ρ) can be estimated from data, the estimated λ tends to be closer to the true one than those that are arbitrarily selected, and thus the power of tests can be improved. However, if we re-use the data (which is used to estimate λ) then the constructed IV's suffer from endogeneity and the procedure fails to control the probability of type I error. Thus, we employ the split-sample technique, which splits the sample into two parts. The first subsample is used to construct IV's and to estimate ρ, and tests can be implemented with the second subsample. Note that we can estimate ρ as well as the optimal IV's using the first subsample. As a result, the number of IV's can be reduced to the number of endogenous variables.

It is a natural thing to replace $\bar{Z}_{-2}\pi$ by $\bar{Z}_{-2}\hat{\pi}$, where $\hat{\pi}$ is an estimator of π. One could use $\hat{\pi} = (\bar{Z}_{-2}'\bar{Z}_{-2})^{-1}\bar{Z}_{-2}'y_{-1}$, the least squares estimate of π based on (2.11). Then we have:

$$y - \phi_0 y_{-1} = \bar{Z}_{-2}\hat{\pi}(\phi - \phi_0) + X\beta + [\xi + \bar{Z}_{-2}(\pi - \hat{\pi})(\phi - \phi_0)] = \hat{y}_{-1}\delta_* + X\beta + \bar{\xi}, \tag{3.26}$$

where

$$\delta_* := (\phi - \phi_0), \quad \hat{y}_{-1} := \bar{Z}_{-2}\hat{\pi}, \quad \bar{\xi} := e + v - \phi_0 v_{-1} + [u_{-1} + \bar{Z}_{-2}(\pi - \hat{\pi})](\phi - \phi_0). \tag{3.27}$$

Again, the null hypothesis $(\phi - \phi_0)$ may be tested by testing $H_0^{**} : \delta_* = 0$ in model (3.26). Here the standard AR-statistic for H_0^{**} is obtained by replacing Z_{-2} by $\hat{y}_{-1}$ in (3.10):

$$AR(\phi_0; \hat{y}_{-1}) = \frac{(y - \phi_0 y_{-1})'(M[X] - M[X, \hat{y}_{-1}])(y - \phi_0 y_{-1})/l}{(y - \phi_0 y_{-1})'M[X, \hat{y}_{-1}](y - \phi_0 y_{-1})/(T - l - k)}. \tag{3.28}$$

This test statistic is valid only when $\phi = \phi_0 = 0$, since in this case, $\hat{y}_{-1}$ and $\bar{\xi}$ are independent and, conditional on $\hat{y}_{-1}$, model (3.26) satisfies all the assumptions of the classical linear model. Thus the null distribution of the statistic $AR(0; \hat{y}_{-1})$ for testing $\phi_0 = 0$ is $F(l, T-l-k)$. Unfortunately, this property does not extend to a more general statistic $AR(\phi_0; \hat{y}_{-1})$ where $\phi_0 \neq 0$ because $\hat{y}_{-1}$ and $\bar{\xi}$ are not independent in this case. In order to deal with more general hypotheses, we need to take care of two things:

1. Get an estimate $\tilde{\pi}$ of π such that $\hat{y}_{-1}$ $(= \bar{Z}_{-2}\tilde{\pi})$ and $\bar{\xi}$ are independent;

2. Since under the null $\bar{\xi}$ is an MA(1) process, we need an estimate of $\rho(\phi_0) = Cov(\bar{\xi}_t \bar{\xi}_{t-1})/Var(\bar{\xi}_t)$ for the GLS-type transformation.

In particular, the split-sample procedure is as follows. Split the sample into subsample (1) with sample size T_1: $y^{(1)}, X^{(1)}, \bar{Z}^{(1)}$ and subsample (2) with sample size T_2: $y^{(2)}, X^{(2)}, \bar{Z}^{(2)}$ where

$$y = \begin{pmatrix} y^{(1)} \\ y^{(2)} \end{pmatrix}, \quad X = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}, \quad Z = \begin{pmatrix} \bar{Z}^{(1)} \\ \bar{Z}^{(2)} \end{pmatrix}. \tag{3.29}$$

Use first subsample to estimate $\tilde{\pi}^{(1)}$ of π such that $\tilde{\pi}^{(1)} = \left(\bar{Z}_{-2}^{(1)\prime} \bar{Z}_{-2}^{(1)}\right)^{-1} \bar{Z}_{-2}^{(1)\prime} y_{-1}^{(1)}$ and use second subsample to construct the following model:

$$y^{(2)} - \phi_0 y_{-1}^{(2)} = \bar{Z}_{-2}^{(2)} \tilde{\pi}^{(1)} (\phi - \phi_0) + X^{(2)}\beta + \bar{\xi}^{(2)} = \hat{y}_{-1}^{(2)} \delta_* + X^{(2)}\beta + \bar{\xi}^{(2)} \tag{3.30}$$

where

$$\delta_* := (\phi - \phi_0), \quad \hat{y}_{-1}^{(2)} := \bar{Z}_{-2}^{(2)} \tilde{\pi}^{(1)}, \quad \bar{\xi}^{(2)} := e^{(2)} + v^{(2)} - \phi_0 v_{-1}^{(2)} + [u_{-1}^{(2)} + \bar{Z}_{-2}^{(2)}(\pi - \hat{\pi})](\phi - \phi_0).$$

In model (3.30), $\hat{y}_{-1}^{(2)}$ and $\bar{\xi}^{(2)}$ are independent. Now under the null hypothesis, when $\phi = \phi_0$, $\bar{\xi}^{(2)} = e^{(2)} + v^{(2)} - \phi_0 v_{-1}^{(2)}$ is an MA(1) process and the variance covariance matrix of $\bar{\xi}^{(2)}$ is $\sigma^2_{\bar{\xi}^{(2)}} \Sigma_{\bar{\xi}^{(2)}}$ where $\sigma^2_{\bar{\xi}^{(2)}} = (1 + \phi_0^2)\sigma^2_{v^{(2)}} + \sigma^2_{e^{(2)}}$ and $\Sigma_{\bar{\xi}^{(2)}}$ is a *Toeplitz* $([1, -\rho^{(2)}, 0, \dots, 0])$ matrix with $\rho^{(2)} = Cov(\bar{\xi}_t^{(2)} \bar{\xi}_{t-1}^{(2)})/Var(\bar{\xi}_t^{(2)})$. Since $\Sigma_{\bar{\xi}^{(2)}}$ is a $T_2 \times T_2$ real symmetric positive-definite matrix, there exists a $T_2 \times T_2$ matrix $C(\rho^{(2)})$, such that $C(\rho^{(2)}) \Sigma_{\bar{\xi}^{(2)}} C'(\rho^{(2)}) = I_{T_2}$. If the $\rho^{(2)}$ is known then we can multiply equation (3.30) by $C(\rho^{(2)})$ to make the error covariance matrix spherical. We use an estimates of $\rho^{(2)}$ from second subsample then the test statistic is no longer F-distributed, it is converges under H_0 to a χ^2 distributed random variable in large sample. In order to solve this problem, we use an estimate of $\rho^{(2)}$ from the first subsample that is $\hat{\rho}^{(1)}$. This transformation gives us the following test statistic:

$$SS(\phi_0; \hat{y}_{-1}^{(2)}, \hat{\rho}^{(1)}) = \frac{y^{(2)}(\phi_0, \hat{\rho}^{(1)})' (M_{C(\hat{\rho}^{(1)})}[X^{(2)}] - M_{C(\hat{\rho}^{(1)})}[X^{(2)}, \hat{y}_{-1}^{(2)}]) y^{(2)}(\phi_0, \hat{\rho}^{(1)})/l}{y^{(2)}(\phi_0, \hat{\rho}^{(1)})' M_{C(\hat{\rho}^{(1)})}[X^{(2)}, \hat{y}_{-1}^{(2)}] y^{(2)}(\phi_0, \hat{\rho}^{(1)})/(T_2 - l - k)}, \tag{3.31}$$

where $y^{(2)}(\phi_0, \hat{\rho}^{(1)}) = C(\hat{\rho}^{(1)})(y^{(2)} - \phi_0 y_{-1}^{(2)})$, $M_{C(\hat{\rho}^{(1)})}[A] = I - A[A'\Sigma(\hat{\rho}^{(1)})^{-1}A]^{-1}A'\Sigma(\hat{\rho}^{(1)})^{-1}$. This test statistic follows a $F(l, T_2 - l - k)$ distribution when $\phi = \phi_0$. Consequently, the critical region $SS(\phi_0; \hat{y}_{-1}^{(2)}, \hat{\rho}^{(1)}) > F_\alpha(l, T_2 - l - k)$ has size α. Furthermore

$$\bar{C}_\phi(\alpha) = \{\phi_0 : SS(\phi_0; \hat{y}_{-1}^{(2)}, \hat{\rho}^{(1)}) \leq F_\alpha(l, T_2 - l - k)\}$$

is a confidence set for ϕ with size $1 - \alpha$ and this confidence set takes a form similar to in Section 3.1. A test statistic of joint test of H_0: $\phi = \phi_0$ and $\rho = \rho_0$ is following:

$$SS(\phi_0, \rho_0; \hat{y}_{-1}^{(2)}) = \frac{y^{(2)}(\phi_0, \rho_0)' (M_{C_0}[X^{(2)}] - M_{C_0}[X^{(2)}, \hat{y}_{-1}^{(2)}]) y^{(2)}(\phi_0, \rho_0)/l}{y^{(2)}(\phi_0, \rho_0)' M_{C_0}[X^{(2)}, \hat{y}_{-1}^{(2)}] y^{(2)}(\phi_0, \rho_0)/(T_2 - l - k)}, \tag{3.32}$$

where $y^{(2)}(\phi_0,\rho_0) = C_0(y^{(2)} - \phi_0 y^{(2)}_{-1})$, $M_{C_0}[A] = I - A[A'\Sigma(\rho_0)^{-1}A]^{-1}A'\Sigma(\rho_0)^{-1}$.

It is noteworthy that we should be careful about the order of the subsamples (1) and (2). The order does not matter in a static model but it does in a dynamic model. If we use the second subsample to estimate parameters, then the estimators include $y^{(2)}$ and $y^{(2)}_{-1}$, which have past errors inside. As a result, $\hat{y}^{(1)}_{-1} = \bar{Z}^{(1)}_{-2}\widetilde{\pi}^{(2)}$ and $\bar{\xi}^{(1)}$ are not independent and the inference procedure does not control its level correctly. Therefore we should use the first part of the sample to get the estimates. A crucial issue in the split-sample tests is how to determine a splitting ratio, $\tau = T_1/T$. SS-type tests are depend on the choice of split ratio, τ and power of these tests are inversely related with τ; see Dufour and Jasiak (2001). However this ratio does not affect the validity of the test.

3.4. Split-sample-type point optimal procedure (SS^*)

We now propose a split-sample version of the point optimal procedure. The split-sample methods gives us additional flexibility for inference since we can estimate the nuisance parameter from the first sample and use it with the second sample to do the inference. Now consider a split-sample version of test statistic similar to (3.23) for $\rho = \rho_0$ against $\rho = \rho_1$:

$$\overline{SS}(\phi_0,\rho_0,\rho_1;\hat{y}^{(2)}_{-1}) = \frac{y^{(2)}(\phi_0,\rho_0)'M_{C_0}[X]y^{(2)}(\phi_0,\rho_0)}{y^{(2)}(\phi_0,\rho_1)'M_{C_1}[X,\hat{y}^{(2)}_{-1}]y^{(2)}(\phi_0,\rho_1)}, \tag{3.33}$$

where $y^{(2)}(\phi_0,\rho_0) = C_0(y^{(2)} - \phi_0 y^{(2)}_{-1})$, $y^{(2)}(\phi_0,\rho_1) = C_1(y^{(2)} - \phi_0 y^{(2)}_{-1})$, $M_{C_i}[A] = I - A[A'\Sigma(\rho_i)^{-1}A]^{-1}A'\Sigma(\rho_i)^{-1}, i = 0,1$. One can also use $\hat{\rho}^{(1)}$ instead of ρ_1 to construct test that controls the level as following

$$SS^*(\phi_0,\rho_0;\hat{y}^{(2)}_{-1},\hat{\rho}^{(1)}) = \frac{y^{(2)}(\phi_0,\rho_0)'M_{C_0}[X]y^{(2)}(\phi_0,\rho_0)}{y^{(2)}(\phi_0,\hat{\rho}^{(1)})'M_{C(\hat{\rho}^{(1)})}[X,\hat{y}^{(2)}_{-1}]y^{(2)}(\phi_0,\hat{\rho}^{(1)})}, \tag{3.34}$$

or, for simple test ($\phi = \phi_0$)

$$SS^*(\phi_0;\hat{y}^{(2)}_{-1},\hat{\rho}^{(1)}) = \frac{y^{(2)}(\phi_0,\hat{\rho}^{(1)})'M_{C(\hat{\rho}^{(1)})}[X]y^{(2)}(\phi_0,\hat{\rho}^{(1)})}{y^{(2)}(\phi_0,\hat{\rho}^{(1)})'M_{C(\hat{\rho}^{(1)})}[X,\hat{y}^{(2)}_{-1}]y^{(2)}(\phi_0,\hat{\rho}^{(1)})}, \tag{3.35}$$

where $y^{(2)}(\phi_0,\rho_0) = C(\rho_0)(y^{(2)} - \phi_0 y^{(2)}_{-1})$, $y^{(2)}(\phi_0,\hat{\rho}^{(1)}) = C(\hat{\rho}^{(1)})(y^{(2)} - \phi_0 y^{(2)}_{-1})$, $M_{C(\hat{\rho}^{(1)})}[A] = I - A[A'\Sigma(\hat{\rho}^{(1)})^{-1}A]^{-1}A'\Sigma(\hat{\rho}^{(1)})^{-1}$.

Again it is difficult to derive the analytical null distribution of (3.33) or (3.34) under the Gaussian assumption, while the MCT method described in Section 4 can be implemented and a confidence set for ϕ and ρ with level $(1-\alpha)$ is obtained by inverting the tests.

Further, note that $\overline{SS}(\phi_0,\rho_0,\rho_1;\hat{y}^{(2)}_{-1})$ can become degenerate in the limit whereas a monotonic transformation of $\overline{SS}(\phi_0,\rho_0,\rho_1;\hat{y}^{(2)}_{-1})$, given by

$$SS^*(\phi_0,\rho_0,\rho_1;\hat{y}^{(2)}_{-1}) = T_2[SS^*(\phi_0,\rho_0,\rho_1;\hat{y}^{(2)}_{-1}) - 1]. \tag{3.36}$$

is more for asymptotic theory. Again, for finite-sample inference, both $\overline{SS}(\phi_0,\rho_0,\rho_1;\hat{y}^{(2)}_{-1})$ and $SS^*(\phi_0,\rho_0,\rho_1;\hat{y}^{(2)}_{-1})$ lead to identical results since a monotonic transformation does not change the rank of the statistic in the MCT method.

3.5. Inference on general transformations

In Section 3.1 - 3.4, we make joint inference on $(\phi, \rho)'$. These tests are based on extensions of Anderson-Rubin statistics and designed for the hypothesis that fixing the entire vector of the endogenous (or unobserved) regressor coefficients. When one is interested in its subsets, or more generally in any functions of the parameters, projection technique can be applied; see Dufour (1989), Dufour and Jasiak (2001), Dufour and Taamouti (2005, 2007).

Let $\theta := (\phi, \rho)'$ for the convenience of notation. A confidence set associated with one of the tests for $H_0(\theta_0) : \theta = \theta_0$ in the previous subsections can be written as

$$C_\alpha(\theta) = \{\theta_0 \mid H_0(\theta_0) \text{ is not rejected}\}. \tag{3.37}$$

If the test has level α, the confidence set $C_\alpha(\theta)$ has level $1-\alpha$. Note that all of the four tests are based on pivotal functions, and all these tests have size α, and thus the confidence sets in (3.37) from these tests have size $1-\alpha$.

Now consider an arbitrary (possibly nonlinear) transformation $\delta = g(\theta)$ of θ, then a confidence set of δ, with the level at least $1-\alpha$, can be constructed as

$$C_\alpha(\delta) = \{\delta_0 \mid \delta_0 = g(\theta) \text{ for some } \theta \in C_\alpha(\theta)\}. \tag{3.38}$$

Since $\theta \in C_\alpha(\theta)$ implies $\delta = g(\theta) \in C_\alpha(\delta)$, and further,

$$Pr[\delta \in C_\alpha(\delta)] \geq Pr[\theta \in C_\alpha(\theta)] \geq 1-\alpha,$$

implies that $C_\alpha(\delta)$ has level $1-\alpha$.

Thus by the duality of tests and confidence sets, a test for $H_0(\delta_0) : \delta = \delta_0$ is to reject the null hypothesis if

$$\delta_0 \notin C_\alpha(\delta),$$

and we get a test of level α.

One can use numerical optimization technique or grid search over economically or statistically plausible parameter space to implement the projection method. However, if the parameter transformation of interest is a linear scalar function, an analytical expression for $C_\alpha(\delta)$ is available in Dufour and Taamouti (2005).

If $\delta = \phi$ where $\theta = (\phi, \rho)'$, then the projection method can be implemented more efficiently. Let $F(\theta_0)$ and c_α denote a test statistic used in confidence set in (3.37) and a corresponding critical value, respectively. Then, the confidence set in (3.38) is rewritten as

$$C_\alpha(\phi) = \{\phi_0 \mid \inf_{\rho \in \bar{\rho}} F(\phi_0, \rho) \leq c_\alpha\}, \tag{3.39}$$

where $\bar{\rho}$ is the parameter space for ρ. An alternative projection technique improves efficiency by restricting ρ. The procedure is

1. Construct $C_{\alpha_1}(\rho \mid \phi_0)$, a confidence set for ρ under $H_0 : \phi = \phi_0$ with level $(1-\alpha_1)$.

2. Reject $H_0 : \phi = \phi_0$ if $C_{\alpha_1}(\rho \mid \phi_0) = \emptyset$, or

$$\inf_{\rho \in C_{\alpha_1}(\rho|\phi_0)} F(\phi_0, \rho) > c_{\alpha_2},$$

   where $\alpha = \alpha_1 + \alpha_2$ and c_{α_2} is a critical value chosen in the same manner as c_α but with α_2 instead of α. By Bonferroni inequality, the test has level α, and it can be inverted to get confidence set for ϕ with level $1-\alpha$.

Since the infimum is computed over $C_{\alpha_1}(\rho \mid \phi_0)$, this procedure is expected to be more efficient. Furthermore, it is worthwhile noting that, even though the simultaneous confidence set $C_\alpha(\theta)$ for θ may be interpreted as a confidence set based on inverting LR-type tests for $\theta = \theta_0$ [see Meeker and Escobar (1995) or Chen and Jennrich (1996)], projection-based confidence sets, such as $C_\alpha(\delta)$, are not (strictly speaking) LR confidence sets. For more details and further discussion about the projection technique, see Dufour (1989, 1990), Chaudhuri, Richardson, Robins and Zivot (2010) and Chaudhuri and Zivot (2011).

4. Finite-sample procedures with possibly non-Gaussian errors

In this section, we extend the exact tests that are proposed in the previous section, by incorporating non-Gaussian distribution. The use of Gaussian assumptions, when the volatility distributions are not normal, can be hazardous; such a practice could lead us to invalid inferences that will create a wrong choice of portfolio, the underestimation of extreme losses, and hugely mispriced derivative products. An apparent reason is that Gaussian errors are not flexible enough to capture the fat-tailedness commonly observed in financial return distributions. In the past, many researchers used non-Gaussian distribution to get better model fits; see Liesenfeld and Jung (2000) and Chib et al. (2002) in the context of SV models and Bollerslev (1987) in the context of GARCH-type models.

Under the non-Gaussian assumptions, we can build an exact test based on the MCT technique. We can take the observed test statistic (derived under Gaussian assumptions) and perform simulations to obtain an exact test. In order to do that, we need the null distribution of the test statistic under non-Gaussian errors. Under the assumption 2.5, the GLS transformed composite error $\xi^* \sim \mathrm{N}(0, \sigma_\xi^2 I_T)$, where $\sigma_\xi^2 = (1+\phi_0^2)\sigma_v^2 + \sigma_e^2$. We need the following assumption about the transformed composite error to get the exact inference under non-Gaussian errors.

Assumption 4.1 CONDITIONAL SCALE MODEL OF TRANSFORMED COMPOSITE ERROR.

$$\xi^* = \sigma_\xi \vartheta, \tag{4.1}$$

where σ_ξ is a (possibly random) scalar such that $P[\sigma_\xi \neq 0] = 1$, and the conditional distribution of ϑ is completely or incompletely specified such that

$$\vartheta \mid \overline{X} := (\vartheta_1, \ldots, \vartheta_T) \sim \mathcal{F}(v), \tag{4.2}$$

where $\mathcal{F}(\cdot)$ represents a known distribution function and $\overline{X} = [X, Z_{-2}]$.

We consider both the case where the error distribution does not involve nuisance parameters,

$$\vartheta \mid \overline{X} \sim \mathcal{F}(v_0), \quad \text{where } v_0 \text{ is specified} \tag{4.3}$$

and the one where it does,

$$\vartheta \mid \overline{X} \sim \mathcal{F}(v), \quad \text{where } v \text{ is unknown.} \tag{4.4}$$

The above assumption includes the Gaussian distribution, all elliptically symmetric distributions, such as the multivariate t, and cases where $\vartheta_1, \ldots, \vartheta_T$ are i.i.d. according to any given non-elliptical distribution.

In the following proposition, we characterize the null distribution of $AR(\phi_0, \rho_0)$ given in (3.22) under the above assumption.

Proposition 4.1 NULL DISTRIBUTION OF AR-TEST STATISTIC UNDER NON-GAUSSIAN ERRORS. *Suppose* *equation* (3.21) *and assumption 4.1 hold. If* $\phi = \phi_0$ *and* $\rho = \rho_0$*, we have*

$$AR(\phi_0, \rho_0) = \kappa \frac{\vartheta'(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta}{\vartheta' M_{C_0}[X, Z_{-2}]\vartheta}, \tag{4.5}$$

where $\kappa = (T - l - k)/l$*,* $\vartheta = y(\phi_0, \rho_0) = C_0(y - \phi_0 y_{-1})$*,* $M_{C_0}[A] = I - A[A'\Sigma(\rho_0)^{-1}A]^{-1}A'\Sigma(\rho_0)^{-1}$*, and the conditional distribution of* ϑ *is given in assumption 4.1.*

Proposition 4.1 covers the null distribution of $AR(\phi_0, \rho_0)$. It is easy to see that the null distribution of the other proposed test statistic under non-Gaussian errors can be derived in the same way upon employing assumption 4.1.

The proposition 4.1 means that the conditional null distribution of $AR(\phi_0, \rho_0)$ given $\overline{X}$, only depends on the distribution of ϑ. If the distribution of $\vartheta \mid \overline{X}$ can be simulated, one can get exact tests based on $AR(\phi_0, \rho_0, \vartheta \mid \overline{X})$ through the MCT method [see Dufour (2006)], even if this distribution is non-Gaussian. Furthermore, the exact test obtained in this way is robust to weak IV's as well as if the distribution does not have moments (*e.g.*, the Cauchy distribution).

The technique of MCT was originally proposed by Dwass (1957) for implementing permutation tests and did not involve nuisance parameters. This technique was also independently proposed by Barnard (1963); for a review, see Dufour and Khalaf (2001) and for a general discussion and proofs, see Dufour (2006). It has the great attraction of providing exact (randomized) tests based on any statistic whose finite-sample distribution may be intractable but can be simulated. Here we have briefly summarized the procedure.

Let $S(Y, \overline{X})$ be a test statistic which can be rewritten in the form

$$S(Y, \overline{X}) = \bar{S}(\vartheta, \overline{X}) \tag{4.6}$$

under the null hypothesis, where ϑ is defined by (4.2) and the distribution of ϑ is known. For example, $S(Y, \overline{X})$ could be the AR-type statistic considered in proposition 4.1. Then the conditional distribution of $S(Y, \overline{X})$, given $\overline{X}$, is completely determined by the matrix $\overline{X}$ and the conditional distribution of ϑ given $\overline{X}$, *i.e.*, $S(Y, \overline{X})$ is pivotal. We can then proceed as follows to obtain an exact critical region.

1. Compute the statistic $S^{(0)}$ (based on data), where $S^{(0)} = AR^{(0)}(\phi_0, \rho_0)$.
2. By Monte Carlo methods, draw N i.i.d. replications of $\vartheta : \vartheta_{(j)} = [\vartheta_1^{(j)}, \ldots, \vartheta_T^{(j)}]$, $j = 1, \ldots, N$.
3. From each simulated error matrix $\vartheta_{(j)}$, compute the statistics, $S^{(j)} = \bar{S}(\vartheta_{(j)}, X)$, $j = 1, \ldots, N$, according to the fully specified distribution of $\vartheta \mid \overline{X}$. For instance, in the case of the AR statistic underlying proposition 4.1, calculate

$$AR^{(j)} := AR(\vartheta_{(j)}) = \frac{\vartheta'_{(j)}(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta_{(j)}}{\vartheta'_{(j)} M_{C_0}[X, Z_{-2}]\vartheta_{(j)}}, \quad 1, \ldots, N. \tag{4.7}$$

4. Compute the MC p-value $\hat{p}_N[S] := p_N(S^{(0)}; S)$, where

$$p_N(x, S) := \frac{NG_N(x; S) + 1}{N + 1}, \tag{4.8}$$

$$G_N(x; S) := \frac{1}{N} \sum_{j=1}^{N} I_{[0,\infty)}(S^{(j)} - x), \quad I_{[0,\infty)}(x) = \begin{cases} 1 & \text{if } x \in [0, \infty) \\ 0 & \text{if } x \notin [0, \infty) \end{cases}. \tag{4.9}$$

In other words, $p_N(S^{(0)}; S) = [NG_N(S^{(0)}; S) + 1]/(N + 1)$ where $NG_N(S^{(0)}; S)$ is the number of simulated values which are greater than or equal to $S^{(0)}$. When $S^{(0)}, S^{(1)}, \ldots, S^{(N)}$ are all distinct [an event with probability one when the vector $(S^{(0)}, S^{(1)}, \ldots, S^{(N)})'$ has an absolutely continuous distribution], $\hat{R}_N(S^{(0)}) = N+1-NG_N(S^{(0)};S)$ is the rank of $S^{(0)}$ in the series $S^{(0)}, S^{(1)}, \ldots, S^{(N)}$.

5. The MC critical region is: $\hat{p}_N[S] \le \alpha$, $0<\alpha<1$. If α^* and N such that $\alpha(N+1)$ is an integer and the distribution of S is continuous under the null hypothesis, then under null,

$$P[\hat{p}_N[S] \le \alpha] = \alpha \tag{4.10}$$

The above algorithm is valid for any fully specified distribution of ϑ and we reject the null hypothesis $H_0(\phi_0, \rho_0)$ at level α when $\hat{p}_N[AR^{(0)}(\phi_0, \rho_0)] \le \alpha$.

Under the null hypothesis $H_\phi(\phi_0, \rho_0)$, $P\big[\hat{p}_N[AR^{(0)}(\phi_0, \rho_0)] \le \alpha\big] = \alpha$, so that we have a test with level α. If the distribution of the test statistic is not continuous, the MC test procedure can easily be adapted by using "tie-breaking" method described in Dufour (2006).[6] Correspondingly, a confidence set with level $1-\alpha$ for (ϕ, ρ) is given by the set of all values (ϕ_0, ρ_0) which are not rejected by the above MC test. More precisely, the set

$$C_{(\phi,\rho)}(\alpha) = \big\{(\phi_0, \rho_0) : \hat{p}_N[AR^{(0)}(\phi_0, \rho_0)] > \alpha\big\} \tag{4.11}$$

is a confidence set with level $1-\alpha$ for (ϕ_0, ρ_0).

Now consider the case where the distribution of ϑ involves a nuisance parameter υ and $\upsilon \in \Phi_0$.

1. Let $S^{(0)}$ be the observed test statistic (based on data).
2. For each $\upsilon \in \Phi_0$, by Monte Carlo methods, draw N i.i.d. replications of $\vartheta : \vartheta_{(j)} = [\vartheta_1^{(j)}, \ldots, \vartheta_T^{(j)}]$, $j = 1, \ldots, N$ and compute the statistics, $S^{(j)}(\upsilon) = \bar{S}(\vartheta_{(j)}(\upsilon), X)$, $j = 1, \ldots, N$.
3. Using these simulations we compute the MC p-value $\hat{p}_N[S] := p_N(S^{(0)}; S)$, where

$$\hat{p}_N[x; S \mid \upsilon] := \frac{N\hat{G}_N[x; S \mid \upsilon] + 1}{N+1}. \tag{4.12}$$

4. The p-value function $\hat{p}_N[S \mid \upsilon]$ as a function of υ is maximized over the parameter values compatible with the Φ_0, and H_0 is rejected if

$$\sup_{\upsilon \in \Phi_0} \hat{p}_N[S \mid \upsilon] \le \alpha. \tag{4.13}$$

If the number of simulated statistics N is chosen such that $\alpha(N+1)$ is an integer, then we have under H_0:

$$P[\sup_{\upsilon \in \Phi_0} \{\hat{p}_N[S \mid \upsilon]\} \le \alpha] \le \alpha, \tag{4.14}$$

The test defined by $\hat{p}_N[S \mid \upsilon] \le \alpha$ has size α for known υ . Treating υ as a nuisance parameter and Φ_0 is a nuisance parameter set consistent with null, the test is *exact at level* α; for a proof, see Dufour (2006).

Because of the maximization in the critical region (4.13) the test is called a *maximized Monte Carlo* (MMC) test. MMC tests provide valid inference under general regularity conditions such as almost-unidentified models or time series processes involving unit-roots. In particular, even though the moment conditions defining the estimator are derived under the stationarity assumption, this does not question in any way the validity of maximized MC tests, unlike the parametric bootstrap whose distributional theory

[6] Without the correction for continuity, the algorithm proposed for statistics with continuous distributions yields a conservative test, *i.e.*, the probability of rejection under the null hypothesis is not larger than the nominal level.

is based on strong regularity conditions. Only the power of MMC tests may be affected. However, the simulated p-value function is not continuous, thus standard gradient-based methods cannot be used to maximize it. But search methods applicable to non-differentiable functions are applicable, *e.g.*, simulated annealing [see Goffe, Ferrier and Rogers (1994)]. A simplified approximate version of the MMC procedure can alleviate its computational load whenever a consistent point or set estimate of υ is available; for further discussion, see Dufour (2006).

