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House Allocation via Deferred-Acceptance*

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Abstract

We study the simple model of assigning indivisible and heterogenous objects (*e.g.*, houses, jobs, offices, *etc.*) to agents. Each agent receives at most one object and monetary compensations are not possible. For this model, known as the house allocation model, we characterize the class of rules satisfying *unavailable object invariance*, *individual rationality*, *weak non-wastefulness*, *resource-monotonicity*, *truncation invariance*, and *strategy-proofness*: any rule with these properties must allocate objects based on (implicitly induced) objects' priorities over agents and the agent-proposing deferred-acceptance-algorithm.

JEL Classification: D63, D70

Keywords: deferred-acceptance-algorithm, indivisible objects allocation, resource-monotonicity, strategy-proofness.

1 Introduction

We study the simple model of assigning indivisible and heterogenous objects (*e.g.*, houses, jobs, offices, *etc.*) to agents. Agents have strict preferences over objects and remaining unassigned. An assignment is an allocation of the objects to the agents such that every agent receives at most one object. A rule associates an assignment to each preference profile. This problem is known as the "house allocation problem" and the search for "good" rules to solve it has been the subject of various contributions (Ehlers, 2002; Ehlers and Klaus, 2004, 2007, 2011; Kesten, 2009; Pápai, 2000).

As Ehlers and Klaus (2004, 2011) and Kesten (2009) we consider situations where resources may change, *i.e.*, it could be that additional objects are available. When the change of the environment is exogenous, it would be unfair if the agents who were not responsible for this change were treated

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unequally. We apply this idea of solidarity and require that if additional resources become available, then all agents (weakly) gain. This requirement is called *resource-monotonicity* (Chun and Thomson, 1988). Ehlers and Klaus (2004, 2011) and Kesten (2009) consider resource-monotonic rules for house allocation and prove that *resource-monotonicity* together with *efficiency* and some other properties characterizes the class of mixed-dictator-pairwise-exchange rules. Here, we only impose the mild efficiency requirement of *weak non-wastefulness*[1] as well as the very basic and intuitive properties of *individual rationality*[2] and *unavailable object invariance*.[3] We also impose the invariance property *truncation invariance*.[4] Our last property is the well-known strategic robustness condition of *strategy-proofness*.[5] We show that these elementary and intuitive properties characterize the class of so-called deferred-acceptance-rules, *i.e.*, any rule with these properties must allocate objects based on (implicitly induced) objects' priorities over agents and the outcome of the agent-proposing deferred-acceptance-algorithm. The class of deferred-acceptance-rules contains the class of mixed-dictator-pairwise-exchange rules as characterized by Ehlers and Klaus (2004, 2011) and Kesten (2009); in fact, our main result implies these previous characterizations (and strengthens one of them).

Related papers are Kojima and Manea (2010) and Ehlers and Klaus (2012). They characterize deferred-acceptance rules based on "responsive" priorities and "acceptant substitutable" priorities for a more general model where several identical copies of an object may be available (whereas in our house allocation model only one copy of each object is available). Example 1 shows that our characterization only holds for house allocation but not for the more general model considered by Kojima and Manea (2010) and Ehlers and Klaus (2012). Note that in all these contributions priorities are derived from a rule via a set of properties. Other papers take exogenous priorities as given and impose properties on the rule using these exogenous priorities. Balinski and Sönmez (1999) and Morrill (2013) then characterize the deferred-acceptance rules based on "responsive" priorities and "substitutable" priorities.

The paper is organized as follows. In Section 2 we introduce the house allocation model, properties of rules, and the class of deferred-acceptance-rules. In Section 3 we state our characterization of the class of deferred-acceptance-rules (Theorem 1) and derive (strengthen) previous characterizations of Ehlers and Klaus (2004, 2011) and Kesten (2009) (Corollary 1). Proofs and the independence of properties in Theorem 1 can be found in the Appendix.

[1] No agent who does not receive any real object (but the so-called null object) would prefer to obtain a real object that is not assigned.

[2] Each agent weakly prefers his allotment to not receiving any object (or to receiving the so-called null object).

[3] The rule only depends on the set of available object types.

[4] "Truncating preferences" by moving the null object below the assigned object does not change the allocation.

[5] No agent can manipulate the allocation to his advantage by lying about his preferences.

2 House Allocation

2.1 The Model and Notation

Our house allocation model is identical to that described in Ehlers and Klaus (2004, 2011).

