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Optimum Forest Rotations of Alternative Tree Species[1]

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Abstract

We solve Faustmann's problem when two tree species are available for planting. The analysis also applies to optimal forest exploitation before an endogenous switch to some alternative land use such as agriculture, housing, or preservation, and vice versa. Each species has its own deterministic growth function and commands a timber price that grows exponentially at a constant rate. Therefore, it may be optimal to first exploit the species whose price is high but grows slowly, and then switch to the alternative species once its price has sufficiently increased relative to the price of the first one. When the land is bare, there exists a threshold of the relative price at which the investor is indifferent between planting either species. When the relative price lies below this switching threshold, it is optimal to plant and harvest the high-price low-rate species repeatedly until the value of the other species warrants its introduction; it is then repeatedly harvested and replanted indefinitely according to the standard Faustmann rule; the rotation does not depend on timber price. Before the switch, the optimal harvest age depends on the relative price; it defines a replanting boundary for relative prices below the switching threshold and a switching boundary for relative prices above the switching threshold. We show that the replanting boundary is a sequence of continuous segments giving the harvest age as function of the relative price; these segments differ depending on the number of harvests of the initial species that remain before the switch. Each segment is first decreasing, then increasing, and crosses Faustmann's rotation twice. On an optimal sequence of harvests, successive rotations are increasingly higher or decreasingly lower than Faustmann's rotation; they may also be constant and equal to Fausmann's rotation.

Key words: Forestry; Land use; Alternative use; Faustmann; Alternative Species; Rotation.

JEL classification: C61, D81, G11, G13, Q23.

1. Introduction

Faustmann (1849) gave forestry economics its foundations by addressing the question: at what age should a stand of even-aged trees be harvested? He did so under the assumption of constant timber prices by comparing the net marginal benefits from letting timber grow further, to the opportunity cost of currently planted trees plus the opportunity cost of the land, itself determined by the optimization of all future harvest decisions. Faustmann's original problem has been refined and generalized in many ways to include for instance a rising timber price (Newman et al., 1985), a constrained harvest rate (Heaps and Neher, 1979), non-timber benefits (Hartman, 1976, and Strang, 1983) and stochastic timber prices (e.g. Brazee and Mendelsohn, 1988; Clarke and Reed, 1989; Thomson, 1992; Reed, 1993; Willassen, 1998; Insley, 2002). Over time, applications have been extended to include more and more problems, such as differentiated timber prices (Forboseh et al., 1996), uneven-aged management (Haight, 1990), multi-species stands under changing growth conditions caused by climate change (Jacobsen and Thorsen, 2003), the value of carbon storage (Ekholm, 2015), and many others referred to in Amacher, Ollikkainen, and Koskela (2009).

In this paper, we reconsider the original Faustmann problem while assuming the availability of two alternative tree species whose prices grow at different rates. The model may just as well be applied to study the conditions of a switch to some alternative land use (e.g. agriculture, residential use, conservation, etc.). As will appear below, such a setup implies that managing the forest under an initial tree species may be desirable only for a finite time, which implies that the optimum management is not a time autonomous problem and has a solution notably different from that of the original Faustmann problem.

When alternative species were considered in the literature, the future stand value was treated as exogenous, independent of the current choice. For instance, Thorsen (1999) analyses the choice of tree species for afforestation as a real option problem, and Thorsen and Malchow-Møller (2003) extend it to a two-option problem with two mutually exclusive options (two tree species), where exercising one option implies losing the other. With uncertain timber prices, Jacobsen (2007) goes one step further: upon harvest, the current stand (of spruce) may be allowed to regenerate naturally and costlessly, or may be replaced with oaks at some cost. Jacobsen studies the optimum harvest age: it is not certain whether it is higher or lower than Faustmann's rotation or Wicksell's single rotation.

We assume that each species timber price is known with certainty and increases at a constant rate that is lower than the discount rate. This price behavior was first analyzed by Lyon (1981) and justified on the ground that there is a dimension of mining in forest exploitation, but that it is moderated by the renewability of the resource. It is generally supported by historical data (Newman et al, 1985). However, at a desegregated level, the prices of various species may grow at different rates because of differences in demand or, e.g., of their different abilities to sequester carbon (Sohngen and Mendelson, 1998). This implies that a switch from one species to the alternative species may be desirable at some point in time while other features of Faustmann's model remain valid.

It turns out that both the nature of the rotation choice problem and its solution are considerably modified by the possibility of switching to an alternative species. Indeed, the absence of regeneration costs or other fixed costs in a Faustmann model makes the rotation independent of the level of the price, assumed constant. Newman et al. (1985) have further studied a model with the price rising at a constant rate. When regeneration costs are absent, the rotation is also independent of the price level and constant from one harvest to the next; this constant rotation is higher, the higher the rate of price rise.

In this paper, the optimal harvest age is shown not to be generally constant from one harvest to the next if a switch is to occur in the future; it depends on the relative price of the species at the time of the harvest. It also depends on the number of harvests remaining before the switch. A similar phenomenon was identified by numerical methods in a single species context by Newman et al (1985) , to whom we also owe some of the analytical apparatus used and adapted to the case of two species in this paper. The solution can be described in the two-dimensional space of tree age and relative species price. In that space there exists a "non maturity" or waiting region delimited by an upper age boundary: given some relative price, one should harvest if the age of the trees equals or exceeds the upper boundary. Furthermore, over some range of relative prices, there also exists a lower boundary to the waiting region: if the age of the trees is higher than the boundary, it is optimal to allow them to grow until they reach maturity (the upper boundary); but if the age of the trees is lower than the lower boundary, they should be cut and the alternative species should be adopted immediately.

Suppose that the land were bare, available for the establishment of one or the other species. There is a critical relative price at which the investor would be indifferent between planting either species. Surprisingly, we show that, in an optimal sequence of harvests, that price never coincides with a harvest, let alone with the switch from one species to the next. If the optimal sequence is such that one species is to be replaced by the other at some date, the former will be last established at a price strictly below the critical indifference price, and the replacement species will first be established at a price strictly higher than the indifference price. In other words, if an optimal program has been followed before the relative price reaches the critical level, the land is not bare when this price is reached and the existing trees are to be allowed to grow further to reach financial maturity. Similarly, if timber producing land is to be reallocated to some alternative use, the switch should occur later, that is at a higher value of the alternative use, than if the land were bare.

The upper boundary is different when it leads to reestablishing the same species than when it leads to a switch; we call "replanting boundary" and "switching boundary" these alternative forms of the upper boundary. The replanting boundary applies when the relative price is below the critical level.[2] It is composed of a succession of segments giving the optimal harvest age as function of the price of the species to be adopted last relative to the price of the species in place. Each of these functions first decreases and then increases, forming a sequence of downward followed by upward sloping segments. Each downward segment indicates the optimal harvest age corresponding to a particular

[2] Without loss of generality one can define the relative price such that it is rising.

number of remaining harvests until the switch to the last species. Upward segments are not reached by any optimal sequence of relative-price tree-age pair; they indicates the age below which it is worth allowing a tree to reach maturity rather than cut it, given the relative price. The downward sloping segments start at an optimal harvest age above the Faustmann rotation and end below it. The upward sloping segments ensure the continuity of the forest value as a function of the relative price despite the decreasing number of further harvests of the initial species. The age difference spanned by the upward sloping segments is higher, the lower the number of further harvests of the initial species before the switch.

Another finding is that, before the switch, harvest ages from one harvest to the next are constant or increasing or decreasing; if constant, they remain equal to the Fautmann rotation; if increasing, they are always higher than the Faustmann rotation; if decreasing, they are always lower than the Faustmann age.

The general setting and assumptions are introduced in Section 2. In Section 3, we establish the optimal harvest age when the land is available for one rotation only (Wicksell setup) and when it is available for multiple rotations (Faustmann setup) in the case where just one tree species exists as in Newman et al. (1985) but with no regeneration cost. These intermediate results are used to study the case where two tree species are available. In Section 4, we extend the Faustmann framework to consider the availability of an alternative tree species. After harvesting, the land may be planted with anyone of the two available tree species. The decision maker must decide at what age the trees of the current stand must be cut, and whether they should be replaced with trees of the same species or of the alternative species. Some properties of the decision rules and value functions are derived analytically and presented in a number of propositions. Section 5 concludes.

2. General setting and assumptions

We study the decision of a forest manager to establish one or the other of two alternative tree Species P and P' on a plot of bare land. We assume that the timber price of Species

P (respectively P') changes over time t at the instantaneous rate μ (respectively μ') as in the one-species model of Newman et al. (1985):

$$p_t = p_0 e^{\mu t}, \tag{1a}$$

$$p'_t = p'_0 e^{\mu' t}. \tag{1b}$$

Newman at al. justify their assumption on empirical grounds, rightly arguing that 'Timber is unique among natural resources in that its price shows a long-term increasing trend relative to the price of other goods.' While explanations for this empirical regularity may have been refined, the same regularity is still observed nowadays (Stavins, 2011). Constant rates of growth is a restrictive assumption for prices; nonetheless it is useful both conceptually (as it encompasses constant prices and is also arguably a useful tool to study rising wood scarcity). In our generalization to two species, it is further a good way to model the progressive loss of appeal of one wood species relative to another.

In the rest of the paper time dependent variables will in general not be indexed and should be considered current unless otherwise mentioned. The relative price $\theta = \frac{p'}{p}$ varies over time; $\delta = r - \mu$ and $\delta' = r - \mu'$ are strictly positive constant parameters[3], where r is the constant discount rate.[4]

Each tree species is characterized by a timber volume growth function with the following properties:

Assumption 1 *There exists $\underline{a} > 0$ and $\underline{a}' > 0$, such that the timber volume functions $V(a)$ and $V'(a)$ are continuous over $[0, +\infty[$, $V(a) = 0$ over $[0, \underline{a}]$, $V'(a) = 0$ over $[0, \underline{a}']$; $V(a)$ and $V'(a)$ are positive, continuous, differentiable and concave over $[\underline{a}, +\infty[$ and $[\underline{a}', +\infty[$ respectively. In addition, $\lim_{a\to+\infty} V_a(a) = 0$ and $\lim_{a\to+\infty} V'_a(a) = 0$.*

Tree volume functions usually have a convex initial part and become concave once the trees have reached some strictly positive age. This implies that it is never optimum

[3] We assume that $\delta > 0$ and $\delta' > 0$; otherwise it would be optimal to delay the decision to cut forever.