5. Asymptotic distributional theory

In this section, we relax assumption 2.5 and show that under weaker assumptions, the proposed procedures remain "asymptotically valid". More precisely, we wish to show that if assumption 4.1 holds with a specific distributional assumption on ξ^*/σ_ξ (*e.g.*, $\xi^*/\sigma_\xi \sim \mathrm{N}(0, I_T)$) yields tests whose probability of type I error converges to the nominal level of the test as $T \to \infty$ under any parameter configuration compatible with the null hypothesis (pointwise asymptotic validity).

All our results up to now have been established for a given sample size T. To formulate asymptotic properties, we need to consider a sequence of tests indexed by T. Consider the following sequence

$$\{S(T) := [y(T), y_{-1}(T), X(T), Z_{-2}(T), \xi(T)], T \geq T_0\}, \tag{5.1}$$

and rewrite the test statistic (3.10) in the following form:

$$AR_T(\phi_0) = \kappa(T)\frac{y_T'(M[Q_{1T}] - M[Q_T])y_T}{y_T'M[Q_T]y_T/T}, \tag{5.2}$$

where $y_T = y(T) - \phi_0 y_{-1}(T)$, $Q_T = [Q_{1T} \vdots Q_{2T}]$, $Q_{1T} = X(T)$, $Q_{2T} = Z_{-2}(T)$, $\kappa(T) = (T - l - k)/lT$, and k and l are the numbers of columns of Q_{1T} and Q_{2T} , respectively.

We examine the asymptotic distribution of $AR_T(\phi_0)$ under the following assumptions (where $\Longrightarrow$ refers to weak convergence as the sample size tends to infinity).

Assumption 5.1 *The sequence* $(S(T), T \geq T_0)$ *given in* (5.1) *belongs to a class* $\mathcal{Z}$ *of stochastic processes such that for each process in* $\mathcal{Z}$ *the following limits hold:*

1. $\frac{\xi'(T)\xi(T)}{T} \underset{T\longrightarrow\infty}{\overset{p}{\longrightarrow}} \sigma_\xi^2 > 0$, *where* σ_ξ^2 *is the same for all processes in* $\mathcal{Z}$*;*
2. *There exists a sequence of* $m \times m$*, nonsingular matrices* D_T *such that:*
   (A) $D_T'Q_T'Q_T D_T \underset{T\longrightarrow\infty}{\overset{p}{\longrightarrow}} \Sigma_{QQ} = \begin{pmatrix} \Sigma_{Q_1Q_1} & \Sigma_{Q_1Q_2} \\ \Sigma_{Q_2Q_1} & \Sigma_{Q_2Q_2} \end{pmatrix}$,
   where Σ_{QQ} *and* $\Sigma_{Q_1Q_1}$ *are* $m \times m$ *and* $k \times k$ *nonsingular matrices, respectively;*
   (B) $D_T'Q_T'\xi(T) \Longrightarrow q \sim \mathrm{N}(0, \sigma_\xi^2 \Sigma_{QQ})$,
   where $q = (q_1', q_2')'$*,* q_1 *and* q_2 *are* $k \times 1$ *and* $l \times 1$ *random vectors, respectively.*

It should be emphasized that the assumption 5.1 satisfies the condition

$$q_2 \mid q_1 \sim \mathrm{N}(\Sigma_{Q_2Q_1}\Sigma_{Q_1Q_1}^{-1}q_1, \sigma_\xi^2\Sigma_{q_2|q_1}),$$

where $\Sigma_{q_2|q_1}) = \Sigma_{Q_2Q_2} - \Sigma_{Q_2Q_1}\Sigma_{Q_1Q_1}^{-1}\Sigma_{Q_2Q_2}$. Thus the asymptotic distribution of $(q'\Sigma_{QQ}^{-1}q - q_1'\Sigma_{Q_1Q_1}^{-1}q_1)/\sigma_\xi^2$ is a $\chi^2_{(l)}$ distributed random variable. Note that the normality of the sub-vector of q_1 is not required, the conditional normality of q_2 given q_1 is sufficient.

Further, in the above assumption 5.1-2, we allow both stationary and nonstationary regressors by adjusting the scaling matrix D_T, which is typical of the form, $D_T = diag[T^{-d_1}, \ldots, T^{-d_m}]$, where $d_i > 0$ for $i = 1, \ldots, m$ relying on the degree of nonstationarity of the regressors. For example, if $X(T)$ and $Z_{-2}(T)$ are stationary then $d_i = 0.5$ for $i = 1, \ldots, m$. However, if $X(T)$ and $Z_{-2}(T)$ are nonstationary and are integrated of order one, then the corresponding d_i should be one. The following proposition establishes the asymptotic validity of the AR procedure.

Proposition 5.1 ASYMPTOTIC VALIDITY OF AR-TYPE TEST. *Under the assumption 5.1 and the null hypothesis in* (3.1)*, the statistic* $AR_T(\phi_0)$ *in* (5.2) *has the same limiting distribution for all processes in* $\mathcal{Z}$*, i.e.,* $AR_T(\phi_0) \Longrightarrow \chi^2_{(l)}/l$.

Similarly, one can show that the joint test defined in (3.22) has the null distribution of $AR_T(\phi_0, \rho_0) \Longrightarrow \chi^2_{(l)}/l$. Now we consider the test statistic of the AR-type point optimal procedure which is rewritten in following form:

$$AR^*_T(\phi_0, \rho_0, \rho_1) = T\left[\frac{y_T(\phi_0, \rho_0)' M[\hat{Q}_{1T}] y_T(\phi_0, \rho_0)}{y_T(\phi_0, \rho_1)' M[\tilde{Q}_T] y_T(\phi_0, \rho_1)} - 1\right], \tag{5.3}$$

where $y_T(\phi_0, \rho_0) = C(\rho_0)(y(T) - \phi_0 y_{-1}(T))$, $y_T(\phi_0, \rho_1) = C(\rho_1)(y(T) - \phi_0 y_{-1}(T))$, $\hat{Q}_{1T} = C(\rho_0)X(T)$, $\tilde{Q}_T = [\tilde{Q}_{1T} \vdots \tilde{Q}_{2T}]$, $\tilde{Q}_{1T} = C(\rho_1)X(T)$, $\tilde{Q}_{2T} = C(\rho_1)Z_{-2}(T)$, k is the numbers of columns in $\hat{Q}_{1T}$ or $\tilde{Q}_{2T}$, l is the numbers of columns in $\tilde{Q}_{2T}$ and $m = l + k$. In order to prove the asymptotic validity of the $AR^*_T(\phi_0, \rho_0, \rho_1)$ that defined in (3.25), we need following assumption:

Assumption 5.2 *The sequence* $(S(T), T \geq T_0)$ *given in* (5.1) *belongs to a class* $\mathcal{Z}$ *of stochastic processes such that for each process in* $\mathcal{Z}$ *the following limits hold:*

1. $\frac{\hat{\xi}'(T)\hat{\xi}(T)}{T} \xrightarrow[T \longrightarrow \infty]{p} \sigma^2_{\xi} > 0$, *where* σ^2_{ξ} *is the same for all processes in* $\mathcal{Z}$*;*
2. $\frac{\tilde{\xi}'(T)\tilde{\xi}(T)}{T} \xrightarrow[T \longrightarrow \infty]{p} \sigma^2_{\xi} > 0$, *where* σ^2_{ξ} *is the same for all processes in* $\mathcal{Z}$*;*
3. *There exists a sequence of* $m \times m$*, nonsingular matrices* D_T *such that:*
   - *(A)* $D'_T \tilde{Q}'_T \tilde{Q}_T D_T \xrightarrow[T \longrightarrow \infty]{p} \Sigma_{\tilde{Q}\tilde{Q}} = \begin{pmatrix} \Sigma_{\tilde{Q}_1\tilde{Q}_1} & \Sigma_{\tilde{Q}_1\tilde{Q}_2} \\ \Sigma_{\tilde{Q}_2\tilde{Q}_1} & \Sigma_{\tilde{Q}_2\tilde{Q}_2} \end{pmatrix}$,
   where $\Sigma_{\tilde{Q}\tilde{Q}}$ *and* $\Sigma_{\tilde{Q}_1\tilde{Q}_1}$ *are* $m \times m$ *and* $k \times k$ *nonsingular matrices, respectively;*
   - *(B)* $D'_{T1} \hat{Q}'_{T1} \hat{Q}_{T1} D_{T1} \xrightarrow[T \longrightarrow \infty]{p} \Sigma_{\hat{Q}_1\hat{Q}_1}$,
   where $\Sigma_{\hat{Q}_1\hat{Q}_1}$ *is a* $k \times k$ *nonsingular matrix;*
   - *(C)* $D'_T \tilde{Q}'_T \tilde{\xi}(T) \Longrightarrow \tilde{q} \sim \mathrm{N}(0, \sigma^2_{\xi} \Sigma_{\tilde{Q}\tilde{Q}})$,
   where $\tilde{q} = (\tilde{q}'_1, \tilde{q}'_2)'$, $\tilde{q}_1$ *and* $\tilde{q}_2$ *are* $k \times 1$ *and* $l \times 1$ *random vectors, respectively.*
   - *(D)* $D'_T \hat{Q}'_{1T} \hat{\xi}(T) \Longrightarrow \hat{q}_1 \sim \mathrm{N}(0, \sigma^2_{\xi} \Sigma_{\hat{Q}_1\hat{Q}_1})$,
   where $\hat{q}_1$ *is a* $k \times 1$ *random vector.*

The following proposition establishes the asymptotic validity of the AR^* optimal procedure.

Proposition 5.2 ASYMPTOTIC VALIDITY OF AR-TYPE POINT OPTIMAL TEST. *Under the assumption 5.2 and the null hypothesis in* (3.19)*, the statistic* $AR^*_T(\phi_0, \rho_0, \rho_1)$ *in* (5.3) *has the same limiting distribution for all processes in* $\mathcal{Z}$*, i.e.,* $AR^*_T(\phi_0, \rho_0, \rho_1) \Longrightarrow \chi^2_{(l)}$.

We consider the following sequences for the split-sample methods, where each element of the sequence (5.1) is split into the first and second subsamples with size T_1 and T_2 ($T = T_1 + T_2$), respectively:

$$\{S^{(1)}(T) := [y^{(1)}(T), y^{(1)}_{-1}(T), X^{(1)}(T), Z^{(1)}_{-2}(T), \bar{\xi}^{(1)}(T)],$$
$$S^{(2)}(T) := [y^{(2)}(T), y^{(2)}_{-1}(T), X^{(2)}(T), Z^{(2)}_{-2}(T), \bar{\xi}^{(2)}(T)], T > T_0\}. \tag{5.4}$$

The split-sample test statistic in (3.32) can be constructed from (5.4) as following:

$$SS_T(\phi_0, \rho_0) = \kappa(T_2)\frac{y^{*\prime}_T(M[Q^*_{1T}] - M[\hat{Q}^*_T])y^*_T}{y^{*\prime}_T M[\hat{Q}^*_T]y^*_T/T_2}, \tag{5.5}$$

where $y^*_T = C(\rho_0)(y^{(2)}(T) - \phi_0 y^{(2)}_{-1}(T))$, $\hat{Q}^*_T = [Q^*_{1T} \vdots \hat{Q}^*_{2T}]$, $Q^*_{1T} = C(\rho_0)X^{(2)}(T)$, $\hat{Q}^*_{2T} = C(\rho_0)\hat{y}^{(2)}_{-1}(T)$, $\hat{y}^{(2)}_{-1} = \bar{Z}^{(2)}_{-2}\tilde{\pi}^{(1)}$, $\tilde{\pi}^{(1)} = \left(\bar{Z}^{(1)\prime}_{-2}\bar{Z}^{(1)}_{-2}\right)^{-1}\bar{Z}^{(1)\prime}_{-2}y^{(1)}_{-1}$, $\kappa(T_2) = (T_2 - l - k)/lT_2$, and k and l are the numbers of columns of Q^*_{1T} and $\hat{Q}^*_{2T}$, respectively. We will examine the asymptotic distribution of $SS_T(\phi_0, \rho_0)$ under the following assumptions.

Assumption 5.3 $T_1/T \longrightarrow \tau \in (0, 1)$ *as* $T \longrightarrow \infty$.

Assumption 5.4 *The sequence* $(S^{(1)}(T), S^{(2)}(T), T > T_0)$ *given in* (5.4) *belongs to a class* $\mathcal{Z}$ *of stochastic processes such that for each process in* $\mathcal{Z}$ *the following limits hold:*

1. $\frac{\bar{\xi}^{*(2)\prime}_T\bar{\xi}^{*(2)}_T}{T_2} \xrightarrow[T_2\longrightarrow\infty]{p} \sigma^2_{\xi} > 0$, *where* $\bar{\xi}^{*(2)}_T := C(\rho_0)\bar{\xi}^{(2)}(T)$, *and* σ^2_{ξ} *is the same for all processes in* $\mathcal{Z}$*;*
2. *Conditional on the first subsample, there exists a sequence of* $m \times m$*, nonsingular matrices* D_T *such that:*

   (A) $D'_T\hat{Q}^{*\prime}_T\hat{Q}^*_T D_T \xrightarrow[T_2\longrightarrow\infty]{p} \Sigma_{\hat{Q}^*\hat{Q}^*} = \begin{pmatrix} \Sigma_{Q^*_1Q^*_1} & \Sigma_{Q^*_1\hat{Q}^*_2} \\ \Sigma_{\hat{Q}^*_2Q^*_1} & \Sigma_{\hat{Q}^*_2\hat{Q}^*_2}\end{pmatrix}$,
   where $\Sigma_{\hat{Q}^*\hat{Q}^*}$ *and* $\Sigma_{Q^*_1Q^*_1}$ *are* $m \times m$ *and* $k \times k$ *nonsingular matrices, respectively;*

   (B) $D'_T\hat{Q}^{*\prime}_T\bar{\xi}^{*(2)}_T \Longrightarrow \hat{q}^* \sim \mathrm{N}(0, \sigma^2_{\xi}\Sigma_{\hat{Q}^*\hat{Q}^*})$,
   where $\hat{q}^* = (q^{*\prime}_1, \hat{q}^{*\prime}_2)'$, q^*_1 *and* $\hat{q}^*_2$ *are* $k \times 1$ *and* $l \times 1$ *random vectors, respectively, such that* $\tilde{q}^* := \hat{q}^{*\prime}\Sigma^{-1}_{\hat{Q}^*\hat{Q}^*}\hat{q}^* - q^{*\prime}_1\Sigma^{-1}_{Q^*_1Q^*_1}q^*_1$ *has an absolutely continuous (non-degenerate) distribution on* $\mathbb{R}$*, which is the same for all processes in* $\mathcal{Z}$*.*

It should be noted that $\hat{Q}^*_{2T}$ is depend on $\tilde{\pi}^{(1)}$, which is estimated from the first subsample. There may be a possible randomness of $\tilde{\pi}^{(1)}$ which is not disappearing even in the limit when IV's are weak. However, conditioning on the first subsample, we can get rid of this unnecessary randomness. Assumption 5.4.2.B implies that $\Sigma_{\hat{Q}^*\hat{Q}^*}$ and $\hat{q}^*$ are depend on $S^{(1)}(T)$, while $\tilde{q}^*$ is not depend on $S^{(1)}(T)$. To see this, we consider an example where $D_T = T_2^{-1/2}I_{T_2}$ and $\hat{q}^* = q(\tilde{\pi}^{(1)})$ follows a normal distribution. Thus, given $\tilde{\pi}^{(1)}$, $\mathrm{plim}_{T_2\to\infty}\frac{Q_T(\tilde{\pi}^{(1)})'Q_T(\tilde{\pi}^{(1)})}{T_2} = \Sigma_{QQ}(\tilde{\pi}^{(1)}) = \Sigma_{\hat{Q}^*\hat{Q}^*}$, $\frac{Q_T(\tilde{\pi}^{(1)})'Q_T(\tilde{\pi}^{(1)})}{\sqrt{T_2}} \Longrightarrow q(\tilde{\pi}^{(1)}) = \hat{q}^* \sim N(0,\ \sigma^2_{\xi}\Sigma_{\hat{Q}^*\hat{Q}^*})$, then, $(\hat{q}^{*\prime}\Sigma^{-1}_{\hat{Q}^*\hat{Q}^*}\hat{q}^*)/\sigma^2_{\xi} \sim \chi^2_m$, where χ^2_m is the χ^2 distribution with m degrees of freedom. As a result, even though $\Sigma_{\hat{Q}^*\hat{Q}^*}$ and $\hat{q}^*$ rely on $\tilde{\pi}^{(1)}$, $\hat{q}^{*\prime}\Sigma^{-1}_{\hat{Q}^*\hat{Q}^*}\hat{q}^*$ does not depend on $\tilde{\pi}^{(1)}$.

For the finite-sample distributional theory for the split-sample procedures, independence of $\bar{\xi}^*_t = C(\rho_0)\bar{\xi}_t$ over $t = 1, \ldots, T$ is assumed. For asymptotic theory, however, a similar restriction on dependence of $\bar{\xi}^*_t$ (for example, α-mixing assumption) is implicitly imposed by assumption 5.4.2.B; if dependence between $\bar{\xi}^{*(1)}_t$ in the estimates and $\bar{\xi}^{*(2)}_t$ is too strong, then the limiting distribution $\hat{q}^*$ would rely on nuisance parameters governing the dependence, and as a result the assumption cannot be satisfied. The following proposition proves the asymptotic validity of SS procedure.

Proposition 5.3 ASYMPTOTIC VALIDITY OF SS-TYPE TEST. *Under the assumptions 5.3-5.4 and the null hypothesis in* (3.19)*, the statistic* $SS_T(\phi_0, \rho_0)$ *in* (5.5) *has the same limiting distribution for all processes in* $\mathcal{Z}$*, i.e.,* $SS_T(\phi_0, \rho_0) \Longrightarrow \chi^2_{(l)}/l$.

Similarly, one can also prove the asymptotic validity of SS^* procedure.

6. Simulation study

In this section, we compare the performance of our proposed tests to the asymptotic t-type test. The standard SV model (given in (2.1-2.2)) has the following state-space representation:

$$w_t = \mu + \phi w_{t-1} + e_t, \qquad y_t = w_t + v_t, \qquad y_t := \log s_t^2 - \mathbb{E}[\log z_t^2], \tag{6.1}$$

where $w_t := \log \sigma_t^2$, and the v_t's and e_t's are i.i.d. $\mathrm{N}(0, \sigma_v^2)$ and $\log(\chi^2_{(1)})$ random variables, respectively and they are orthogonal to each other with $\sigma_v^2 = \pi^2/2$. With an instrument equation, the DGP in (6.1) is as follows:

$$y_t = \mu + \phi y_{t-1} + \xi_t, \quad \xi_t := e_t + v_t - \phi v_{t-1}, \quad e_t \sim \text{i.i.d. } N(0, \sigma_e^2), \quad v_t \sim \text{i.i.d. } \log(\chi^2_{(1)}) \tag{6.2}$$

$$y_{t-1} = Z_{t-2}\pi_1 + \eta_{t-1}, \quad \eta_{t-1} := v_{t-1} + u_{t-1}, \quad u_t \sim \text{i.i.d. } N(0, \sigma_u^2), \tag{6.3}$$

where π_1 is an l-vector of first-stage coefficients, Z_{t-2} is an l-vector of independent $\mathrm{N}(0,1)$ variables, and the vector (ξ_t, η_{t-1}) has a bivariate distribution with zero means, $Var(\xi_t) = (1+\phi^2)\sigma_v^2 + \sigma_e^2$, $Var(\eta_{t-1}) = \sigma_v^2 + \sigma_u^2$, and $Cov(\xi_t, \eta_{t-1}) = -\phi\sigma_v^2$. We construct π_1 as:

$$\pi_1 = \frac{||\bar{\lambda}||\sqrt{(\sigma_v^2 + \sigma_u^2)}}{\sqrt{Tl}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{6.4}$$

so that $||\bar{\lambda}^2|| = \frac{T\pi_1'\pi_1}{\sigma_v^2 + \sigma_u^2}$. Since $Var(Z_{t-2}) = I_l$ and $Var(\eta) = \sigma_v^2 + \sigma_u^2$, $||\bar{\lambda}^2||$ is the concentration parameter in this model. Note that, the DGP given in (6.2 - 6.3) is a GSV model with no exogenous explanatory variable.

In all experiments we set: $\sigma_v^2 = \pi^2/2$. Hence, μ, ϕ and σ_e, are the only parameters that will vary. We use 10,000 replication to compute the empirical level and powers and employ 99 replications for the PO MCT tests. For all tests, the nominal level is fixed at 5%. Thus, under the null hypothesis, the rejection rates should be less than (or close to) 5% for tests to be valid. Except for the analysis of asymptotic tests (Section 6.1), the sample sizes are $T = 100$, 200. For all split-sample type tests, we employ split ratio, $\tau = 0.2$ and use OLS to construct the instrument set. Note that, SS-type tests are depend on the choice of split ratio and power of these tests are inversely related with τ [see Dufour and Jasiak (2001)], therefore we set $\tau = 0.2$ to gain relatively more power.

6.1. Asymptotic size distortion

In this section, we evaluate the performance the asymptotic t-type test of $H_0 : \phi = \phi_0$. The simulated DGP is (6.2) [the standard SV model] with $\mu = 2$, $\sigma_e = 2$, and $\phi \in [0,\ 1]$. For sample sizes, T = {100, 200, 300, 400, 500, 1000, 2000, 5000} are used.

Table 2 reported the size of asymptotic t-type tests for $H_0(\phi) : \phi = \phi_0$. The test statistic is calculated using the simple winsorized estimator of Ahsan and Dufour (2019) [equation 3.9 with $J = 10$]. This estimator is more efficient compared to conventional methods (QMLE, GMM) and as efficient as the Bayesian procedure. In addition to that, it is extremely time-efficient and it produces empirical estimates which are similar to the Bayesian estimates. For the details of this asymptotic t-test, see Section 6.1 of Ahsan and Dufour (2019).

We can see from the results that the t-test (which is based on the asymptotic standard error) failed to control the level when $\phi \longrightarrow 1$. Size distortions are severe and equal to 100% when $\phi \longrightarrow 1$. These size distortions do not go away even in larger samples ($T = 2000,\ 5000$), especially when $\phi = 0.9999$, *i.e.*, ϕ is close to the unit circle.

6.2. Performance of the proposed tests

We will now examine the performance of the tests proposed in Section 3.1 - 3.4. To simplify the exposition, we focus on four setups:

(i) The model is correctly specified under assumption 2.4-2.5;

(ii) The model is incorrectly specified under assumption 2.5, *i.e.*, the error distribution is misspecified;

(iii) The model is incorrectly specified under assumption 2.4-2.5 and instrument set includes past lags of the LF volatility proxy;

(iv) The model is incorrectly specified under assumption 2.4-2.5, and the instrument set includes HF IV's, *i.e.*, realized volatility.

For (i), (ii) and (iii), we consider the joint tests $[H_0 : (\phi,\ \rho) = (\phi_0,\ \rho_0)]$. We do not use any plug-in estimator for ρ in these tests, and consider test (3.22) for AR, (3.25) for AR^*, (3.32) for SS, and (3.36) for SS^*.

For (iv), we consider the simple tests $[H_0 : \phi = \phi_0]$. We use a plug-in estimator for ρ in these tests, and consider test (3.22) with $\rho_0 = \rho_1 = \hat{\rho}$ for AR, (3.25) with $\rho_0 = \rho_1 = \hat{\rho}$ for AR^*, (3.31) for SS, and (3.35) for SS^*.

For the weak IV's robustness check (in Section 6.2.1 and 6.2.2), we simulate the DGP given in (6.2 - 6.3) with $\mu = 2$. We consider the concentration parameter $\bar{\lambda}^2 \in \{0, 0.1, 10\}$ with $\sigma_v^2 = \pi^2/2$ and $\sigma_u^2 = 0.1$. Thus, given $\bar{\lambda}^2 \in \{0, 0.1, 10\}$, the corresponding values of the first stage coefficients $\pi_1[1, i] = \{0, 0.05, 0.50\}, i = \{1, \cdots, l\}$ for $T = 100$ and $\pi_1[1, l] = \{0, 0.04, 0.35\}, i = \{1, \cdots, l\}$ for $T = 200$. The simulated DGP's use different values of ϕ and ρ. These values are $\phi = \{0.50, 0.75, 0.90, 1.00\}$ and $\rho = \{0.1, 0.2, 0.3\}$. Thus, given $\rho = 0.1$ and $\phi = \{0.50, 0.75, 0.90, 1.00\}$, the corresponding values of λ $[= \rho/(\phi - \rho(1+\phi)^2]$ are $\lambda = \{0.27, 0.17, 0.14, 0.13\}$. Since we also fix $\sigma_v^2 = \pi^2/2$, given $\lambda = \{0.27, 0.17, 0.14, 0.13\}$ the corresponding values of σ_e $[= \sqrt{\sigma_v^2/\lambda}]$ are $\sigma_e = \{4.30, 5.41, 5.96, 6.28\}$. Similarly, for $\rho = \{0.2, 0.3\}$, we have different set of values for λ and σ_e. As a result, a restriction on ρ implies a restriction on λ or σ_e. For example, a joint null $(\phi_0, \rho_0) = (0.5, 0.1)$ is same as $(\phi_0, \lambda_0) = (0.5, 0.27)$ or $(\phi_0, \sigma_{e0}) = (0.5, 4.30)$. In power comparison, for PO tests, we set the alternative: $\rho_1 = 0.30$.

6.2.1. Size and power when the model is correctly specified

We simulate the DGP given in (6.2 - 6.3) with $v_t \sim$ i.i.d. $N(0, \pi^2/2)$. The generated instrument set Z_{-2} was independent of the error distribution of e and v. The simulated DGP is correctly specified under the assumption 2.4-2.5. The results are presented in Tables 3 - 4 and confirm the theoretical contributions of Section 3.1 - 3.4.

First, from Table 3, all the proposed tests (AR, AR^*, SS, SS^*) controls the levels very well (rejection frequencies are less than (or close to) 5%). This result holds whether the identification is completely failed

[$\bar{\lambda}^2 = 0$], weak [$\bar{\lambda}^2 \in \{0, 0.1\}$], partial [$\bar{\lambda}^2 \in \{0.1, 10\}$], or strong [$\bar{\lambda}^2 = 10$]. This represents a substantial improvement over the asymptotic test. However, the *AR* test shows some size distortion in some cases. Further, the *SS* controls the level correctly, but in most cases this test is undersized and it increases with the number of IV's. The optimal tests perfectly control the level.

Second, from Table 4, all these tests exhibit excellent power as long as identification is not very weak. Note that, in our joint tests, we have an additional restriction under the null hypothesis on the parameter of the error distribution. This restriction works as an additional source of gaining power. We also see that in all cases [weak or strong IV's], the AR^* and SS^* tests have more power compare to their counterpart the *AR* and *SS*. As expected, the power of these tests increases with the sample size and the concentration parameter and decreases as the number of IV's increases.

6.2.2. Size and power when the model is incorrectly specified

We simulate the DGP given in (6.2 - 6.3) and generate the instrument set Z_{-2} such that it is independent of the error distribution of e and v. We consider $v_t \sim$ i.i.d. $\log(\chi^2_{(1)})$, so the simulated DGP is incorrectly specified under the assumption 2.5. This DGP represents an SV model with an instrument equation. The results are presented in Tables 5 - 6. The Results confirm that the tests proposed in Section 3.1 - 3.4 are valid and insensitive under model misspecification, *i.e.*, misspecification of the error distribution.

First, from Table 5, the empirical levels of all proposed tests are almost identical when the model is correctly specified [compare Table 3 with 5] – rejection frequencies are less than (or close to) 5%, whether identification is completely failed [$\bar{\lambda}^2 = 0$], weak [$\bar{\lambda}^2 \in \{0, 0.1\}$], partial [$\bar{\lambda}^2 \in \{0.1, 10\}$], or strong [$\bar{\lambda}^2 = 10$], for all sample sizes considered. The optimal tests based MCT method have better level control.

Second, from Table 6, the misspecification of the error distributions does not affect the power of these tests [compare Table 6 with Table 4]. *Third*, as the sample size increases, the rejection frequencies of these tests increase and in many cases, reach 100%. Overall, these tests appear to be reasonably robust to a misspecification of the error distribution, even with a small sample size.

6.2.3. Size and power when the model is incorrectly specified with low-frequency instruments

We simulate the DGP given in (6.2) with $\mu = 2$, $\phi \in (0.5,\ 1]$ and $\sigma_e \in (0.94,\ 3.14)$. This DGP corresponds to the standard log-normal SV model. We use past lags of y_{t-1} as IV's, as a result the instrument set Z_{-2} is not independent of the error distribution of e and v. As a result, the simulated DGP is incorrectly specified under the assumptions 2.4-2.5. In power comparison, for PO tests, we set the alternative: $\rho_1 = 0.35$. The results are presented in Table 7.

First, in both samples ($T = 100, 200$), all the proposed tests (AR, AR^*, SS , SS^*) control levels quite well, even when $\phi = 1$. As the number of IV's increases, the *SS* test under-reject down to 0% (when $l = 10$).

Second, all these tests exhibit excellent power (see from the second part of Table 7). Since, we set the alternative hypothesis $(\phi,\ \rho) = (0.5,\ 0.35)$, as a result the point optimal tests can gain power from the differences in covariance structure, *i.e.*, when $\rho = 0.25, 0.30$. From the results, in all cases, *AR* and *SS* tests have more power compare to their counterpart AR^* and SS^* when $l = 1$. Again, as expected, the power of these tests increases with the sample size and decreases as the number of IV's increases.

Third, we also simulate the same DGP with $v_t \sim$ i.i.d. $N(0, \pi^2/2)$ and results are almost identical [compare Table 8 with Table 7] – rejection frequencies are comparable.

6.2.4. Size and power when the model is incorrectly specified with high-frequency instruments

In this simulation experiment, we consider HF IV's. Simulating the DGP [given in (6.2)] with higher frequency has some limitations, in particular, the HF model parameters are different than the LF model parameters. Therefore, making an inference requires unique functional relationships between HF and LF parameters under temporal aggregation, *e.g.*, $\phi_{lf} = f(\phi_{hf})$, where ϕ_{hf} and ϕ_{lf} are the HF and LF parameter, receptively.