Let N denote a finite set of agents, $|N| \geq 2$. Let K denote a set of potential real objects (or real houses). Not receiving any real object is called "receiving the null object." Let 0 represent the *null object*. Each agent $i \in N$ is equipped with a preference relation R_i over all objects $K \cup \{0\}$. Given $x, y \in K \cup \{0\}$, $x\, R_i\, y$ means that agent i weakly prefers x to y, and $x\, P_i\, y$ means that agent i strictly prefers x to y. We assume that R_i is strict, *i.e.*, R_i is a linear order over $K \cup \{0\}$. Let $\mathcal{R}$ denote the class of all linear orders over $K \cup \{0\}$, and $\mathcal{R}^N$ the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. Given $i \in N$ and $R_i \in \mathcal{R}$, object $x \in K$ is *acceptable under* R_i if $x\, P_i\, 0$. Let $A(R_i) = \{x \in K : x\, P_i\, 0\}$ denote the *set of acceptable objects under* R_i. Given $K' \subseteq K \cup \{0\}$, let $R_i|_{K'}$ denote the restriction of R_i to K' and $R|_{K'} = (R_i|_{K'})_{i \in N}$. Given $R \in \mathcal{R}^N$ and $M \subseteq N$, let R_M denote the profile $(R_i)_{i \in M}$. It is the restriction of R to the set of agents M. We also use the notation $R_{-M} = R_{N \setminus M}$ and $R_{-i} = R_{N \setminus \{i\}}$.

An *allocation* is a list $a = (a_i)_{i \in N}$ such that for all $i \in N$, $a_i \in K \cup \{0\}$ and none of the real objects in K is assigned to more than one agent. Note that 0, the null object, can be assigned to any number of agents and that not all real objects have to be assigned. Let $\mathcal{A}$ denote the set of all allocations. Let $\mathcal{H}$ denote the set of all non-empty subsets H of K. A *(house allocation) problem* consists of a preference profile $R \in \mathcal{R}^N$ and a set of real objects $H \in \mathcal{H}$. Note that the associated set of *available objects* $H \cup \{0\}$ always includes the null object. An *(allocation) rule* is a function $\varphi : \mathcal{R}^N \times \mathcal{H} \to \mathcal{A}$ such that for all problems $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, $\varphi(R, H) \in \mathcal{A}$ is *feasible*, *i.e.*, for all $i \in N$, $\varphi_i(R, H) \in H \cup \{0\}$. By feasibility, each agent receives an available object. Given $i \in N$, we call $\varphi_i(R, H)$ the *allotment* of agent i at $\varphi(R, H)$.

2.2 Properties of Rules

A natural requirement for a rule is that the chosen allocation depends only on preferences over the set of available objects.

Unavailable Object Invariance:[6] For all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ and all $R' \in \mathcal{R}^N$ such that $R|_{H \cup \{0\}} = R'|_{H \cup \{0\}}$, $\varphi(R, H) = \varphi(R', H)$.

By *individual rationality* each agent should weakly prefer his allotment to the null object.

Individual Rationality: For all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ and all $i \in N$, $\varphi_i(R, H)\, R_i\, 0$.

Next, we introduce two properties that require a rule to not waste any resources. First, *non-wastefulness* (Balinski and Sönmez, 1999) requires that no agent prefers an available real object that is not assigned to his allotment. *Non-wastefulness* is a weak efficiency requirement.

[6] In Ehlers and Klaus (2004, 2011) we call this property *independence of irrelevant objects*.

Non-Wastefulness: For all $(R,H) \in \mathcal{R}^N \times \mathcal{H}$, all $x \in H$, and all $i \in N$, if $x\ P_i\ \varphi_i(R,H)$, then there exists $j \in N$ such that $\varphi_j(R,H) = x$.

Next, we weaken *non-wastefulness* by requiring that no agent receives the null object while he prefers an available real object that is not assigned.

Weak Non-Wastefulness: For all $(R,H) \in \mathcal{R}^N \times \mathcal{H}$, all $x \in H$, and all $i \in N$, if $x\ P_i\ \varphi_i(R,H)$ and $\varphi_i(R,H) = 0$, then there exists $j \in N$ such that $\varphi_j(R,H) = x$.

Weak non-wastefulness is a limited efficiency requirement that only applies to agents who receive the null object.

Of course, no resources are wasted if a rule is *(Pareto) efficient*.

Efficiency: For all $(R,H) \in \mathcal{R}^N \times \mathcal{H}$, there exists no feasible allocation $a \in \mathcal{A}$ such that for all $i \in N$, $a_i\ R_i\ \varphi_i(R,H)$, and for some $j \in N$, $a_j\ P_j\ \varphi_j(R,H)$.

Note that *efficiency* implies *individual rationality* and *(weak) non-wastefulness*.

When the set of objects varies, another natural requirement is *resource-monotonicity*. As already explained in the Introduction, this is a widely used solidarity property introduced by Chun and Thomson (1988) and it describes the effect of a change in the available resources on the welfare of the agents. A rule is *resource-monotonic* if the availability of more real objects has a (weakly) positive effect on all agents.

Resource-Monotonicity: For all $R \in \mathcal{R}^N$ and all $H, H' \in \mathcal{H}$, if $H \subseteq H'$, then for all $i \in N$, $\varphi_i(R,H')\ R_i\ \varphi_i(R,H)$.

Many rules that are used in real life ignore agents' preferences below their allotments (*e.g.*, any rule based on or equivalent to the famous deferred-acceptance-algorithm or the so-called priority rules, Roth and Sotomayor, 1990, Sections 5.4.1 and 5.5.1). That is, an allocation does not change if an agents changes his reported preferences below his allotment. We formulate a weaker version of this invariance property by restricting agents' changes below their allotments to truncations.