[4] Throughout the paper the notation " ' " applied to a function will refer to the alternative species while first or second derivatives of functions will be denoted by indices. Thus, for variables a and p, $G_a(a, p)$ denotes the partial derivative of the function $G(a, p)$ with respect to a and $V'_{aa}(a)$ denotes the second derivative of the function $V'(a)$.

to harvest at an arbitrarily low age. Assumption 1 ensures that this stylized property is satisfied while avoiding delicate and economically uninteresting complications associated with the non convexity of the volume functions at low tree ages [5]. This assumption also ensures that the volume function is bounded.

We assume that operational costs (while trees are growing) and harvesting costs are either nil or accounted for in prices p and p'. A more consequential assumption is that the regeneration cost is nil. Under that assumption the optimal harvest age is independent of the timber price in one-species models with constant price or exponentially changing price. Adopting it in our two-species model allows us to better identify the cause of a change in the relationship between price and optimal harvest age.

3. Optimal rotation without an alternative tree species

In this section, we recall the determination of Wicksell's and Faustmann's rotations when only one tree species is available whose timber price is given by Equation ($1a$). These results will be useful in the remaining sections to establish the tree harvest age when two species coexist. The Wicksellian tree harvest problem refers to the problem of choosing the age at which a stand of even-aged trees will be harvested. One single harvest is considered and the impact of this unique harvest on land value is neglected. Thus, the investor maximizes the harvest revenue $e^{-rT}V(T)pe^{\mu T}$ by choosing the harvest date T. The optimal harvest age a_w is determined by Equation (2) similar to the well-known Wicksellian rule except that the discount rate r is replaced by $\delta = r - \mu$ to account for the exponential change in the timber price. As in the case of constant timber price, the optimal age is chosen in such a way that the marginal value growth of the trees is equal to the opportunity cost of holding on to them:

$$\frac{V_a(a_w)}{V(a_w)} = \delta. \tag{2}$$

Faustmann's rotation refers to the optimal harvest age when harvesting is immediately followed by replanting, in an indefinite succession of rotations. Thus, the in-

[5]While this assumption is applied to the tree volume functions here, it is applied to the timber revenue function in Heaps and Neher (1979).

vestor maximizes the harvest revenues over an infinite number of equal rotations solution to $\max_T \sum_{n=1}^{+\infty} e^{-rnT} V(T) p e^{\mu n T}$.[6] The optimal harvest age a_f is determined by Equation ($3a$), formalizing the well-known Faustmann rule (1849) where the discount rate r is again replaced by $\delta = r - \mu$. As in the case where the timber price is constant, the optimal harvest age a_f is constant from one rotation to another. In this case, the land value is equal to cp where the coefficient c is given by Equation ($3b$):

$$\frac{V_a(a_f)}{V(a_f)} = \frac{\delta}{1 - e^{-\delta a_f}}, \tag{3a}$$

$$c = V(a_f) \frac{e^{-\delta a_f}}{1 - e^{-\delta a_f}}. \tag{3b}$$

In the rest of the paper, we refer to a_f as the Faustmann rotation age and a_w as the Wicksell rotation age. It is clear that $a_f < a_w$.

4. Optimal rotation when an alternative tree species is available

In the remaining sections, we assume that two tree species are available at any time, whose timber prices are given by Equations ($1a$) and ($1b$). First, note that when $\mu = \mu'$, the relative price θ remains constant. If the right species is in place, then it will continue to be planted and harvested successively forever. Switching from one species to the alternative is conceivable in that case only if the problem starts with an initial stand that would not have been introduced as a result of a rational decision. In that case one should switch to the appropriate species, either immediately or after allowing existing trees to become economically mature. We do not consider this possibility any further.

When $\mu \neq \mu'$ a switch from one species to the other may become desirable as part of an optimal succession of harvests. Without loss of generality, assume from now on that $\overline{\mu} = \mu' - \mu > 0$ so that the growth rate of the price of the species in place, say Species P, is smaller than that of the alternative species P'. When p' is still relatively low relative to p, it is optimum to exploit Species P until the price of Species P' has sufficiently increased. Since the growth rate of p' is higher than that of p, a shift to Species P' will

[6] Harvest revenues are then a geometric series:
$\sum_{n=1}^{+\infty} e^{-rnT} V(T) p e^{\mu n T} = V(T) \sum_{n=1}^{+\infty} e^{-\delta n T} = V(T) \frac{e^{-\delta T}}{1 - e^{-\delta T}}$

eventually be desirable. Suppose that Species P is replanted n times until the switch to P' occurs. At that moment, Species P will be cut for the last time and Species P' will be planted thereafter forever after each harvest, as if it were the sole available species. The assumption $\mu' > \mu$ implies that the switch to P' is permanent, so that the problem to be solved from then on is the standard one-species problem of Faustmann. Once species P' is planted, the land value is thus equal to $c'p'$ and the rotation is the constant a'_f defined by adapting ($3a$) and ($3b$) to the case of Species P':

$$\frac{V'_a(a'_f)}{V'(a'_f)} = \frac{\delta'}{1-e^{-\delta' a'_f}}, \tag{4a}$$

$$c' = V'(a'_f)\frac{e^{-\delta' a'_f}}{1-e^{-\delta' a'_f}}. \tag{4b}$$

Since $\frac{\delta}{1-e^{-\delta a}}$ is increasing in δ for any $a > 0$ and $\frac{V_a}{V}(a)$ is decreasing in a, the Faustmann age of a given species is higher, the higher the growth rate of its price as verified numerically by Newman et al. (1985).[7]

Before the switch, at relatively low values of θ, the situation is different from the standard single-species case that defines the Faustmann rotation. The number of harvests of Species P to be carried out before the switch diminishes at each harvest. In such a non-autonomous problem, it cannot be assumed that the harvest age is the same at each harvest. As will become clear further below, there are intervals of θ values over which there remains only one harvest of Species P, or two harvests, three harvests, etc.; furthermore, the optimum harvest age depends on θ within each of these intervals and this functional dependency differs according to the number of remaining harvests. Thus the problem at hand is to find the optimal harvest age of Species P trees as a function of the value of θ at harvest time. This function not only depends on the number of remaining harvests, which is itself endogenous, but also on the initial conditions of the problem. In other words we will find that the optimal harvest age when k harvests of Species P remain before the switch to P' depends on the initial value of θ and on the age of the trees in place at that time.

[7] It is not possible in general to compare a_f and a'_f as species P and P' may have different volume functions.

In short, the problem does not collapse to the static problem of finding the unique optimal rotation that applies at all dates as is the case with the standard Faustmann formulation. It is dynamic, non autonomous, and its solution is an optimum trajectory over time of the tree age and species for successive generations. This solution will be expressed in the state space of relative species price and tree age (θ, a) for each generation of trees. As hinted already, it takes a different form after the switch to Species P'; it also depends on the initial value of the (θ, a) pair. Since the sole actions are to cut trees and to choose the replacement species, a convenient way to represent the solution in state space will be to define boundaries between loci where it is desirable to harvest the trees, and loci where it is desirable to allow existing trees to grow further.

It will be convenient to index harvest ages in reverse chronological order from the date of the switch. Thus $a_n(\theta)$ stands for the harvest age when the relative price is θ and the land is to be replanted n more times with the same species P, while $a_0(\theta)$ is the age at which Species P is cut for the last time and is replaced with the alternative species forever. The reason to count rotations backward from n to 0 is that whatever the current relative timber price is, the switch to the alternative species will take place when the stand age is $a_0(\theta)$ for a certain relative price θ at the harvest time. However, as will be clear, the price at which the switch to Species P' occurs is not unique; it depends on the initial state (relative price and tree age) of the problem.

Harvest ages $a_n, .., a_0$ are chosen to maximize the value of the forest $G(p, p', a)$, which is the sum of the value of the current stand and the endogenous value of the land resulting from subsequent harvests:

$$G(p, p', a) = \max_{a_0 \dots a_n} \left\{ p_n V(a_n) e^{-r(a_n - a)} + \dots + p_0 V(a_0) e^{-r(a_n + \dots + a_0 - a)} + c' p'_0 e^{-r(a_n + \dots + a_0 - a)} \right\}$$

where $p_i = p e^{\mu(a_n + \dots + a_i - a)}$ for $i = 0, .., n$ and $p'_0 = p' e^{\mu'(a_n + \dots + a_0 - a)}$. The first $n+1$ terms in the maximand give the present-value contribution of each harvest of P trees, where the last one, at price p_0 and age a_0 is followed by the establishment of Species P'; the last term gives the present value of the infinite sequence of harvests of Species P' that follows, with c' given by $(4a)$. This implies that

$$G(p,p',a) = p \max_{a_0 \dots a_n} \left\{ V(a_n)e^{-\delta(a_n-a)} + \dots + V(a_0)e^{-\delta(a_n+\dots+a_0-a)} + c'\theta e^{-\delta'(a_n+\dots+a_0-a)} \right\}$$

so that $G(p,p',a) = pg(\theta,a)$ where

$$g(\theta,a) = \max_{a_0 \dots a_n} \left\{ \left[V(a_n)e^{-\delta a_n} + \dots + V(a_0)e^{-\delta(a_n+\dots+a_0)} \right] e^{\delta a} + c'\theta e^{-\delta'(a_n+\dots+a_0)} e^{\delta' a} \right\}. \quad (5)$$

Hence, the forest value function $G(p,p',a)$ admits a reduced (price intensive) form $g(\theta,a)$,[8] a function of the relative price and the stand age. By solving the maximization in (5), one can see that each optimum a_i is a function of a single variable, the relative price $\varphi = \theta e^{\overline{\mu}(\Sigma a_i - a)}$ current at the date of the harvest, which is itself uniquely determined by the relative price θ and tree age a applying at the date where the maximization is carried out.