We assume stationarity of the latent volatility process, therefore under the assumptions 2.1 - 2.2, the process y_t in (6.2) admits an non-Gaussian ARMA(1, 1) representation [see Proposition 3.1 of Ahsan and Dufour (2019)] and given by

$$y_t = \mu + \phi y_{t-1} + \bar{\eta}_t - \theta\bar{\eta}_{t-1}, \tag{6.5}$$

with $\bar{\eta}_t - \theta\bar{\eta}_{t-1} = e_t + v_t - \phi v_{t-1}$. The moving average ($\theta$) and conditional variance ($\sigma^2_{\bar{\eta}}$) parameters can be related to ϕ, σ^2_e, and σ^2_v by the following non-linear equations

$$(1+\theta^2)\sigma^2_{\bar{\eta}} = \sigma^2_e + (1+\phi^2)\sigma^2_v, \qquad -\theta\sigma^2_{\bar{\eta}} = -\phi\sigma^2_v. \tag{6.6}$$

Equating coefficients and making substitutions leads to $\sigma^2_{\bar{\eta}} = \sigma^2_v\phi/\theta$ and θ is a solution to the quadratic equation $\theta^2 - \theta k + 1 = 0$ where $k = (\sigma^2_e + \sigma^2_v(1+\phi^2))/(\sigma^2_v\phi)$. It can be shown that $k^2 - 4 = (k-2)(k+2)$ is positive since $k > 2$ is equivalent to $\sigma^2_e + \sigma^2_v(1-\phi)^2 > 0$. The induced model is invertible if $|\theta| < 1$ which after some algebra is shown to be true for the root $(k + (k^2-4)^{1/2})/2$ when $0 < \phi < 1$ and for the root $(k - (k^2-4)^{1/2})/2$ when $-1 < \phi < 0$. Thus for temporal aggregation, we can exploit the well-known results for ARMA process.

Under time-aggregation, a m-period nonoverlapping aggregates of y_t defined by

$$Y_T = \sum_{t=m(T-1)+1}^{mT} y_t = (1 + B + \cdots + B^{m-1})y_{mT} = \sum_{j=0}^{m-1} B^j y_{mT}, \tag{6.7}$$

where m is the fixed order of aggregation and T is the aggregate time unit. The time series y_t and Y_T will be called the basic and the aggregate time series, respectively [$m = 1$ implies no aggregation]. If the basic time series y_t follow the ARMA(1, 1) model, then the aggregate series Y_T in (6.7) follows an ARMA(1, 1) model but the relationship between the parameters of both models is extremely complicated. It has been derived by Ahsanullah and Wei (1984) and result is as follows. Since y_t follows an ARMA(1, 1) model of type

$$(1 - \phi_{hf}B)y_t = (1 - \theta_{hf}B)\bar{\eta}_t, \tag{6.8}$$

then Y_T follows an ARMA(1, 1) model of type:

$$(1 - \phi_{lf}B)Y_T = (1 - \theta_{lf}B)\zeta_T, \tag{6.9}$$

with

$$\phi_{lf} = \phi^m_{hf}, \tag{6.10}$$

and θ_{lf} is the root of the quadratic equation $\theta^2_{lf} + \psi_3\theta_{lf} + 1 = 0$ with $\psi_3 = \psi_1/\psi_2$, where

$$\psi_1 = \sum_{i=0}^{m-1}\Big(1 + (\phi_{hf}-\theta_{hf})\sum_{j=0}^{i-1}\phi^j_{hf}\Big)^2 + \sum_{i=m}^{2(m-1)}\Big((\phi_{hf}-\theta_{hf})\sum_{j=i-m}^{m-2}\phi^j_{hf} - \theta_{hf}\phi^{m-1}_{hf}\Big)^2 + \Big(\theta_{hf}\phi^{m-1}_{hf}\Big)^2$$

$$\psi_2 = \sum_{i=0}^{m-2}\Big(1 + (\phi_{hf}-\theta_{hf})\sum_{j=0}^{i-1}\phi^j_{hf}\Big)\Big((\phi_{hf}-\theta_{hf})\sum_{j=i}^{m-2}\phi^j_{hf} - \theta_{hf}\phi^{m-1}_{hf}\Big) - \Big(1 + (\phi_{hf}-\theta_{hf})\sum_{j=0}^{m-2}\phi^j_{hf}\Big)\theta_{hf}\phi^{m-1}_{hf}$$

i.e., $\theta_{lf} = (-\psi_3 \pm \sqrt{\psi^2_3 - 4})/2$ such that $|\theta_{lf}| < 1$ to ensure invertibility of the aggregate model. Further, $\sigma^2_\zeta = \psi_1\sigma^2_{\bar{\eta}}/(1+\theta^2_{lf})$.

Since the nuisance parameter ρ_{lf} has no unique solution, we use a plug-in estimator for it and consider the simple test $H_0: \phi = \phi_0$. Note that, using a plug-in estimator for ρ may lead to some inconsequential size distortion.

We simulate model (6.2) with $\mu = 10^{-6}$, $\sigma_v^2 = \pi^2/2$, and different values of ϕ and σ_e. We consider LF values of $\phi = \{0, 0.1, 0.3, 0.5, 0.7, 0.8, 0.9, 0.95, 0.999\}$ and $\sigma_e = 0.25$. We consider equal-spaced HF intraday data with frequency = { 30s, 1m, 5m, 10m}, where s and m stand for second and minute. Therefore, within a day (trading hours = 6.5) the number of HF observations are $N\Delta t = \{780, 390, 78, 39\}$. For each frequency, we generate data from (6.2) model with sample size equal to $T \times N\Delta t$ and HF parameters $\phi_{hf} = \phi_{lf}^{N\Delta t}$ and $\sigma_{e,\,hf} = \sigma_e / N\Delta t$. For example, for 1m frequency, HF values of ϕ_{hf} are related to LF by $\phi_{hf} = \phi_{lf}^{390}$. Thus, in order to generate nearly nonstationary LF volatility process, we employ very high values of ϕ_{hf}. Tables 9 - 10 report the results of simple test.

First, we see from Table 9 that in all cases of HF IV's (these are the log of realized volatilities), all the proposed tests (AR, AR^*, SS, SS^*) controls the levels very well [rejection frequencies are less than (or close to) 5%]. This results holds whether sample sizes are different [$T = 100, 200$], or the instrument set contains different number of IV's [$l = 1, 5, 10$].

Second, from Table 10 that in all cases of HF IV's [30s, 1m, 5m, 10m], AR, AR^*, SS, and SS^* tests have excellent power against alternative [$\phi = 0$] – up to 99.2%, 99%, 95.9%, and 96.6%, respectively and the power of these tests increases with the sample size, and decreases as the number of IV's increases.

Third, all these tests have similar power across different sampling frequency HF IV's and the split sample tests have relatively less power.

7. Application to stock prices

In this section, we construct confidence intervals for the volatility persistence parameter using our proposed methods with real data.

7.1. Data description

The HF price data are taken from the TAQ (Trade and Quote) database, and LF prices are obtained from the CRSP database. We also consider model-free option implied volatilities as IV's. The options data are sourced from the OptionMetrics database. The access to these databases is done through the Wharton Research Data Services. The sample period is from January 1, 2009, to December 31, 2013 (1258 trading days). Initially, we consider daily IV's of nine stocks: General Electric Company (GE), IBM Common Stock (IBM), JPMorgan Chase & Co. (JPM), The Coca-Cola Co (KO), Pfizer Inc. (PFE), Exxon Mobil Corporation (XOM) and (2) The Procter and Gamble Company (PG), AT&T Inc. (T) and Walmart Inc. (WMT). Daily IV's are constructed from log squared daily returns. After examining the strength of daily IV's, we proceed with IBM stock and consider realized measures and implied volatilities as IV's. We check the strength of HF IV's and implement the proposed tests with various IV's and construct confidence intervals for ϕ by inverting the tests.

7.2. High-frequency instruments of asset price variability

We consider the HF volatility measures as the choice of IV's for the daily volatility. As we all know that based on the sampling frequency and different techniques, we could have different HF realized measures of volatility. The HF realized volatility was introduced by Andersen and Bollerslev (1998), who documented that the sum of squared intraday returns, known as the realized variance, provides an accurate measurement of daily volatility. The theoretical foundation of realized variance was developed in Andersen, Bollerslev, Diebold and Labys (2001) and Barndorff-Nielsen, Nicolato and Shephard (2002).

Let $p_t = \log S_t$ denote the logarithmic price where S_t is the observed price (at time t) and $r_t = p_t - p_{t-1}$ denote the continuously compounded return from time $t-1$ to t. Assume that the logarithmic price process, p_t, may exhibit both stochastic volatility and jumps. It could belong to the class of continuous-time jump diffusion processes,

$$dp_t = \mu_t d_t + \sigma_t dW_t + \kappa_t dq_t, \quad 0 \le t \le T \tag{7.1}$$

where μ_t is a continuous and locally bounded variation process and σ_t is a stochastic volatility process; W_t is the standard Brownian motion; dq_t is a counting process such that $dq_t = 1$ represents a jump at time t (and $dq_t = 0$ if no jump) with jump intensity λ_t. If p_{t-} denotes the price immediately prior to the jump at time t, then $\kappa_t = \Delta p_t = p_t - p_{t-}$. The process p_t is consists of a continuous component and a pure jump component. The quadratic variation (QV) of this process is defined by

$$[r,r]_t = \int_0^t \sigma_s^2 dW_s + \sum_{0<s\le t} \kappa_s^2, \tag{7.2}$$

where the first component, called integrated volatility, comes from the continuous component of (7.1), and the second term is the contribution from discrete jumps. In the absence of jumps, the second term on the right-hand side disappears, and the quadratic variation is simply equal to the integrated volatility (IVol). Several classes of HF IV's are considered and descriptions of these classes are given below.

7.2.1. Classes of realized measures not robust to jumps

These classes of realized measures have been proposed to provide robustness to various types of market microstructure effects (bid-ask bounce, stale quotes, mis-reported prices) and to improve the efficiency of estimates of volatility. We consider four broad classes of realized measures, which are consistent estimators of the QV in the absence of jumps.

7.2.1.1. Realized volatility. The realized volatility (RV) is defined as the sum of squared intraday returns. By dividing some interval of time, *e.g.*, $[T_0, T_1]$ say, into n subintervals, $T_0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = T_1$, we can define the intraday returns, $r_{i,n} = p_{t_{i,n}} - p_{t_{i-1,n}}$, then the $RV_t = \sum_{i=1}^n r_{i,n}^2$. Andersen et al. (2001) showed that the RV is a consistent estimator for the QV, which is just equal to IVol in the absence of jumps, i.e,

$$RV_t \xrightarrow{p} IVol_t = \int_0^t \sigma_s^2 dW_s.$$

7.2.1.2. RV with optimal sampling. RV with optimal sampling (RVbr) is proposed by Bandi and Russell (2008), where the sampling frequency is chosen optimally with a standard RV estimator, RV with optimal sampling (RVbr). This sampling frequency is calculated using estimates of integrated quarticity and variance of the microstructure noise. We use RVbr with the estimated optimal sampling frequency, which is the key feature of this estimator.

7.2.1.3. Multi-scales RV. Multi-scales RV by Zhang (2006) uses a combination of several high and lower frequencies to remove the noise and estimate the volatility. It is a generalization of two-scales RV [Zhang et al. (2005)] and can be defined for $1 \le J < K \le n$ as

$$MSRV_t = [r,r]_t^{(K)} - \frac{\bar{n}_K}{\bar{n}_J}[r,r]_t^{(J)} \xrightarrow{p} IVol_t,$$

by combining the time scales J and K. Here $\bar{n}_K = (n-K+1)/K$ and similarly for $\bar{n}_J$.

7.2.1.4. Realized kernels. The realized kernel (RK) by Barndorff-Nielsen, Hansen, Lunde and Shephard (2008) is a robust measures of volatility which ensures consistency and positive semi-definiteness, and several generalization to handle more lags and various shapes of autocorrelation function are derived in Barndorff-Nielsen, Hansen, Lunde and Shephard (2011). In this paper, we use the latter variant, which is given by

$$RK = \sum_{h=-H}^{H} k\left(\frac{h}{H+1}\right)\gamma_h,$$

where $k(x)$ is the kernel function and $\gamma_h = \sum_{i=|h|+1}^{n} r_{i,n} r_{i-h,n}$. We consider four types of kernel functions: (1) Bartlett kernel [RKbart: $k(x) = 1-x$, flat-top, $n^{1/6}$ rate], (2) Cubic kernel [RKcub: $k(x) = 1-3x^2+2x^3$, flat-top, $n^{1/4}$ rate], (3) Parzen kernel [RKnfp: $k(x) = \{1-6x^2+6x^3 \text{ if } 0 \le x \le 1/2\ , 2(1-x)^3 \text{ if } 1/2 \le x \le 1\}$, non-flat-top, $n^{1/5}$ rate], (4) Tukey-Hanning kernel with power 2 [RKth2: $k(x) = \sin^2\{\pi/2(1-x)^2\}$, flat-top, $n^{1/4}$ rate].

7.2.1.5. Realized range RV. Realized range RV (RRV) by Christensen and Podolskij (2007) uses sum of normalized squared high-low ranges for intra-daily periods rather than sum of squared returns. As a result, it is based on extremes from the entire price path and, thus, provides more information than returns sampled at fixed time intervals. Decomposing the daily time interval into K *non-overlapping* intervals of size m_K, the estimator is given by:

$$RRV^{(m_K,K)} = \frac{1}{\lambda_{2,m_K}} \sum_{i=1}^{K} s_i^{(m_K)^2} \xrightarrow{p} IVol,$$

where the range of the price process over the ith interval is given by $s_i^{(m_K)} = \max_{0\le h,l \le m_K}\left(p_{\frac{i-1+h/mK}{K}} - p_{\frac{i-1+l/mK}{K}}\right)$, $i = 1,\dots,K$, and $\lambda_{2,m_K} = E[\max_{0\le h,l\le m_K}(W_{h/m_K} - W_{l/m_K})^2]$ is the second moment of the range of a standard Brownian motion over the unit interval with m_K observed increments.

7.2.2. Classes of realized measures robust to jumps

In the presence of jumps, RV is a consistent estimator of the QV (see Andersen and Bollerslev (1998), Andersen et al. (2001), Barndorff-Nielsen et al. (2002)), which is now a combination of IVol and jump variation (JV):

$$RV_t \xrightarrow{p} \underbrace{\int_0^t \sigma_s^2 dW_s}_{IVol_t} + \underbrace{\sum_{0<s\le t} \kappa_s^2}_{JV_t}. \tag{7.3}$$

We consider two classes of jump-robust realized measures:

7.2.2.1. Bipower variation. The most widely used estimator of IVol in the presence of jumps is the Bipower variation (BV) measure of Barndorff-Nielsen and Shephard (2004). It is the sum of adjacent absolute returns, *i.e.*,

$$BV_t := \frac{\pi}{2}\sum_{i=2}^{n} |r_{i-1,n}||r_{i,n}| \xrightarrow{p} IVol_t = \int_0^t \sigma_s^2 dW_s. \tag{7.4}$$

7.2.2.2. Nearest neighbor truncated RV. Andersen, Dobrev and Schaumburg (2012) used nearest neighbor truncation approach to estimate the integrated volatility, where the median (MedRV) and minimum (MinRV) realized volatility estimators use min or median of blocks of returns (MinRV with blocks of two returns and MedRV with blocks of three returns). The proposed estimators are:

$$MinRV_n = \frac{\pi}{\pi-2}\left(\frac{n}{n-1}\right)\sum_{i=1}^{n-1}[\min(|r_{i,n}|,\ |r_{i+1,n}|)]^2,$$

$$MedRV_n = \frac{\pi}{6-4\sqrt{3}+\pi}\left(\frac{n}{n-2}\right)\sum_{i=2}^{n-2}[\text{med}(|r_{i-1,n}|,\ |r_{i,n}|,\ |r_{i+1,n}|)]^2.$$

7.2.3. Aditional HF measures and jump variations

We also consider realized semivariance, jump variation, and signed jump variation as IV's for true latent daily volatility. Further, we exploit the information content of log squared jump variation and signed jump variation.

7.2.3.1. Jump variation. As noted by Barndorff-Nielsen and Shephard (2006), combining the results in equations (7.3) and (7.4), the contribution of the jump process in the quadratic variation can be estimated by the difference between these two variations can be consistently estimated by

$$JV_t := RV_t - BV_t \xrightarrow{p} \sum_{0<s\le t} \kappa_s^2. \tag{7.5}$$

7.2.3.2. Realized semivariance. Barndorff-Nielsen, Kinnebrock and Shephard (2010) proposed realized semivariance (RSV) estimators that can capture the variation only due to negative or positive returns using the realized semivariance estimator. These estimators are defined as

$$RSV_t^+ := \sum_{j=1}^{n} r_{t_j}^2 \mathbf{1}_{\{r_{t_j}>0\}} \xrightarrow{p} \frac{1}{2}\int_0^t \sigma_s^2 dW_s + \sum_{0\le s\le t} \kappa_s^2 \mathbf{1}_{\{\kappa_s>0\}}, \tag{7.6}$$

$$RSV_t^- := \sum_{j=1}^{n} r_{t_j}^2 \mathbf{1}_{\{r_{t_j}<0\}} \xrightarrow{p} \frac{1}{2}\int_0^t \sigma_s^2 dW_s + \sum_{0\le s\le t} \kappa_s^2 \mathbf{1}_{\{\kappa_s<0\}}, \tag{7.7}$$

where the first term in the limit of both RSV^+ and RSV^- is one-half of the integrated variance. These estimators provide a complete decomposition of RV, in the sense that $RV = RSV^+ + RSV^-$.

7.2.3.3. Signed jump variation. The variation due to the continuous component can be removed by simply subtracting one RSV from the other, without the need to estimate it separately. The remaining part can be defined as the signed jump variation:

$$SJV_t := \lim_{n\to\infty}(RSV_t^+ - RSV_t^-) = \sum_{0\le s\le t} \kappa_s^2 \mathbf{1}_{\{\kappa_s>0\}} - \sum_{0\le s\le t} \kappa_s^2 \mathbf{1}_{\{\kappa_s<0\}}. \tag{7.8}$$

7.3. Final instrument set

Our final instrument set also includes implied volatilities (ImV), principal component factors (PCF), and daily log volatility. These ImV are calculated from IBM American options, and three classes of ImV are considered: (1) call options; (2) put options; (3) both call and put options. For each class, we use all implied volatilities at a given date to construct six ImV subclasses, which are mean, minimum, maximum, and three quantiles (q1, q2, q3). Three largest principal component factors are extracted from HF IV's; see Table 11 for details. Formally, Kapetanios, Khalaf and Marcellino (2016) propose PCF-based identification-robust inference in the context of instrumental-variable regressions to deal with the problem of many IV's.

We consider one hundred and seventy-five IV's, which can be divided into 22 classes. The description of all IV's are given in Table 11. The HF subclass includes different sampling frequencies [tick, second and minute], sampling scheme [tick or business], and sub-sampling. We use 1-minute sub-sampling [ss] in the calculation of several HF measures.

7.4. Results

7.4.1. Strength of IV's

We investigate the strength of daily IV's since a pressing concern with an instrumental-variable approach is the possible use of weak IV's, which can produce biased estimators [bias towards OLS estimates] and hypothesis tests with large size distortion. The existing econometric literature defines weak IV's based on the strength of the first-stage equation (*e.g.*, Bekker (1994), Staiger and Stock (1997), and Stock and Yogo (2005). Following Stock and Yogo (2005), we employ the first-stage F-statistic to detect whether IV's are weak or not.

Tests based on F-statistics that whether daily IV's [past lags of the endogenous variable] all have zero coefficients are reported in Table 12 with corresponding critical values associated with the desired maximum level of size distortion. From the table, we can see that many F-statistics are less than the corresponding critical value associated with the maximum asymptotic size of a Wald test [these critical values are obtained using weak-IV asymptotic distributions]. These results suggest that instrumental-variable estimates are biased towards OLS estimates, and we need to use weak instrument robust inference methods.

Now, we wish to check if the HF and others IV's are weak or not. We consider IBM stock and different classes of IV's. Results with other stocks are qualitatively similar and omitted to conserve space. Table 13 reports the first-stage F-statistics of all IV's. Most of the HF IV's are strong for IBM. Exceptions are JV and SJV HF classes, ImV-mean subclass, and daily IV's. However, if we consider multiple IV's then Wald-type tests fail to control the level in many cases. Further, in most cases, F-statistic (that measures the strength of IV's) is maximized when we consider only one instrument irrespective of it is weak or strong.

7.4.2. Projection-based confidence sets

To construct a projection-based confidence interval for the volatility persistence parameter ϕ, we first construct a $(1-\alpha_1)$ confidence interval for λ, denoted as $C_{\alpha_1}(\lambda)$. We parametrize the noise ratio λ rather than ρ since this is the more natural choice. We set $\alpha_1 = 0.05$, and compute λ using the simple method proposed by Ahsan and Dufour (2019). We use equation 3.8 with $J = 10$ of Ahsan and Dufour (2019) to estimate σ_e^2 and the corresponding standard error (SE). By setting $\sigma_v^2 = \pi^2/2$, the SE of $\hat{\lambda} = \sigma_v^2/\hat{\sigma_e}^2$ is computed using the delta method. The estimated 95% confidence interval for the nuisance parameter λ is $C_{0.05}(\lambda) = [33.943,\ 61.154]$ with $\hat{\lambda} = 47.548$ and $\text{SE}(\hat{\lambda}) = 6.935$. For each value of λ in the confidence interval $C_{\alpha_1}(\lambda)$, we then construct $(1-\alpha_2)$ confidence intervals for ϕ given λ [denoted as $C_{\alpha_2}(\phi|\lambda)$] by inverting a test robust to weak IV's, proposed in Section 3.1-3.4. We use $\alpha_2 = 0.05$, and by Bonferroni's inequality, this confidence interval has coverage of at least $100(1-\alpha)\%$, where $\alpha = \alpha_1 + \alpha_2 = 0.10$. A 90% confidence interval for ϕ that does not depend on λ can be obtained by

$$C_{0.10}(\phi) = \bigcup_{\lambda \in C_{0.05}(\lambda)} C_{0.05}(\phi|\lambda).$$

The projection method is thoroughly discussed in Section 3.5. Note that, we employ grid testing during the test inversion, in which a series of tests $[H_0 : \phi = \phi_0,\ \lambda = \lambda_0$, where $\phi_0 \in [0,\ 1],\ \lambda_0 \in C_{\alpha_1}(\lambda)]$ performed. Note that we restrict ϕ_0 in the most relevant part of the parameter space, *i.e.*, $\phi_0 \in [0,\ 1]$.

We use $\alpha_1 = \alpha_2$, which is the rule typically employed in the literature on simultaneous inference (*e.g.*, in Bonferroni-type procedures) and test combination; see Miller (1981), Savin (1984). Cavanagh, Elliott and Stock (1995) suggest a refinement of the Bonferroni method that makes it less conservative than the basic approach. The idea is to shrink the confidence interval for λ so that the refined interval is a subset of the original (unrefined) interval. This consequently shrinks the Bonferroni confidence interval for ϕ, achieving an exact test of desired significance level. However, it is important to note that α should be selected a priori, not on the basis of the results yielded by different choices of α_1 for a given sample.

As pointed out by Dufour (1997), when IV's can be arbitrary weak, then a confidence set with correct coverage probability must have an infinite length with positive probability.[7] As a result, the length of a weak instrument robust confidence interval can summarize the identification strength of the corresponding instrument. Since we restrict $\phi_0 \in [0,\ 1]$, an irrelevant (no identification) instrument for the regressor should produce a confidence interval with length equal to 1.

From an identification-robust confidence set, we define the instrument i's strength by

$$d_i := 1 - (ub_i - lb_i), \tag{7.9}$$

where ub and lb are the upper and lower bound of the confidence set, and $ub - lb$ is the length of the confidence set. The definition d_i implies that if i is a weak instrument then it will produce d_i close to 0 and if i is a strong instrument then it will produce d_i close to 1.

Figure 1 shows the identification strength d_i of different classes of IV's, where the instrument set consists of a constant and a lag of the corresponding instrument. We use log squared transformation for the first 13 classes of IV's (RV-RSVP); see Table 11. For each class, we consider the average, median, minimum, and maximum strength across the proposed inference methods $[AR,\ AR^*,\ SS,\ SS^*]$. These inference procedures are proposed in Section 3.1-3.4. We use $\tau = 0.2$ for SS-type tests and employ 99 Monte Carlo replications for point optimal type procedures.

The following inferences emerge from Figure 1. *First*, except for JV and SJV classes, the identification strength of all HF classes are strong, *i.e.*, these classes produce very high d_i values. These results hold in all strength measures and across four inference methods. *Second*, JV and SJV classes have many weak and irrelevant (no identification) IV's, since average strength measures are low and median strength measures are zero. These results suggest that JV and SJV classes have no or very little predictive power regarding the latent daily volatility. However, log squared JV and SJV have strong identification strength. This finding suggests that the second moment of jumps or signed jumps is correlated with the latent daily volatility proxy. *Third*, both PCF and ImV classes have some relevant IV's. However, all ImV classes include some weak IV's. *Fourth*, according to SS-type tests, the daily instrument is uninformative regarding the latent daily volatility proxy. That is, SS and SS^* produce d_i equal to 0.047 and 0, indicating weak and no identification, respectively.

Figure 2 shows the identification strength of different subclasses of HF IV's. On average, all HF subclasses produce confidence intervals with similar lengths, *e.g.*, on average, both 1s and 5m produce almost similar identification-robust confidence intervals. It is worth noting that the equation (3.4) is the ultimate instrument equation, which connects the daily volatility proxy to the instrument set. With an instrument set which contains a constant and a lag of HF instrument, (3.4) yields

$$y_t = \pi_0 + \pi_1 Z_{t-1} + \eta_t,$$

where y_t is the daily volatility proxy and Z_{t-1} is the selected RV instrument. The constant term π_0 captures the bias in the RV estimate due to the non-trading hours and microstructure noise. If the bias-correction term π_0 is negative, RV has an upward bias that may be due to the market microstructure noise, and if π_0 is positive, it has a downward bias due to the non-trading hours; see Takahashi et al. (2009). Due to this bias-correction term, even with a very high sampling frequency, the proposed inference methods produce confidence intervals robust to the non-trading hours and microstructure noise.

To formalize, we define the notion of the average strength of an instrument set i over the proposed inference methods by

$$\bar{d}_{i,s} := \sum_{i=1}^{S} d_i, \tag{7.10}$$

[7] Dufour (1997) showed that if the IV's are not correlated with the regressor [irrelevant IV's], then parameter ϕ is not identified, and any value of ϕ is consistent with data. A valid confidence set in such a case must be infinite, at least with probability equal to the coverage. Most empirical applications use the conventional Wald confidence interval, which is always finite. As a result, the Wald confidence interval has a low coverage probability and should not be used when IV's are weak.

where $s \in S$ and S is the set of identification-robust inference methods. We use this measure to rank the strength of instruments. Table 14 reports the projection-based 90% confidence intervals for ϕ using strong IV's, *i.e.*, based on $\bar{d}_{i,s}$. Panel A includes superior IV's while panel B and C include IV's which produce slightly larger confidence sets than IV's in panel A. Panel A mostly includes HF IV's, and 70% of these are 5m subclass. This finding proves that HF RV does provide an additional gain in predicting the LF volatility proxy. The best instrument is the RSVN-5m-ss. The average implied volatility that extracts from IBM call options is also a strong instrument. This finding is in line with Christensen and Prabhala (1998), who find that implied volatility has important explanatory power regarding past volatility. We also find that confidence sets with 30s RVs [Panel C: RSVN-30s, RV-30s, BV-30s, MSRV-30s] are spacious than confidence sets with 5m RVs [Panel A and B] and conclude that the effect of market microstructure noise leads to a slightly wider confidence set. Since it is well-known in the RV literature that the market microstructure noise becomes progressively more dominant as the sampling frequency increases; see Zhang et al. (2005), Bandi and Russell (2008), and Hansen and Lunde (2006). This result suggests that the proposed inference methods produce valid confidence sets even with noisy RVs. Further, 85% of the time, Panel A and B include IV's with frequency 1m, 5m, and 10m. These confidence sets are less sensitive to the market microstructure noise.

Table 14 also gives several other conclusions. *First*, we can infer from these confidence sets that the persistence parameter lies roughly between 0.85 and 1.0 for IBM. This outcome indicates that the volatility process is highly persistent, close to unit-root, consistent with the empirical literature. These confidence sets include $\phi = 1$, implying that these sets are also robust to nonstationarity. *Second*, in all cases, simulation-based point-optimal confidence sets [AR^* and SS^*] are conservative compared to the corresponding AR-type confidence sets [AR and SS].

Table 15 presents the projection-based 90% confidence intervals for ϕ using weak IV's, *i.e.*, based on $\bar{d}_{i,s}$. Panel A of Table 15 contains IV's with no identification, as a result, these IV's produce unbounded confidence intervals (these confidence intervals cover the entire set of $\phi \in [0,\ 1]$). Panel A comprises mostly by JV and SJV HF classes and ImV-max subclass. Note that under no identification, all values of ϕ are observationally equivalent, which implies that the proposed test statistics yield valid confidence sets that are unbounded with a non-zero probability. Consequently, the proposed tests are robust to weak identification. From Panel C, we find that the LF daily instrument produces a valid confidence set, although the length of this set is larger compare to HF confidence sets given in Table 14. We also see that in some cases, SS-type tests produce confidence intervals that are entirely different than those that are provided by AR-type tests. This finding could be because SS-type tests are computed from the second part of the sample, and may be affected by an unmodeled structural change.