Let $i \in N$ and $R_i \in \mathcal{R}$. Then, a truncation of a preference relation R_i is a preference relation $\bar{R}_i$ that ranks the real objects in the same way as the corresponding original preference relation and each real object which is acceptable under the truncation is also acceptable under the original preference relation. Formally, preference relation $\bar{R}_i \in \mathcal{R}$ is a *truncation of* R_i if (a) $\bar{R}_i|_K = R_i|_K$ and (b) $A(\bar{R}_i) \subseteq A(R_i)$. Loosely speaking, a truncation strategy of R_i is obtained by moving the null object "up."

If an agent truncates his preference relation in a way such that his allotment remains acceptable under the truncated preference relation, then *truncation invariance* requires that the allocation is the same under both profiles. The property is quite natural on its own in the sense that the chosen allocations do not depend on where any agent, who receives an individually rational real object, ranks the null object below his allotment.

Truncation Invariance: For all $(R,H) \in \mathcal{R}^N \times \mathcal{H}$, all $i \in N$, and all $\bar{R}_i \in \mathcal{R}_i$, if $\bar{R}_i$ is a truncation of R_i and $\varphi_i(R,H)$ is acceptable under $\bar{R}_i$ (*i.e.*, $\varphi_i(R,H) \in A(\bar{R}_i)$), then $\varphi((\bar{R}_i, R_{-i}), H) = \varphi(R,H)$.

The well-known non-manipulability property *strategy-proofness* requires that no agent can ever benefit from misrepresenting his preferences.

Strategy-Proofness: For all $(R,H) \in \mathcal{R}^N \times \mathcal{H}$, all $i \in N$, and all $\bar{R}_i \in \mathcal{R}$, $\varphi_i(R,H)\ R_i\ \varphi_i((\bar{R}_i, R_{-i}), H)$.

The following strengthening of *strategy-proofness* requires that no group of agents can ever benefit by misrepresenting their preferences.

Group Strategy-Proofness: For all $(R,H) \in \mathcal{R}^N \times \mathcal{H}$, all $M \subseteq N$, and all $\bar{R}_M \in \mathcal{R}^M$, if for all $i \in M$, $\varphi_i((\bar{R}_M, R_{-M}), H)\ R_i\ \varphi_i(R,H)$, then for all $i \in M$, $\varphi_i((\bar{R}_M, R_{-M}), H) = \varphi_i(R,H)$.

2.3 Priority Structures, Marriage Markets, and Stability

Given object $x \in K$, let $\succ_x$ denote a *priority ordering on* N, *e.g.*, $\succ_x$: $1\ 2\ \dots\ n$ means that agent 1 has higher priority for object x than agent 2, who has higher priority for object x than agent 3, *etc.* Let $\succ \equiv (\succ_x)_{x \in K}$ denote a *priority structure*. Then, given a priority structure $\succ$ and a problem (R,H), we can interpret $(R,(H,\succ))$ as a *marriage market* (Gale and Shapley, 1962; Roth and Sotomayor, 1990) where the set of agents N, for instance, corresponds to the set of women, the set of objects H corresponds to the set of available men, preferences $R|_{H \cup \{0\}}$ correspond to women's preferences over available men, and the priority structure $(\succ_x)_{x \in H}$ corresponds to the available men's preferences over women. Stability is an important requirement for many real-life matching markets and it will turn out to be essential in our context of allocating indivisible objects to agents as well.

Stability under $\succ$: Given $(R,H) \in \mathcal{R}^N \times \mathcal{H}$, a feasible *allocation* $a \in \mathcal{A}$ *is stable under* $\succ$ if there exists no agent-object pair $(i,x) \in N \times H \cup \{0\}$ such that (a) $x\ P_i\ a_i$ and (b) either $[x=0]$ or [for all $j \in N$, $a_j \neq x$] or [there exists $k \in N$ such that $a_k = x$ and $i \succ_x k$].

Furthermore, *rule* φ *is stable* if there exists a priority structure $\succ$ such that for each problem $(R,H) \in \mathcal{R}^N \times \mathcal{H}$, $\varphi(R,H)$ is stable under $\succ$.

Note that *stability* implies *individual rationality* and *(weak) non wastefulness*, but it does not imply *efficiency*.

For any marriage market $(R,(H,\succ))$, we denote by $DA^{\succ}(R,H)$ the agent-optimal stable allocation that is obtained by using Gale and Shapley's (1962) agent-proposing deferred-acceptance-algorithm: let $(R,(H,\succ))$ be given. Then,

- at the first step of the deferred-acceptance-algorithm, every agent applies to his favorite available object in $H \cup \{0\}$. For each available real object $x \in H$, the applicant who has the highest priority for x is placed on the waiting list of x, and all others are rejected. The null object 0 accepts all agents.

- At the r-th step of the deferred-acceptance-algorithm, those applicants who were rejected at step $r-1$ apply to their next best available object in $H \cup \{0\}$. For each available real object $x \in H$, the applicant among the new applicants and the one on the waiting list who has the highest priority for x is placed on the (updated) waiting list of x, and all others are rejected. The null object 0 accepts all agents.