The land value $F(p,p') = G(p,p',0)$ can similarly be written $F(p,p') = pf(\theta)$ where the reduced land value $f(\theta)$ depends on the relative price θ only:

$$f(\theta) = \max_{s} \left[V(s)e^{-\delta s} + f(\theta_s)e^{-\delta s} \right]. \quad (6)$$

In order to characterize the harvest age and species choice decisions, we follow Newman et al. (1985) and consider the two positive, strictly decreasing functions $K(a) = \frac{V_a(a)}{\delta} - V(a)$ and $L(a) = \frac{V_a(a)}{\delta e^{\delta a}}$ respectively defined on $[\underline{a}, a_w]$ and $[\underline{a}, +\infty[$. As $L(a)$ is strictly decreasing it is invertible. Let $R = L^{-1} \circ K$; Figure 1 illustrates how its values are established. The functions K and L illustrated in Figure 1 and function R will be used repeatedly to characterize the solution at various stages. Precisely, it will be shown that if it is optimal to harvest and replant stand P at age a, then the next rotation is $R(a)$; furthermore, the only fixed point $R(a) = a$ is the Faustmann age a_f.

The next proposition characterizes the relative price at which the land, if bare, could indifferently be planted with Species P or P'.

[8] The reduced function $g(\theta,a)$ could be called 'price intensive' by analogy with the standard 'factor intensive' form of a constant returns neoclassical production function. The reduced function $g(\theta,a)$ gives the value of the stand (including land) in terms of a numeraire, chosen to be Species P.

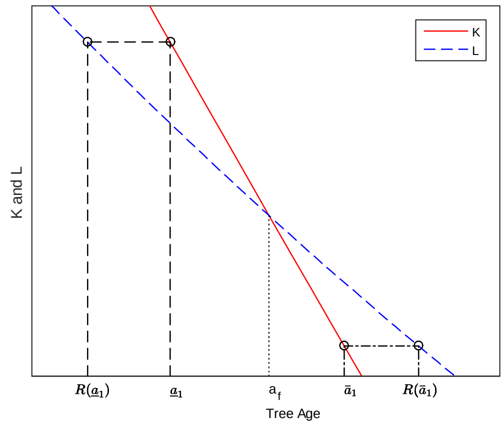

Figure 1: Successive rotations before the switch

Proposition 1 *There exists a value of the relative price $\theta_0 > 0$ such that, if the land is bare, it is equivalent to plant Species P' repeatedly forever or to plant Species P for one rotation of duration $\underline{a}_0$ followed by a permanent switch to Species P'; θ_0 and $\underline{a}_0$ are determined by*

$$\frac{V_a}{V}(\underline{a}_0) = \delta + \frac{\delta'}{e^{\delta' \underline{a}_0} - 1}, \tag{7a}$$

$$K(\underline{a}_0) = \frac{\delta'}{\delta} c' \theta_0 e^{\overline{\mu} \underline{a}_0}. \tag{7b}$$

Furthermore, $\underline{a}_0$ satisfies $\underline{a} < \underline{a}_0 < a_f$.

Proof. See the Appendix. ■

At relative prices equal to or higher than the switching threshold θ_0, the reduced land value $f(\theta)$ equals $c'\theta$, the form that applies when the sole species is P'. When the relative price tends to zero, the reduced land value tends to c, the form that applies when the sole species is P. For all levels of θ, the land value when both species are available is equal to or higher than the value generated by one species only.

Equation ($7b$) will be generalized to apply to the upper switching boundary in Proposition 3. The following proposition is used to compute the reduced land value function depicted in Figure 2.

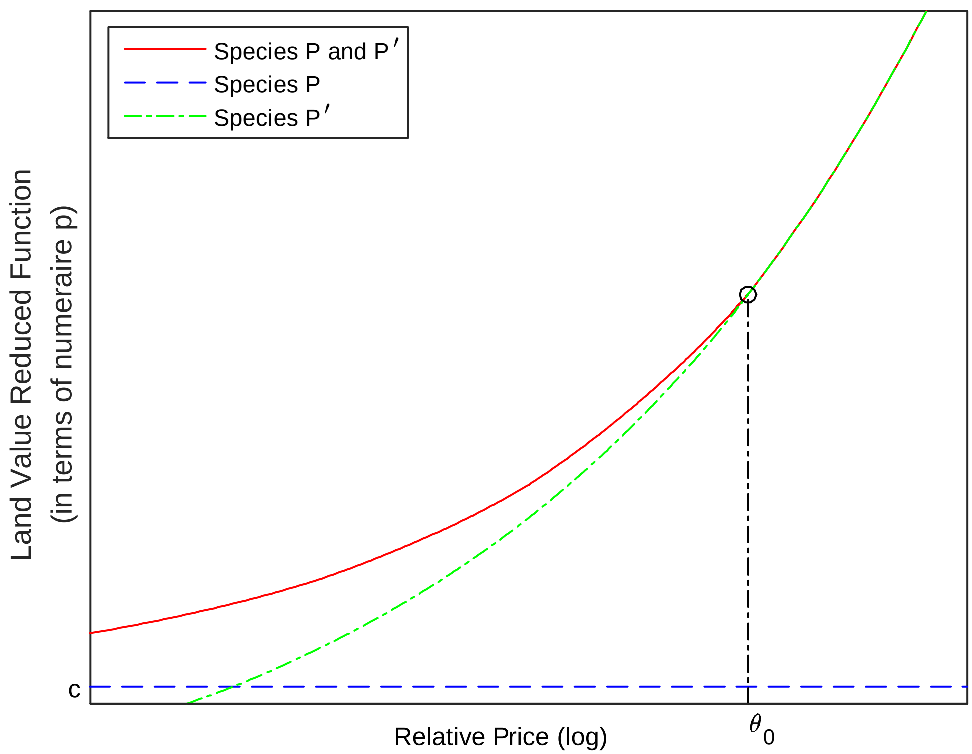

Figure 2: Land Value Function with Two Species

Proposition 2 *The reduced land value function $f(\theta)$ can be computed recursively as:*

$$f(\theta) = \begin{cases} \max_{\hat{\theta} \geq \theta} \left(\frac{\hat{\theta}}{\theta}\right)^{-\frac{\delta}{\delta-\delta'}} \left[V(\frac{\log(\frac{\hat{\theta}}{\theta})}{\mu'-\mu}) + f(\hat{\theta})\right], \ \theta \leq \theta_0, \\ c'\theta, \ \theta \geq \theta_0. \end{cases} \tag{8}$$

Proof. See the Appendix. ■

Figure 2 gives the alternative values of the same plot of land using Species P as numeraire when alternatively a single Species P or P', or both Species, can be established on the land.

When $\theta > \theta_0$, Species P' needs to be established if the land is bare. If the land is already planted with trees of Species P', the optimum decision is to allow them to grow until they reach economic maturity as *per* (4a) and (4b), harvest them and replant the same Species P' forever. However, if the land is planted with trees of Species P aged a, two possibilities arise. The first one is to harvest immediately and plant P'. This is clearly optimal if p' is so superior to p that postponing a new harvest of P' cannot be justified by any optimization of the value derived from harvesting the current P trees, whatever their age. The second possibility arises if the relative price of P' is not too far above the threshold θ_0 and the current P trees are relatively old. It is then preferable to allow them to grow further, harvest them when they reach economic maturity, and only then switch to P'. This alternative defines the upper switching boundary $a^+(.)$ for relative prices higher than θ_0. Nonetheless, at such values of θ, it is optimal to plant P' if the land is bare; consequently there must exist some age for P trees below which it is preferable to replace them immediately rather than allowing them to reach maturity. In such case, unlike the standard Faustmann problem, there is a lower switching boundary $a^-(.)$ to the continuation region in addition to the upper switching boundary $a^+(.)$. We call these boundaries switching boundaries to distinguish them from the replanting boundary $a(.)$ that applies for relative prices below θ_0 and defines the age at which trees of Species P are to be harvested and replaced with trees of the same species. The next two propositions define respectively the upper switching boundary $a^+(.)$ and the lower switching boundary $a^-(.)$. We postpone until Propositions 5 and 6 a discussion of the optimum harvest age relative to the Faustmann rotation.

Proposition 3 *An upper switching boundary $a^+(\theta)$ exists for $\theta_0 \leq \theta \leq \overline{\theta}$. It is the set of pairs (θ, a) such that $\theta_0 \leq \theta \leq \overline{\theta}$, $\underline{a} \leq a \leq \overline{a}_0$ and $K(a) = \frac{\delta'}{\delta}c'\theta$, where $\overline{\theta}$ and $\overline{a}_0$ are respectively the unique solutions of:*

$$V_a(\underline{a}) \quad = \quad \delta' c' \overline{\theta} e^{\overline{\mu}\underline{a}} \tag{9a}$$

$$\textit{and} \tag{9b}$$

$$K(\overline{a}_0) \quad = \quad \frac{\delta'}{\delta} c' \theta_0. \tag{9c}$$

The upper switching boundary is strictly decreasing on $\left[\theta_0,\ \overline{\theta}\right]$, with $a^+(\theta_0) \equiv \overline{a}_0 < a_w$ and $a^+(\overline{\theta}) = \underline{a}$.

Proof. See the Appendix. ■

Equation $K(a) = \frac{\delta'}{\delta}c'\theta$ at point (θ, a) of the upper switching boundary is equivalent to $V_a(a) = \delta V(a) + \delta' c'\theta$. It states that it is optimal to harvest Species P when the change in the marginal revenue equals the opportunity cost of postponing the current harvest revenue augmented by the opportunity cost of the land, whose value stems from harvesting Species P' forever.