In Table 16, we report the estimated confidence intervals, where the instrument set includes a constant and several lags of an instrument, $l = 1,\ 3,\ 5$. In this setup, we use the first set of strong IV's [Table 14 - Panel A], ImV-C-q3, and 1-day. In most cases, we find that all (AR, AR^*, SS, SS^*) confidence intervals for ϕ are getting wider as l increases. The average length of confidence intervals is larger. Therefore, we do not see any apparent gains by adding more lags in the instrument set. The only exception is the LF daily instrument, where the average length of confidence intervals is shorter. This result implies that we should use more daily lags as IV's to get a smaller confidence set. We also construct several confidence sets where the instrument set includes a constant and various combinations of strong IV's. We report these confidence sets in Table 17. The conclusion is similar to Table 16, *i.e.*, no apparent gains from combining strong IV's.

Finally, these confidence sets can be extended to allow for non-Gaussian error distribution [where the conditional distribution of scale transformed error has non-Gaussian error distribution] using the procedure described in Section 4. Furthermore, these confidence intervals formed from several values due to grid testing; as a result, it is easy to get a nonparametric estimate of ϕ by applying the Hodges-Lehman principal.

8. Conclusion

In this paper, we have introduced a novel class of generalized stochastic volatility models, which can use high-frequency information content, and accommodate nonstationary volatility process. We employ instrumental variable methods to provide a unified framework for the analysis of GSV models.

Using the proposed GSV class models, we have studied the problem of testing hypotheses and building confidence sets for the volatility persistence parameter. This parameter has an intrinsic interest because it measures the persistence of the latent volatility process, *i.e.*, "volatility clustering of asset returns". We proposed more reliable identification-robust finite-sample procedures. These inferential procedures are not only robust to weak IV's but also robust to nonstationarity of the latent volatility process. We also showed that these finite-sample procedures (based on a Gaussian assumption on the errors) remain asymptotically valid under weaker distributional assumptions. We then study the statistical properties of the proposed tests in simulation experiments. These tests outperform the asymptotic t-type test in terms of size and exhibit excellent power.

We applied these methods to IBM's price and option data and observed several empirical facts. The superior instrument set constitutes of HF realized measures and call option implied volatilities, and these IV's produce confidence sets, which show that the latent volatility process of IBM is close to unit-root. We find RVs with higher frequency produces spacious confidence interval compared to RVs with slightly lower frequency, pointing out that these confidence intervals adjust to incorporate the microstructure noise. We also find jumps and signed jumps have no or little information content regarding the low-frequency volatility, whereas their log squared versions have a strong identification strength. When we consider irrelevant or weak instruments, the proposed procedures give unbounded confidence intervals.

Finally, it is easy to see that the inference methods used in this paper can be adapted to other situations, *e.g.*, measurement error in ARMA-type models, or noisy realized measures in HAR volatility modeling. The extension to multivariate models is a topic of ongoing research.

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A. Appendix

A.1. Proofs

PROOF OF PROPOSITION 4.1 When $\phi = \phi_0$ and $\rho = \rho_0$, on multiplying the two sides of (3.21) by $M_{C_0}[X] - M_{C_0}[X, Z_{-2}]$ and $M_{C_0}[X]$, we see that:

$$\begin{aligned} (M_{C_0}[X] - M_{C_0}[X, Z_{-2}])C_0(y - \phi_0 y_{-1}) &= \sigma_\xi (M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta, \\ M_{C_0}[X]C_0(y - \phi_0 y_{-1}) &= \sigma_\xi M_{C_0}[X]\vartheta. \end{aligned} \tag{A.1}$$

Thus, the AR-statistic in (3.22) can be rewritten as:

$$AR(\phi_0, \rho_0) = \frac{\sigma_\xi^2 \vartheta'(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta / l}{\sigma_\xi^2 \vartheta' M_{C_0}[X]\vartheta/(T - l - k)} = \frac{\vartheta'(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta / l}{\vartheta' M_{C_0}[X]\vartheta/(T - l - k)}.$$

Hence, the null conditional distribution of $AR(\phi_0, \rho_0)$, given $\overline{X}$, only depends on distribution of ϑ. If normality holds conditional on $\overline{X}$, *i.e.*, $\vartheta \mid X \sim \mathrm{N}(0, I_T)$,we have $\vartheta'(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta \sim \chi^2_{(l)}$ and $\vartheta' M_{C_0}[X]\vartheta \sim \chi^2_{(T-l-k)}$. Since $M_{C_0}[X, Z_{-2}](M_{C_0}[X] - M_{C_0}[X, Z_{-2}]) = 0$, hence $\vartheta'(M_{C_0}[X] - M_{C_0}[X, Z_{-2}])\vartheta$ and $\vartheta' M_{C_0}[X]\vartheta$ are independent conditional on $\overline{X}$. Consequently, $AR(\phi_0, \rho_0) \sim F(l, T - l - k)$. □

PROOF OF PROPOSITION 5.1 Under the null hypothesis $\phi = \phi_0$,

$$AR_T(\phi_0) = \kappa(T)\frac{\Lambda_{1T} - \Lambda_{2T}}{\Lambda_{2T}/T}, \tag{A.2}$$

where

$$\Lambda_{1T} := \xi(T)' M[Q_{1T}]\xi(T), \quad \Lambda_{2T} := \xi(T)' M[Q_T]\xi(T), \quad \kappa(T) := \frac{T - l - k}{lT}.$$

Under assumption (5.1), we have

$$\kappa(T) \underset{T\longrightarrow\infty}{\longrightarrow} \frac{1}{l}, \tag{A.3}$$

$$q_2 \mid q_1 \sim \mathrm{N}(\Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} q_1, \sigma_\xi^2 \Sigma_{q_2|q_1}), \tag{A.4}$$

where $\Sigma_{q_2|q_1} = \Sigma_{Q_2Q_2} - \Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1}\Sigma_{Q_1Q_2}$. Then

$$(q_2 - \Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} q_1)'\Sigma^{-1}_{q_2|q_1}(q_2 - \Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} q_1) \sim \sigma_\xi^2 \chi^2_{(l)}. \tag{A.5}$$

$$\begin{aligned} \Lambda_{1T} - \Lambda_{2T} &= \xi(T)' M[Q_{1T}]\xi(T) - \xi(T)' M[Q_T]\xi(T) \\ &= \xi(T)'(I - P[Q_{1T}])\xi(T) - \xi(T)'(I - P[Q_T])\xi(T) \\ &= \xi(T)' Q_T (Q_T' Q_T)^{-1} Q_T' \xi(T) - \xi(T)' Q_{1T}(Q_{1T}' Q_{1T})^{-1} Q_{1T}' \xi(T) \\ &= \xi(T)' Q_T D_T (D_T' Q_T' Q_T D_T)^{-1} D_T' Q_T' \xi(T) - \xi(T)' D_{1T} Q_{1T}(D_{1T}' Q_{1T}' Q_{1T} D_{1T})^{-1} D_{1T}' Q_{1T}' \xi(T) \\ &\Longrightarrow q'\Sigma^{-1}_{QQ} q - q_1'\Sigma^{-1}_{Q_1Q_1} q_1. \end{aligned} \tag{A.6}$$

Now using standard formulas of a partitioned matrix inverse for Σ_{QQ} and setting $S = q'\Sigma^{-1}_{QQ} q - q_1'\Sigma^{-1}_{Q_1Q_1} q_1$ (see Gentle (2007), section 3.4.1), we have

$$\begin{aligned} S &= q'\Sigma^{-1}_{QQ} q - q_1'\Sigma^{-1}_{Q_1Q_1} q_1 \\ &= (q_1', q_2')' \begin{bmatrix} \Sigma^{-1}_{Q_1Q_1} + \Sigma^{-1}_{Q_1Q_1}\Sigma_{Q_1Q_2}\Sigma^{-1}_{q_2|q_1}\Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} & -\Sigma^{-1}_{Q_1Q_1}\Sigma_{Q_1Q_2}\Sigma^{-1}_{q_2|q_1} \\ -\Sigma^{-1}_{q_2|q_1}\Sigma_{Q_2Q_1}\Sigma^{-1}_{Q_1Q_1} & \Sigma^{-1}_{q_2|q_1} \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&- q_1'\Sigma_{Q_1Q_1}^{-1}q_1 \\
&= q_1'\Sigma_{Q_1Q_1}^{-1}q_1 + q_1'\Sigma_{Q_1Q_1}^{-1}\Sigma_{Q_1Q_2}\Sigma_{q_2|q_1}^{-1}\Sigma_{Q_2Q_1}\Sigma_{Q_1Q_1}^{-1}q_1 - 2q_2'\Sigma_{Q_1Q_1}^{-1}\Sigma_{Q_1Q_2}\Sigma_{q_2|q_1}^{-1}q_2 + q_2'\Sigma_{q_2|q_1}^{-1}q_2 \\
&- q_1'\Sigma_{Q_1Q_1}^{-1}q_1 \\
&= q_1'\Sigma_{Q_1Q_1}^{-1}\Sigma_{Q_1Q_2}\Sigma_{q_2|q_1}^{-1}\Sigma_{Q_2Q_1}\Sigma_{Q_1Q_1}^{-1}q_1 - 2q_2'\Sigma_{Q_1Q_1}^{-1}\Sigma_{Q_1Q_2}\Sigma_{q_2|q_1}^{-1}q_2 + q_2'\Sigma_{q_2|q_1}^{-1}q_2 \\
&= (q_2 - \Sigma_{Q_2Q_1}\Sigma_{Q_1Q_1}^{-1}q_1)'\Sigma_{q_2|q_1}^{-1}(q_2 - \Sigma_{Q_2Q_1}\Sigma_{Q_1Q_1}^{-1}q_1).
\end{aligned} \tag{A.7}
$$

Thus, from (A.5), (A.6), and (A.7), we have

$$
\Lambda_{1T} - \Lambda_{2T} \Longrightarrow \sigma_\xi^2\chi_{(l)}^2, \tag{A.8}
$$

and

$$
\frac{\Lambda_{2T}}{T} \xrightarrow[T\longrightarrow\infty]{p} \sigma_\xi^2, \tag{A.9}
$$

hence

$$
AR_T(\phi_0) \Longrightarrow \frac{\chi_{(l)}^2}{l}.
$$

□

PROOF OF PROPOSITION 5.2 Under the null hypothesis $\phi = \phi_0, \rho = \rho_0$,

$$
AR_T^*(\phi_0, \rho_0, \rho_1) = \frac{\Lambda_{1T} - \Lambda_{2T}}{\Lambda_{2T}/T}, \tag{A.10}
$$

where

$$
\Lambda_{1T} := \hat{\xi}(T)'M[\hat{Q}_{1T}]\hat{\xi}(T), \quad \Lambda_{2T} = \tilde{\xi}(T)'M[\tilde{Q}_T]\tilde{\xi}(T).
$$

Under assumption 5.2, we have

$$
\begin{aligned}
\Lambda_{2T}/T &= \tilde{\xi}(T)'\tilde{\xi}(T)/T - \tilde{\xi}(T)'P[\tilde{Q}_T]\tilde{\xi}(T)/T \\
&= \tilde{\xi}(T)'\tilde{\xi}(T)/T - \tilde{\xi}(T)'\tilde{Q}_T D_T(D_T'\tilde{Q}_T'\tilde{Q}_T D_T)^{-1}D_T'\tilde{Q}_T'\tilde{\xi}(T)/T \\
&= \tilde{\xi}(T)'\tilde{\xi}(T)/T - \tilde{\xi}(T)'\tilde{Q}_T D_T(D_T'\tilde{Q}_T'\tilde{Q}_T D_T)^{-1}D_T'\tilde{Q}_T'\tilde{\xi}(T)/T \\
&\xrightarrow[T\longrightarrow\infty]{p} \sigma_\xi^2,
\end{aligned} \tag{A.11}
$$

where the last equality follows from

$$
\tilde{\xi}(T)'\tilde{\xi}(T)/T \xrightarrow[T\longrightarrow\infty]{p} \sigma_\xi^2, \quad \tilde{\xi}(T)'\tilde{Q}_T D_T(D_T'\tilde{Q}_T'\tilde{Q}_T D_T)^{-1}D_T'\tilde{Q}_T'\tilde{\xi}(T)/T \Longrightarrow \frac{\sigma_\xi^2\chi_{(l+k)}^2}{T} \xrightarrow[T\longrightarrow\infty]{} 0.
$$

Now using restrictions under the null and alternative that $\hat{\xi}(T) = \tilde{\xi}(T) := \xi_T^* \sim N(0, I_T)$, we have

$$
\begin{aligned}
\Lambda_{1T} - \Lambda_{2T} &= \hat{\xi}(T)'M[\hat{Q}_{1T}]\hat{\xi}(T) - \tilde{\xi}(T)'M[\tilde{Q}_T]\tilde{\xi}(T) \\
&= \xi_T^{*\prime}M[\hat{Q}_{1T}]\xi_T^* - \xi_T^{*\prime}M[\tilde{Q}_T]\xi_T^* \\
&= \left[\xi_T^{*\prime}\xi_T^* - \xi_T^{*\prime}\xi_T^*\right] + \left[\xi_T^{*\prime}P[\tilde{Q}_T]\xi_T^* - \xi_T^{*\prime}P[\hat{Q}_{1T}]\xi_T^*\right] \\
&= \xi_T^{*\prime}P[\tilde{Q}_T]\xi_T^* - \xi_T^{*\prime}P[\hat{Q}_{1T}]\xi_T^* \\
&= \xi_T^{*\prime}\tilde{Q}_T(\tilde{Q}_T'\tilde{Q}_T)^{-1}\tilde{Q}_T'\xi_T^* - \xi_T^{*\prime}\hat{Q}_{1T}(\hat{Q}_{1T}'\hat{Q}_{1T})^{-1}\hat{Q}_{1T}'\xi_T^* \\
&= \xi_T^{*\prime}Q_T[Q_T'\Sigma(\rho_1)^{-1}Q_T]^{-1}Q_T'\Sigma(\rho_1)^{-1}\xi_T^* - \xi_T^{*\prime}Q_{1T}[Q_{1T}'\Sigma(\rho_0)^{-1}Q_{1T}]^{-1}Q_{1T}'\Sigma(\rho_0)^{-1}\xi_T^* \\
&= \xi_T^{*\prime}Q_T D_T[D_T'Q_T'\Sigma(\rho_1)^{-1}Q_T D_T]^{-1}D_T'Q_T'\Sigma(\rho_1)^{-1}\xi_T^*
\end{aligned}
$$

$$
\begin{aligned}
&- \xi_T^{*\prime} Q_{1T} D_{1T} [D_{1T}' Q_{1T}' \Sigma(\rho_0)^{-1} Q_{1T} D_{1T}]^{-1} D_{1T}' Q_{1T}' \Sigma(\rho_0)^{-1} \xi_T^* \\
&= \xi_T^{*\prime} \overline{\Lambda}_1 \xi_T^* - \xi_T^{*\prime} \overline{\Lambda}_0 \xi_T^* \\
&= \overline{\overline{\Lambda}}_1 - \overline{\overline{\Lambda}}_0, \qquad (A.12)
\end{aligned}
$$

where $Q_T = [Q_{1T} \vdots Q_{2T}]$, $Q_{1T} = X(T)$, $Q_{2T} = Z_{-2}(T)$, and

$$
\begin{aligned}
\overline{\Lambda}_1 &:= Q_T D_T [D_T' Q_T' \Sigma(\rho_1)^{-1} Q_T D_T]^{-1} D_T' Q_T' \Sigma(\rho_1)^{-1}, \\
\overline{\Lambda}_0 &:= Q_{1T} D_{1T} [D_{1T}' Q_{1T}' \Sigma(\rho_0)^{-1} Q_{1T} D_{1T}]^{-1} D_{1T}' Q_{1T}' \Sigma(\rho_0)^{-1}, \\
\overline{\overline{\Lambda}}_1 &:= \xi_T^{*\prime} \overline{\Lambda}_1 \xi_T^*, \qquad \overline{\overline{\Lambda}}_0 := \xi_T^{*\prime} \overline{\Lambda}_0 \xi_T^*.
\end{aligned}
$$

Under assumption 5.2, we have

$$
\begin{aligned}
\overline{\overline{\Lambda}}_1 &= \xi_T^{*\prime} \overline{\Lambda}_1 \xi_T^* \Longrightarrow \sigma_\xi^2 \chi^2_{(l+k)}, \\
\overline{\overline{\Lambda}}_0 &= \xi_T^{*\prime} \overline{\Lambda}_0 \xi_T^* \Longrightarrow \sigma_\xi^2 \chi^2_{(k)}.
\end{aligned}
$$

Further, from the properties of quadratic forms [see Hogg and Craig (1958)], if $\overline{\overline{\Lambda}}_1 - \overline{\overline{\Lambda}}_0 \geq 0$, then

$$
\overline{\overline{\Lambda}}_1 - \overline{\overline{\Lambda}}_0 \Longrightarrow \sigma_\xi^2 \chi^2_{(l)}. \qquad (A.13)
$$

Since $\overline{\Lambda}_1$ is a projection onto $[D_{1T}X(T),\ D_{2T}Z_{-2}(T)]$ plane and $\overline{\Lambda}_0$ is a projection onto $D_{1T}X(T)$, $\overline{\Lambda}_1 - \overline{\Lambda}_0$ is a projection onto $D_{2T}Z_{-2}(T)$, *i.e.*, it is a projection onto the orthogonal complement of $D_{1T}X(T)$ within $[D_{1T}X(T),\ D_{2T}Z_{-2}(T)]$. As a result, $\overline{\Lambda}_1 - \overline{\Lambda}_0$ is an idempotent and positive-semidefinite matrix. This implies

$$
\overline{\overline{\Lambda}}_1 - \overline{\overline{\Lambda}}_0 = \xi_T^{*\prime} (\overline{\Lambda}_1 - \overline{\Lambda}_0) \xi_T^* \geq 0, \qquad (A.14)
$$

and therefore

$$
\overline{\overline{\Lambda}}_1 - \overline{\overline{\Lambda}}_0 \Longrightarrow \sigma_\xi^2 \chi^2_{(l)}. \qquad (A.15)
$$

Hence from (A.11) and (A.15), we have

$$
AR_T^*(\phi_0, \rho_0, \rho_1) \Longrightarrow \chi^2_{(l)}.
$$

□

PROOF OF PROPOSITION 5.3 Under H_0: $\phi = \phi_0$ and $\rho = \rho_0$,

$$
SS_T(\phi_0, \rho_0) = \kappa(T_2) \frac{\Lambda_{1T}^* - \Lambda_{2T}^*}{\Lambda_{2T}^* / T_2},
$$

where $\Lambda_{1T}^* = \bar{\xi}_T^{*(2)\prime} M[Q_{1T}^*] \bar{\xi}_T^{*(2)}$ and $\Lambda_{2T}^* = \bar{\xi}_T^{*(2)\prime} M[\hat{Q}_T^*] \bar{\xi}_T^{*(2)}$. Under assumption 5.4,

$$
\kappa(T_2) \underset{T \longrightarrow \infty}{\longrightarrow} \frac{1}{l}, \qquad (A.16)
$$

$$
\hat{q}_2^* \mid q_1^* \sim \mathrm{N}(\Sigma_{\hat{Q}_2^* Q_1^*} \Sigma_{Q_1^* Q_1^*}^{-1} q_1^*, \sigma_\xi^2 \Sigma_{\hat{q}_2^* | q_1^*}), \qquad (A.17)
$$

where $\Sigma_{\hat{q}_2^* | q_1^*} = \Sigma_{\hat{Q}_2^* \hat{Q}_2^*} - \Sigma_{\hat{Q}_2^* Q_1^*} \Sigma_{Q_1^* Q_1^*}^{-1} \Sigma_{Q_1^* \hat{Q}_2^*}$. Then

$$
(\hat{q}_2^* - \Sigma_{\hat{Q}_2^* Q_1^*} \Sigma_{Q_1^* Q_1^*}^{-1} q_1^*)' \Sigma_{\hat{q}_2^* | q_1^*}^{-1} (\hat{q}_2^* - \Sigma_{\hat{Q}_2^* Q_1^*} \Sigma_{Q_1^* Q_1^*}^{-1} q_1^*) \sim \sigma_\xi^2 \chi^2_{(l)}. \qquad (A.18)
$$

$$\begin{aligned}
\Lambda^*_{1T} - \Lambda^*_{2T} &= \bar{\xi}^{*(2)'}_T M[Q^*_{1T}]\bar{\xi}^{*(2)}_T - \bar{\xi}^{*(2)'}_T M[\hat{Q}^*_T]\bar{\xi}^{*(2)}_T \\
&= \bar{\xi}^{*(2)'}_T (I - P[Q^*_{1T}])\bar{\xi}^{*(2)}_T - \bar{\xi}^{*(2)'}_T (I - P[\hat{Q}^*_T])\bar{\xi}^{*(2)}_T \\
&= \bar{\xi}^{*(2)'}_T \hat{Q}^*_T(\hat{Q}^{*\prime}_T\hat{Q}^*_T)^{-1}\hat{Q}^{*\prime}_T\bar{\xi}^{*(2)}_T - \bar{\xi}^{*(2)'}_T Q^*_{1T}(Q^{*\prime}_{1T}Q^*_{1T})^{-1}Q^{*\prime}_{1T}\bar{\xi}^{*(2)}_T \\
&= \bar{\xi}^{*(2)'}_T \hat{Q}^*_T D_T(D'_T\hat{Q}^{*\prime}_T\hat{Q}^*_T D_T)^{-1}D'_T\hat{Q}^{*\prime}_T\bar{\xi}^{*(2)}_T - \bar{\xi}^{*(2)'}_T D_{1T}Q^*_{1T}(D'_{1T}Q^{*\prime}_{1T}Q^*_{1T}D_{1T})^{-1}D'_{1T}Q^{*\prime}_{1T}\bar{\xi}^{*(2)}_T \\
&\Longrightarrow \hat{q}^{*\prime}\Sigma^{-1}_{\hat{Q}^*\hat{Q}^*}\hat{q}^* - q_1^{*\prime}\Sigma^{-1}_{Q_1^*Q_1^*}q_1^*. \qquad \text{(A.19)}
\end{aligned}$$

Now using standard formulas of a partitioned matrix inverse for $\Sigma_{\hat{Q}^*\hat{Q}^*}$ and setting $\tilde{q}^* := \hat{q}^{*\prime}\Sigma^{-1}_{\hat{Q}^*\hat{Q}^*}\hat{q}^* - q_1^{*\prime}\Sigma^{-1}_{Q_1^*Q_1^*}q_1^*$ (see Gentle (2007), section 3.4.1), we have

$$\begin{aligned}
\tilde{q}^* &= \hat{q}^{*\prime}\Sigma^{-1}_{\hat{Q}^*\hat{Q}^*}\hat{q}^* - q_1^{*\prime}\Sigma^{-1}_{Q_1^*Q_1^*}q_1^* \\
&= (q_1^{*\prime}, \hat{q}_2^{*\prime})' \begin{bmatrix} \Sigma^{-1}_{Q_1^*Q_1^*} + \Sigma^{-1}_{Q_1^*Q_1^*}\Sigma_{Q_1^*\hat{Q}_2^*}\Sigma^{-1}_{\hat{q}_2^*|q_1^*}\Sigma_{\hat{Q}_2^*Q_1^*}\Sigma^{-1}_{Q_1^*Q_1^*} & -\Sigma^{-1}_{Q_1^*Q_1^*}\Sigma_{Q_1^*\hat{Q}_2^*}\Sigma^{-1}_{\hat{q}_2^*|q_1^*} \\ -\Sigma^{-1}_{\hat{q}_2^*|q_1^*}\Sigma_{\hat{Q}_2^*Q_1^*}\Sigma^{-1}_{Q_1^*Q_1^*} & \Sigma^{-1}_{\hat{q}_2^*|q_1^*} \end{bmatrix} \begin{pmatrix} q_1^* \\ \hat{q}_2^* \end{pmatrix} \\
&\quad - q_1^{*\prime}\Sigma^{-1}_{Q_1^*Q_1^*}q_1^* \\
&= q_1^{*\prime}\Sigma^{-1}_{Q_1^*Q_1^*}q_1^* + q_1^{*\prime}\Sigma^{-1}_{Q_1^*Q_1^*}\Sigma_{Q_1^*\hat{Q}_2^*}\Sigma^{-1}_{\hat{q}_2^*|q_1^*}\Sigma_{\hat{Q}_2^*Q_1^*}\Sigma^{-1}_{Q_1^*Q_1^*}q_1^* - 2\hat{q}_2^{*\prime}\Sigma^{-1}_{Q_1^*Q_1^*}\Sigma_{Q_1^*\hat{Q}_2^*}\Sigma^{-1}_{\hat{q}_2^*|q_1^*}\hat{q}_2^* + \hat{q}_2^{*\prime}\Sigma^{-1}_{\hat{q}_2^*|q_1^*}\hat{q}_2^* \\
&\quad - q_1^{*\prime}\Sigma^{-1}_{Q_1^*Q_1^*}q_1^* \\
&= q_1^{*\prime}\Sigma^{-1}_{Q_1^*Q_1^*}\Sigma_{Q_1^*\hat{Q}_2^*}\Sigma^{-1}_{\hat{q}_2^*|q_1^*}\Sigma_{\hat{Q}_2^*Q_1^*}\Sigma^{-1}_{Q_1^*Q_1^*}q_1^* - 2\hat{q}_2^{*\prime}\Sigma^{-1}_{Q_1^*Q_1^*}\Sigma_{Q_1^*\hat{Q}_2^*}\Sigma^{-1}_{\hat{q}_2^*|q_1^*}\hat{q}_2^* + \hat{q}_2^{*\prime}\Sigma^{-1}_{\hat{q}_2^*|q_1^*}\hat{q}_2^* \\
&= (\hat{q}_2^* - \Sigma_{\hat{Q}_2^*Q_1^*}\Sigma^{-1}_{Q_1^*Q_1^*}q_1^*)'\Sigma^{-1}_{\hat{q}_2^*|q_1^*}(\hat{q}_2^* - \Sigma_{\hat{Q}_2^*Q_1^*}\Sigma^{-1}_{Q_1^*Q_1^*}q_1^*). \qquad \text{(A.20)}
\end{aligned}$$

Thus, from (A.18), (A.19), and (A.20), we have

$$\Lambda^*_{1T} - \Lambda^*_{2T} \Longrightarrow \sigma^2_\xi \chi^2_{(l)}, \qquad \text{(A.21)}$$

and

$$\frac{\Lambda^*_{2T}}{T_2} \xrightarrow[T_2\longrightarrow\infty]{p} \sigma^2_\xi, \qquad \text{(A.22)}$$

hence

$$SS_T(\phi_0, \rho_0) \Longrightarrow \frac{\chi^2_{(l)}}{l}.$$

□

A.2. Figures

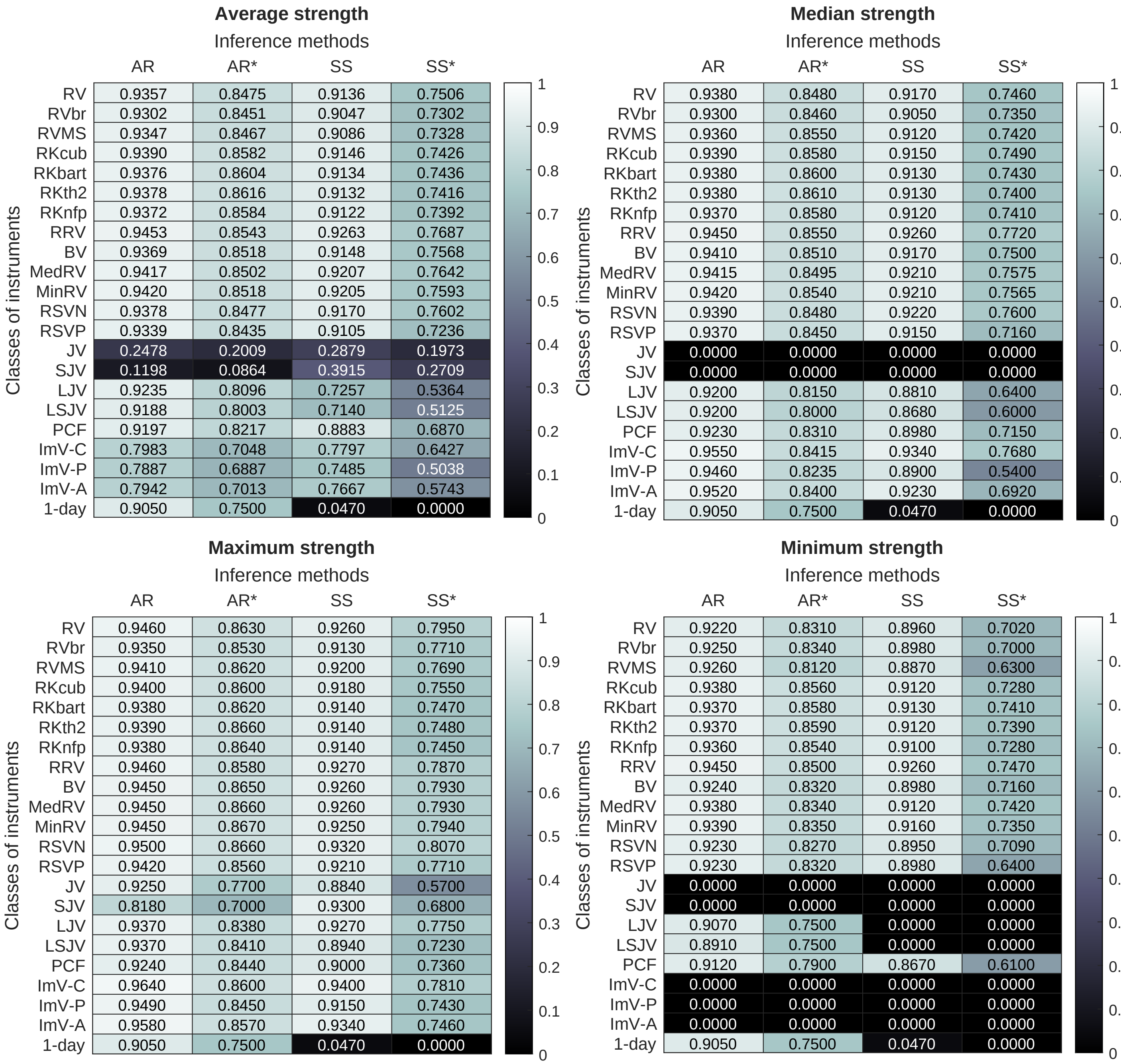

Figure 1. IBM: 2009-2013: Identification strength of different classes of instruments.