The deferred-acceptance-algorithm terminates when every agent is on a waiting list.[7] Once the algorithm ends, available real objects are assigned to the agents on the waiting lists (all other agents were accepted by and receive the null object) and the resulting allocation is the agent-optimal stable allocation for the marriage problem $(R, (H, \succ))$, denoted by $DA^{\succ}(R, H)$.

Deferred-Acceptance-Rules: A rule φ is a deferred-acceptance-rule if there exists a priority structure $\succ$ such that for each $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, $\varphi(R, H) = DA^{\succ}(R, H)$.

3 Characterizations of Deferred Acceptance

We first present our main characterization.

Theorem 1. *Deferred-acceptance-rules are the only rules satisfying unavailable object invariance, individual rationality, weak non-wastefulness, resource-monotonicity, truncation invariance, and strategy-proofness.*

The proof of Theorem 1 in Section A.1 reveals the following additional result, which is based on the same properties as used in Theorem 1 except for *strategy-proofness*.

Proposition 1. *If a rule satisfies unavailable object invariance, individual rationality, weak non-wastefulness, resource-monotonicity, and truncation invariance, then it is stable.*

Next, we show how Theorem 1 implies a previous characterization result (Ehlers and Klaus, 2011, Theorem 2) and strengthens another one (Ehlers and Klaus, 2011, Corollary 1; Kesten , 2009, Theorem 1). To this end we introduce an acyclicity condition which can be derived by applying Ergin's (2002) acyclicity condition to house allocation problems.

Cycles and Acyclicity: Given a priority structure $\succ$ a *cycle* consists of distinct $x, y \in K$ and $i, j, k \in N$ such that $i \succ_x j \succ_x k$ and $k \succ_y i$. A priority structure $\succ$ is *acyclic* if it has no cycles.

[7] Note that the null object has unlimited capacity and eventually any agent is put on the waiting list of either a real object or accepted by the null object.

A deferred-acceptance-rule is *acyclic* if the associated priority structure is acyclic.

Ergin (2002, Theorem 1) shows that the acyclicity of the priority structure $\succ$ is equivalent to *efficiency* or *group strategy-proofness* of the induced deferred-acceptance-rule $DA^{\succ}$. Furthermore, with Kesten's (2009, Theorem 1) result it follows that for house allocation the class of *efficient* deferred-acceptance-rules equals the class of mixed-dictator-pairwise-exchange-rules characterized in Ehlers and Klaus (2004, 2011). Then, Theorem 1 implies the following characterizations of the subclass of acyclic deferred-acceptance-rules.

Corollary 1.
(a) *Deferred-acceptance-rules with acyclic priority structures are the only rules satisfying unavailable object invariance, efficiency, resource-monotonicity, and truncation invariance;*
(b) *Deferred-acceptance-rules with acyclic priority structures are the only rules satisfying individual rationality, weak non-wastefulness, resource-monotonicity, and group strategy-proofness.*

Corollary 1(a) equals Ehlers and Klaus's (2011) Theorem 2 (a correction of Ehlers and Klaus, 2004, Theorem 1). Corollary 1(b) strengthens Ehlers and Klaus's (2004) Corollary 1 and Kesten's (2009) Theorem 1 by replacing *efficiency* by the weaker properties *individual rationality* and *weak non-wastefulness*.

The following example demonstrates that Theorem 1 and Proposition 1 do not hold for the more general model where more than one copy of each object might be available.

Example 1. Let $N = \{1,2,3\}$, $K = \{x,y\}$, and there are two copies of x and one copy of y available. We then denote a problem by (R,q), where $q = (q_x, q_y)$ denotes the availability of objects x and y, *e.g.*, $q = (2,0)$ denotes that two copies of x and no copy of y are available. Let $O_+(q) = \{x \in K : q_x > 0\}$ denote the set of *available real objects* under q. All properties are easily adapted to this more general allocation situation.

The following rule f satisfies *unavailable object invariance*, *individual rationality*, *weak non-wastefulness*, *resource-monotonicity*, *truncation invariance*, and *strategy-proofness*.

Let $\succ_x$: 1 2 3, $\succ'_x$: 1 3 2, and $\succ_y$: 1 2 3. Let $\succ = (\succ_x, \succ_y)$ and $\succ' = (\succ'_x, \succ_y)$. Then, for each problem (R,q),

$$f(R,q) = \begin{cases} DA^{\succ'}(R,q) & \text{if } q_x = 2 \text{ and } x \text{ is agent 1's favorite object in } O_+(q) \text{ and} \\ DA^{\succ}(R,q) & \text{otherwise.} \end{cases}$$

Note that f is not stable (and therefore, f is not a deferred-acceptance-rule): suppose that f is stable with respect to $\hat{\succ}$; if $q_x = 2$ and each agent's favorite object is x, then agents 1 and 3 receive x and agent 2 receives 0 (not the desired x), implying $3 \mathrel{\hat{\succ}_x} 2$; if $q_x = 1$, agent 2's and agent 3's favorite object is x, and agent 1's favorite object is 0, then agent 2 receives x and agent 3 receive 0 (not the desired x), implying $2 \mathrel{\hat{\succ}_x} 3$; a contradiction.