Proposition 4 *A lower switching boundary $a^-(\theta)$ exists for $\theta \in [\theta_0, \overline{\theta}]$. It is the set of pairs (θ, a), with $\theta_0 \leq \theta \leq \overline{\theta}$ and $0 \leq a \leq \underline{a}$, such that, for some $s \geq 0$:*

$$\frac{V_a}{V}(a+s) = \delta + \frac{\delta'}{e^{\delta' s} - 1}, \tag{10a}$$

$$K(a+s) = \frac{\delta'}{\delta}c'\theta e^{\overline{\mu} s}. \tag{10b}$$

The lower switching boundary is strictly increasing on $[\theta_0, \overline{\theta}]$.

Proof. See the Appendix. ∎

The last two propositions describe the rich and novel features of the harvest decision in the upper vicinity of the switching price.

First the lower switching boundary is a notion that appears only when at least two alternative species are in competition: the existence of an alternative imposes an opportunity cost on the species in place; if that opportunity cost is high enough the trees in place should be replaced without giving them the time to reach financial maturity. Thus when P trees are in place, if their age is lower than indicated by the lower boundary, they should be cut and replaced with P' trees. The lower switching boundary is rising from age zero at θ_0 to $\underline{a}$ at $\overline{\theta}$.

Second, whereas the Faustmann rotation is independent of the price in the absence of regeneration costs when only one species is available, the upper switching boundary here depends on the relative price. Precisely it decreases from $\overline{a}_0$ at θ_0 to $\underline{a}$ at $\overline{\theta}$ where it connects with the lower boundary.

The next two propositions also describe a form of solution not encountered in previous versions of the Faustmann problem, although Newman et al (1985) discovered a similar behvior in a single species context. They characterize the replanting boundary $a(.)$, that is to say the harvest boundary that applies at relative prices lower than θ_0. Instead of the constant harvest age found in regular Faustmann problems, the replanting boundary turns out to be a sequence of alternatively downward and upward sloping functions of θ, with one particular downward sloping segment followed by an upward sloping segment for each possible number k of harvests remaining before the switch to Species P'.

Proposition 5 describes the succession of segments constituting the harvest boundary for each of the remaining harvests. Further below, Proposition 6 will focus on the monotonicity properties of the segments and precisely compare optimum harvest ages with the Faustmann rotation.

It turns out that not only the number of harvests remaining before the switch, but also the age at which the trees will be cut at each harvest and the relative price at which each harvest will take place, depend on the initial state of the problem, that is to say on the age of the trees in place and the relative price of the two species at the initial time.[9]

The problem of choosing the number of remaining harvests and the corresponding ages and relative prices can be somewhat simplified by noting that the initial state (θ, a) can be summarized by a single variable φ which is the relative price when trees were planted. If the initial state is (θ, a) then the price that prevailed a time units earlier is $\varphi = \theta e^{-\overline{\mu}a}$ and the hypothetical state at that time would be $(\varphi, 0)$. Consider the problem with initial state $(\varphi, 0)$ rather than (θ, a); if its solution does not involve any harvest during the period over which the price moves from φ to θ, then the state (θ, a) is reached and the continuation gives the solution to the problem starting at (θ, a). Proposition 5 characterizes the solution for this special case. Its generalization to any initial age will be provided with the discussion of the solution.

Proposition 5 *Suppose that the land is bare and that the relative price is $\varphi \leq \theta_0$. Let $n(\varphi)$ denote the number of times that Species P is to be planted before the permanent switch to the alternative Species P'. Then,*

[9] As mentioned earlier the maintained assumptions in this part of the paper are such that Species P is in place; its relative price diminishes but is still sufficiently high to justify that P be harvested and replanted an appropriate number of times n before the switch to Species P'.

1. *Species P is harvested - and immediately replanted - at ages $a_k(\varphi_k)$ and relative prices $\varphi_k(\varphi) = \varphi e^{\overline{\mu}[a_n(\varphi_n)+...+a_k(\varphi_k)]} < \theta_0$, $k = n(\varphi), ..., 1$.*

2. *The last harvest of Species P occurs at tree age $a_0(\varphi_0)$ and relative price $\varphi_0(\varphi) = \varphi e^{\overline{\mu}[a_n(\varphi_n)+...+a_0(\varphi_0)]} > \theta_0$ on the upper switching boundary $a^+(.)$ and is followed immediately by a permanent switch to Species P'.*

3. *The harvest ages $a_k(\varphi_k)$, $k = 1, ..., n$ are determined recursively by the series of Equations $(11a)$ and $a_0(.)$ is determined by Equation $(11b)$:*

$$R[a_k(\varphi_k)] = a_{k-1}(\varphi_{k-1}), \ \varphi_k(\varphi) = \varphi e^{\overline{\mu}[a_n(\varphi_n)+...+a_k(\varphi_k)]}, \ k = 1, ..., n(\varphi), \tag{11a}$$

$$K[a_0(\varphi_0)] = \frac{\delta'}{\delta} c' \varphi_0, \ \varphi_0(\varphi) = \varphi e^{\overline{\mu}[a_n(\varphi_n)+...+a_0(\varphi_0)]}. \tag{11b}$$

where $n(\varphi)$ is determined by the condition that $\varphi_1 < \theta_0 < \varphi_0$.

4. *For $k = 1, ..., n(\varphi)$, the successive harvest ages $a_k(\varphi_k)$ are increasing and higher than the Faustmann age or decreasing and lower than the Faustmann rotation. They may also be constant and equal to the Faustmann rotation. The Faustmann rotation a_f is optimal at each harvest if and only if the harvests occur at relative prices $\theta_{fk} = \frac{K(a_f)}{\frac{\delta'}{\delta} c' e^{(k+1)\overline{\mu} a_f}}$, that is to say if $\varphi = \varphi_f = \frac{K(a_f)}{\frac{\delta'}{\delta} c' e^{(n+1)\overline{\mu} a_f}}$.*

Proof. See the Appendix. ■

Note that the k^{th} first-order condition for the maximization problem (5) that defines the forest value $g(\varphi, 0)$ in terms of Species P is equivalent to

$$V_a(a_k) e^{-\delta(a_k+...+a_n)} = \delta \sum_{i=0}^{k} V(a_i) e^{-\delta(a_i+...+a_n)} + \delta' c' \varphi e^{-\delta'(a_0+...+a_n)}. \tag{12}$$

This condition equates at harvest time the marginal increase per unit of time of the standing trees (left-hand side), with the opportunity cost of holding standing trees and of occupying land (right-hand side). Indeed, the value of the standing trees is the first element $(i = 0)$ in the summation term on the right-hand side; the value of the bare land results from the k subsequent harvests of Species P $(i = 1, ..., k)$ in the summation term, plus the indefinite sequence of harvests of Species P' corresponding to the last term on the right-hand side. Thus the interpretation of the harvest rule implied by (12) is the standard textbook interpretation of Faustmann's rotation (Amacher et al., 2009), with the important difference that the rotations are not constant here: a_k depends on $\varphi_k(\varphi)$.

Whereas condition (12) links the current harvest to the entire succession of remaining harvests, conditions (11a) each focuses on two successive harvests of the same species, specifying the condition for replanting the same species. Conditions (11a) are equivalent to $V_a(a_k) - \delta V(a_k) = V_a(a_{k-1})e^{-\delta a_{k-1}}$. Substituting δ for $r - \mu$ gives

$$V_a(a_k) + \mu V(a_k) = e^{\mu a_{k-1}} V_a(a_{k-1}) e^{-r a_{k-1}} + rV(a_k),$$

Using Species P as numeraire, this equation shows that the harvest must be delayed until the marginal revenue (left-hand side) from timber volume change ($V_a(a_k)$) and price change ($\mu V(a_k)$) equals the opportunity cost of doing so (right-hand side). This opportunity cost has two components. The first one is equal to the discounted value of the lost growth incurred because the next harvest is delayed ($e^{\mu a_{k-1}} V_a(a_{k-1}) e^{-r a_{k-1}}$). The second one is the opportunity cost form harvesting later ($rV(a_k)$).

Proposition 5 partly defines the replanting boundary, whose description will be completed in Proposition 6. Figure 3 represents that boundary, which applies at relative prices lower than θ_0, together with the higher and lower switching boundaries defined in Propositions 3 and 4, which apply at relative prices above θ_0. These three boundaries look like a saw with its teeth upward. The body of the blade corresponds to situations where it is optimal to wait before harvesting and the upper side of the blade (the teeth) correspond to optimal harvest ages. The right end sector of the blade corresponds to the last harvest of Species P, occurring at a relative price above θ_0. Relative price-age pairs (θ, a) grow deterministically along rays such as the dashed line from point θ_1. While the horizontal axis belongs to the blade for $\theta \leq \theta_0$, the bottom of the blade is truncated to the right of θ_0 so that P trees younger than indicated by the lower switching boundary must be cut immediately.