Note: The instrument set consists of a constant and a lag of instrument, $l = 1$. We use log squared transformation for the first 13 classes of instruments (RV-RSVP) given in Table 11. The strength of instrument i is defined as $d_i = 1 - (ub_i - lb_i)$. For each class, we consider the average, median, minimum, and maximum strength across the proposed inference methods [AR, AR^*, SS and SS^*]. These inference procedures are described in Section 3.1-3.4. We use $\tau = 0.2$ for SS-type tests and employ 99 Monte Carlo replications for point optimal type procedures.

Average strength

Inference methods

Subclasses of instruments	AR	AR*	SS	SS*
1s	0.9360	0.8415	0.9129	0.7449
5s	0.9356	0.8361	0.9169	0.7500
30s	0.9399	0.8485	0.9201	0.7685
1m	0.9313	0.8424	0.9053	0.7339
5m	0.9312	0.8404	0.9091	0.7336
10m	0.9336	0.8491	0.9094	0.7429
5m-ss	0.9360	0.8313	0.9063	0.6978
10m-ss	0.9237	0.8148	0.6803	0.5320
1t	0.9314	0.8483	0.9050	0.7346
5t	0.9314	0.8492	0.9068	0.7309
10t	0.9324	0.8537	0.9072	0.7345
20t	0.9336	0.8528	0.9075	0.7234
50t	0.9339	0.8502	0.9105	0.7292

Median strength

Inference methods

Subclasses of instruments	AR	AR*	SS	SS*
1s	0.9380	0.8410	0.9165	0.7455
5s	0.9380	0.8345	0.9210	0.7505
30s	0.9415	0.8470	0.9210	0.7665
1m	0.9310	0.8360	0.9030	0.7340
5m	0.9300	0.8380	0.9050	0.7380
10m	0.9300	0.8480	0.9070	0.7310
5m-ss	0.9370	0.8445	0.9080	0.7240
10m-ss	0.9315	0.8330	0.9010	0.6900
1t	0.9340	0.8540	0.9050	0.7375
5t	0.9335	0.8530	0.9085	0.7375
10t	0.9330	0.8555	0.9100	0.7355
20t	0.9330	0.8525	0.9065	0.7165
50t	0.9340	0.8510	0.9115	0.7315

Maximum strength

Inference methods

Subclasses of instruments	AR	AR*	SS	SS*
1s	0.9420	0.8580	0.9180	0.7510
5s	0.9390	0.8410	0.9220	0.7600
30s	0.9440	0.8580	0.9240	0.7850
1m	0.9420	0.8580	0.9180	0.7510
5m	0.9410	0.8560	0.9220	0.7600
10m	0.9440	0.8560	0.9240	0.7850
5m-ss	0.9450	0.8660	0.9250	0.7930
10m-ss	0.9420	0.8530	0.9190	0.7480
1t	0.9400	0.8660	0.9150	0.7550
5t	0.9410	0.8600	0.9140	0.7460
10t	0.9390	0.8610	0.9150	0.7510
20t	0.9400	0.8620	0.9180	0.7490
50t	0.9400	0.8590	0.9180	0.7530

Minimum strength

Inference methods

Subclasses of instruments	AR	AR*	SS	SS*
1s	0.9250	0.8190	0.9010	0.7350
5s	0.9260	0.8320	0.9030	0.7380
30s	0.9310	0.8450	0.9130	0.7570
1m	0.9220	0.8270	0.8950	0.7140
5m	0.9230	0.8330	0.8980	0.7000
10m	0.9260	0.8440	0.8980	0.7090
5m-ss	0.9250	0.7700	0.8840	0.5500
10m-ss	0.8900	0.7400	0.0000	0.0000
1t	0.9220	0.8270	0.8950	0.7140
5t	0.9230	0.8330	0.8980	0.7000
10t	0.9260	0.8440	0.8980	0.7090
20t	0.9290	0.8420	0.9010	0.7020
50t	0.9290	0.8340	0.9000	0.7020

Figure 2. IBM: 2009-2013: Identification strength of different subclasses of HF instruments.

Note: The instrument set consists of a constant and a lag of instrument, $l = 1$. We use log squared transformation for the first 13 classes of instruments (RV-RSVP) given in Table 11. The strength of instrument i is defined as $d_i = 1 - (ub_i - lb_i)$. For each class, we consider the average, median, minimum, and maximum strength across the proposed inference methods [AR, AR^*, SS and SS^*]. These inference procedures are described in Section 3.1-3.4. We use $\tau = 0.2$ for SS-type tests and employ 99 Monte Carlo replications for point optimal type procedures.

A.3. Tables

Table 1. Corresponding values of λ_0

ϕ_0 \ ρ_0	-0.5	-0.4999	-0.45	-0.4	-0.35	-0.3	-0.25	-0.2	-0.15	-0.1	-0.05	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.4999	0.5
-1	-	2499.50	4.50	2.00	1.17	0.75	0.50	0.33	0.21	0.13	0.06	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.9999	-	2499.81	4.50	2.00	1.17	0.75	0.50	0.33	0.21	0.13	0.06	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.95	-	-	4.79	2.12	1.23	0.79	0.53	0.35	0.23	0.13	0.06	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.9	-	-	5.26	2.27	1.31	0.84	0.56	0.37	0.24	0.14	0.06	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.8	-	-	7.26	2.78	1.55	0.97	0.64	0.42	0.27	0.16	0.07	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.7	-	-	15.25	3.85	1.96	1.19	0.76	0.50	0.31	0.18	0.08	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.6	-	-	-	7.14	2.82	1.56	0.96	0.61	0.38	0.22	0.09	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.5	-	-	-	-	5.60	2.40	1.33	0.80	0.48	0.27	0.11	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.4	-	-	-	-	-	5.77	2.27	1.19	0.66	0.35	0.15	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.3	-	-	-	-	-	-	9.09	2.44	1.10	0.52	0.20	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.2	-	-	-	-	-	-	-	-	3.41	1.04	0.34	0.00	-	-	-	-	-	-	-	-	-	-	-
-0.1	-	-	-	-	-	-	-	-	-	-	1.01	0.00	-	-	-	-	-	-	-	-	-	-	-
0	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
0.1	-	-	-	-	-	-	-	-	-	-	-	0.00	1.01	-	-	-	-	-	-	-	-	-	-
0.2	-	-	-	-	-	-	-	-	-	-	-	0.00	0.34	1.04	3.41	-	-	-	-	-	-	-	-
0.3	-	-	-	-	-	-	-	-	-	-	-	0.00	0.20	0.52	1.10	2.44	9.09	-	-	-	-	-	-
0.4	-	-	-	-	-	-	-	-	-	-	-	0.00	0.15	0.35	0.66	1.19	2.27	5.77	-	-	-	-	-
0.5	-	-	-	-	-	-	-	-	-	-	-	0.00	0.11	0.27	0.48	0.80	1.33	2.40	5.60	∞	-	-	-
0.6	-	-	-	-	-	-	-	-	-	-	-	0.00	0.09	0.22	0.38	0.61	0.96	1.56	2.82	7.14	-	-	-
0.7	-	-	-	-	-	-	-	-	-	-	-	0.00	0.08	0.18	0.31	0.50	0.76	1.19	1.96	3.85	15.25	-	-
0.8	-	-	-	-	-	-	-	-	-	-	-	0.00	0.07	0.16	0.27	0.42	0.64	0.97	1.55	2.78	7.26	-	-
0.9	-	-	-	-	-	-	-	-	-	-	-	0.00	0.06	0.14	0.24	0.37	0.56	0.84	1.31	2.27	5.26	-	-
0.95	-	-	-	-	-	-	-	-	-	-	-	0.00	0.06	0.13	0.23	0.35	0.53	0.79	1.23	2.12	4.79	-	-
0.9999	-	-	-	-	-	-	-	-	-	-	-	0.00	0.06	0.13	0.21	0.33	0.50	0.75	1.17	2.00	4.50	2499.81	-
1	-	-	-	-	-	-	-	-	-	-	-	0.00	0.06	0.13	0.21	0.33	0.50	0.75	1.17	2.00	4.50	2499.50	∞

Table 2: Size of asymptotic t-type test for $H_0 : \phi = \phi_0$ (nominal level: 5%)

ϕ	$T = 100$	$T = 200$	$T = 300$	$T = 400$	$T = 500$	$T = 1000$	$T = 2000$	$T = 5000$
0.0000	0.21	0.06	0.01	0.01	0.01	0.01	0.00	0.00
0.1000	0.20	0.06	0.02	0.03	0.01	0.03	0.01	0.11
0.2000	0.19	0.07	0.03	0.03	0.05	0.18	0.44	1.73
0.3000	0.22	0.14	0.02	0.14	0.22	0.99	1.72	2.64
0.4000	0.21	0.27	0.42	0.73	0.97	2.22	2.44	2.61
0.5000	0.34	0.85	1.25	1.57	1.87	2.41	2.32	2.09
0.6000	0.85	1.74	2.00	2.18	2.35	2.17	2.02	1.68
0.7000	1.68	2.30	2.36	2.36	2.27	1.78	1.69	1.49
0.8000	2.62	2.44	2.63	2.25	2.29	2.02	1.92	1.57
0.9000	4.43	3.41	3.69	3.61	3.21	2.75	2.91	2.80
0.9500	7.38	5.56	5.12	4.69	4.35	3.36	3.27	3.42
0.9800	4.47	7.61	6.86	6.21	6.02	4.49	3.82	3.58
0.9850	2.00	5.36	6.99	7.03	6.89	5.07	4.23	3.86
0.9900	1.29	0.87	3.24	5.80	6.88	6.29	5.07	4.06
0.9950	25.64	0.38	0.02	0.03	0.20	6.78	7.03	4.97
0.9990	97.50	99.95	99.99	100.00	99.88	23.70	0.00	3.63
0.9999	99.54	100.00	100.00	100.00	100.00	100.00	100.00	100.00
1.0000	99.35	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Table 3. Size comparison of joint tests for $H_0: \phi = \phi_0, \rho = \rho_0$ (nominal level: 5%)

		$T=100$																																			
	ρ	0.1												0.2												0.3											
	ϕ	0.50			0.75			0.90			1.00			0.50			0.75			0.90			1.00			0.50			0.75			0.90			1.00		
	λ	0.27			0.17			0.14			0.13			0.80			0.46			0.37			0.33			2.40			1.07			0.84			0.75		
	σ_e	4.30			5.41			5.96			6.28			2.48			3.29			3.64			3.85			1.43			2.15			2.42			2.57		
	$\bar{\lambda}^2$	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10
$l=2$	$\pi_1[1,1]$	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50
	AR	5.1	5.0	5.4	5.0	5.0	4.6	5.1	5.1	4.5	5.2	5.1	8.3	4.9	5.0	8.7	5.1	5.0	5.9	5.1	5.0	4.0	5.0	5.0	6.5	4.8	5.1	12.3	5.0	5.2	9.4	5.1	5.2	4.8	5.3	5.2	5.5
	AR^*	4.8	4.7	4.7	4.7	4.7	3.9	4.6	4.7	3.8	4.6	4.7	7.0	4.1	4.1	5.8	3.9	4.0	3.5	3.8	3.9	2.3	3.8	3.9	3.9	2.6	2.6	5.7	2.7	2.7	3.4	2.5	2.5	1.3	2.6	2.7	1.7
	SS	1.4	1.4	1.3	1.3	1.4	1.2	1.3	1.4	1.2	1.3	1.2	2.5	1.3	1.2	2.8	1.3	1.3	1.7	1.2	1.2	0.9	1.3	1.4	1.7	1.2	1.3	3.8	1.1	1.2	2.8	1.3	1.2	0.9	1.5	1.5	1.3
	SS^*	4.6	4.7	4.7	4.6	4.6	3.5	4.7	4.4	3.7	4.7	4.7	6.3	4.0	4.1	5.7	3.8	4.1	3.4	3.9	3.9	2.2	4.5	4.4	4.0	2.9	3.0	5.4	2.8	3.0	3.3	2.8	2.8	1.2	3.5	3.4	1.8
$l=5$	$\pi_1[1,1]$	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32
	AR	5.5	5.4	5.5	5.3	5.3	5.2	5.4	5.5	5.2	4.6	4.6	6.5	5.6	5.6	7.1	5.3	5.3	5.8	5.4	5.4	5.2	4.7	4.7	5.6	5.4	5.4	8.6	5.5	5.6	7.4	5.6	5.5	5.4	5.1	5.1	5.0
	AR^*	5.1	5.0	5.0	5.1	5.0	4.7	5.0	4.9	4.5	4.2	4.3	5.7	4.0	4.1	4.8	3.8	3.9	3.7	3.9	3.8	3.1	3.2	3.2	3.2	2.1	2.1	3.4	2.2	2.1	2.3	2.2	2.1	1.5	1.9	1.8	1.4
	SS	0.1	0.1	0.1	0.1	0.1	0.0	0.1	0.1	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.1	0.1	0.1	0.1	0.1	0.2	0.1	0.1	0.1	0.1	0.1	0.0	0.1	0.1	0.1
	SS^*	4.6	4.7	4.6	4.5	4.6	4.0	4.7	4.7	4.3	4.8	4.9	5.9	4.1	4.2	5.2	4.1	4.0	3.7	4.2	4.2	3.0	4.0	4.1	4.0	3.1	3.0	4.3	2.9	2.9	3.3	3.0	3.2	1.8	3.2	3.2	2.4
$l=10$	$\pi_1[1,1]$	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22
	AR	4.8	4.8	4.8	5.0	5.0	4.8	5.1	5.1	4.7	5.0	5.1	6.4	5.0	4.9	5.8	4.9	4.9	5.1	4.8	4.8	4.6	5.0	5.1	5.9	4.5	4.5	6.2	4.9	4.9	5.8	4.8	4.9	5.0	5.3	5.3	5.7
	AR^*	4.4	4.4	4.3	4.5	4.5	4.2	4.5	4.5	4.3	4.4	4.4	5.6	3.0	3.0	3.7	3.2	3.2	3.2	3.1	3.1	2.8	3.2	3.2	3.7	1.4	1.4	1.8	1.4	1.4	1.5	1.4	1.4	1.1	1.4	1.4	1.4
	SS	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	SS^*	4.9	4.9	4.6	4.5	4.5	4.4	4.7	4.6	4.5	5.2	5.2	5.6	4.4	4.3	4.8	4.1	4.0	4.0	4.0	4.0	3.5	4.8	4.8	4.5	3.5	3.5	3.8	3.2	3.1	3.2	3.1	3.0	2.1	3.4	3.4	3.0
		$T=200$																																			
	ρ	0.1												0.2												0.3											
	ϕ	0.50			0.75			0.90			1.00			0.50			0.75			0.90			1.00			0.50			0.75			0.90			1.00		
	λ	0.27			0.17			0.14			0.13			0.80			0.46			0.37			0.33			2.40			1.07			0.84			0.75		
	σ_e	4.30			5.41			5.96			6.28			2.48			3.29			3.64			3.85			1.43			2.15			2.42			2.57		
	$\bar{\lambda}^2$	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10
$l=2$	$\pi_1[1,1]$	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35
	AR	5.4	5.4	6.6	5.2	5.2	5.7	5.2	5.2	4.8	5.1	5.1	7.1	5.5	5.4	11.3	5.4	5.4	8.7	5.4	5.3	5.6	5.3	5.3	6.3	5.3	5.6	16.0	5.2	5.3	14.6	5.3	5.3	7.6	5.4	5.5	5.7
	AR^*	5.0	5.0	5.9	5.0	4.9	4.9	4.9	4.8	4.3	4.8	4.8	6.4	4.0	4.1	8.3	4.0	4.0	6.1	3.9	3.9	3.6	4.1	4.1	4.1	2.7	2.6	8.5	2.5	2.6	7.0	2.6	2.6	2.7	2.9	2.9	2.1
	SS	1.7	1.6	2.0	1.7	1.7	1.6	1.7	1.8	1.4	1.4	1.4	1.9	1.8	1.7	4.0	1.6	1.6	3.0	1.5	1.7	1.4	1.5	1.5	1.7	1.6	1.6	5.5	1.6	1.6	5.1	1.5	1.5	2.1	1.6	1.7	1.5
	SS^*	5.0	4.9	5.8	5.1	5.0	4.9	5.2	5.1	4.1	4.8	4.7	6.0	4.5	4.2	7.8	4.5	4.5	5.8	4.4	4.3	3.6	4.2	4.3	4.1	3.2	3.0	7.9	3.2	3.1	6.8	3.1	3.1	2.8	3.3	3.3	2.3
$l=5$	$\pi_1[1,1]$	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22
	AR	4.9	4.9	5.8	5.0	4.9	5.1	5.0	5.0	4.8	4.9	5.0	6.2	4.9	5.0	8.4	5.1	5.1	6.9	5.0	5.1	5.3	4.8	4.8	5.4	5.0	5.1	10.5	5.1	5.1	9.7	5.1	5.2	6.4	5.0	5.1	5.2
	AR^*	4.6	4.5	5.2	4.6	4.6	4.6	4.6	4.6	4.3	4.6	4.7	5.7	3.4	3.4	5.7	3.6	3.5	4.3	3.5	3.6	3.4	3.3	3.4	3.6	1.8	1.9	4.3	1.8	1.9	3.7	2.1	2.0	2.2	1.9	1.8	1.7
	SS	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.1	0.1	0.1	0.1	0.1	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.3	0.1	0.1	0.3	0.1	0.1	0.1	0.2	0.2	0.1
	SS^*	5.1	4.9	5.3	4.8	4.9	4.4	5.0	4.7	4.0	4.7	4.6	5.7	4.1	4.1	6.2	4.4	4.2	5.0	4.3	4.0	3.3	4.1	4.2	4.5	3.2	3.2	5.5	3.2	3.3	4.9	3.3	3.1	2.7	2.9	3.1	2.9
$l=10$	$\pi_1[1,1]$	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16
	AR	5.1	5.0	5.4	5.1	5.1	5.1	5.1	5.1	4.8	5.2	5.2	5.9	5.1	5.1	6.8	5.0	5.0	6.0	5.0	5.0	5.1	5.3	5.2	5.7	5.0	5.1	8.1	5.2	5.1	7.6	5.2	5.1	5.7	5.3	5.4	5.6
	AR^*	4.5	4.6	4.8	4.3	4.4	4.5	4.5	4.5	4.4	4.9	4.9	5.2	3.0	3.0	4.0	3.2	3.2	3.7	3.3	3.3	3.1	3.5	3.5	3.5	1.4	1.4	2.3	1.4	1.4	1.9	1.5	1.5	1.4	1.5	1.5	1.4
	SS	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	SS^*	4.6	4.7	4.8	4.7	4.8	4.4	4.9	4.9	4.1	4.4	4.3	4.9	4.0	3.9	5.3	4.0	4.0	4.7	4.5	4.4	3.7	3.9	3.9	4.1	2.8	2.8	4.3	3.1	3.0	4.3	3.1	3.0	3.0	2.8	3.0	2.7

Table 4. Power comparison of joint tests for $H_0: \phi_0 = 0.50, \rho_0 = 0.30$ (nominal level: 5%)

$T = 100$	ρ	0.1												0.2												0.3											
	ϕ	0.50			0.75			0.90			1.00			0.50			0.75			0.90			1.00			0.50			0.75			0.90			1.00		
	λ	0.27			0.17			0.14			0.13			0.80			0.46			0.37			0.33			2.40			1.07			0.84			0.75		
	σ_e	4.30			5.41			5.96			6.28			2.48			3.29			3.64			3.85			1.43			2.15			2.42			2.57		
	$\bar{\lambda}^2$	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10
	$\pi_1[1,1]$	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50
$l = 2$	AR	7.6	8.0	29.9	12.6	14.7	91.4	16.4	20.8	99.7	20.2	26.8	100.0	6.4	6.6	21.8	10.9	12.8	86.9	15.5	19.4	99.4	19.7	26.5	100.0	4.8	5.1	12.3	8.9	10.1	76.5	13.3	17.1	98.6	19.1	25.1	100.0
	AR^*	0.8	0.8	1.5	63.7	64.0	75.3	98.4	98.5	99.3	100.0	100.0	100.0	0.3	0.3	1.1	25.7	26.3	64.9	89.6	89.5	98.4	99.5	99.5	100.0	2.6	2.6	5.7	5.1	5.8	62.1	8.1	10.4	96.4	12.5	17.2	99.8
	SS	2.1	2.3	13.6	3.9	4.6	69.8	5.5	6.8	90.7	7.2	9.3	89.7	1.6	1.6	8.6	3.1	3.9	61.2	5.2	6.0	88.0	6.7	8.7	88.6	1.2	1.3	3.8	2.4	2.8	46.1	4.4	5.3	81.9	6.5	8.5	86.2
	SS^*	0.8	0.8	1.7	57.4	57.5	67.2	96.5	96.4	97.7	99.7	99.7	99.8	0.4	0.4	1.4	24.5	25.1	54.8	84.9	85.1	94.4	98.5	98.5	99.3	2.9	3.0	5.4	4.6	5.3	48.0	7.4	8.4	83.0	9.9	11.8	87.3
	$\pi_1[1,1]$	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32
$l = 5$	AR	9.4	9.6	21.3	17.4	18.5	81.6	24.5	27.0	98.7	29.5	34.1	99.9	7.4	7.4	15.0	14.6	15.9	74.7	22.2	25.0	97.8	28.7	32.8	99.9	5.4	5.4	8.6	11.6	12.4	61.0	19.0	21.3	95.8	27.8	31.2	99.8
	AR^*	1.3	1.3	1.8	56.4	56.7	70.4	97.3	97.5	98.9	99.9	99.9	100.0	0.6	0.7	1.0	19.2	19.8	53.2	78.5	79.1	96.5	98.0	98.0	99.8	2.1	2.1	3.4	5.3	5.8	43.1	10.5	11.8	90.4	16.4	20.0	99.4
	SS	0.1	0.1	1.3	0.4	0.5	21.6	0.9	1.0	51.2	1.2	1.5	54.0	0.1	0.1	0.6	0.4	0.4	15.5	0.7	0.9	44.8	1.2	1.5	51.0	0.1	0.1	0.2	0.3	0.3	8.4	0.6	0.7	34.8	0.9	1.1	47.0
	SS^*	0.9	0.9	1.4	57.5	57.5	64.0	96.5	96.5	97.3	99.6	99.6	99.7	0.5	0.5	1.0	24.9	25.2	44.1	85.5	85.6	92.5	98.1	98.1	99.0	3.1	3.0	4.3	4.9	5.1	30.0	7.3	7.7	61.0	9.3	10.0	67.1
	$\pi_1[1,1]$	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22
$l = 10$	AR	10.1	10.3	17.8	23.1	24.0	74.5	35.3	37.1	97.5	43.2	46.6	99.8	7.1	7.2	12.0	18.9	19.6	65.1	31.6	33.5	96.1	41.9	45.4	99.8	4.5	4.5	6.2	13.9	14.2	48.6	26.3	28.0	91.6	39.6	42.9	99.6
	AR^*	1.4	1.3	1.6	46.3	46.6	62.1	94.6	94.9	97.9	99.7	99.7	99.9	0.7	0.7	0.9	15.6	15.9	41.3	64.5	65.4	93.4	94.1	94.4	99.6	1.4	1.4	1.8	5.2	5.4	28.1	12.7	14.0	81.5	22.6	24.7	98.3
	SS	0.0	0.0	0.0	0.0	0.0	1.7	0.0	0.0	10.3	0.1	0.1	14.0	0.0	0.0	0.0	0.0	0.0	1.0	0.1	0.1	7.6	0.1	0.1	12.8	0.0	0.0	0.0	0.0	0.0	0.4	0.0	0.0	4.9	0.0	0.1	10.6
	SS^*	0.8	0.8	1.3	57.3	57.4	61.0	96.3	96.3	96.7	99.6	99.6	99.7	0.4	0.4	0.7	25.2	25.2	36.1	84.9	84.9	89.1	98.1	98.1	98.7	3.5	3.5	3.8	5.2	5.0	18.4	7.6	8.0	40.0	9.3	10.0	47.6

$T = 200$	ρ	0.1												0.2												0.3											
	ϕ	0.50			0.75			0.90			1.00			0.50			0.75			0.90			1.00			0.50			0.75			0.90			1.00		
	λ	0.27			0.17			0.14			0.13			0.80			0.46			0.37			0.33			2.40			1.07			0.84			0.75		
	σ_e	4.30			5.41			5.96			6.28			2.48			3.29			3.64			3.85			1.43			2.15			2.42			2.57		
	$\bar{\lambda}^2$	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10
	$\pi_1[1,1]$	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35
$l = 2$	AR	7.9	8.3	38.2	12.8	15.2	97.0	17.3	22.1	100.0	20.7	27.5	100.0	6.6	7.0	28.3	11.2	13.3	94.5	16.3	20.6	100.0	20.8	27.0	100.0	5.3	5.6	16.0	9.5	10.7	88.1	14.4	18.3	99.9	20.2	26.5	100.0
	AR^*	0.2	0.2	0.6	90.8	90.7	94.2	100.0	100.0	100.0	100.0	100.0	100.0	0.0	0.0	0.2	52.3	52.8	82.6	99.7	99.7	100.0	100.0	100.0	100.0	2.7	2.6	8.5	5.1	6.3	78.6	8.7	11.9	99.6	13.2	18.8	100.0
	SS	2.6	2.5	19.0	4.3	4.9	82.7	6.2	7.9	95.9	7.3	9.7	89.9	2.2	2.1	12.3	3.7	4.1	75.6	5.6	7.4	94.6	7.3	9.5	89.7	1.6	1.6	5.5	3.0	3.1	62.5	5.0	6.4	92.1	7.4	9.3	88.4
	SS^*	0.3	0.3	0.7	85.1	85.2	89.0	99.9	99.9	100.0	100.0	100.0	100.0	0.0	0.0	0.3	47.9	48.4	73.6	99.3	99.3	99.8	100.0	100.0	100.0	3.2	3.0	7.9	5.3	5.5	66.5	7.9	9.5	92.8	10.8	13.0	89.6
	$\pi_1[1,1]$	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22
$l = 5$	AR	9.1	9.4	28.4	17.9	19.2	92.2	26.1	29.0	99.9	32.7	37.5	100.0	7.2	7.3	19.6	15.5	16.6	87.6	24.0	27.2	99.7	32.2	37.4	100.0	5.0	5.1	10.5	12.0	12.8	76.3	20.9	23.6	99.4	31.4	36.4	100.0
	AR^*	0.3	0.3	0.7	87.9	87.9	92.2	100.0	100.0	100.0	100.0	100.0	100.0	0.1	0.1	0.3	40.4	41.0	74.7	99.4	99.5	99.9	100.0	100.0	100.0	1.8	1.9	4.3	5.6	6.0	60.6	11.1	13.2	98.0	19.2	22.9	100.0
	SS	0.2	0.2	2.3	0.5	0.6	38.8	0.9	1.2	71.9	1.3	1.6	63.5	0.2	0.2	1.2	0.5	0.5	29.7	0.8	0.9	67.5	1.2	1.6	61.8	0.1	0.1	0.3	0.3	0.3	16.9	0.7	0.8	58.2	1.3	1.6	59.1
	SS^*	0.3	0.3	0.6	85.3	85.2	87.9	100.0	100.0	100.0	100.0	100.0	100.0	0.1	0.1	0.2	47.5	47.6	66.8	99.3	99.4	99.7	100.0	100.0	100.0	3.2	3.2	5.5	5.4	5.6	46.7	7.7	8.2	79.1	10.1	11.2	74.7
	$\pi_1[1,1]$	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16
$l = 10$	AR	11.1	11.1	24.0	24.8	26.0	88.0	38.2	40.9	99.8	48.3	51.8	100.0	8.0	7.8	15.8	20.6	21.8	81.7	34.7	37.3	99.5	47.7	51.3	100.0	5.0	5.1	8.1	15.1	15.9	67.7	29.3	31.4	98.7	46.4	50.1	100.0
	AR^*	0.4	0.4	0.6	84.4	84.5	89.9	100.0	100.0	100.0	100.0	100.0	100.0	0.1	0.1	0.3	33.5	33.9	66.4	98.1	98.3	99.8	100.0	100.0	100.0	1.4	1.4	2.3	5.5	5.8	45.4	14.0	15.5	95.5	26.8	30.2	99.9
	SS	0.0	0.0	0.1	0.0	0.0	6.3	0.1	0.1	28.1	0.2	0.2	26.5	0.0	0.0	0.0	0.0	0.0	3.8	0.1	0.1	23.2	0.2	0.2	25.0	0.0	0.0	0.0	0.0	0.0	1.4	0.1	0.1	15.7	0.1	0.2	23.1
	SS^*	0.3	0.3	0.4	85.5	85.6	87.3	100.0	100.0	100.0	100.0	100.0	100.0	0.1	0.1	0.1	47.6	47.6	61.1	99.4	99.4	99.6	100.0	100.0	100.0	2.8	2.8	4.3	5.4	5.3	31.2	8.1	8.5	62.5	10.5	11.2	59.5

Table 5. Size comparison of joint tests for $H_0: \phi = \phi_0, \rho = \rho_0$ (nominal level: 5%): misspecified model

$T = 100$

	ρ	0.1											
	ϕ	0.50			0.75			0.90			1.00		
	λ	0.27			0.17			0.14			0.13		
	σ_e	4.30			5.41			5.96			6.28		
	$\bar{\lambda}^2$	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10
$l=2$	$\pi_1[1,1]$	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50
	AR	5.3	5.4	5.5	5.4	5.4	4.9	5.3	5.4	4.8	5.4	5.3	8.3
	AR^*	4.8	4.8	4.9	4.8	4.8	4.2	4.8	4.9	4.1	5.0	4.7	7.0
	SS	1.4	1.4	1.5	1.4	1.4	1.2	1.4	1.3	1.2	1.4	1.4	2.3
	SS^*	5.1	4.8	4.6	4.7	4.8	3.8	4.7	4.6	3.7	5.1	4.9	6.7
$l=5$	$\pi_1[1,1]$	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32
	AR	5.1	5.1	5.2	5.1	5.1	4.6	5.2	5.1	4.5	4.9	5.0	6.5
	AR^*	4.7	4.7	4.6	4.6	4.6	4.2	4.5	4.7	4.2	4.4	4.4	5.4
	SS	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.0	0.1	0.1
	SS^*	5.2	5.0	4.9	4.9	4.7	4.3	5.0	5.0	4.2	4.8	4.8	5.8
$l=10$	$\pi_1[1,1]$	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22
	AR	5.1	5.0	5.1	5.0	5.0	4.9	4.9	4.9	4.9	5.0	5.0	6.2
	AR^*	4.6	4.5	4.6	4.7	4.7	4.3	4.6	4.7	4.3	4.3	4.4	5.2
	SS	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	SS^*	5.0	5.0	5.1	4.9	4.9	4.7	4.8	4.7	4.5	4.8	4.9	5.8