It is easy to see that f satisfies *unavailable object invariance*, *individual rationality*, *weak non-wastefulness*, *truncation invariance*, and *strategy-proofness*.

For *resource-monotonicity*, let $R \in \mathcal{R}^N$ and q, q' be such that for all $z \in K$, $q_z \leq q'_z$. It suffices to check for possible violations of *resource-monotonicity* when f uses a different priority structure for (R, q) and (R, q'). Hence, $q'_x = 2$.

If $q_x = 2$, then $q'_y = 1$, $q_y = 0$, x is agent 1's favorite object in $O_+(q)$, and y is agent 1's favorite object in $O_+(q')$ (otherwise the priority structure for (R, q) and (R, q') would not change). Furthermore, $\succ$ is used for problem (R, q) and $\succ'$ is used for problem (R, q'). Then, $f_1(R, q) = x$ and $f_1(R, q') = y$. But now two copies of object x are available for agents 2 and 3 for problem (R, q') and *resource-monotonicity* is satisfied.

Otherwise, $q_x < 2$ and x must be agent 1's favorite object in $O_+(q')$ (otherwise the priority structure for (R, q) and (R, q') would not change).

If $q_x = 0$, then the only violation of *resource-monotonicity* could be that an agent receives y at $f(R, q)$ and 0 at $f(R, q')$. However, since object y is allocated according to the same priority ordering under $\succ$ and $\succ'$, this cannot happen.

If $q_x = 1$, then $f_1(R, q) = x$ and none of the agents 2 or 3 can obtain x. Hence, in terms of allocating object x is does not matter if priority ordering $\succ_x$ or $\succ'_x$ is used; it is as if $\succ'$ is used for both problems and no violation of *resource-monotonicity* occurs. $\diamond$

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A Appendix

A.1 Proof of Proposition 1 and Theorem 1

First, note that all deferred-acceptance rules are stable and that *stability* implies *individual rationality* and *weak non-wastefulness*. Furthermore, it is easy to check that all deferred-acceptance rules satisfy *unavailable object invariance* and *truncation invariance*. Dubins and Freedman (1981) and Roth (1982) proved *strategy-proofness* of all deferred-acceptance-rules. Crawford (1991) studied comparative statics of deferred-acceptance-rules. From his results it follows that all deferred-acceptance-rules are *resource-monotonic*.

Second, let φ be a rule satisfying the properties of Theorem 1. First, we "calibrate/construct the priority structure using maximal conflict preference profiles."

We denote a preference relation with only one acceptable object $x \in K$ by R^x, i.e., $A(R^x) = \{x\}$. We denote the set of all preference relations that have $x \in K$ as the unique acceptable object by $\mathcal{R}^x$. Let $R^0 \in \mathcal{R}$ be such that $A(R^0) = 0$.

For any $S \subseteq N$, let $R^x_S = (R^x_i)_{i \in S}$ such that for all $i \in S$, $R^x_i = R^x$, and similarly $R^0_S = (R^0_i)_{i \in S}$ such that for all $i \in S$, $R^0_i = R^0$.

Consider the problem $(R^x_N, \{x\})$. By *weak non-wastefulness*, for some $j \in N$, $\varphi_j(R^x_N, \{x\}) = x$, say $j = 1$. Then, for all $i \in N \setminus \{1\}$, we set $1 \succ_x i$.

Next consider the problem $((R^0_1, R^x_{-1}), \{x\})$. By *weak non-wastefulness* and *individual rationality*, for some $j \in N \setminus \{1\}$, $\varphi_j((R^0_1, R^x_{-1}), \{x\}) = x$, say $j = 2$. Then, for all $i \in N \setminus \{1, 2\}$, we set $2 \succ_x i$.

By induction, we obtain $\succ_x$ for any object x and thus a priority structure $\succ = (\succ_x)_{x \in K}$.

Lemma 1. *For all $R \in \mathcal{R}^N$ and all $x \in K$, if for some $j \in N$, $\varphi_j(R, \{x\}) = x$, then for all $i \in N \setminus \{j\}$, $x \in A(R_i)$ implies $j \succ_x i$.*

Proof. Let $R \in \mathcal{R}^N$ and $x \in K$. Without loss of generality, suppose $1 \succ_x 2 \succ_x \cdots \succ_x n$. Let $S = \{i \in N : x \in A(R_i)\}$ and let $j = \min S$. We prove Lemma 1 by showing that $\varphi_j(R, \{x\}) = x$.

Note that for all $i \in N \setminus S$, $0 P_i x$. We partition the set $N \setminus S$ into the "lower" set $L = \{1, \ldots, j-1\}$ (possibly $L = \emptyset$) and the "upper" set $U = N \setminus (L \cup S)$ (possibly $U = \emptyset$). Note that by *unavailable object invariance*, $\varphi(R, \{x\}) = \varphi((R^0_L, R^x_S, R^0_U), \{x\})$.

By the construction of $\succ_x$, $\varphi_j((R^0_L, R^x_{S \cup U}), \{x\}) = x$. Hence, if $U = 0$, then $\varphi_j(R, \{x\}) = x$ and for all $i \in N$, $x \in A(R_i)$ implies $j \succ_x i$.