Figure 3 can be given a dynamic interpretation. Since θ grows at rate $\bar{\mu}$, any tra-

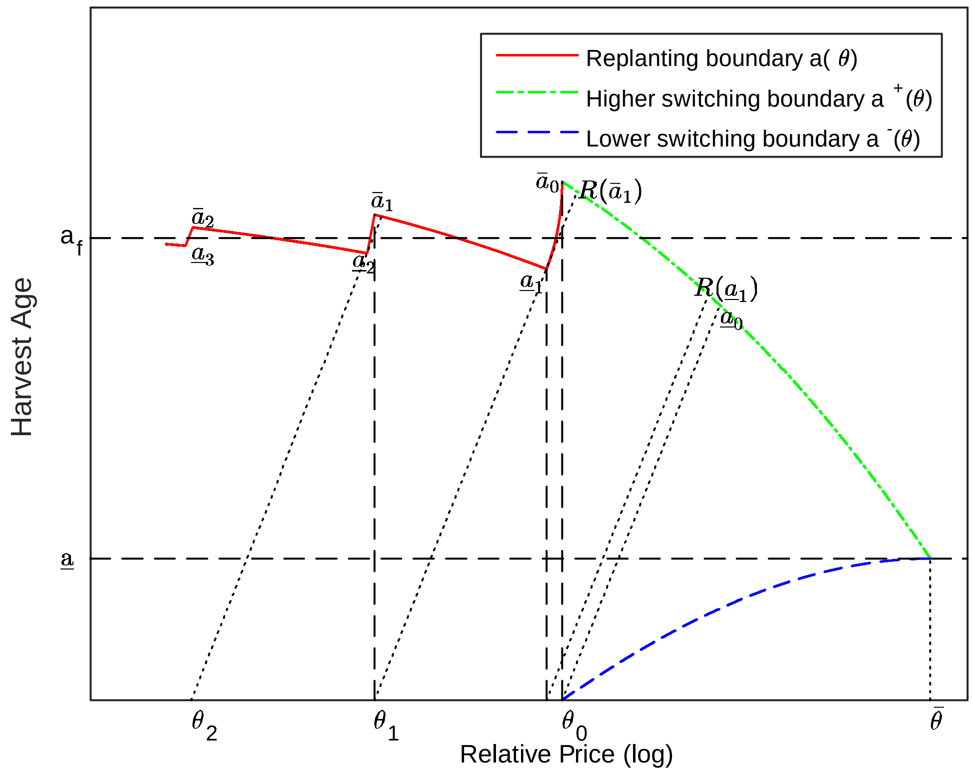

Figure 3: Optimal Harvest Age when an Alternative Species is Available

jectory over time of a price-age pair (θ, a) can be represented by one of the dotted lines rising obliquely. On such a trajectory, when trees are cut, a is reset to zero so that the trajectory starts again rising from the horizontal axis.[10] For example, when the land is bare and the price φ is between θ_1 and θ_0, which corresponds to a point on the horizontal axis to the left of θ_0, a new stand of P trees is planted and is to be harvested when the trajectory of the (θ, a) pair rising obliquely from that point reaches the higher switching boundary on segment $\bar{a}_0\underline{a}_0$. At that time θ is above θ_0 so that the switch to P' occurs. Similarly, if the initial condition is a point $(\varphi, 0)$ such that $\theta_2 \leq \varphi < \theta_1$, the second last harvest is to take place when the (θ, a) trajectory rising obliquely from $(\varphi, 0)$ hits the replanting boundary on the $\bar{a}_1\underline{a}_1$ segment. This signals the last time P is planted $(k = 1)$.

The key details involved in constructing and interpreting Figure 3 are the following. Proposition 5 implies the existence of a strictly decreasing[11] sequence $(\theta_k)_{k\in\mathbb{N}}$ such that, if the land is bare when the relative price is $\theta \in [\theta_{k+1}, \theta_k]$, it is optimal to plant Species P exactly $k+1$ more times before switching to Species P'. Any possible initial value $\varphi \leq \theta_0$ then belongs to one of the intervals in conformity with Proposition 5. Furthermore, if $\varphi = \theta_k$, then $\varphi \in [\theta_{k+1}, \theta_k]$ and $\varphi \in [\theta_k, \theta_{k-1}]$; by definition, it is therefore optimal to plant and harvest Species P indifferently k times or $k-1$ times before the switch to P'. The bounds θ_k can thus be computed by recurrence, starting from the value of θ_0 defined in Proposition 1. For example, by definition of θ_1, when the relative timber price is equal to θ_1 it is equivalent to plant Species P, to harvest at age $R(\bar{a}_1)$ and then switch to Species P', or to harvest at age $\underline{a}_1$ and replant the same species one time. Combining this condition of value equivalence with the characteristics of a_k and $R(a_k)$ given in Proposition 5 for $k = 1$ gives the values of $R(\bar{a}_1)$ and $\underline{a}_1$, and so on for $k = 2, ...\infty$.

The replanting boundary turns out to be a sequence of alternatively downward segments $\bar{a}_k\underline{a}_k$ and upward segments $\underline{a}_k\bar{a}_{k-1}$[12] for $k = 1, ..., +\infty$. The upward sloping

[10] The trajectories are straight lines because the horizontal units are logarithmic.

[11] The index k diminishes as time increases, since $k = 0$ corresponds to the last harvest of Species P. The maintained assumption is that θ is increasing over time. Hence the sequence is decreasing in k.

[12] The strict monotonicity of each particular function $a_k(.)$ can be established by recurrence using

segments will be explained shortly. The segment $\bar{a}_k\underline{a}_k$ is the image of $a_k(.)$ defined in Proposition 5 when the value of φ_k spans the interval $[\theta_k, \hat{\theta}_{k-1}]$, with $\hat{\theta}_{k-1} < \theta_{k-1}$. Thus Point $\bar{a}_1$ has coordinates $(\theta_1, a_1(\varphi_1))$ with $\varphi_1 = \theta_1$ where $a_1(\varphi_1)$ is given by $(11a)$ using $(11b)$ to substitute for $a_0(\varphi_0)$; $a_0(\varphi_0)$ is equal to $R(\bar{a}_1)$ since $\varphi_0 = \theta_1$ in that instance, as illustrated in Figure 3. Point $\bar{a}_0$, for its part, is determined in Proposition 3 as the left most point on the upper switching boundary.

Let us focus now on the right-hand ends of segments $\bar{a}_k\underline{a}_k$ (points such as $\underline{a}_1$).[13] As Proposition 5.4 indicates that each $\bar{a}_k\underline{a}_k$ segment contains a_f and is strictly downward sloping, it follows that they are necessarily disconnected, with $\underline{a}_k < \underline{a}_{k-1}$ for $k = 1, .., +\infty$.

On the other hand, it can be shown by contradiction that the forest value function g must be continuous in both θ and a at each state (θ, a). A discontinuity in the replanting boundary would cause a discontinuity in g. To ensure the continuity of the replanting boundary, any two successive downward sloping segments $\bar{a}_k\underline{a}_k$ and $\bar{a}_{k-1}\underline{a}_{k-1}$ must be linked by an upward segment $\underline{a}_k\bar{a}_{k-1}$.[14]

The upward sloping segments of the replanting boundary are not characterized in Proposition 5. This is because, with the exception of their lowest end, Point $\underline{a}_1$ for example, they cannot be reached by any optimal (θ, a) trajectory under the condition imposed in the Proposition that the land be bare initially. Indeed, consider the upward sloping segment $\underline{a}_1\bar{a}_0$; it can be shown that it lies above the dotted trajectory-line rising from $(\theta_1, 0)$ through Point $\underline{a}_1$ as drawn.[15] An optimum trajectory starting at $(\theta_1, 0)$ reaches the boundary at Point $\underline{a}_1$ whose coordinates are $(\theta_1 e^{\bar{\mu}\underline{a}_1}, \underline{a}_1)$; a harvest occurs at

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$(11a)$ and $(11b)$, starting with $a_0(.)$. A detailed proof is provided in the Proof of Proposition **6.**

[13] This will also clarify the meaning of the upwward sloping segments; unlike the downward sloping segments, these segments are not defined in Proposition 5.

[14] That the link is an upward sloping segment rather than a vertical line or a backward bending curve is a logical necessity. A backward bending curve would imply that part of the $\bar{a}_k\underline{a}_k$ segment has absissa exceeding θ_k (i.e. $\hat{\theta}_k > \theta_k$); this would imply that the segment corresponds to both k more harvests and $k-1$ more harvests. A vertical line would imply that the function g is constant in a at ages a_k when the volume function is strictly rising.

[15] To prove that segment $\underline{a}_1\bar{a}_0$, lies above the price line rising from Point $\bar{a}_1$, note that the condition that defines Point $\underline{a}_1$, indifference between cutting the current trees at age $\underline{a}_1$ and plant P for one more harvest of P, or wait for the current trees to reach age $R(\bar{a}_1)$ to harvest them and switch to P', implies that $g^1(\theta_1 e^{\bar{\mu}\underline{a}_1}, \underline{a}_1) = g^0(\theta_1 e^{\bar{\mu}\underline{a}_1}, \underline{a}_1)$, where g^k is the value function defined by (5) when the number of

that point so that the trajectory is reset slightly to the left of $(\theta_0, 0)$; its continuation along an oblique price line reaches the higher switching boundary at Point $R(\underline{a}_1)$ and thus never reaches the segment $\underline{a}_1\bar{a}_0$. Similarly, no optimum trajectory from any initial point to the left or to the right of $(\theta_1, 0)$ reaches segment $\underline{a}_1\bar{a}_0$, given that it lies above the price line rising from Point $\bar{a}_1$ as assumed. This implies that, on an optimal trajectory defined by Proposition 5, no harvest takes place when the relative price is strictly inside the intervals delimited by the absissas of Points $\underline{a}_k$ and $R(\bar{a}_k)$. In turn this implies that such portions of the boundary segments as $R(\bar{a}_1)\bar{a}_0$, $R(\bar{a}_2)\bar{a}_1, \ldots$, are not reached by any optimal trajectory.