	ρ	0.2											
	ϕ	0.50			0.75			0.90			1.00		
	λ	0.80			0.46			0.37			0.33		
	σ_e	2.48			3.29			3.64			3.85		
	$\bar{\lambda}^2$	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10
$l=2$	$\pi_1[1,1]$	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50
	AR	5.1	5.2	8.8	5.3	5.3	6.5	5.3	5.3	4.2	5.4	5.3	6.5
	AR^*	4.0	4.1	6.2	4.0	4.1	4.1	4.0	4.0	2.6	4.3	4.4	4.0
	SS	1.4	1.4	2.6	1.4	1.4	1.6	1.5	1.4	1.0	1.4	1.3	1.5
	SS^*	4.2	4.2	5.7	4.2	4.3	3.3	4.1	4.1	2.2	4.4	4.5	3.9
$l=5$	$\pi_1[1,1]$	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32
	AR	5.1	5.2	6.6	5.1	5.1	5.4	5.0	5.0	4.4	5.0	5.0	5.6
	AR^*	3.8	3.7	4.5	3.5	3.6	3.4	3.4	3.5	2.7	3.4	3.4	3.2
	SS	0.0	0.1	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
	SS^*	4.6	4.6	5.1	4.4	4.2	3.7	4.6	4.5	3.1	4.2	4.2	4.0
$l=10$	$\pi_1[1,1]$	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22
	AR	5.2	5.1	5.9	4.7	4.8	5.1	4.8	4.8	4.5	5.2	5.1	5.9
	AR^*	3.3	3.3	3.6	3.3	3.3	3.1	3.3	3.3	2.8	3.3	3.3	3.4
	SS	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	SS^*	4.6	4.5	5.1	4.2	4.3	4.0	4.3	4.5	3.4	4.3	4.2	4.5

	ρ	0.3											
	ϕ	0.50			0.75			0.90			1.00		
	λ	2.40			1.07			0.84			0.75		
	σ_e	1.43			2.15			2.42			2.57		
	$\bar{\lambda}^2$	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10
$l=2$	$\pi_1[1,1]$	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50
	AR	5.4	5.7	12.7	5.3	5.3	9.9	5.2	5.2	4.8	5.7	5.7	5.5
	AR^*	2.7	2.8	6.1	2.8	2.7	3.7	2.7	2.6	1.5	2.9	2.9	1.8
	SS	1.5	1.5	4.1	1.5	1.5	2.9	1.6	1.6	0.9	1.3	1.3	1.2
	SS^*	3.1	3.2	5.8	3.0	3.3	3.4	3.2	3.2	1.3	3.2	3.2	1.9
$l=5$	$\pi_1[1,1]$	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32
	AR	5.0	5.1	8.8	5.1	5.2	7.1	5.2	5.3	5.0	5.1	5.1	5.2
	AR^*	2.2	2.3	3.3	2.0	2.0	2.4	2.0	2.1	1.4	2.0	2.1	1.6
	SS	0.1	0.1	0.2	0.1	0.1	0.2	0.1	0.1	0.1	0.1	0.1	0.1
	SS^*	3.1	3.2	4.3	3.3	3.3	3.3	3.3	3.3	1.9	3.2	3.1	2.5
$l=10$	$\pi_1[1,1]$	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22
	AR	5.2	5.2	6.7	5.0	4.9	5.7	4.7	4.7	4.6	5.4	5.4	5.9
	AR^*	1.6	1.6	2.0	1.6	1.6	1.6	1.6	1.5	1.4	1.8	1.7	1.4
	SS	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	SS^*	3.2	3.3	3.9	3.3	3.4	3.4	3.2	3.3	2.5	3.3	3.3	3.0

$T = 200$

	ρ	0.1											
	ϕ	0.50			0.75			0.90			1.00		
	λ	0.27			0.17			0.14			0.13		
	σ_e	4.30			5.41			5.96			6.28		
	$\bar{\lambda}^2$	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10
$l=2$	$\pi_1[1,1]$	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35
	AR	4.9	4.8	6.2	4.7	4.8	5.3	4.8	4.8	4.6	4.7	4.7	6.8
	AR^*	4.7	4.7	5.5	4.6	4.5	4.6	4.6	4.6	4.0	4.7	4.7	6.0
	SS	1.4	1.5	2.0	1.5	1.5	1.6	1.6	1.6	1.2	1.3	1.4	2.1
	SS^*	4.8	4.8	5.8	4.7	4.9	4.7	4.7	4.6	4.1	4.7	4.6	5.9
$l=5$	$\pi_1[1,1]$	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22
	AR	4.8	4.8	5.6	4.7	4.8	5.2	4.7	4.7	4.7	5.1	5.1	6.3
	AR^*	4.4	4.4	4.9	4.4	4.4	4.5	4.3	4.4	4.3	4.5	4.5	5.7
	SS	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
	SS^*	4.8	4.8	5.4	4.7	4.6	4.5	4.6	4.4	4.1	4.8	4.8	5.9
$l=10$	$\pi_1[1,1]$	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16
	AR	5.2	5.2	5.6	5.3	5.2	5.2	5.2	5.2	5.1	5.0	5.0	5.9
	AR^*	4.9	4.9	5.1	4.9	4.9	4.8	4.7	4.7	4.5	4.6	4.6	5.3
	SS	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	SS^*	4.8	4.9	5.1	4.8	4.8	4.8	5.0	4.9	4.4	4.9	4.8	5.0

	ρ	0.2											
	ϕ	0.50			0.75			0.90			1.00		
	λ	0.80			0.46			0.37			0.33		
	σ_e	2.48			3.29			3.64			3.85		
	$\bar{\lambda}^2$	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10
$l=2$	$\pi_1[1,1]$	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35
	AR	5.1	4.9	11.2	5.0	5.1	8.5	5.0	5.0	5.4	4.7	4.8	6.0
	AR^*	3.8	4.0	8.2	3.9	4.1	5.6	3.9	4.0	3.6	3.7	3.8	3.9
	SS	1.3	1.4	4.1	1.7	1.5	2.9	1.7	1.6	1.7	1.3	1.3	1.8
	SS^*	3.9	4.3	8.5	4.1	4.2	5.8	3.8	4.0	3.4	4.0	4.0	3.9
$l=5$	$\pi_1[1,1]$	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22
	AR	5.0	5.0	7.7	4.9	4.9	6.3	4.9	4.9	5.0	5.0	5.0	5.5
	AR^*	3.5	3.5	5.5	3.4	3.4	4.3	3.3	3.4	3.1	3.6	3.5	3.5
	SS	0.1	0.1	0.3	0.1	0.1	0.2	0.1	0.1	0.1	0.1	0.1	0.1
	SS^*	4.3	4.3	6.4	4.1	4.0	5.2	4.0	4.1	3.5	4.1	4.2	4.3
$l=10$	$\pi_1[1,1]$	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16
	AR	5.2	5.3	7.2	5.5	5.5	6.5	5.3	5.3	5.4	5.1	5.2	5.6
	AR^*	3.4	3.4	4.4	3.4	3.4	3.7	3.4	3.3	3.1	3.1	3.3	3.2
	SS	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	SS^*	4.2	4.2	5.5	4.4	4.2	4.6	4.5	4.4	3.8	4.3	4.4	4.2

	ρ	0.3											
	ϕ	0.50			0.75			0.90			1.00		
	λ	2.40			1.07			0.84			0.75		
	σ_e	1.43			2.15			2.42			2.57		
	$\bar{\lambda}^2$	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10
$l=2$	$\pi_1[1,1]$	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35
	AR	5.1	5.2	16.1	5.0	5.0	14.4	5.1	5.1	7.5	4.8	4.8	5.4
	AR^*	2.5	2.5	8.8	2.4	2.6	6.8	2.5	2.5	2.8	2.4	2.3	1.9
	SS	1.3	1.5	5.9	1.6	1.6	5.4	1.8	1.7	2.2	1.4	1.4	1.4
	SS^*	2.9	3.1	8.4	3.2	3.1	7.1	3.3	3.4	2.8	2.8	2.7	2.2
$l=5$	$\pi_1[1,1]$	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22
	AR	5.0	5.1	10.7	4.9	4.9	9.5	4.9	4.9	6.0	5.2	5.2	5.3
	AR^*	1.9	1.8	4.1	1.8	1.8	3.5	1.9	1.9	2.0	1.9	1.9	1.6
	SS	0.1	0.1	0.4	0.0	0.1	0.4	0.1	0.1	0.1	0.1	0.1	0.0
	SS^*	3.0	3.1	5.7	3.0	3.0	5.3	2.9	2.8	2.8	2.9	2.9	2.5
$l=10$	$\pi_1[1,1]$	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16
	AR	4.9	4.9	8.2	5.4	5.4	8.2	5.5	5.6	6.1	5.6	5.7	5.8
	AR^*	1.5	1.5	2.3	1.6	1.5	2.1	1.5	1.5	1.5	1.4	1.4	1.3
	SS	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	SS^*	3.0	2.9	4.2	3.0	3.0	4.4	3.3	3.3	2.8	3.2	3.3	2.9

Table 6. Power comparison of joint tests for $H_0 : \phi_0 = 0.50, \rho_0 = 0.30$ (nominal level: 5%): misspecified model

$T = 100$

	ρ	0.1												0.2												0.3											
	ϕ	0.50			0.75			0.90			1.00			0.50			0.75			0.90			1.00			0.50			0.75			0.90			1.00		
	λ	0.27			0.17			0.14			0.13			0.80			0.46			0.37			0.33			2.40			1.07			0.84			0.75		
	σ_e	4.30			5.41			5.96			6.28			2.48			3.29			3.64			3.85			1.43			2.15			2.42			2.57		
	$\bar{\lambda}^2$	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10
$l=2$	$\pi_1[1,1]$	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50	0	0.05	0.50
	AR	7.8	8.3	29.4	12.7	14.6	91.1	16.5	20.7	99.7	20.2	27.0	100.0	6.9	7.1	21.3	11.4	13.0	86.8	15.4	19.3	99.5	19.9	26.4	100.0	5.4	5.7	12.7	9.5	11.1	76.5	14.0	17.2	98.5	18.8	25.1	100.0
	AR^*	1.0	1.0	1.9	64.4	64.3	75.7	98.4	98.4	99.3	99.9	99.9	100.0	0.5	0.5	1.6	26.3	27.1	64.7	89.1	89.7	98.4	99.4	99.4	99.9	2.7	2.8	6.1	5.4	6.1	62.0	8.4	11.0	96.5	12.5	17.2	99.8
	SS	2.4	2.3	12.6	4.0	4.5	69.5	5.7	6.7	90.7	7.1	9.3	89.4	2.0	2.0	8.5	3.3	3.9	60.6	5.2	6.1	88.1	7.0	8.9	88.0	1.5	1.5	4.1	2.6	3.1	46.9	4.5	5.6	82.2	6.8	8.2	86.1
	SS^*	1.0	0.9	1.7	57.6	57.7	67.4	96.5	96.4	97.9	99.7	99.7	99.8	0.5	0.5	1.6	25.6	25.8	55.0	85.0	85.2	94.8	98.2	98.3	99.2	3.1	3.2	5.8	5.0	5.7	48.8	7.3	8.6	83.3	9.9	11.8	87.3
$l=5$	$\pi_1[1,1]$	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32	0	0.03	0.32
	AR	9.1	9.4	20.7	17.3	18.4	81.8	24.4	27.0	98.8	29.7	34.2	99.9	7.2	7.2	14.5	14.5	15.6	74.4	22.3	24.4	98.0	28.5	33.0	99.9	5.0	5.1	8.8	11.3	11.9	60.6	19.1	21.2	95.7	27.2	31.7	99.7
	AR^*	1.3	1.3	1.7	56.4	56.6	69.8	97.2	97.3	98.8	99.9	99.9	100.0	0.7	0.7	1.2	19.5	20.1	52.8	77.7	78.2	96.4	97.9	97.9	99.9	2.2	2.3	3.3	5.4	5.6	42.7	10.3	11.8	90.2	16.3	19.2	99.3
	SS	0.1	0.1	1.5	0.4	0.5	21.8	0.9	1.0	51.0	1.2	1.6	54.0	0.1	0.1	0.6	0.4	0.4	15.6	0.8	0.8	45.1	1.2	1.5	51.3	0.1	0.1	0.2	0.3	0.3	8.3	0.6	0.8	35.1	1.2	1.5	47.5
	SS^*	1.0	1.0	1.4	56.9	57.2	64.1	96.2	96.2	97.1	99.6	99.7	99.7	0.4	0.5	1.1	25.7	25.7	44.7	85.6	85.6	91.8	98.3	98.4	98.9	3.1	3.2	4.3	5.1	5.1	30.2	7.4	7.8	61.3	9.4	10.3	67.4
$l=10$	$\pi_1[1,1]$	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22
	AR	10.8	10.8	17.9	23.4	24.5	74.8	35.5	37.5	97.4	43.9	46.6	99.8	7.7	7.6	11.9	19.4	20.2	65.8	31.8	33.9	95.7	42.1	45.4	99.8	5.2	5.2	6.7	14.2	14.8	50.9	26.9	28.9	92.2	40.1	43.2	99.5
	AR^*	1.5	1.5	1.8	46.7	47.0	62.2	94.6	94.6	97.7	99.7	99.7	99.9	0.8	0.8	1.1	15.8	16.3	42.8	65.0	66.3	93.1	94.1	94.1	99.6	1.6	1.6	2.0	5.5	5.7	29.6	13.1	14.0	82.3	22.3	24.9	98.3
	SS	0.0	0.0	0.0	0.0	0.0	1.9	0.0	0.0	10.1	0.1	0.2	13.7	0.0	0.0	0.0	0.0	0.0	1.1	0.0	0.0	7.8	0.1	0.1	12.6	0.0	0.0	0.0	0.0	0.0	0.4	0.0	0.0	4.9	0.1	0.1	10.5
	SS^*	0.9	0.9	1.2	58.0	57.6	61.6	96.2	96.2	96.6	99.7	99.7	99.7	0.3	0.4	0.7	25.4	25.3	37.0	84.5	84.5	89.2	98.2	98.2	98.7	3.2	3.3	3.9	5.1	5.1	19.1	7.1	7.2	40.1	9.3	9.6	48.3

$T = 200$

	ρ	0.1												0.2												0.3											
	ϕ	0.50			0.75			0.90			1.00			0.50			0.75			0.90			1.00			0.50			0.75			0.90			1.00		
	λ	0.27			0.17			0.14			0.13			0.80			0.46			0.37			0.33			2.40			1.07			0.84			0.75		
	σ_e	4.30			5.41			5.96			6.28			2.48			3.29			3.64			3.85			1.43			2.15			2.42			2.57		
	$\bar{\lambda}^2$	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10	0	0.1	10
$l=2$	$\pi_1[1,1]$	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35	0	0.04	0.35
	AR	7.5	7.7	38.5	12.7	14.4	96.8	17.1	22.0	100.0	20.7	27.4	100.0	6.4	6.4	28.2	11.1	12.6	94.3	16.0	20.7	100.0	20.4	26.9	100.0	5.1	5.2	16.1	9.2	10.3	87.9	14.4	18.5	99.9	19.8	26.5	100.0
	AR^*	0.3	0.4	0.7	90.5	90.6	94.0	100.0	100.0	100.0	100.0	100.0	100.0	0.0	0.0	0.4	52.7	53.4	82.3	99.7	99.8	100.0	100.0	100.0	100.0	2.5	2.5	8.8	4.8	6.1	78.5	8.8	11.8	99.6	13.2	18.7	100.0
	SS	2.3	2.4	19.5	4.1	5.0	82.5	6.5	8.0	96.0	7.4	9.7	90.2	1.8	1.9	12.8	3.6	4.2	75.4	5.8	7.6	94.4	7.2	9.4	89.8	1.3	1.5	5.9	2.9	3.3	62.5	5.1	6.4	91.6	6.9	9.0	88.6
	SS^*	0.4	0.4	0.7	85.4	85.5	89.0	100.0	100.0	100.0	100.0	100.0	100.0	0.1	0.1	0.3	48.6	49.1	73.5	99.2	99.2	99.8	100.0	100.0	100.0	2.9	3.1	8.4	5.3	5.4	65.7	7.9	9.5	92.6	10.3	13.1	89.9
$l=5$	$\pi_1[1,1]$	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22	0	0.02	0.22
	AR	9.0	9.1	28.5	17.8	19.3	92.5	25.8	29.1	99.9	32.6	37.5	100.0	6.9	7.0	19.7	15.5	16.5	87.4	23.9	27.2	99.9	32.0	37.0	100.0	5.0	5.1	10.7	12.0	12.6	76.3	20.6	23.6	99.4	31.4	36.3	100.0
	AR^*	0.4	0.4	0.7	87.7	87.8	92.2	100.0	100.0	100.0	100.0	100.0	100.0	0.1	0.1	0.4	41.3	41.8	74.2	99.2	99.2	100.0	100.0	100.0	100.0	1.9	1.8	4.1	5.3	5.7	60.2	10.9	12.6	98.0	19.2	23.0	100.0
	SS	0.2	0.1	2.5	0.5	0.5	39.0	1.0	1.2	71.8	1.3	1.9	63.4	0.1	0.1	1.2	0.4	0.5	29.8	0.9	1.1	66.8	1.3	1.7	61.5	0.1	0.1	0.4	0.3	0.3	17.6	0.7	0.9	57.8	1.3	1.7	59.0
	SS^*	0.3	0.3	0.5	84.6	84.6	87.3	99.9	99.9	100.0	100.0	100.0	100.0	0.0	0.0	0.3	48.1	48.3	66.4	99.2	99.2	99.6	100.0	100.0	100.0	3.0	3.1	5.7	4.9	5.1	46.8	7.3	8.0	78.8	10.1	11.2	75.0
$l=10$	$\pi_1[1,1]$	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16	0	0.02	0.16
	AR	11.1	11.1	24.3	24.9	26.3	88.0	38.3	40.8	99.7	48.6	51.9	100.0	7.9	8.0	16.1	20.9	21.7	81.6	35.0	37.2	99.4	47.6	51.0	100.0	4.9	4.9	8.2	14.9	15.7	67.7	30.0	31.8	98.6	46.4	50.0	100.0
	AR^*	0.6	0.6	0.9	84.0	83.9	89.9	100.0	100.0	100.0	100.0	100.0	100.0	0.2	0.2	0.4	33.6	34.0	66.2	98.3	98.3	99.8	100.0	100.0	100.0	1.5	1.5	2.3	5.9	6.3	45.4	14.2	15.5	95.3	27.3	30.7	99.9
	SS	0.0	0.0	0.1	0.1	0.0	6.4	0.1	0.1	27.9	0.2	0.2	26.8	0.0	0.0	0.0	0.0	0.0	3.8	0.1	0.1	22.5	0.1	0.2	25.3	0.0	0.0	0.0	0.0	0.0	1.4	0.1	0.1	16.1	0.1	0.1	23.3
	SS^*	0.3	0.3	0.5	85.2	85.2	87.4	99.9	99.9	100.0	100.0	100.0	100.0	0.0	0.1	0.2	47.5	47.5	60.9	99.4	99.4	99.7	100.0	100.0	100.0	3.0	2.9	4.2	5.0	5.2	31.7	7.8	8.0	63.2	10.7	10.8	59.5

Table 7. Size and power comparison of joint tests for the misspecified model with past lags as instruments (nominal level: 5%)

Size

				T = 100												T = 200											
				$l=1$				$l=5$				$l=10$				$l=1$				$l=5$				$l=10$			
ρ	ϕ	λ	σ_e	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*
0.25	0.50	1.33	1.92	4.8	2.8	4.6	2.9	4.6	2.8	0.1	3.3	3.5	2.3	0.0	3.0	5.0	2.6	5.0	2.6	4.7	3.1	0.1	3.1	4.0	2.8	0.0	3.1
	0.60	0.96	2.27	5.1	2.5	4.9	2.6	4.6	2.8	0.1	2.7	3.6	2.3	0.0	2.6	5.0	2.3	4.8	2.5	4.7	3.0	0.1	2.9	4.1	2.6	0.0	2.9
	0.70	0.76	2.54	5.2	2.4	5.2	2.4	4.8	2.6	0.1	2.5	3.6	2.3	0.0	2.4	5.1	2.0	5.3	2.4	4.8	2.8	0.1	2.4	4.1	2.5	0.0	2.5
	0.80	0.64	2.77	5.6	2.3	6.0	2.6	4.8	2.5	0.1	2.4	4.0	2.3	0.0	2.2	5.5	1.9	5.8	2.2	5.0	2.8	0.2	2.1	4.2	2.5	0.0	2.0
	0.90	0.56	2.97	7.1	2.8	7.6	3.2	5.4	2.6	0.1	2.5	4.4	2.4	0.0	2.3	6.3	2.1	6.8	2.4	5.4	2.9	0.1	2.5	4.4	2.5	0.0	1.9
	1.00	0.50	3.14	5.9	1.6	6.5	1.6	5.2	2.3	0.2	1.7	5.0	2.7	0.0	1.7	5.9	4.2	6.3	4.3	4.7	4.7	0.1	3.8	4.5	4.9	0.0	3.8
0.3	0.50	2.40	1.43	4.8	2.3	4.6	2.4	4.3	2.4	0.0	2.9	3.6	1.8	0.0	2.7	4.8	2.3	4.9	2.3	4.5	2.5	0.1	3.0	3.9	2.3	0.0	3.0
	0.60	1.56	1.78	4.9	2.0	4.7	2.2	4.7	2.2	0.1	2.5	3.4	1.8	0.0	2.3	4.8	2.0	4.9	2.1	4.9	2.4	0.1	2.5	4.1	2.1	0.0	2.4
	0.70	1.19	2.04	5.2	1.9	5.3	2.1	4.9	2.1	0.1	2.1	3.7	1.8	0.0	2.0	5.2	1.7	5.4	1.8	4.9	2.3	0.1	2.1	4.1	2.2	0.0	2.0
	0.80	0.97	2.25	5.7	1.9	6.1	2.1	4.9	1.9	0.1	1.8	4.0	1.8	0.0	1.8	5.7	1.6	5.9	1.6	5.0	2.2	0.1	1.8	4.3	2.0	0.0	1.6
	0.90	0.84	2.42	7.1	2.2	7.7	2.3	5.4	2.0	0.1	2.0	4.4	2.0	0.0	1.8	6.4	1.5	6.7	1.7	5.4	2.2	0.2	1.8	4.3	1.8	0.0	1.5
	1.00	0.75	2.57	6.0	1.0	6.5	1.0	5.4	1.8	0.2	1.1	4.9	2.0	0.0	1.2	6.2	2.0	6.7	2.0	4.7	2.5	0.1	1.8	4.7	2.8	0.0	1.9
0.35	0.50	5.60	0.94	4.6	2.1	4.4	2.0	4.1	1.9	0.1	2.5	3.2	1.4	0.0	2.0	4.8	1.9	5.2	1.9	4.5	1.9	0.1	2.9	3.9	1.6	0.0	2.4
	0.60	2.82	1.32	4.9	1.6	4.6	1.9	4.3	1.7	0.0	2.5	3.5	1.3	0.0	2.0	4.8	1.6	5.0	1.7	4.7	1.8	0.1	2.1	4.0	1.6	0.0	2.1
	0.70	1.96	1.59	5.2	1.5	5.2	1.8	4.8	1.6	0.0	1.9	3.5	1.2	0.0	1.7	5.2	1.3	5.5	1.4	5.0	1.8	0.1	1.8	4.1	1.5	0.0	1.5
	0.80	1.55	1.79	5.9	1.5	6.1	1.7	4.9	1.4	0.1	1.5	3.9	1.3	0.0	1.4	5.7	1.1	5.8	1.2	4.9	1.6	0.1	1.3	4.3	1.3	0.0	1.2
	0.90	1.31	1.94	7.4	1.5	7.8	1.6	5.4	1.3	0.1	1.4	4.4	1.4	0.0	1.4	6.6	1.0	6.8	1.1	5.3	1.4	0.2	1.1	4.3	1.2	0.0	1.1
	1.00	1.17	2.06	6.6	0.7	6.9	0.5	5.7	1.2	0.2	0.6	5.0	1.3	0.0	0.7	7.0	0.8	7.3	0.9	4.9	1.2	0.2	0.7	4.7	1.4	0.0	0.7

Power for $H_0: \phi_0 = 0.50, \rho_0 = 0.35$

				T = 100												T = 200											
				$l=1$				$l=5$				$l=10$				$l=1$				$l=5$				$l=10$			
ρ	ϕ	λ	σ_e	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*
0.25	0.50	1.33	1.92	7.4	0.1	6.8	0.1	6.1	0.2	0.2	0.2	4.6	0.3	0.0	0.3	11.5	0.0	9.9	0.0	7.6	0.1	0.3	0.0	6.3	0.0	0.0	0.0
	0.60	0.96	2.27	22.4	2.1	17.8	2.0	13.8	2.0	0.9	1.7	9.3	1.5	0.1	1.3	45.3	1.7	36.8	2.0	28.2	1.3	3.5	1.2	19.7	1.2	0.2	1.0
	0.70	0.76	2.54	57.4	21.0	46.4	16.6	38.5	15.8	5.8	13.0	26.4	10.9	0.3	8.9	88.8	36.1	80.7	31.5	74.9	30.4	29.2	25.6	62.4	25.8	6.5	22.1
	0.80	0.64	2.77	89.9	68.8	80.8	58.0	77.3	57.8	28.5	49.0	62.9	45.4	3.8	37.7	99.6	92.8	98.4	87.0	98.3	89.8	80.6	83.8	96.4	85.4	48.0	80.1
	0.90	0.56	2.97	99.0	96.3	96.8	91.0	97.1	92.6	68.2	86.0	92.4	86.9	24.4	77.6	100.0	99.9	100.0	99.7	100.0	99.9	98.4	99.4	100.0	99.8	92.2	99.4
	1.00	0.50	3.14	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.6	100.0	97.2	100.0	97.2	100.0	97.5	100.0	97.5	100.0	97.4	100.0	97.4	100.0
0.3	0.50	2.40	1.43	5.6	0.1	5.2	0.2	4.7	0.4	0.1	0.4	3.9	0.4	0.0	0.3	6.6	0.0	5.8	0.1	5.3	0.1	0.1	0.0	4.4	0.1	0.0	0.0
	0.60	1.56	1.78	15.5	1.5	12.1	1.4	9.3	1.5	0.4	1.4	6.6	1.1	0.0	1.0	31.0	1.5	24.3	1.6	16.9	1.1	1.3	0.9	12.2	1.1	0.0	0.8
	0.70	1.19	2.04	46.4	16.3	36.9	12.5	28.8	10.9	3.3	8.3	18.8	7.1	0.1	5.4	80.0	31.5	69.7	26.2	60.4	23.4	16.9	18.8	47.5	17.8	2.5	15.2
	0.80	0.97	2.25	84.6	61.3	74.1	49.3	68.3	47.3	20.3	36.1	52.5	34.7	2.1	23.1	99.1	90.2	96.7	83.3	96.2	84.4	69.7	76.1	92.2	77.5	33.9	68.9
	0.90	0.84	2.42	98.4	94.4	95.2	87.3	95.0	88.9	60.4	77.0	88.9	80.5	19.0	60.2	100.0	99.9	100.0	99.5	100.0	99.7	97.1	98.9	99.9	99.4	87.4	98.3
	1.00	0.75	2.57	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.8	100.0	99.0	100.0	99.0	100.0	99.1	100.0	99.1	100.0	99.0	100.0	99.0	100.0
0.35	0.50	5.60	0.94	4.6	2.1	4.4	2.0	4.1	1.9	0.1	2.5	3.2	1.4	0.0	2.0	4.8	1.9	5.2	1.9	4.5	1.9	0.1	2.9	3.9	1.6	0.0	2.4
	0.60	2.82	1.32	8.8	2.6	7.6	2.1	5.8	2.3	0.2	2.8	4.2	1.8	0.0	2.5	15.2	5.3	12.6	4.0	8.6	3.6	0.3	3.7	6.5	2.7	0.0	3.4
	0.70	1.96	1.59	32.4	14.8	25.5	9.9	18.2	8.3	1.4	6.5	11.8	5.3	0.0	4.3	62.6	39.3	51.8	29.2	39.9	24.5	7.0	18.5	29.1	16.5	0.6	14.2
	0.80	1.55	1.79	75.4	54.7	63.6	41.0	54.4	37.8	12.3	24.9	39.5	25.0	1.0	13.7	97.0	90.7	92.3	81.9	89.6	80.9	52.1	67.2	81.9	70.7	18.4	55.7
	0.90	1.31	1.94	97.1	91.9	92.3	82.1	90.7	83.4	50.0	62.5	82.1	71.5	13.2	39.4	100.0	99.9	99.8	99.4	99.8	99.6	94.3	96.5	99.6	99.0	78.2	92.0
	1.00	1.17	2.06	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.9	99.9	99.8	100.0	99.8	100.0	99.8	100.0	99.8	100.0	99.8	100.0	99.8	100.0

Table 8. Size and power comparison of joint tests for the correctly specified model with past lags as instruments (nominal level: 5%)