Step 1: Let $k \in U$. We prove that $\varphi_j((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R^0_k), \{x\}) = \varphi_j((R^0_L, R^x_{S \cup U}), \{x\}) = x$.

Let $y \in K \setminus \{x\}$ and $R'_k \in \mathcal{R}$ be such that $R'_k : y\ x\ 0\ \ldots$. By *unavailable object invariance*, $\varphi_j((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R'_k), \{x\}) = \varphi_j((R^0_L, R^x_{S \cup U}), \{x\}) = x$. Now by *resource-monotonicity*, $\varphi_j((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R'_k), \{x, y\}) = x$. By *weak non-wastefulness* and *individual rationality*, $\varphi_k((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R'_k), \{x, y\}) = y$.

On the other hand, suppose that $\varphi_j((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R^0_k), \{x\}) \neq x$. Then, by *individual rationality*, $\varphi_k((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R^0_k), \{x\}) \neq x$. Let $R''_k \in \mathcal{R}$ be such that $R''_k : y\, 0\, x\ \ldots$ and $R''_k|_K = R'_k|_K$. By *unavailable object invariance*, $\varphi((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R''_k), \{x\}) = \varphi((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R^0_k), \{x\})$. Thus, by *weak non-wastefulness* and *individual rationality*, for some $l \in S \cup (U \setminus \{j, k\})$, $\varphi_l((R^0_L, R^x_{S \cup (U \setminus \{j\})}, R''_k), \{x\}) = x$.

Now by *resource-monotonicity*, $\varphi_l((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R''_k), \{x, y\}) = x$. By *weak non-wastefulness* and *individual rationality*, $\varphi_k((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R''_k), \{x, y\}) = y$. Now R''_k is a truncation of R'_k and both $y \in A(R''_k)$ and $\varphi_k((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R'_k), \{x, y\}) = y$. Since $\varphi_j((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R''_k), \{x, y\}) \neq \varphi_j((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R'_k), \{x, y\})$ (because $j \neq l$), this is a contradiction to *truncation invariance*. Hence, $\varphi_j((R^0_L, R^x_{S \cup (U \setminus \{k\})}, R^0_k), \{x\}) = x$.

Steps 2, …: Let $U = \{k_1, \ldots, k_l\}$. Then using the same arguments as above, it follows that $x = \varphi_j((R^0_L, R^x_{S \cup U}), \{x\}) = \varphi_j((R^0_L, R^x_{S \cup (U \setminus \{k_1\})}, R^0_{k_1}), \{x\}) = \varphi_j((R^0_L, R^x_{S \cup (U \setminus \{k_1, k_2\})}, R^0_{\{k_1, k_2\}}), \{x\}) = \ldots = \varphi_j((R^0_L, R^x_{S \cup \{k_l\}}, R^0_{U \setminus \{k_l\}}), \{x\}) = \varphi_j((R^0_L, R^x_S, R^0_U), \{x\}) = \varphi_j(R, \{x\})$. Hence, we obtain the desired result that $\varphi_j(R, \{x\}) = x$. $\square$

With the following lemma (note that so far we have not used *strategy-proofness*), Proposition 1 follows.

Lemma 2. *For all $(R,H) \in \mathcal{R}^N \times \mathcal{H}$, $\varphi(R,H)$ is stable under $\succ$.*

Proof. Let $(R,H) \in \mathcal{R}^N \times \mathcal{H}$. Assume that $\varphi(R,H)$ is not stable under $\succ$. Then, there exists an agent-object pair $(i,x) \in N \times (H \cup \{0\})$ such that (a) $x \, P_i \, \varphi_i(R,H)$ and (b) either $[x = 0]$ or [for all $j \in N$, $\varphi_j(R,H) \neq x$] or [there exists $k \in N$ such that $\varphi_k(R,H) = x$ and $i \succ_x k$]. By *individual rationality*, $x \neq 0$. Hence, [for all $j \in N$, $\varphi_j(R,H) \neq x$] or [there exists $k \in N$ such that $\varphi_k(R,H) = x$ and $i \succ_x k$].

Let $\bar{R} \in \mathcal{R}^N$ be such that (i) for all $j \in N$ such that $\varphi_j(R,H) \neq 0$, $\bar{R}_j$ is a truncation of R_j such that there exists no $y \in K \setminus \{\varphi_j(R,H)\}$ with $\varphi_j(R,H) \, \bar{R}_j \, y \, \bar{R}_j \, 0$ and (ii) for all $j \in N$ such that $\varphi_j(R,H) = 0$, $\bar{R}_j = R_j$. (By *individual rationality*, $\bar{R}_j$ in (i) is well-defined as truncation of R_j.) By *truncation invariance*, $\varphi(\bar{R},H) = \varphi(R,H)$ and $(i,x) \in N \times H$ is such that $x \, \bar{P}_i \, \varphi_i(\bar{R},H)$ and [for all $j \in N$, $\varphi_j(\bar{R},H) \neq x$] or [there exists $k \in N$ such that $\varphi_k(\bar{R},H) = x$ and $i \succ_x k$].

Let $S = \{j \in N : x \, \bar{P}_j \, \varphi_j(\bar{R},H)\}$. Note that $i \in S$.