However, allow initial conditions $a > 0$ that are ruled out in Proposition 5. In initial configurations such that $k \geq 1$ and the initial point is both above the price line rising from Point $\underline{a}_k$ and below segment $\underline{a}_k\bar{a}_{k-1}$, the optimal trajectory reaches some point on Segment $\bar{a}_k R(\bar{a}_{k-1})$. For example any trajectory starting at $a > \underline{a}_1$ and between the dotted line $R(\bar{a}_1)\underline{a}_1$ but below segment $\underline{a}_1\bar{a}_0$ hits the boundary on segment $R(\bar{a}_1)\bar{a}_0$. All other possible initial conditions lead to some state $(\theta, 0)$ already covered in Proposition 5, either because there is immediate harvest (initial state above the replanting boundary) or because the price trajectory hits one of the $\bar{a}_k\underline{a}_k$ segments. In other words, the

remaining harvests beyond the current trees is k. Precisely,

$$g^1(\theta, a) = \max_{a_0, a_1 \geq a} \left\{ V(a_1)e^{-\delta(a_1 - a)} + V(a_0)e^{-\delta(a_1 + a_0 - a)} + c'\theta e^{-\delta'(a_1 + a_0 - a)} \right\} \tag{13}$$

$$g^0(\theta, a) = \max_{a_0 \geq a} \left\{ V(a_0)e^{-\delta(a_0 - a)} + c'\theta e^{-\delta'(a_0 - a)} \right\}. \tag{14}$$

Trajectories emanating from $(\theta_1, 0)$ have coordinates $(\theta_1 e^{\bar{\mu}a}, a)$; for any value of a such that $0 \leq a \leq \underline{a}_1$, the constraint on the choice of a_1 in (13) is not binding so that the equality of $g^1(\theta_1 e^{\bar{\mu}\underline{a}_1}, \underline{a}_1)$ and $g^0(\theta_1 e^{\bar{\mu}\underline{a}_1}, \underline{a}_1)$ implies that the same equality holds at any pair $(\theta_1 e^{\bar{\mu}a}, a)$; however, when $\underline{a}_1 \leq a \leq R(\bar{a}_1)$, the constraint $a_1 \geq a$ is binding in (13), implying immediate harvest ($a_1 = a$ in (13)), and $g^1(\theta_1 e^{\bar{\mu}a}, a) < g^1(\theta_1 e^{\bar{\mu}\underline{a}_1}, \underline{a}_1)$; meanwhile the maximization in (14) remains unconstrained. It follows that $g^1(\theta_1 e^{\bar{\mu}a}, a) < g^0(\theta_1 e^{\bar{\mu}a}, a)$ for $\underline{a}_1 \leq a \leq R(\bar{a}_1)$. This means that, on the trajectory represented by the dotted curve above Point $\bar{a}_1$, the actions implied by (13), immediate harvest of existing trees followed by the establishment of a new crop of P, produce less value than the decision implied by (14), which is to allow the existing trees to reach age $R(\bar{a}_1)$. This proves that the boundary that signals equality between the value implied by immediate harvest and the value implied by allowing further growth is strictly above the dotted line rising from $(\theta_1, 0)$ through Point $\underline{a}_1$ as postulated. Segment $\underline{a}_1\bar{a}_0$ represents such a boundary. On the left of $\underline{a}_1\bar{a}_0$ it is preferable to harvest immediately and replant P. A similar analysis applies to all upward sloping segments $\underline{a}_{k+1}\bar{a}_k$, adapting the value equivalence condition for $k = 2, \ldots$ as follows: harvesting at age $R(\bar{a}_k)$ is not accompanied by a switch to P' as when $k = 1$ but involves scheduling one fewer further harvests of P than harvesting at age $\underline{a}_k$.

replanting boundary and the higher switching boundary defined above and represented in Figure 3 apply to all possible initial conditions.

The above discussion completes the characterization of the replanting boundary $a\,(.)$. The results are gathered in the following proposition.

Proposition 6 *1. There exists values θ_k of the relative price such that $\theta_{k+1} < \theta_k < \theta_0$ and, if the land is bare and $\theta = \theta_k$, it is equivalent to plant Species P repeatedly k times or $k-1$ times before the switch to Species P'. In the first instance the first remaining harvest takes place at age $\underline{a}_k$ and price $\theta_k e^{\overline{\mu}\ \underline{a}_k}$; in the second instance it takes place at age $a = R\,(\overline{a}_k)$ and price $\theta_k e^{\overline{\mu}\ R(\overline{a}_k)}$. θ_k is obtained implicitly by recurrence by solving for θ the equality of function g (defined by (5)) for n and for $n-1$, with $a = 0$, starting with $n = 1$ and with an initial value θ_0 given by Proposition 1.*

2. The replanting boundary $a(.)$ for relative prices below θ_0 is continuous and consists of an infinite sequence of one downward sloping segment $\overline{a}_k\underline{a}_k$ followed by one upward segment $\underline{a}_k\overline{a}_{k-1}$, defined on $[\theta_k, \theta_{k-1}]$, and giving the harvest age as function of the relative price when the number of times Species P will be replanted before the switch to Species P' is k.

3. The upward sloping segments $\underline{a}_k\overline{a}_{k-1}$ are strictly increasing from $(\theta_k e^{\overline{\mu}\ \underline{a}_k}, \underline{a}_k)$ to $(\theta_{k-1}, \overline{a}_{k-1})$. They are defined by the condition of equivalence between immediate harvest followed by k harvest of Species P, or harvest when the current stand reaches the age indicated on the downward sloping segment $\overline{a}_{k-1}\underline{a}_{k-1}$, followed by $k-1$ more harvests of P.

Proof. Established in the text preceeding the Proposition. ∎

5. Further comments

Propositions 5.4 and 6.3 indicate that the optimal harvest age before the switch may be higher or lower than Faustmann's rotation, but in a very precise way: if it is higher than

Faustmann's rotation at one harvest, it is higher than Faustmann's rotation at all harvests before the switch, and is furthermore increasing at each harvest by Proposition 6.4. Vice versa if it is once lower than Faustmann's rotation, it is lower than Faustmann's rotation at all harvests and decreasing from one harvest to the next one. Thus, while the optimum harvest age is defined on downward sloping parts of the replanting boundary or on the downward sloping switching boundary, successive harvest ages are not necessarily diminishing as price increases. As a matter of fact, successive harvests may optimally take place at a constant age, which further equals Faustmann's rotation [16]. According to Proposition 5.4 this happens if and only if, as k diminishes over time from n to zero, the trees reach that age precisely each time the relative price reaches θ_{fk}. This is so if and only if the initial state is $\left(\varphi_f, 0\right)$.

Figure 3 shows that a situation where P trees are cut before financial maturity to be replaced by P' trees cannot occur as the continuation of an optimum program. Indeed, since the oblique dotted lines from the horizontal axis represent the price age trajectories, trees of Species P that reach any relative price-age combination corresponding to a point below the lower switching boundary must have been planted when the relative price was higher than θ_0, in violation of Proposition 1. In other words, points below the lower switching boundary (below the body of the blade on the right-hand side of θ_0) can be conceived of only as initial states, not as resulting from previous optimum decisions.

The same argument implies that the right-most point on the upper switching boundary that can be reached as the continuation of an optimum program involving an initially bare land and an initial price $\theta < \theta_0$ is the point labeled $R\left(\underline{a}_1\right)$, to the left of Point $\underline{a}_0$. This is so because the right-most point of the segment $\left[\underline{a}_1, \overline{a}_1\right]$ corresponds to a last plantation of P trees, at price $\theta_1 e^{r\bar{\mu}\ \underline{a}_1}$; Point $R\left(\underline{a}_1\right)$ signals the harvest of these trees.

Another implication of Proposition 6 illustrated in Figure 3 is that the longest possible rotation $\overline{a}_0$, which may occur only if the switch to Species P' takes place at price

[16] These properties, proved in 6, are illustrated graphically in Figure 1:
$\underline{a}_1 < a_f \iff R\left(\underline{a}_1\right) < \underline{a}_1 < a_f$,
$\overline{a}_1 > a_f \iff R\left(\overline{a}_1\right) > \overline{a}_1 > a_f$,
$R\left(a\right) = a \iff a = a_f$.

θ_0, cannot be observed as the continuation of an optimum trajectory. It can occur only if the initial state, inherited from an irrational past, is precisely Point $(\theta_0, \overline{a}_0)$.

Consequently, if trees P have been planted rationally for at least one rotation then the highest possible switching age is $R(\overline{a}_1) < \overline{a}_0$ and the lowest is $R(\underline{a}_1) > \underline{a}_0$. Similarly, if trees P have been planted rationally for at least two rotations then the highest possible switching age is $R^2(\overline{a}_2) < R(\overline{a}_1) < \overline{a}_0$ and the lowest is $R^2(\underline{a}_2) > R(\underline{a}_1) > \underline{a}_0$, etc. Hence, the longer trees P have been planted rationally, the closer the switching age to the Faustmann rotation.

6. Conclusion

We have extended the conventional forestry economics model to include two alternative species. This entails more sophisticated planting and harvesting decisions than had been considered before. When the decision maker has the opportunity to exploit the forest land for an indefinite number of rotations, she must decide at what age the current stand should be harvested and whether the same species, or the alternative species, should be planted. We have characterized explicitly the value functions and the optimal management strategy that apply when each species has its own, non stochastic, growth function and has a unit price that grows at some specific exponential rate. Starting from a situation where the price that grows more slowly is initially high, so that the corresponding species, say Species P, is initially to be exploited, the relative price of the alternative species P' gradually increases so that it becomes desirable to replace P with P' at some point in time. We have shown that the second phase is equivalent to a situation where only one species is available as in any standard Faustmann model with exponentially rising price. In the initial phase, the situation is completely different. The model with two species does not reduce to a time autonomous problem as the conventional Faustmann model, and its solution does not reduce to a single optimal rotation.

The species whose price is high but growing more slowly is harvested and replanted repeatedly for a finite number of rotations before the switch to the alternative species.

The optimal harvest age is generally different from one harvest to the next and depends on the prices of both species despite the fact that the harvest value does not depend on the price of the alternative species directly.

The solution is described in the state space by boundaries giving the optimal harvest ages as functions of relative prices. The boundaries differ when the harvest is followed by replanting of the same species rather than by a switch to the alternative species. Furthermore, at relative prices above the switching price but lower than some higher threshold, there is a lower boundary indicating that trees should be allowed to reach the upper boundary provided they are already big enough; otherwise they should be cut immediately and replaced with the alternative species. More specifically, under well known assumptions where the one-species Faustmann rotation is independent of price, we showed that the boundaries, whether the replanting boundary or the switching boundary, are locally decreasing with respect to the relative price of timber. However, when Species P has the most attractive price and is replanted after each harvest, the replanting boundary is discontinuous, with a sawtooth shape, so that each of its downward sloping segments entails the possibility of a harvest at age a_f, the Faustmann rotation. In fact we showed that depending on the initial relative price and age of the trees, successive harvests occur at progressively higher ages, all above a_f or at progressively lower ages, all below a_f, until the switch to Species P'.