Size

				$T=100$												$T=200$											
				$l=1$				$l=5$				$l=10$				$l=1$				$l=5$				$l=10$			
ρ	ϕ	λ	σ_e	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*
0.25	0.50	1.33	1.92	5.3	3.0	5.0	2.8	4.3	2.6	0.1	3.3	3.4	2.3	0.0	3.0	4.6	2.6	4.8	2.8	5.0	3.4	0.1	3.3	4.5	3.0	0.0	3.1
	0.60	0.96	2.27	5.1	2.7	5.0	2.5	4.2	2.6	0.1	2.8	3.5	2.1	0.0	2.8	4.8	2.2	4.7	2.6	5.2	3.3	0.1	2.8	4.4	2.9	0.0	2.8
	0.70	0.76	2.54	5.2	2.4	5.2	2.4	4.5	2.4	0.0	2.5	3.7	2.0	0.0	2.4	4.9	2.1	5.0	2.3	5.4	3.1	0.1	2.3	4.4	2.8	0.0	2.5
	0.80	0.64	2.77	5.7	2.3	5.9	2.4	4.6	2.3	0.0	2.4	3.8	2.1	0.0	2.3	5.2	2.1	5.3	2.1	5.5	2.9	0.1	2.1	4.5	2.7	0.0	2.1
	0.90	0.56	2.97	7.1	2.6	7.3	3.0	5.1	2.5	0.1	2.6	4.2	2.1	0.0	2.5	6.4	2.2	6.5	2.3	5.6	2.9	0.2	2.2	4.7	2.7	0.0	2.0
	1.00	0.50	3.14	5.9	1.4	6.4	1.5	5.5	2.3	0.1	1.7	4.5	2.1	0.0	1.6	5.8	4.4	6.1	4.5	4.4	4.7	0.1	3.9	4.8	5.5	0.0	4.1
0.3	0.50	2.40	1.43	5.1	2.6	5.1	2.6	4.5	2.3	0.1	3.1	3.4	1.8	0.0	2.5	4.9	2.3	5.0	2.4	4.9	2.9	0.1	3.2	4.2	2.4	0.0	2.9
	0.60	1.56	1.78	5.2	2.3	5.2	2.3	4.3	2.1	0.1	2.7	3.5	1.8	0.0	2.2	4.8	1.8	4.9	2.1	5.1	2.7	0.1	2.5	4.3	2.3	0.0	2.4
	0.70	1.19	2.04	5.3	2.0	5.4	2.0	4.4	1.9	0.0	2.4	3.7	1.7	0.0	2.0	4.9	1.6	4.9	1.9	5.3	2.3	0.1	1.9	4.3	2.1	0.0	2.1
	0.80	0.97	2.25	5.7	1.8	5.9	1.9	4.7	1.8	0.1	2.0	3.8	1.5	0.0	1.8	5.2	1.6	5.4	1.8	5.5	2.2	0.1	1.6	4.3	2.0	0.0	1.6
	0.90	0.84	2.42	7.1	2.0	7.4	2.3	5.3	1.9	0.1	2.1	4.2	1.6	0.0	1.8	6.4	1.6	6.6	1.8	5.6	2.2	0.2	1.6	4.6	2.1	0.0	1.5
	1.00	0.75	2.57	6.0	0.8	6.3	1.0	5.3	1.6	0.2	1.2	4.6	1.6	0.0	1.1	6.1	2.0	6.5	2.0	4.7	2.5	0.1	1.7	4.9	3.1	0.0	1.9
0.35	0.50	5.60	0.94	5.5	2.3	5.2	2.3	4.5	2.1	0.0	2.9	3.6	1.4	0.0	2.3	5.0	1.9	5.2	2.2	4.9	2.2	0.1	3.0	4.4	1.9	0.0	2.3
	0.60	2.82	1.32	5.3	2.0	5.3	1.9	4.6	1.8	0.1	2.7	3.5	1.3	0.0	1.9	4.9	1.4	5.1	1.6	5.2	2.1	0.1	2.4	4.3	1.7	0.0	2.1
	0.70	1.96	1.59	5.4	1.7	5.7	1.6	4.4	1.4	0.1	2.2	3.7	1.2	0.0	1.6	4.7	1.2	5.0	1.4	5.3	1.8	0.1	1.7	4.3	1.6	0.0	1.8
	0.80	1.55	1.79	6.1	1.4	6.1	1.4	4.7	1.3	0.1	1.8	3.8	1.0	0.0	1.3	5.4	1.1	5.3	1.3	5.5	1.6	0.1	1.2	4.4	1.5	0.0	1.1
	0.90	1.31	1.94	7.2	1.5	7.5	1.5	5.3	1.3	0.1	1.7	4.4	1.1	0.0	1.3	6.5	1.1	6.7	1.3	5.6	1.5	0.2	1.0	4.5	1.3	0.0	1.0
	1.00	1.17	2.06	6.4	0.5	6.8	0.4	5.4	1.0	0.2	0.6	4.7	1.1	0.0	0.7	6.8	0.8	6.9	0.9	5.0	1.1	0.2	0.8	5.0	1.4	0.0	0.7

Power for $H_0: \phi_0 = 0.50, \rho_0 = 0.35$

				$T=100$												$T=200$											
				$l=1$				$l=5$				$l=10$				$l=1$				$l=5$				$l=10$			
ρ	ϕ	λ	σ_e	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*
0.25	0.50	1.33	1.92	7.6	0.1	6.5	0.1	6.0	0.2	0.2	0.1	4.6	0.2	0.0	0.2	11.5	0.0	10.1	0.0	8.6	0.0	0.3	0.0	6.6	0.0	0.0	0.0
	0.60	0.96	2.27	22.6	2.0	17.2	1.9	13.8	1.7	0.8	1.6	8.9	1.2	0.0	1.3	46.0	1.4	37.0	1.6	28.3	1.4	3.4	1.0	20.0	1.1	0.3	1.0
	0.70	0.76	2.54	57.4	21.1	46.3	16.5	38.5	15.4	5.8	12.1	25.9	10.0	0.3	8.8	89.3	36.0	80.8	30.5	75.3	29.8	29.4	24.9	62.6	25.3	6.3	21.4
	0.80	0.64	2.77	89.7	68.7	80.8	57.9	77.2	57.7	27.8	47.9	62.8	45.6	3.5	36.9	99.7	93.1	98.5	87.2	98.5	90.3	80.7	84.1	96.1	86.0	47.1	80.4
	0.90	0.56	2.97	99.2	96.1	96.8	90.9	97.2	93.0	67.9	85.9	92.4	87.0	25.0	77.4	100.0	99.9	100.0	99.7	100.0	99.9	98.6	99.5	100.0	99.7	92.1	99.6
	1.00	0.50	3.14	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.6	100.0	97.0	100.0	97.0	100.0	97.4	100.0	97.4	100.0	97.2	100.0	97.2	100.0
0.3	0.50	2.40	1.43	5.7	0.1	5.6	0.2	4.9	0.4	0.1	0.3	3.7	0.3	0.0	0.4	6.7	0.0	6.1	0.0	5.9	0.0	0.1	0.0	5.0	0.1	0.0	0.0
	0.60	1.56	1.78	15.0	1.4	11.4	1.4	9.4	1.2	0.4	1.2	6.2	1.0	0.0	1.1	31.3	1.3	24.9	1.3	17.7	1.1	1.2	0.7	12.3	1.0	0.1	0.7
	0.70	1.19	2.04	46.7	16.4	36.5	11.7	28.6	10.8	3.2	7.9	18.2	6.3	0.1	5.0	80.1	31.2	70.0	25.4	60.6	22.9	16.8	17.9	47.2	17.1	2.7	14.0
	0.80	0.97	2.25	84.5	60.8	74.1	49.0	67.6	46.7	19.2	35.0	53.0	34.2	1.9	22.8	99.1	90.6	97.1	83.3	96.3	84.9	69.2	76.1	91.9	77.5	32.8	68.4
	0.90	0.84	2.42	98.5	94.5	95.4	87.4	95.1	89.0	60.2	76.6	89.0	80.3	19.0	59.8	100.0	99.9	99.9	99.5	100.0	99.7	97.4	99.0	99.9	99.5	87.4	98.5
	1.00	0.75	2.57	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.8	100.0	98.9	100.0	98.9	100.0	99.0	100.0	99.0	100.0	98.9	100.0	98.9	100.0
0.35	0.50	5.60	0.94	5.5	2.3	5.2	2.3	4.5	2.1	0.0	2.9	3.6	1.4	0.0	2.3	5.0	1.9	5.2	2.2	4.9	2.2	0.1	3.0	4.4	1.9	0.0	2.3
	0.60	2.82	1.32	8.8	2.7	7.3	2.1	6.0	2.0	0.1	3.0	4.3	1.6	0.0	2.5	15.1	5.2	12.6	3.8	9.3	3.9	0.3	3.5	6.6	2.9	0.0	3.3
	0.70	1.96	1.59	32.5	14.4	24.7	9.2	18.2	8.3	1.2	6.6	10.9	4.6	0.1	4.3	62.2	39.3	51.7	29.2	40.1	25.2	6.4	18.4	28.7	16.7	0.6	13.5
	0.80	1.55	1.79	74.7	54.2	63.6	40.4	53.6	36.9	11.4	23.9	39.5	23.9	0.8	13.2	97.1	91.2	92.8	82.1	89.8	81.1	51.3	67.0	81.8	70.4	17.6	54.2
	0.90	1.31	1.94	97.0	92.2	92.2	82.5	90.9	83.6	48.7	61.8	82.3	71.6	12.9	39.6	100.0	99.9	99.9	99.4	99.9	99.5	94.6	96.6	99.6	99.0	78.0	91.9
	1.00	1.17	2.06	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.9	100.0	99.7	100.0	99.7	100.0	99.8	100.0	99.8	100.0	99.7	100.0	99.7	100.0

Table 9. Size comparison of simple tests for the misspecified model with high-frequency instruments (nominal level: 5%)

				Size																							
				$T=100$												$T=200$											
				$l=1$				$l=5$				$l=10$				$l=1$				$l=5$				$l=10$			
Frequency	ϕ_{lf}	ϕ_{hf}	σ_e	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*	AR	AR^*	SS	SS^*
30-sec	0.000	0.000000	0.00032	5.5	5.4	8.6	5.4	5.2	5.4	0.7	4.9	5.1	5.8	0.2	5.2	5.0	5.3	7.6	4.9	5.3	5.4	0.6	5.0	4.8	5.0	0.1	4.9
	0.100	0.997052		7.4	5.1	7.9	4.8	7.7	5.8	0.8	5.0	7.8	5.4	0.3	5.3	7.2	4.9	7.1	4.7	7.2	5.3	0.5	4.7	7.1	5.2	0.2	4.7
	0.300	0.998458		5.2	3.2	5.6	3.2	6.7	5.1	0.7	4.6	7.6	5.2	0.2	4.6	4.8	3.0	4.7	3.1	6.6	4.7	0.5	4.2	6.7	5.0	0.2	4.1
	0.500	0.999112		2.5	1.6	3.2	1.7	5.9	4.5	0.6	4.1	6.8	5.1	0.2	4.0	2.2	1.4	2.3	1.5	6.0	4.5	0.5	3.9	6.3	4.6	0.1	3.8
	0.700	0.999543		0.7	0.4	1.1	0.8	5.6	4.1	0.5	3.6	6.6	5.1	0.1	3.8	0.9	0.5	0.7	0.6	5.7	4.4	0.6	3.4	5.9	4.4	0.2	3.6
	0.800	0.999714		0.5	0.2	0.5	0.5	5.1	3.9	0.5	3.2	6.4	4.7	0.1	3.4	0.4	0.2	0.3	0.3	5.3	4.1	0.6	3.1	5.6	4.2	0.1	3.2
	0.900	0.999865		0.5	0.1	0.3	0.3	4.6	3.3	0.5	2.6	6.0	4.3	0.1	3.0	0.8	0.1	0.1	0.1	5.1	3.2	0.5	2.7	5.6	3.4	0.1	2.9
	0.950	0.999934		0.6	0.0	0.1	0.2	4.7	2.7	0.4	2.3	5.7	3.5	0.1	2.7	1.4	0.0	0.0	0.1	5.8	1.9	0.4	2.0	6.2	2.1	0.1	2.5
	0.999	0.999999		0.3	0.0	0.1	0.1	4.2	1.6	0.3	1.9	5.1	2.5	0.1	2.3	0.3	0.0	0.0	0.1	7.8	0.4	0.3	1.4	7.0	0.5	0.0	1.9
1-min	0.000	0.000000	0.00064	4.8	5.0	8.0	4.9	5.0	5.1	0.6	4.9	5.5	5.4	0.2	5.3	4.8	4.9	7.7	4.8	4.8	4.9	0.6	4.9	4.9	4.8	0.1	4.7
	0.100	0.994113		7.1	4.7	7.9	4.4	7.1	5.5	0.8	4.9	7.9	5.5	0.1	4.8	7.0	4.6	7.5	4.6	6.9	5.3	0.5	4.8	7.1	5.3	0.1	5.1
	0.300	0.996918		5.2	3.4	5.8	3.1	6.8	4.9	0.7	4.7	7.5	5.2	0.1	4.4	5.3	3.6	5.2	3.4	6.8	4.9	0.4	4.4	7.1	5.3	0.1	4.6
	0.500	0.998224		2.7	1.9	3.0	1.7	6.2	4.7	0.6	3.9	6.9	5.0	0.1	3.9	2.6	1.8	2.9	2.0	5.9	4.5	0.4	3.8	6.9	5.2	0.0	4.1
	0.700	0.999086		0.9	0.5	1.1	0.8	5.4	4.2	0.5	3.3	6.6	4.7	0.1	3.6	0.9	0.6	0.9	0.7	5.2	4.0	0.4	3.1	6.6	4.9	0.0	3.9
	0.800	0.999428		0.6	0.3	0.6	0.5	5.1	4.0	0.4	3.0	6.3	4.6	0.1	3.4	0.5	0.2	0.5	0.4	5.0	3.8	0.3	2.8	6.4	4.5	0.0	3.5
	0.900	0.999730		0.4	0.1	0.3	0.2	4.8	3.3	0.4	2.7	6.1	4.2	0.1	2.8	0.7	0.1	0.2	0.2	4.7	2.9	0.2	2.5	6.1	4.0	0.0	2.9
	0.950	0.999868		0.4	0.0	0.2	0.1	4.7	2.7	0.4	2.3	5.7	3.5	0.1	2.5	1.3	0.0	0.1	0.1	5.6	1.6	0.2	1.9	6.7	2.5	0.0	2.5
	0.999	0.999997		0.3	0.0	0.1	0.1	4.3	1.6	0.3	1.9	5.0	2.3	0.0	2.1	0.2	0.0	0.0	0.0	7.3	0.4	0.1	1.4	7.3	0.6	0.0	1.7
5-min	0.000	0.000000	0.00321	4.9	5.1	8.1	5.0	5.1	5.3	0.7	4.7	5.3	5.4	0.1	4.9	5.0	5.5	8.0	5.2	5.1	5.2	0.6	5.1	5.0	5.2	0.1	5.3
	0.100	0.970911		7.2	4.9	8.0	5.2	7.4	5.3	0.7	4.7	7.8	5.3	0.1	4.3	7.6	5.1	8.0	5.1	7.2	5.3	0.6	4.6	7.4	5.5	0.2	5.2
	0.300	0.984683		6.2	3.9	6.7	4.2	7.1	4.9	0.7	4.6	7.7	5.3	0.1	4.0	6.7	4.5	6.9	4.4	6.8	5.0	0.5	4.5	7.4	5.5	0.2	5.0
	0.500	0.991153		4.3	2.8	4.6	2.8	6.7	4.8	0.7	4.1	7.8	5.5	0.1	4.1	4.6	3.0	4.8	3.0	6.3	4.7	0.4	4.1	7.6	5.6	0.1	4.7
	0.700	0.995438		2.1	1.5	2.4	1.6	6.3	4.7	0.6	3.7	7.3	5.3	0.1	3.7	2.3	1.5	2.0	1.6	5.9	4.5	0.4	3.7	7.3	5.6	0.1	4.3
	0.800	0.997143		1.0	0.8	1.4	1.1	6.0	4.6	0.5	3.5	7.0	5.3	0.1	3.4	1.3	0.8	1.1	1.0	5.8	4.2	0.4	3.3	7.3	5.4	0.2	4.2
	0.900	0.998650		0.5	0.4	0.7	0.7	5.6	4.0	0.5	3.0	6.6	4.8	0.1	2.9	0.9	0.3	0.5	0.5	5.3	3.3	0.4	2.9	6.5	4.4	0.1	3.7
	0.950	0.999343		0.4	0.1	0.4	0.4	5.5	3.4	0.4	2.7	6.2	3.8	0.1	2.7	1.3	0.0	0.2	0.3	5.6	1.7	0.3	2.3	6.6	2.6	0.1	3.1
	0.999	0.999987		0.4	0.1	0.3	0.3	4.6	1.9	0.3	2.5	5.5	2.5	0.0	2.7	0.4	0.0	0.1	0.1	7.2	0.5	0.2	1.7	6.6	0.7	0.1	2.4
10-min	0.000	0.000000	0.00641	5.2	5.4	9.4	5.9	5.0	5.4	0.8	5.0	5.4	5.6	0.1	4.9	4.9	5.3	8.6	5.2	5.0	5.4	0.7	5.0	5.1	5.6	0.2	5.1
	0.100	0.942668		8.6	5.6	9.0	5.3	7.4	5.4	0.6	4.9	8.0	5.7	0.2	4.6	7.7	5.0	8.4	5.2	7.8	5.7	0.7	5.2	7.8	5.8	0.2	5.3
	0.300	0.969601		7.8	5.0	8.1	4.9	7.4	5.2	0.5	4.5	8.0	5.9	0.2	4.2	7.0	4.5	7.6	4.7	7.4	5.5	0.7	5.0	7.6	5.8	0.1	5.2
	0.500	0.982384		6.1	3.8	5.9	3.5	7.5	5.3	0.5	4.1	7.8	5.9	0.2	4.0	5.4	3.4	5.5	3.6	7.2	5.3	0.6	4.4	7.6	5.8	0.1	4.8
	0.700	0.990896		3.3	2.1	3.5	2.2	7.2	5.0	0.5	4.0	7.5	5.9	0.2	3.8	2.9	2.0	3.0	2.2	6.5	4.8	0.6	3.8	7.7	5.8	0.1	4.5
	0.800	0.994295		2.2	1.6	2.4	1.7	6.7	4.9	0.5	3.9	7.5	5.7	0.2	3.5	1.9	1.4	2.0	1.5	6.2	4.5	0.5	3.7	7.3	5.3	0.1	4.3
	0.900	0.997302		1.1	0.8	1.2	0.9	6.6	4.4	0.4	3.4	6.8	5.2	0.2	3.1	1.1	0.5	0.9	0.8	5.6	3.4	0.5	3.1	6.8	4.6	0.1	3.7
	0.950	0.998686		0.8	0.4	0.7	0.6	6.0	3.7	0.4	3.0	6.5	4.3	0.2	2.9	1.5	0.2	0.4	0.4	6.1	2.1	0.4	2.5	6.7	2.5	0.1	3.1
	0.999	0.999974		0.4	0.2	0.5	0.4	5.2	2.2	0.4	2.6	5.6	2.7	0.1	2.5	0.3	0.0	0.2	0.2	6.4	0.5	0.3	1.7	6.7	0.6	0.1	2.5

Table 10. Power comparison of simple tests for the misspecified model with high-frequency instruments (nominal level: 5%)

				Level adjusted power for $H_0: \phi_0 = 0$																							
				$T = 100$												$T = 200$											
				$l = 1$				$l = 5$				$l = 10$				$l = 1$				$l = 5$				$l = 10$			
Frequency	ϕ_{lf}	ϕ_{hf}	σ_e	*AR*	*AR**	*SS*	*SS**	*AR*	*AR**	*SS*	*SS**	*AR*	*AR**	*SS*	*SS**	*AR*	*AR**	*SS*	*SS**	*AR*	*AR**	*SS*	*SS**	*AR*	*AR**	*SS*	*SS**
30-sec	0.000	0.000000	0.00032	5.0	5.0	5.0	5.0	5.0	5.0	0.7	4.9	5.0	5.0	0.2	5.0	5.0	5.0	5.0	4.9	5.0	5.0	0.6	5.0	4.8	5.0	0.1	4.9
	0.100	0.997052		2.9	5.4	5.3	5.1	2.8	5.2	0.9	5.0	2.4	5.1	0.2	5.3	2.8	5.2	5.6	5.0	3.3	5.0	0.5	4.8	2.9	5.4	0.2	4.9
	0.300	0.998458		4.9	5.1	7.6	5.0	3.6	5.6	0.9	5.1	2.6	5.1	0.3	5.3	4.7	5.3	7.5	5.0	3.6	5.3	0.5	4.9	3.3	5.4	0.1	4.7
	0.500	0.999112		5.6	5.8	9.2	5.4	4.6	6.2	0.9	5.4	3.4	5.9	0.2	5.4	6.3	6.8	9.3	6.1	4.8	6.0	0.7	5.6	4.1	5.7	0.1	5.1
	0.700	0.999543		9.5	9.4	12.3	8.5	6.3	7.6	1.1	6.1	4.7	7.2	0.2	5.8	15.1	15.1	16.8	12.5	8.4	10.5	1.4	7.8	7.0	8.8	0.2	6.1
	0.800	0.999714		16.7	16.6	17.7	13.2	9.5	11.1	1.5	7.5	6.5	9.7	0.4	6.8	31.8	31.1	28.9	24.1	16.4	19.3	3.2	12.3	12.2	15.4	0.5	8.1
	0.900	0.999865		38.1	36.5	31.8	28.3	21.2	24.5	3.7	13.7	13.4	18.9	0.8	10.6	69.7	67.8	58.0	56.5	46.6	49.8	11.2	27.9	33.8	41.0	2.7	18.4
	0.950	0.999934		60.3	58.3	47.6	45.6	38.6	42.6	8.4	22.3	26.0	33.7	2.1	16.8	91.6	90.7	80.4	81.5	76.7	78.8	27.2	48.5	64.3	71.4	9.7	34.4
	0.999	0.999999		82.2	80.7	67.5	67.7	66.0	68.9	19.8	38.6	52.2	59.4	7.0	30.7	99.1	99.0	95.9	96.6	94.1	96.8	61.0	76.5	91.8	95.3	38.2	64.9
1-min	0.000	0.000000	0.00064	4.8	5.0	5.0	4.9	5.0	5.0	0.6	4.9	5.0	5.0	0.2	5.0	4.8	4.9	5.0	4.8	4.8	4.9	0.6	4.9	4.9	4.8	0.1	4.7
	0.100	0.994113		2.7	5.1	5.4	4.7	2.8	4.9	0.9	5.1	2.5	4.9	0.1	5.0	2.6	4.9	5.3	4.8	3.1	5.2	0.5	4.8	3.2	5.2	0.1	4.9
	0.300	0.996918		4.8	5.2	7.3	4.9	3.2	5.6	0.7	5.3	2.9	5.4	0.1	5.0	4.6	5.1	7.7	5.1	3.3	5.4	0.5	5.0	3.2	5.2	0.1	5.2
	0.500	0.998224		5.5	5.9	8.6	5.5	4.3	5.9	0.8	5.5	3.7	5.7	0.1	5.0	6.3	6.8	9.6	6.6	4.9	6.3	0.7	5.2	3.8	5.6	0.1	5.4
	0.700	0.999086		9.3	9.4	11.7	8.1	6.2	7.6	1.0	6.1	4.7	6.6	0.1	5.7	14.9	15.3	17.0	13.0	8.9	10.5	1.4	7.2	6.3	8.7	0.1	6.4
	0.800	0.999428		16.6	16.1	17.4	12.9	9.1	10.8	1.7	7.8	6.8	9.3	0.2	6.8	30.7	29.9	29.2	24.3	16.6	19.0	2.8	11.2	11.2	15.3	0.4	8.6
	0.900	0.999730		38.6	36.8	32.2	28.4	20.1	23.3	3.9	13.5	13.9	18.7	0.7	10.6	69.8	67.6	57.5	56.2	45.6	48.7	9.8	26.1	32.9	40.6	2.5	18.6
	0.950	0.999868		60.1	57.9	47.4	45.4	36.9	41.2	8.5	22.5	26.3	34.0	2.0	17.1	90.7	89.9	80.0	80.5	76.2	78.5	26.1	47.4	63.8	71.8	9.8	35.2
	0.999	0.999997		81.6	80.3	66.3	66.6	64.0	67.2	19.5	38.7	51.8	59.0	7.4	30.8	99.2	99.0	95.4	96.0	94.6	97.1	60.9	76.5	91.3	95.2	37.9	64.7
5-min	0.000	0.000000	0.00321	4.9	5.0	5.0	5.0	5.0	5.0	0.7	4.7	5.0	5.0	0.1	4.9	5.0	5.0	5.0	5.0	5.0	5.0	0.6	5.0	5.0	5.0	0.1	5.0
	0.100	0.970911		2.4	5.0	5.3	5.1	2.8	5.1	0.7	4.8	2.2	5.0	0.1	4.6	2.4	5.2	5.0	5.1	3.0	5.0	0.6	4.9	2.6	4.9	0.2	5.0
	0.300	0.984683		3.3	5.0	6.8	5.3	3.2	5.5	0.8	5.1	2.3	5.0	0.1	4.5	3.4	5.7	6.5	5.5	3.2	5.3	0.6	5.0	2.7	4.9	0.2	5.3
	0.500	0.991153		4.9	5.5	9.0	5.5	3.6	5.8	0.8	5.0	2.4	5.3	0.1	4.5	6.6	7.1	9.9	6.2	3.9	5.7	0.7	5.2	2.6	5.3	0.2	5.2
	0.700	0.995438		8.0	8.7	12.0	7.6	5.1	7.6	1.1	5.9	3.8	6.6	0.2	5.1	13.5	14.6	16.9	11.8	7.0	9.7	1.2	6.7	4.5	7.5	0.3	6.0
	0.800	0.997143		14.9	15.2	17.5	12.0	7.9	10.7	1.6	7.5	5.6	9.2	0.2	6.2	28.5	28.0	28.4	22.5	14.6	18.1	2.7	10.4	9.1	14.0	0.5	8.1
	0.900	0.998650		35.8	34.6	31.6	26.7	18.4	23.2	4.1	13.0	12.2	18.1	0.7	9.7	66.6	65.1	56.5	53.3	42.4	47.1	9.9	25.4	29.7	39.1	2.5	17.6
	0.950	0.999343		58.3	56.6	46.9	43.1	35.8	41.1	8.3	21.7	24.4	32.7	2.0	15.4	89.9	88.6	79.3	78.8	74.2	77.2	25.4	46.2	61.3	70.3	9.5	33.7
	0.999	0.999987		80.6	79.2	66.9	65.1	63.8	67.7	19.1	37.4	50.5	58.9	7.1	28.6	99.0	98.6	95.4	95.5	94.1	96.6	59.9	75.4	91.7	95.0	37.5	64.7
10-min	0.000	0.000000	0.00641	5.0	5.0	5.0	5.0	5.0	5.0	0.8	5.0	5.0	5.0	0.1	4.9	4.9	5.0	5.0	5.0	5.0	5.0	0.7	5.0	5.0	5.0	0.2	5.0
	0.100	0.942668		1.5	4.9	5.1	5.1	2.5	5.1	0.6	5.0	2.6	4.9	0.2	4.8	2.2	5.0	4.9	5.1	2.3	5.0	0.7	5.0	2.5	4.8	0.1	4.9
	0.300	0.969601		2.3	5.5	5.9	5.6	2.4	5.1	0.7	5.1	2.5	4.6	0.2	4.8	2.8	5.2	6.2	5.4	2.4	4.9	0.7	5.3	2.7	5.1	0.1	5.2
	0.500	0.982384		4.3	6.2	8.8	6.0	2.6	5.6	0.7	4.9	2.5	4.6	0.2	5.0	5.4	6.4	9.6	6.4	3.0	5.6	0.8	5.4	2.8	5.4	0.1	5.4
	0.700	0.990896		8.2	9.2	12.4	7.9	3.9	7.1	0.9	5.6	3.4	5.4	0.2	5.3	11.3	13.1	16.3	10.9	5.7	9.2	1.3	6.9	4.1	7.7	0.2	6.1
	0.800	0.994295		14.0	15.2	17.3	12.2	6.9	10.5	1.6	7.1	4.8	7.9	0.3	6.0	24.5	25.8	26.6	20.5	11.9	16.7	2.8	10.5	8.3	13.2	0.5	8.1
	0.900	0.997302		33.7	33.8	30.6	25.1	17.0	22.9	3.8	12.2	11.3	17.0	0.7	8.7	62.3	61.8	54.7	49.5	39.8	46.3	9.4	24.6	28.4	38.6	2.7	17.6
	0.950	0.998686		55.8	55.2	46.0	41.4	34.5	40.1	7.3	20.1	22.6	32.1	1.8	14.7	88.5	87.8	77.8	76.6	71.8	76.7	24.8	45.5	60.1	70.3	9.4	33.2
	0.999	0.999974		78.6	77.9	64.8	63.3	61.9	66.4	18.9	36.4	48.3	57.4	6.7	27.6	98.9	98.7	95.2	95.1	94.7	96.6	58.5	75.1	90.9	94.8	36.5	63.1

Table 11. Description of instruments

No	Classes of instruments		Subclasses
		HF realized measures not robust to jumps	
1-13	RV	Realized volatility	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t, 5m-ss, 10m-ss
14-24	RVbr	Realized volatility with optimal sampling	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t
25-35	MSRV	Multi-scales realized volatility	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t
36-40	Rkcub	Realized Kernel with fat-top cubic kernel	1t, 5t, 10t, 20t, 50t
41-45	Rkbart	Realized Kernel with fat-top Bartlett kernel	1t, 5t, 10t, 20t, 50t
46-50	RKth2	Realized Kernel with fat-top Tukey-Hanning kernel (power 2)	1t, 5t, 10t, 20t, 50t
51-55	RKnfp	Realized Kernel with non-fat-top Parzen kernel	1t, 5t, 10t, 20t, 50t
56-58	RRV	Realized range realized volatility	1m, 5m, 10m
		HF realized measures robust to jumps	
59-71	BV	Bipower variation	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t, 5m-ss, 10m-ss
72-77	MedRV	Nearest neighbor truncated median RV	1s, 5s, 30s, 1m, 5m, 10m
78-83	MinRV	Nearest neighbor truncated minimum RV	1s, 5s, 30s, 1m, 5m, 10m
		Aditional HF measures and jump variations	
84-96	RSVN	Realized semivariance due to negative returns	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t, 5m-ss, 10m-ss
97-109	RSVP	Realized semivariance due to positive returns	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t, 5m-ss, 10m-ss
110-120	JV	Jump variation	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t
121-131	SJV	Signed jump variation	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t
132-142	LJV	Log squared jump variation	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t
143-153	LSJV	Log squared signed jump variation	1s, 5s, 30s, 1m, 5m, 10m, 1t, 5t, 10t, 20t, 50t
154-156	PCF	HF principal component factor	1, 2, 3
		Other instruments	
157-162	ImV-C	Implied volatility (call option)	mean, min, max, q1, q2, q3
163-168	ImV-P	Implied volatility (put option)	mean, min, max, q1, q2, q3
169-174	ImV-A	Implied volatility (both call and put option)	mean, min, max, q1, q2, q53
175	1-day	Daily realized volatility	

Notes:

1. Sampling frequencies are tick, second and minute, *e.g.*, 1t stands for 1-tick, 1s stands for 1-second and 1m stands for 1-minute.
2. The use of 1-minute subsamples in the calculation of realized measures is denoted by ss.
3. Three principal component factors are extracted from HF instruments (1-109). PCF-1 stands for the largest factor.
4. Implied volatilities (ImV) are calculated from American options. We consider three classes: (1) only call options, (2) only put options, and (3) both call and put options. For each class, we use all implied volatilities at a given date to construct six ImV subclasses, which are mean, min, max, and three quantiles (q1, q2, q3).