If for $j \in S$, $\varphi_j(\bar{R},\{x\}) = x$, then by *resource-monotonicity*, $\varphi_j(\bar{R},H) \, \bar{R}_j \, x$, contradicting $x \, \bar{P}_j \, \varphi_j(\bar{R},H)$. Hence, for all $j \in S$, $\varphi_j(\bar{R},\{x\}) = 0$. If $S = \{j \in N : x \, \bar{P}_j \, 0\}$, then this contradicts *weak non-wastefulness*. Thus, $S \subsetneq \{j \in N : x \, \bar{P}_j \, 0\}$.

By construction of $\bar{R}$, for all $j \in N$ we have either (i) $x \, \bar{P}_j \, \varphi_j(\bar{R},H)$ (and $j \in S$) or (ii) $\varphi_j(\bar{R},H) = x$ or (iii) $0 \, \bar{P}_j \, x$. Thus, there exists $k \in N \setminus S$ such that $S \cup \{k\} = \{j \in N : x \, \bar{P}_j \, 0\}$ and $\varphi_k(\bar{R},H) = x$. But then for $(\bar{R},H)$ we have neither $[x = 0]$ nor [for all $j \in N$, $\varphi_j(\bar{R},H) \neq x$] and it holds $\varphi_k(\bar{R},H) = x$ and $i \succ_x k$.

Hence, by the fact that for all $j \in S$, $\varphi_j(\bar{R},\{x\}) = 0$, *weak non-wastefulness* and *resource-monotonicity*, it follows that $\varphi_k(\bar{R},\{x\}) = x$. By Lemma 1, $k \succ_x i$. However, at the same time $i \succ_x k$; a contradiction. $\square$

So far we have established that for any rule φ that satisfies the properties of Theorem 1, there exists a priority ordering $\succ$ such that for any $(R,H) \in \mathcal{R}^N \times \mathcal{H}$, $\varphi(R,H)$ is stable under $\succ$. Hence, in the terminology of two-sided matching, the rule φ picks a stable matching for the marriage market where objects have preferences over agents who consume the objects based on the priority structure $\succ$ and agents have strict preferences over objects based on preferences R (see Roth and Sotomayor, 1990, Chapter 5). For these markets it is well-known that the deferred-acceptance-rule is the only *strategy-proof* stable matching rule. Hence, Theorem 1 follows immediately from Proposition 1.

A.2 Proof of Corollary 1

Proof. Deferred-acceptance rules (in particular those with acyclic priority structures) satisfy all the properties listed in the corollary.

(a) Let φ be a rule satisfying *unavailable object invariance*, *efficiency*, *resource-monotonicity*, and *truncation invariance*. Since *efficiency* implies *individual rationality* and *weak non-wastefulness*, by Proposition 1, there exists a priority structure $\succ$ such that φ is *stable* under $\succ$. Since the deferred-acceptance-rule Pareto dominates any other stable rule (Balinski and Sönmez, 1999, Theorem 2), efficiency implies that $\varphi = DA^{\succ}$. Finally, by *efficiency* and Ergin (2002, Theorem 1), $\succ$ must be acyclic.

(b) Let φ be a rule satisfying *individual rationality*, *weak non-wastefulness*, *resource-monotonicity*, and *group strategy-proofness*. By Ehlers and Klaus (2004, Lemma 1), *group strategy-proofness* implies *unavailable object invariance*. Furthermore, *group strategy-proofness* implies *truncation invariance* and *strategy-proofness*. Hence, by Theorem 1, there exists a priority structure $\succ$ such that $\varphi = DA^{\succ}$. Finally, by *group strategy-proofness* and Ergin (2002, Theorem 1), $\succ$ must be acyclic. $\square$

A.3 Independence of Properties in Theorem 1

For any strict order π of agents in N, we denote the corresponding *serial dictatorship rule* by f^{π}; for example, if $\pi : 1\ 2\ \ldots\ (n-1)\ n$, then f^{π} works as follows: for each problem (R, H), first agent 1 chooses his preferred object in H, then agent 2 chooses his preferred object from the remaining objects $H \setminus \{f_1^{\pi}(R, H)\}$, *etc.* Note that for each strict order π of N, $f^{\pi} = DA^{\succ^{\pi}}$ where $\succ^{\pi}$ equals the priority order such that for all $x \in K$, $\succ_x^{\pi} = \pi$. Thus, each serial dictatorship rule f^{π} satisfies *unavailable object invariance*, *individual rationality*, *weak non-wastefulness*, *resource-monotonicity*, *truncation invariance*, and *strategy-proofness*.

The following examples establish the independence of the properties (properties not mentioned in the examples follow easily) in Theorem 1.