Surprisingly, although there exists a critical price at which the manager would be indifferent between planting either species when the land is bare, the land must not be bare when that price is reached. If both the land is bare and the price is at that threshold, the harvest sequencing problem must be at its initial state. We also showed that the switching age is closer to the Faustmann rotation when the trees have been planted rationally for a longer period.

The model can be readily extended to include positive planting costs as is often done in Faustmann models. Contrasts with the one species model would not be as sharp because one species models with positive planting costs imply that the optimum rotation depends on the price. Furthermore the algebra would be more involved because the homogeneity of the value function would be lost so that one could not replace individual timber prices by a unique relative price to describe the harvest boundaries in a two-dimensional plane.

The model also describes the conditions of a switch of land use from forestry (the first species) to some alternative use. Indeed, since the second species was shown to be exploited as in a single species Faustmann model, the value of the land after the switch is described as the solution of a time autonomous problem whether it remains allocated to forestry or to some alternative use. In particular our results imply that there is a price of timber relative to the value of the alternative use at which, if the land is bare, it is equivalent to plant timber or to switch to the alternative use immediately. We have shown that switching from timber exploitation to the alternative use when that relative price is reached would be premature if the land were not already bare.

For further research, the model might be extended to involve stochastic timber prices. The solution would also involve harvest boundaries conditional on the species in place. Unlike the certain exponential prices used in this paper, stochastic prices would possibly allow multiple switches from one species to the other.

7. Appendix

Proof of Proposition 1

Proof. The set of $\theta > 0$ for which it is optimal to plant P is not empty as for θ sufficiently low, it is optimal to plant P. This set is bounded as it is optimal to plant P' for θ sufficiently high. Being not empty and bounded, the set of $\theta > 0$ for which it is optimal to plant P has a finite maximum θ_0. The unicity of θ_0 results from continuity. At $\theta = \theta_0$, the manager by definition is indifferent between planting P' forever or planting P for just one rotation of duration $\underline{a}_0$; consequently, $c'\theta_0 = \max_s \left[V(s)e^{-\delta s} + c'\theta_0 e^{-\delta' s}\right]$ where $\underline{a}_0 = \arg\max_s \left[V(s)e^{-\delta s} + c'\theta_0 e^{-\delta' s}\right]$. The first-order condition for this maximization problem is equivalent to Equation $(7b)$. The second-order condition is $K_a(\underline{a}_0) - \overline{\mu}K(\underline{a}_0) < 0$, satisfied for $\underline{a}_0 > \underline{a}$. The first-order condition, together with $c'\theta_0 = V(\underline{a}_0)e^{-\delta \underline{a}_0} + c'\theta_0 e^{-\delta' \underline{a}_0}$, gives Equation $(7a)$ that allows to determine $\underline{a}_0$ unambiguously whereas $(7b)$ determines θ_0. As $\frac{V_a(a)}{V(a)}$ is decreasing on $[\underline{a}, +\infty[$ and $\lim_{a\to\underline{a}} \frac{V_a(a)}{V(a)} = +\infty$ then $\underline{a}_0 > \underline{a}$. Faustmann's rotation a_f is determined by Equation $(3a)$, which is equivalent to $\frac{V_a(a_f)}{V(a_f)} = \delta + \frac{\delta}{e^{\delta a_f} - 1}$. As for any $a > 0$, $\frac{\delta}{e^{\delta a}-1} < \frac{\delta'}{e^{\delta' a}-1}$ then $\frac{V_a(a_f)}{V(a_f)} < \frac{V_a(\underline{a}_0)}{V(\underline{a}_0)}$, which implies $\underline{a}_0 < a_f$. ■

Proof of Proposition 2

Proof. When the current price of Species P is p and $\theta \leq \theta_0$, it is optimal to plant Species P. According to Equation (6), the reduced land value function is $f(\theta) = \max_{s\geq 0} e^{-\delta s}\left[V(s) + f(\theta_s)\right]$ where s is replaced by $\frac{1}{\mu'-\mu}\log(\frac{\theta_s}{\theta})$ to obtain the first line on the right-hand side of Expression (8). For $\theta \geq \theta_0$, it is optimal to plant Species P' forever; therefore the reduced land value function is $c'\theta$. ■

Proof of Proposition 3

Proof. Consider a stand of P characterized by (θ, a) with $\theta \geq \theta_0$ and $a \geq 0$. Existence is shown by construction. The maximization problem is $\max_{s\geq 0}\left[V(a+s)e^{-\delta s} + c'\theta e^{-\delta' s}\right]$ whose first-order condition for an interior solution is equivalent to $K(a+s) = c'\frac{\delta'}{\delta}\theta e^{\overline{\mu} s}$

whereas the second-order condition is $K_a(a+s) - \overline{\mu}K(a+s) < 0$; the solution is interior when $a + s \in]\underline{a}, a_w[$. Therefore, the upper boundary $a^+(\theta)$ is the solution to $K(a^+(\theta)) = c' \frac{\delta'}{\delta}\theta$. The proof of monotonicity is immediate as $K(a)$ is strictly decreasing in a. The highest value of θ compatible with $a \geq \underline{a}$ defines $\overline{\theta}$ with $K(\underline{a}) = \frac{V_a(\underline{a})}{\delta} = \frac{\delta'}{\delta}c'\overline{\theta}e^{\overline{\mu}\underline{a}}$ or $V_a(\underline{a}) = \delta' c' \overline{\theta} e^{\overline{\mu}\underline{a}}$. The highest value of a compatible with $\theta \geq \theta_0$ is $\overline{a}_0$ where $\overline{a}_0 < a_w$ as $K(\overline{a}_0) = c' \frac{\delta'}{\delta}\theta_0 > K(a_w) = 0$. ■

Proof of Proposition 4

Proof. Existence is shown by construction. Assume that $a \in [0, \underline{a}]$ so that the maximization of the current forest value has an interior solution. The set of pairs (θ, a) for which the decision maker is indifferent between harvesting immediately to earn $p\,[c'\theta + V(a)]$, or harvesting after a time period s that maximizes $e^{-rs}[p_s V(a+s) + c' p_s']$, defines the lower switching boundary of the waiting region. It is characterized as the set of pairs (θ, a) that solve $c'\theta + V(a) = e^{-\delta s}V(a+s) + e^{-\delta' s}c'\theta$, where $V(a) = 0$, and Equation (10b), $K(a+s) = \frac{\delta'}{\delta}c'\theta e^{\overline{\mu}s}$, with $\underline{a} \leq a+s \leq \underline{a}_0$ as harvesting later will take place at $(\theta e^{\overline{\mu}s}, a+s)$ on the upper switching boundary where $a+s$ is the age at which the trees will be cut if they are not cut immediately, and a is their age on the lower boundary. Pairs (θ, a) below or to the right of the lower boundary command immediate harvest; while pairs above the lower boundary but below the upper boundary belong to the continuation region. The two last equations lead to Equation (10a).

For a given age $a \in [0, \underline{a}]$ and for $s \in [\underline{a} - a, \underline{a}_0 - a]$, $\frac{V_a(a+s)}{V(a+s)}$ decreases in s from infinity to $\frac{V_a(\underline{a}_0)}{V(\underline{a}_0)}$ whereas $\delta + \frac{\delta'}{e^{\delta' s}-1}$ decreases in s from a finite positive value $\delta + \frac{\delta'}{e^{\delta'(\underline{a}-a)}-1}$ to $\delta + \frac{\delta'}{e^{\delta'(\underline{a}_0 - a)}-1} > \delta + \frac{\delta'}{e^{\delta' \underline{a}_0}-1} = \frac{V_a(\underline{a}_0)}{V(\underline{a}_0)}$ where the last equality follows from (7a). Therefore, for any $a \in [0, \underline{a}]$, there exists a unique $s(a) \in [\underline{a} - a, \underline{a}_0 - a]$ such that $\frac{V_a(a+s)}{V(a+s)} = \delta + \frac{\delta'}{e^{\delta' s}-1}$. Given a and $s\,(a)$, the equation $K(a+s) = \frac{\delta'}{\delta}c'\theta e^{\overline{\mu}s}$ in Proposition 4 determines θ, denoted $\theta(a)$. Let us show that $\theta_a(a) > 0$. Differentiate $\left(1 - e^{-\delta' s}\right)c'\theta = e^{-\delta s}V(a+s)$ with respect to a to obtain $\delta' e^{-\delta' s}c'\theta s_a + \left(1 - e^{-\delta' s}\right)c'\theta_a = -\delta e^{-\delta s}V(a+s)s_a + e^{-\delta s}(1+s_a)V_a(a+s)$ or $\left(1 - e^{-\delta' s}\right)c'\theta_a = -\delta e^{-\delta s}V(a+s)s_a + e^{-\delta s}(1+s_a)V_a(a+s) - \delta e^{-\delta s}s_a K(a+s)$. As $K(a+s) = \frac{V_a(a+s)}{\delta} - V(a+s)$ equals $\frac{\delta'}{\delta}c'\theta e^{\overline{\mu}s}$ then $\left(1 - e^{-\delta' s}\right)c'\theta_a = e^{-\delta s}V_a(a+s)$ which implies that $\theta_a(a) > 0$.