Table 12. Strength comparison with daily past lags as instruments
(F-statistics from first-stage regression)

January 2009 - December 2013, $T = 1258$							
	# of instruments						
Ticker	1	2	3	4	5	6	7
GE	23.64	25.00	21.10	20.73	18.46	16.85	14.81
IBM	9.22	10.08	10.08	9.72	8.63	7.73	6.87
JPM	41.08	38.42	34.99	28.34	24.71	23.22	20.79
KO	6.19	10.24	8.82	9.00	8.31	7.08	7.00
PFE	14.99	11.17	7.53	7.43	7.41	7.45	7.06
PG	3.57	4.28	5.38	4.88	5.76	5.14	6.56
T	5.36	13.65	9.62	7.04	6.76	6.07	5.37
WMT	15.24	11.01	7.71	6.10	5.45	5.36	5.63
XOM	9.48	7.80	7.87	6.97	5.86	6.08	5.69
$CV\_Size(0.10)$	16.38	19.93	22.30	24.58	26.87	29.18	31.50

Notes:

1. The critical value (CV) is a function of the number of included endogenous regressors (one), the number of instrumental variables, and the desired maximal size (10%) of a 5% Wald test of $\phi = \phi_0$, for further details, see Table 5.2 of Stock and Yogo (2005).

2. Instruments are deemed weak if the first-stage F-statistics is less than CV associated with the corresponding column.

Table 13. Strength comparison of all IV's
(F-statistics from first-stage regression)
Ticker: IBM, January 2009 - December 2013, $T = 1258$

No	Instruments	$l=1$	$l=3$	$l=5$	No	Instruments	$l=1$	$l=3$	$l=5$	No	Instruments	$l=1$	$l=3$	$l=5$	No	Instruments	$l=1$	$l=3$	$l=5$
1	RV-1s	70.4	29.2	17.5	45	RKbart-50t	139.0	46.3	27.7	89	RSVN-10m	79.3	31.7	19.3	133	LJV-5s	46.2	23.4	13.9
2	RV-5s	69.3	29.9	17.7	46	RKth2-1t	132.5	46.6	27.9	90	RSVN-1t	99.3	34.2	20.7	134	LJV-30s	9.4	8.3	6.0
3	RV-30s	95.7	34.1	20.5	47	RKth2-5t	139.7	46.3	27.9	91	RSVN-5t	103.6	34.9	21.4	135	LJV-1m	24.2	13.4	9.0
4	RV-1m	99.4	35.0	21.1	48	RKth2-10t	142.7	47.3	28.3	92	RSVN-10t	106.1	37.2	22.7	136	LJV-5m	16.2	12.5	8.3
5	RV-5m	96.8	34.5	21.6	49	RKth2-20t	143.5	47.7	28.6	93	RSVN-20t	122.0	42.4	26.3	137	LJV-10m	23.2	11.8	8.8
6	RV-10m	92.0	33.0	20.4	50	RKth2-50t	138.8	46.1	27.7	94	RSVN-50t	110.3	40.0	24.1	138	LJV-1t	92.5	32.1	19.2
7	RV-1t	99.9	34.5	20.8	51	RKnfp-1t	142.3	47.4	28.3	95	RSVN-5m-ss	93.6	35.2	21.2	139	LJV-5t	62.8	22.0	13.8
8	RV-5t	106.3	35.6	21.9	52	RKnfp-5t	139.7	47.0	28.0	96	RSVN-10m-ss	89.6	34.3	20.6	140	LJV-10t	71.8	27.5	17.7
9	RV-10t	110.7	38.5	23.5	53	RKnfp-10t	136.7	46.0	27.4	97	RSVP-1s	70.0	28.9	17.3	141	LJV-20t	44.1	19.6	14.4
10	RV-20t	128.3	43.6	27.3	54	RKnfp-20t	139.6	46.4	27.7	98	RSVP-5s	68.7	29.3	17.4	142	LJV-50t	90.1	30.6	19.9
11	RV-50t	117.5	40.9	24.7	55	RKnfp-50t	135.4	45.1	27.0	99	RSVP-30s	93.4	33.0	19.8	143	LSJV-1s	24.0	13.8	11.6
12	RV-5m-ss	104.8	36.4	22.0	56	RRV-1m	96.6	34.4	20.7	100	RSVP-1m	95.4	33.2	19.9	144	LSJV-5s	10.4	16.2	11.7
13	RV-10m-ss	101.4	35.4	21.3	57	RRV-5m	85.3	33.5	20.5	101	RSVP-5m	81.5	29.8	18.6	145	LSJV-30s	19.7	17.7	13.7
14	RVbr-1s	84.5	31.0	19.1	58	RRV-10m	80.5	32.6	20.1	102	RSVP-10m	69.1	26.4	16.6	146	LSJV-1m	16.5	13.7	9.4
15	RVbr-5s	81.0	29.8	18.5	59	BV-1s	80.2	30.2	18.3	103	RSVP-1t	99.7	34.6	20.8	147	LSJV-5m	22.6	14.3	10.1
16	RVbr-30s	71.5	27.4	17.5	60	BV-5s	71.6	29.2	17.8	104	RSVP-5t	106.1	35.6	22.0	148	LSJV-10m	13.4	12.3	9.7
17	RVbr-1m	76.5	29.5	18.4	61	BV-30s	97.6	34.5	20.8	105	RSVP-10t	109.6	38.2	23.4	149	LSJV-1t	40.1	17.0	11.8
18	RVbr-5m	87.7	35.1	21.9	62	BV-1m	100.5	35.1	21.1	106	RSVP-20t	125.3	42.5	26.9	150	LSJV-5t	35.4	14.6	10.1
19	RVbr-10m	61.8	27.7	17.3	63	BV-5m	95.5	34.5	21.2	107	RSVP-50t	111.9	38.6	23.5	151	LSJV-10t	38.1	20.6	13.6
20	RVbr-1t	99.4	36.4	21.7	64	BV-10m	87.6	31.3	19.3	108	RSVP-5m-ss	94.2	33.5	20.3	152	LSJV-20t	33.5	12.7	8.3
21	RVbr-5t	93.0	33.8	20.2	65	BV-1t	99.8	34.4	20.9	109	RSVP-10m-ss	82.5	30.6	18.5	153	LSJV-50t	37.3	16.5	10.9
22	RVbr-10t	93.8	34.2	21.5	66	BV-5t	106.8	35.8	22.1	110	JV-1s	0.6	0.5	0.8	154	PCF-1	102.7	35.3	21.4
23	RVbr-20t	95.0	34.1	20.6	67	BV-10t	105.3	36.9	22.4	111	JV-5s	0.7	0.5	0.7	155	PCF-2	98.5	34.1	20.6
24	RVbr-50t	92.3	33.2	20.7	68	BV-20t	129.0	43.8	27.4	112	JV-30s	0.0	2.3	2.2	156	PCF-3	67.4	24.8	15.8
25	MSRV-1s	99.6	34.6	21.2	69	BV-50t	120.6	42.1	25.6	113	JV-1m	2.9	5.7	4.2	157	ImV-C-mean	23.4	18.3	12.3
26	MSRV-5s	92.9	32.3	20.4	70	BV-5m-ss	95.5	34.5	21.2	114	JV-5m	9.1	9.0	7.2	158	ImV-C-min	84.8	29.3	17.4
27	MSRV-30s	94.1	34.2	21.7	71	BV-10m-ss	95.5	34.5	21.2	115	JV-10m	15.4	10.8	6.9	159	ImV-C-max	1.3	1.3	0.9
28	MSRV-1m	98.0	36.1	22.4	72	MedRV-1s	72.5	29.6	17.9	116	JV-1t	0.5	0.9	1.2	160	ImV-C-q1	87.5	29.2	17.8
29	MSRV-5m	83.2	33.2	20.9	73	MedRV-5s	62.9	28.4	16.9	117	JV-5t	0.6	1.2	1.3	161	ImV-C-q2	80.5	29.6	17.6
30	MSRV-10m	81.4	30.6	18.6	74	MedRV-30s	94.0	33.6	20.2	118	JV-10t	0.3	0.7	1.0	162	ImV-C-q3	25.1	18.1	12.1
31	MSRV-1t	123.9	43.2	25.9	75	MedRV-1m	97.6	34.3	20.8	119	JV-20t	0.1	3.6	2.5	163	ImV-P-mean	27.5	12.4	9.2
32	MSRV-5t	123.2	43.7	26.0	76	MedRV-5m	95.9	34.6	21.1	120	JV-50t	0.6	1.3	1.3	164	ImV-P-min	63.1	21.1	13.1
33	MSRV-10t	128.3	44.1	26.3	77	MedRV-10m	91.3	32.5	20.1	121	SJV-1s	0.9	1.3	0.8	165	ImV-P-max	0.2	1.0	1.0
34	MSRV-20t	126.0	42.8	26.3	78	MinRV-1s	74.3	29.1	17.8	122	SJV-5s	0.2	0.7	1.4	166	ImV-P-q1	72.4	27.4	16.6
35	MSRV-50t	142.3	47.3	28.9	79	MinRV-5s	62.1	26.8	16.3	123	SJV-30s	0.8	1.6	3.4	167	ImV-P-q2	71.4	25.4	15.4
36	RKcub-1t	102.8	40.2	24.4	80	MinRV-30s	93.9	33.6	20.2	124	SJV-1m	0.5	2.0	2.5	168	ImV-P-q3	44.0	15.9	10.6
37	RKcub-5t	127.7	42.7	25.6	81	MinRV-1m	97.2	34.1	20.6	125	SJV-5m	0.2	1.9	2.8	169	ImV-A-mean	35.1	17.9	12.0
38	RKcub-10t	145.2	48.2	28.9	82	MinRV-5m	92.1	34.2	20.9	126	SJV-10m	0.4	1.8	2.0	170	ImV-A-min	68.8	22.7	13.7
39	RKcub-20t	136.4	45.5	27.2	83	MinRV-10m	79.7	29.2	18.0	127	SJV-1t	0.7	11.6	7.1	171	ImV-A-max	1.1	1.6	1.1
40	RKcub-50t	134.3	44.8	26.8	84	RSVN-1s	70.5	29.5	17.6	128	SJV-5t	0.0	1.4	1.3	172	ImV-A-q1	83.8	31.3	19.1
41	RKbart-1t	133.9	45.2	27.0	85	RSVN-5s	69.3	30.3	18.0	129	SJV-10t	0.4	0.7	0.9	173	ImV-A-q2	82.3	28.0	17.0
42	RKbart-5t	139.9	46.3	27.9	86	RSVN-30s	92.9	34.2	20.5	130	SJV-20t	0.0	0.7	0.7	174	ImV-A-q3	51.7	21.8	13.5
43	RKbart-10t	141.9	47.0	28.2	87	RSVN-1m	95.0	35.2	21.2	131	SJV-50t	0.0	0.5	0.6	175	1-day	9.2	10.1	8.6
44	RKbart-20t	143.8	47.8	28.6	88	RSVN-5m	88.1	35.1	21.6	132	LJV-1s	56.7	26.6	15.9	$CV_{Size,0.10}$		16.4	22.3	26.9

Notes:

1. The critical value (CV) is a function of the number of included endogenous regressors (one), the number of instrumental variables, and the desired maximal size (10%) of a 5% Wald test of $\phi = \phi_0$, for further details, see Table 5.2 of Stock and Yogo (2005).

2. Instruments are deemed weak if the first-stage F-statistics is less than CV associated with the corresponding column.

Table 14. Projection-based 90% confidence intervals for the volatility persistence parameter ϕ
Strong instruments
Ticker: IBM, January 2009 - December 2013, $T = 1258$

Panel A						
No	Instruments	$\bar{d}_{i,s}$	AR	AR^*	SS	SS^*
1	RSVN-5m-ss	0.8860	[0.950, 1.000]	[0.866, 1.000]	[0.932, 1.000]	[0.796, 1.000]
2	RSVN-5m	0.8855	[0.948, 1.000]	[0.864, 1.000]	[0.931, 1.000]	[0.799, 1.000]
3	RSVN-1m	0.8848	[0.947, 1.000]	[0.856, 1.000]	[0.929, 1.000]	[0.807, 1.000]
4	ImV-C-mean	0.8830	[0.964, 1.000]	[0.852, 1.000]	[0.937, 1.000]	[0.779, 1.000]
5	MinRV-5m	0.8828	[0.945, 1.000]	[0.867, 1.000]	[0.925, 1.000]	[0.794, 1.000]
6	RV-5m-ss	0.8825	[0.946, 1.000]	[0.863, 1.000]	[0.926, 1.000]	[0.795, 1.000]
7	BV-5m	0.8823	[0.945, 1.000]	[0.865, 1.000]	[0.926, 1.000]	[0.793, 1.000]
8	BV-5m-ss	0.8823	[0.945, 1.000]	[0.865, 1.000]	[0.926, 1.000]	[0.793, 1.000]
9	BV-10m-ss	0.8823	[0.945, 1.000]	[0.865, 1.000]	[0.926, 1.000]	[0.793, 1.000]
10	MedRV-5m	0.8823	[0.945, 1.000]	[0.866, 1.000]	[0.925, 1.000]	[0.793, 1.000]
Panel B						
No	Instruments	$\bar{d}_{i,s}$	AR	AR^*	SS	SS^*
11	ImV-C-q3	0.8805	[0.964, 1.000]	[0.843, 1.000]	[0.940, 1.000]	[0.775, 1.000]
12	RV-1m	0.8800	[0.944, 1.000]	[0.857, 1.000]	[0.925, 1.000]	[0.794, 1.000]
13	ImV-C-q2	0.8795	[0.958, 1.000]	[0.860, 1.000]	[0.940, 1.000]	[0.760, 1.000]
14	RRV-1m	0.8790	[0.945, 1.000]	[0.858, 1.000]	[0.926, 1.000]	[0.787, 1.000]
15	MedRV-1m	0.8785	[0.944, 1.000]	[0.857, 1.000]	[0.926, 1.000]	[0.787, 1.000]
16	RV-5m	0.8783	[0.943, 1.000]	[0.858, 1.000]	[0.923, 1.000]	[0.789, 1.000]
17	BV-1m	0.8775	[0.944, 1.000]	[0.857, 1.000]	[0.925, 1.000]	[0.784, 1.000]
18	RSVN-10m-ss	0.8775	[0.949, 1.000]	[0.858, 1.000]	[0.931, 1.000]	[0.772, 1.000]
19	RSVN-10m	0.8760	[0.946, 1.000]	[0.861, 1.000]	[0.927, 1.000]	[0.770, 1.000]
20	RV-10m-ss	0.8758	[0.944, 1.000]	[0.857, 1.000]	[0.924, 1.000]	[0.778, 1.000]
Panel C						
No	Instruments	$\bar{d}_{i,s}$	AR	AR^*	SS	SS^*
21	RSVN-30s	0.8753	[0.944, 1.000]	[0.848, 1.000]	[0.924, 1.000]	[0.785, 1.000]
22	RRV-5m	0.8750	[0.946, 1.000]	[0.855, 1.000]	[0.927, 1.000]	[0.772, 1.000]
23	MinRV-1m	0.8745	[0.943, 1.000]	[0.855, 1.000]	[0.924, 1.000]	[0.776, 1.000]
24	ImV-C-min	0.8743	[0.952, 1.000]	[0.834, 1.000]	[0.930, 1.000]	[0.781, 1.000]
25	MSRV-1m	0.8723	[0.939, 1.000]	[0.862, 1.000]	[0.920, 1.000]	[0.768, 1.000]
26	RSVP-1m	0.8715	[0.942, 1.000]	[0.852, 1.000]	[0.921, 1.000]	[0.771, 1.000]
27	RV-30s	0.8713	[0.942, 1.000]	[0.847, 1.000]	[0.922, 1.000]	[0.774, 1.000]
28	BV-30s	0.8713	[0.943, 1.000]	[0.847, 1.000]	[0.923, 1.000]	[0.772, 1.000]
29	ImV-C-q1	0.8710	[0.952, 1.000]	[0.840, 1.000]	[0.931, 1.000]	[0.761, 1.000]
30	MSRV-30s	0.8698	[0.935, 1.000]	[0.858, 1.000]	[0.917, 1.000]	[0.769, 1.000]

Notes:

1. The instrument set consists of a constant and a lag of instrument, $l = 1$.
2. We use log squared transformation for the first 13 classes of instruments (RV-RSVP) given in Table 11.
3. The confidence intervals are constructed by projection technique described in Section 3.5. The corresponding 95% confidence interval for the nuisance parameter λ is [33.943, 61.154] with $\hat{\lambda} = 47.548$ and $\text{SE}(\hat{\lambda}) = 6.935$.
4. The proposed inference procedures [AR, AR^*, SS, SS^*] are described in Section 3.1 - 3.4.
5. We use $\tau = 0.2$ for SS-type tests and employ 99 Monte Carlo replications for point optimal type procedures.
6. An instrument set i's average strength over inference methods is measured by $\bar{d}_{i,s}$.

Table 15. Projection-based 90% confidence intervals for the volatility persistence parameter ϕ
Weak instruments
Ticker: IBM, January 2009 - December 2013, $T = 1258$

Panel A						
No	Instruments	$\bar{d}_{i,s}$	AR	AR^*	SS	SS^*
1	JV-1s	0.0000	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]
2	JV-5s	0.0000	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]
3	JV-30s	0.0000	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]
4	JV-1t	0.0000	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]
5	SJV-1s	0.0000	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]
6	SJV-5s	0.0000	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]
7	SJV-10t	0.0000	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]
8	SJV-20t	0.0000	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]
9	SJV-50t	0.0000	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]
10	ImV-C-max	0.0000	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]
11	ImV-P-max	0.0000	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]
12	ImV-A-max	0.0000	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]	[0.000, 1.000]
Panel B						
No	Instruments	$\bar{d}_{i,s}$	AR	AR^*	SS	SS^*
13	JV-20t	0.0038	[0.000, 1.000]	[0.000, 1.000]	[0.000, 0.985]	[0.000, 1.000]
14	SJV-5t	0.1875	[0.500, 1.000]	[0.250, 1.000]	[0.000, 1.000]	[0.000, 1.000]
Panel C						
No	Instruments	$\bar{d}_{i,s}$	AR	AR^*	SS	SS^*
15	JV-10t	0.3130	[0.000, 1.000]	[0.000, 1.000]	[0.745, 0.993]	[0.500, 1.000]
16	JV-50t	0.3283	[0.000, 1.000]	[0.000, 1.000]	[0.763, 1.000]	[0.550, 1.000]
17	JV-5t	0.3308	[0.000, 1.000]	[0.000, 1.000]	[0.753, 1.000]	[0.570, 1.000]
18	SJV-30s	0.3400	[0.000, 1.000]	[0.000, 1.000]	[0.860, 1.000]	[0.500, 1.000]
19	SJV-1m	0.3845	[0.000, 1.000]	[0.000, 1.000]	[0.898, 1.000]	[0.640, 1.000]
20	SJV-10m	0.3975	[0.000, 1.000]	[0.000, 1.000]	[0.930, 1.000]	[0.660, 1.000]
21	SJV-5m	0.3998	[0.000, 1.000]	[0.000, 1.000]	[0.919, 1.000]	[0.680, 1.000]
22	JV-1m	0.4028	[0.911, 1.000]	[0.700, 1.000]	[0.000, 1.000]	[0.000, 1.000]
23	JV-10m	0.4075	[0.890, 1.000]	[0.740, 1.000]	[0.000, 1.000]	[0.000, 1.000]
24	LSJV-20t	0.4108	[0.883, 0.992]	[0.750, 1.000]	[0.000, 0.998]	[0.000, 1.000]
25	LSJV-5t	0.4208	[0.913, 1.000]	[0.770, 1.000]	[0.000, 1.000]	[0.000, 1.000]
26	1-day	0.4255	[0.870, 0.965]	[0.750, 1.000]	[0.000, 0.953]	[0.000, 1.000]
27	LJV-30s	0.4268	[0.927, 1.000]	[0.780, 1.000]	[0.000, 1.000]	[0.000, 1.000]
28	LJV-1m	0.4373	[0.933, 1.000]	[0.816, 1.000]	[0.000, 1.000]	[0.000, 1.000]
29	SJV-1t	0.6795	[0.810, 0.992]	[0.700, 1.000]	[0.700, 1.000]	[0.500, 1.000]
30	LJV-10m	0.7465	[0.905, 0.998]	[0.750, 1.000]	[0.829, 1.000]	[0.500, 1.000]

Notes:

1. The instrument set consists of a constant and a lag of instrument, $l = 1$.
2. We use log squared transformation for the first 13 classes of instruments (RV-RSVP) given in Table 11.
3. The confidence intervals are constructed by projection technique described in Section 3.5. The corresponding 95% confidence interval for the nuisance parameter λ is [33.943, 61.154] with $\hat{\lambda} = 47.548$ and $\text{SE}(\hat{\lambda}) = 6.935$.
4. The proposed inference procedures [AR, AR^*, SS, SS^*] are described in Section 3.1 - 3.4.
5. We use $\tau = 0.2$ for SS-type tests and employ 99 Monte Carlo replications for point optimal type procedures.
6. An instrument set i's average strength over inference methods is measured by $\bar{d}_{i,s}$.

Table 16. Projection-based 90% confidence intervals for the volatility persistence parameter ϕ
Strong instruments (Several lags)
Ticker: IBM, January 2009 - December 2013, $T = 1258$

	$l=1$					$l=3$					$l=5$				
Instruments	$\bar{d}_{i,s}$	AR	AR^*	SS	SS^*	$\bar{d}_{i,s}$	AR	AR^*	SS	SS^*	$\bar{d}_{i,s}$	AR	AR^*	SS	SS^*
RSVN-5m-ss	0.8860	[0.950, 1.0]	[0.866, 1.0]	[0.932, 1.0]	[0.796, 1.0]	0.8618	[0.949, 1.0]	[0.838, 1.0]	[0.895, 1.0]	[0.765, 1.0]	0.8603	[0.941, 1.0]	[0.845, 1.0]	[0.874, 1.0]	[0.781, 1.0]
RSVN-5m	0.8855	[0.948, 1.0]	[0.864, 1.0]	[0.931, 1.0]	[0.799, 1.0]	0.8668	[0.955, 1.0]	[0.845, 1.0]	[0.892, 1.0]	[0.775, 1.0]	0.8695	[0.960, 1.0]	[0.856, 1.0]	[0.871, 1.0]	[0.791, 1.0]
RSVN-1m	0.8848	[0.947, 1.0]	[0.856, 1.0]	[0.929, 1.0]	[0.807, 1.0]	0.8570	[0.951, 1.0]	[0.851, 1.0]	[0.887, 1.0]	[0.739, 1.0]	0.8555	[0.946, 1.0]	[0.845, 1.0]	[0.861, 1.0]	[0.770, 1.0]
ImV-C-mean	0.8830	[0.964, 1.0]	[0.852, 1.0]	[0.937, 1.0]	[0.779, 1.0]	0.8218	[0.970, 1.0]	[0.828, 1.0]	[0.849, 1.0]	[0.640, 1.0]	0.8078	[0.950, 1.0]	[0.813, 1.0]	[0.757, 1.0]	[0.711, 1.0]
MinRV-5m	0.8828	[0.945, 1.0]	[0.867, 1.0]	[0.925, 1.0]	[0.794, 1.0]	0.8493	[0.935, 1.0]	[0.837, 1.0]	[0.886, 1.0]	[0.739, 1.0]	0.8465	[0.928, 1.0]	[0.825, 1.0]	[0.856, 1.0]	[0.777, 1.0]
RV-5m-ss	0.8825	[0.946, 1.0]	[0.863, 1.0]	[0.926, 1.0]	[0.795, 1.0]	0.8560	[0.939, 1.0]	[0.837, 1.0]	[0.889, 1.0]	[0.759, 1.0]	0.8523	[0.932, 1.0]	[0.840, 1.0]	[0.866, 1.0]	[0.771, 1.0]
BV-5m	0.8823	[0.945, 1.0]	[0.865, 1.0]	[0.926, 1.0]	[0.793, 1.0]	0.8508	[0.935, 1.0]	[0.835, 1.0]	[0.887, 1.0]	[0.746, 1.0]	0.8500	[0.932, 1.0]	[0.827, 1.0]	[0.861, 1.0]	[0.780, 1.0]
BV-5m-ss	0.8823	[0.945, 1.0]	[0.865, 1.0]	[0.926, 1.0]	[0.793, 1.0]	0.8508	[0.935, 1.0]	[0.835, 1.0]	[0.887, 1.0]	[0.746, 1.0]	0.8500	[0.932, 1.0]	[0.827, 1.0]	[0.861, 1.0]	[0.780, 1.0]
BV-10m-ss	0.8823	[0.945, 1.0]	[0.865, 1.0]	[0.926, 1.0]	[0.793, 1.0]	0.8508	[0.935, 1.0]	[0.835, 1.0]	[0.887, 1.0]	[0.746, 1.0]	0.8500	[0.932, 1.0]	[0.827, 1.0]	[0.861, 1.0]	[0.780, 1.0]
MedRV-5m	0.8823	[0.945, 1.0]	[0.866, 1.0]	[0.925, 1.0]	[0.793, 1.0]	0.8493	[0.936, 1.0]	[0.833, 1.0]	[0.886, 1.0]	[0.742, 1.0]	0.8515	[0.930, 1.0]	[0.826, 1.0]	[0.859, 1.0]	[0.791, 1.0]
1-day	0.4255	[0.870, 0.965]	[0.750, 1.0]	[0.000, 0.953]	[0.000, 1.0]	0.7490	[0.855, 0.974]	[0.760, 1.0]	[0.782, 0.977]	[0.550, 1.0]	0.7485	[0.838, 0.979]	[0.770, 1.0]	[0.741, 0.986]	[0.610, 1.0]

Notes:

1. The instrument set consists of a constant and lags of instrument, $l = 1, 3, 5$.
2. We use log squared transformation for the first 13 classes of instruments (RV-RSVP) given in Table 11.
3. The confidence intervals are constructed by projection technique described in Section 3.5. The corresponding 95% confidence interval for the nuisance parameter λ is $[33.943, 61.154]$ with $\hat{\lambda} = 47.548$ and $\text{SE}(\hat{\lambda}) = 6.935$.
4. The proposed inference procedures $[AR, AR^*, SS, SS^*]$ are described in Section 3.1 - 3.4.
5. We use $\tau = 0.2$ for SS-type tests and employ 99 Monte Carlo replications for point optimal type procedures.
6. An instrument set i's average strength over inference methods is measured by $\bar{d}_{i,s}$.

Table 17. Projection-based 90% confidence intervals for the volatility persistence parameter ϕ
Different combinations of strong instruments
Ticker: IBM, January 2009 - December 2013, $T = 1258$

Instrument set	$\bar{d}_{i,s}$	# of Instruments	AR	AR^*	SS	SS^*
RV-5m-ss, ImV-C-q3	0.8748	2	[0.954, 1.000]	[0.848, 1.000]	[0.910, 1.000]	[0.787, 1.000]
BV-5m-ss, ImV-C-q3	0.8775	2	[0.954, 1.000]	[0.850, 1.000]	[0.912, 1.000]	[0.794, 1.000]
RSVN-5m, ImV-C-q3	0.8820	2	[0.953, 1.000]	[0.857, 1.000]	[0.916, 1.000]	[0.802, 1.000]
RKcub-10t, ImV-C-q3	0.8583	2	[0.958, 0.995]	[0.854, 1.000]	[0.892, 1.000]	[0.724, 1.000]
BV-5m-ss, LJV-5s	0.8650	2	[0.936, 1.000]	[0.841, 1.000]	[0.908, 1.000]	[0.775, 1.000]
RKcub-10t, PCF-1, ImV-C-q3	0.8555	3	[0.967, 0.991]	[0.842, 1.000]	[0.878, 1.000]	[0.726, 1.000]
BV-5m-ss, LJV-5s, ImV-C-q3	0.8645	3	[0.946, 1.000]	[0.838, 1.000]	[0.898, 1.000]	[0.776, 1.000]
BV-5m-ss, LJV-5s, PCF-1	0.8568	3	[0.960, 0.999]	[0.844, 1.000]	[0.888, 1.000]	[0.734, 1.000]
BV-5m-ss, LJV-5s, PCF-1, ImV-C-q3	0.8553	4	[0.966, 0.996]	[0.829, 1.000]	[0.881, 1.000]	[0.741, 1.000]
BV-5m-ss, LJV-5s, LSJV-10t, PCF-1, ImV-C-q3	0.8493	5	[0.959, 0.997]	[0.820, 1.000]	[0.872, 1.000]	[0.743, 1.000]

Notes:

1. The instrument set includes a constant and lags of instruments given in column 3.
2. We use log squared transformation for the first 13 classes of instruments (RV-RSVP) given in Table 11.
3. The confidence intervals are constructed by projection technique described in Section 3.5. The corresponding 95% confidence interval for the nuisance parameter λ is [33.943, 61.154] with $\hat{\lambda} = 47.548$ and $\mathrm{SE}(\hat{\lambda}) = 6.935$.
4. The proposed inference procedures [AR, AR^*, SS, SS^*] are described in Section 3.1 - 3.4.
5. We use $\tau = 0.2$ for SS-type tests and employ 99 Monte Carlo replications for point optimal type procedures.
6. An instrument set i's average strength over inference methods is measured by $\bar{d}_{i,s}$.