Not *unavailable object invariant*: Let $n \geq 3$ and $\pi : 1\ 2\ 3\ \ldots\ (n-1)\ n$ and $\pi' : 1\ n\ (n-1)\ \ldots\ 3\ 2$. Then, for each problem $(R, H) \in \mathcal{R}^N \times \mathcal{H}$,

$$\varphi(R, H) = \begin{cases} f^{\pi}(R, H) & \text{if } A(R_1) = \emptyset \text{ and} \\ f^{\pi'}(R, H) & \text{otherwise.} \end{cases}$$

Not *individually rational*: Let $\pi : 1\ 2\ \ldots\ (n-1)\ n$. For each $R_n \in \mathcal{R}$, let $\hat{R}_n$ be such that $A(\hat{R}_n) = K$ and $\hat{R}_n|K = R_n|K$. Then, for each problem $(R, H) \in \mathcal{R}^N \times \mathcal{H}$,

$$\varphi(R, H) = f^{\pi}((R_{-n}, \hat{R}_n), H).$$

Not *weakly non-wasteful*: Fix an object $y \in K$ and $\pi : 1\ 2\ \ldots\ (n-1)\ n$. Then, for each problem $(R, H) \in \mathcal{R}^N \times \mathcal{H}$,

$$\varphi(R, H) = f^{\pi}(R, H \setminus \{y\}).$$

Not *resource-monotonic*: Let π and π' be two distinct strict orders of agents in N. Then, for each problem $(R, H) \in \mathcal{R}^N \times \mathcal{H}$,

$$\varphi(R,H) = \begin{cases} f^{\pi}(R,H) & \text{if } H = K \text{ and} \\ f^{\pi'}(R,H) & \text{otherwise.} \end{cases}$$

Not *strategy-proof*: Let $\succ$ be a priority structure. Then, the deferred-acceptance-rule based on the object-optimal allocation that is obtained by using Gale and Shapley's (1962) object-proposing deferred-acceptance-algorithm satisfies all properties except *strategy-proofness*.[8]

Not *truncation invariant*: Let $N = \{1,2,3\}$, $K = \{x,y\}$, $\succ_x$: 3 1 2, $\succ'_x$: 3 2 1, and $\succ_y$: 3 1 2. Let $\succ = (\succ_x, \succ_y)$ and $\succ' = (\succ'_x, \succ_y)$. Then, for each problem $(R,H) \in \mathcal{R}^N \times \mathcal{H}$,

$$\varphi(R,H) = \begin{cases} DA^{\succ}(R,H) & \text{if } 0 \, P_3 \, x \text{ and } x \in H \text{ and} \\ DA^{\succ'}(R,H) & \text{otherwise.} \end{cases}$$

Let $R_1 : x \; 0 \; y$, $R_2 : x \; 0 \; y$, $R_3 : y \; 0 \; x$, and $R'_3 : y \; x \; 0$. Let $R = (R_1, R_2, R_3)$ and $R' = (R_1, R_2, R'_3)$. Note that R_3 is a truncation of R'_3 and $\varphi_3(R, \{x,y\}) = y = \varphi_3(R', \{x,y\})$. However, $\varphi_1(R, \{x,y\}) = x$ and $\varphi_2(R', \{x,y\}) = x$; a contradiction of *truncation invariance*. Next, we show *strategy-proofness* and *resource-monotonicity* for this rule.

For *strategy-proofness*, note that agents 1 and 2 cannot change the priority structure by reporting a false preference relation. Agent 3 always receives his most preferred object in H for any problem (R,H). Thus, agent 3 cannot profitably manipulate by reporting a false preference relation.

For *resource-monotonicity*, let $|H| = 1$ and $R \in \mathcal{R}^N$. If there is a violation of *resource-monotonicity*, then φ must use different priority structures for (R,H) and $(R,\{x,y\})$. But then we must have $H = \{y\}$ and both $\varphi(R,y) = DA^{\succ'}(R,y)$ and $\varphi(R,\{x,y\}) = DA^{\succ}(R,\{x,y\})$.

If yP_30, then $\varphi_3(R,y) = y$ and $\varphi_1(R,y) = \varphi_2(R,y) = 0$, and all agents weakly prefer $\varphi(R,\{x,y\})$ to $\varphi(R,y)$.

If $0 \, P_3 \, y$, then $\varphi_3(R,y) = 0$. Since $\varphi(R,\{x,y\}) = DA^{\succ}(R,\{x,y\})$, we have $0 \, P_3 \, x$ and by *individual rationality*, $\varphi_3(R,\{x,y\}) = 0$. Note that $1 \succ_x 2$, $1 \succ_y 2$, and $1 \succ'_y 2$ (the latter because $\succ_y = \succ'_y$). Let $\pi : 1 \; 2 \; 3$. Then, $\varphi(R,y) = DA^{\succ'}(R,y) = f^{\pi}(R,y)$ and $\varphi(R,\{x,y\}) = DA^{\succ}(R,\{x,y\}) = f^{\pi}(R,\{x,y\})$. Hence, *resource-monotonicity* is satisfied.

[8]Given $\succ$, there exist many "artificial" stable rules satisfying all properties except *strategy-proofness*. For instance, let $v \in \{1, \ldots, |K|-1\}$. For all $(R,H) \in \mathcal{R}^N \times \mathcal{H}$, let the rule ϕ choose the object-optimal stable allocation (for $\succ$) if $|H| \leq v$ and the agent-optimal stable allocation (for $\succ$) if $|H| > v$. The rule ϕ satisfies all properties except *strategy-proofness*.

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