The pair $(\theta_0, 0)$ satisfies the condition in Proposition 4 with $s = \underline{a}_0$ and $a = 0$. The pair $(\overline{\theta}, \underline{a})$ satisfies this condition with $s = 0$ and $a = \underline{a}$. The lower switching boundary $a^-(\theta)$ is then a strictly increasing curve in the plane (θ, a) stretching between $(\theta_0, 0)$ and $(\overline{\theta}, \underline{a})$. ■

Proof of Proposition 5

1, 2, and 3. Consider the maximization problem (5) $g(\varphi, 0) = \max_{a_0,\ldots a_n} W(\varphi, a_n, \ldots, a_0)$ where $W(\varphi, a_n, \ldots, a_0)$ denotes $V(a_n)e^{-\delta a_n} + \ldots + V(a_0)e^{-\delta(a_n+\ldots+a_0)} + c'\theta e^{-\delta'(a_n+\ldots+a_0)}$. The $n+1$ first-order conditions are $V_a(a_k)e^{-\delta(a_k+\ldots+a_n)} - \delta\sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+\ldots+a_n)} - \delta' c'\varphi e^{-\delta'(a_0+\ldots+a_n)} = 0$ for $k = 0, \ldots, n$. We show further below that the second-order condition is also satisfied. This set of $n+1$ equations can be equivalently simplified into another set of $n+1$ equations obtained by keeping the first-order condition for $k = 0$, and for $k = 1, \ldots, n$, substituting the k^{th} first-order condition for the equation obtained by subtracting the $(k-1)^{th}$ first-order condition from the k^{th} one. This transformation leads to the following equivalent set of $n+1$ equations: $V_a(a_0) - \delta V(a_0) - \delta' c'\varphi e^{\overline{\mu}(a_n+\ldots+a_0)} = 0$ or $K(a_0) = \frac{\delta'}{\delta}c'\varphi_0$ where $\varphi_0 = \varphi e^{\overline{\mu}(a_n+\ldots+a_0)}$, i.e. Equation $(11b)$, and $V_a(a_k)e^{-\delta(a_n+\ldots+a_k)} - \delta V(a_k)e^{-\delta(a_n+\ldots+a_k)} = V_a(a_{k-1})e^{-\delta(a_n+\ldots+a_{k-1})}$ or $K(a_k) = L(a_{k-1})$ for $k = 1, \ldots, n$, i.e. $(11a)$. As L is strictly decreasing, it can be inverted; thus $a_{k-1} = R(a_k)$ where $R = L^{-1} \circ K$ for $k = 1, \ldots, n$. Equation $(11a)$ links any rotation a_k on the replanting boundary with the next rotation a_{k-1} for $k = 1, \ldots, +\infty$, whereas Equation $(11b)$ allows to compute rotation a_0.

4. Both functions K and L are decreasing and intersect at a_f; furthermore, K is more steeply downward sloping than L. Consequently, as illustrated in Figure 1, successive rotations a_k and a_{k-1} satisfy $a_k = a_{k-1} = a_f$, or $a_f < a_k < a_{k-1}$, or $a_f > a_k > a_{k-1}$. As a_f satisfies $R(a_f) = a_f$, then the pair (θ_{fk}, a_f) with $\theta_{fk} = \frac{K(a_f)}{\frac{\delta'}{\delta}C'e^{(k+1)\overline{\mu}a_f}}$ satisfies equations 11a for $k = 1, \ldots, +\infty$ and satisfies equation 11b for $k = 0$; consequently, it belongs to the harvest boundary $a_k(.)$ for $k = 0, 1, \ldots, +\infty$. Equivalently, if $\varphi = \theta_{fk}$ then $a_k = a_f \ \forall\ k$.

Second-order condition for problem (5)**.** To show now that the second-order condition is satisfied, we will show that the Hessian matrix $\left[\frac{\partial^2 W}{\partial a_i \partial a_j}\right]_{1\leq i,j\leq n+1}$ associated with $W(\varphi, a_n, ..., a_0)$ is twice continuously differentiable with respect to $(a_n, ..., a_0)$ and is negative definite on $]\underline{a}, +\infty[^{n+1}$. To do so, we will show that its leading principal minor of order k has the sign of $(-1)^k$ for $k = 1, ..., n+1$.

First determine $W_{kk} = \frac{\partial^2 W}{\partial a_k \partial a_k}$ for $1 \leq k \leq n+1$, and $W_{kl} = \frac{\partial^2 W}{\partial a_l \partial a_k}$ for l such that $k < l \leq n+1$:

$W_{kl} = -\delta V_a(a_k)e^{-\delta(a_k+...+a_n)} + \delta^2 \sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)} + \delta'^2 c'\varphi e^{-\delta'(a_0+...+a_n)}$

$W_{kk} = V_{aa}(a_k)e^{-\delta(a_k+...+a_n)} - \delta V_a(a_k)e^{-\delta(a_k+...+a_n)} - \delta V_a(a_k)e^{-\delta(a_k+...+a_n)} + \delta^2 \sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)} + \delta'^2 c'\varphi e^{-\delta'(a_0+...+a_n)}$. Note that $W_{kk} = W_{kl} + \delta e^{-\delta(a_k+...+a_n)} K_a(a_k)$.

Let us first show that W_{kl} is independent of k and l, and that $W_{kl} < 0$:

$W_{kl} = -\delta V_a(a_k)e^{-\delta(a_k+...+a_n)} + \delta^2 \sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)} + \delta'^2 c'\varphi e^{-\delta'(a_0+...+a_n)}$, where

$\delta' c'\varphi e^{-\delta'(a_0+...+a_n)} = V_a(a_k)e^{-\delta(a_k+...+a_n)} - \delta \sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)}$ from Equation (12),

then

$$
\begin{aligned}
W_{kl} &= -\delta V_a(a_k)e^{-\delta(a_k+...+a_n)} + \delta^2 \sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)} \\
&+\delta' \left[V_a(a_k)e^{-\delta(a_k+...+a_n)} - \delta \sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)}\right] \\
&= -\overline{\mu} V_a(a_k)e^{-\delta(a_k+...+a_n)} + \delta\overline{\mu} \sum_{i=0}^{k} V(a_i)e^{-\delta(a_i+...+a_n)} \\
&= -\delta\overline{\mu} e^{-\delta(a_k+...+a_n)} K(a_k) + \delta\overline{\mu} \sum_{i=0}^{k-1} V(a_i)e^{-\delta(a_i+...+a_n)}
\end{aligned}
$$

Using the first-order conditions $K(a_k) = L(a_{k-1})$, we obtain

$W_{kl} = -\delta\overline{\mu} e^{-\delta(a_k+...+a_n)} L(a_{k-1}) + \delta\overline{\mu} \sum_{i=0}^{k-1} V(a_i)e^{-\delta(a_i+...+a_n)}$

$W_{kl} = -\delta\overline{\mu} K(a_{k-1})e^{-\delta(a_{k-1}+...+a_n)} + \delta\overline{\mu} \sum_{i=0}^{k-2} V(a_i)e^{-\delta(a_i+...+a_n)}$

We continue so and show that $W_{kl} = -\delta\overline{\mu} e^{-\delta(a_0+...+a_n)} K(a_0) < 0$ as $\overline{\mu} > 0$ and $K(a_0) > 0$.

Denote $W_{kl} = \beta < 0$ and $\alpha_k = \delta e^{-\delta(a_k+...+a_n)} K_a(a_k) < 0$ then $W_{kk} = \alpha_k + \beta$. The k^{th} leading principal minor is therefore:

$H_k(\alpha_1, ..., \alpha_k, \beta) = \begin{vmatrix} \alpha_1+\beta & \beta & ... & \beta \\ \beta & . & ... & . \\ . & . & ... & . \\ \beta & \beta & ... & \alpha_k+\beta \end{vmatrix}$, for $k = 1, ..., n+1$. By subtracting the second line from the first one and then developing the minor along the first line, we obtain

$$H_k(\alpha_1, ..., \alpha_k, \beta) = \begin{vmatrix} \alpha_1 & -\alpha_2 & ... & 0 \\ \beta & \alpha_2 + \beta & ... & \beta \\ . & \beta & ... & . \\ \beta & \beta & ... & \alpha_k + \beta \end{vmatrix}$$

$$= \alpha_1 H_{k-1}(\alpha_2, ..., \alpha_k, \beta) + \alpha_2 \begin{vmatrix} \beta & \beta & ... & . & \beta \\ \beta & \alpha_3 + \beta & ... & . & . \\ . & \beta & ... & . & . \\ . & . & ... & \alpha_{k-1} + \beta & \beta \\ \beta & \beta & ... & \beta & \alpha_k + \beta \end{vmatrix}$$

To compute the determinant, subtract its second line from its first one to obtain:

$$H_k = \alpha_1 H_{k-1}(\alpha_2, ..., \alpha_k, \beta) + \alpha_2 \alpha_3 \begin{vmatrix} \beta & \beta & ... & \beta \\ . & \alpha_3 + \beta & ... & . \\ . & . & ... & . \\ \beta & \beta & ... & \alpha_k + \beta \end{vmatrix}$$

$$= \alpha_1 H_{k-1}(\alpha_2, ..., \alpha_k, \beta) + \alpha_2 \alpha_3 ... \alpha_{k-1} \begin{vmatrix} \beta & \beta \\ \beta & \alpha_k + \beta \end{vmatrix}, \text{ and then,}$$

$$H_k(\alpha_1, ..., \alpha_k, \beta) = \alpha_1 H_{k-1}(\alpha_2, ..., \alpha_k, \beta) + \beta \alpha_2 \alpha_3 ... \alpha_k \tag{15}$$

Now, it is possible to show recursively that the k^{th} leading principal minor has the sign of $(-1)^k$. Indeed, the proposition in true for $k = 1$ as $H_1(\alpha_1, \beta) = \alpha_1 + \beta < 0$, and true for $k = 2$ as $H_2(\alpha_1, \alpha_2, \beta) = (\alpha_1 + \beta)(\alpha_2 + \beta) - \beta^2 = \alpha_1 \alpha_2 + (\alpha_1 + \alpha_2) \beta > 0$. Assume now that the leading principal minor of order $k-1$, $2 \leqslant k \leqslant n+1$, has the sign of $(-1)^{k-1}$, then $H_{k-1}(\alpha_2, ..., \alpha_k, \beta)$ has the sign of $(-1)^{k-1}$ and $\alpha_1 H_{k-1}(\alpha_2, ..., \alpha_k, \beta)$ has the sign of $(-1)^k$ as $\alpha_1 < 0$. As $\beta \alpha_2 \alpha_3 ... \alpha_k$ has the sign of $(-1)^k$ too, then according to Equation (15), the k^{th} leading principal minor $H_k(\alpha_1, ..., \alpha_k, \beta)$ has the sign of $(-1)^k$. We conclude that the second-order condition for problem (5) is satisfied.

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