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# A Model of Dynamic Liquidity Contracts\*

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## Abstract

I study long-term financial contracts between lenders and borrowers in the absence of perfect enforceability and when both parties are credit constrained. Borrowers repeatedly have projects to undertake and need external financing. Lenders can commit to contractual agreements whereas borrowers can renege any period. I show that equilibrium contracts feature interesting dynamics: the economy exhibits *efficient* investment cycles; absence of perfect enforcement and shortage of capital skew the cycles toward states of liquidity drought; credit is rationed if either the lender has too little capital or if the borrower has too little collateral. This paper's technical contribution is its demonstration of the existence and characterization of financial contracts that are solutions to a non-convex dynamic programming problem.

*Journal of Economic Literature* Classification Numbers: C6, C7, D9, G2

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## 1 Introduction

Financial intermediaries are a veil in a standard Arrow-Debreu model. When modeled, in finance, they are usually taken as deep-pocketed. Empirical evidence tells otherwise as we witnessed recently during the most damaging financial storm of our lives. Some of the biggest banks, which were thought to be the untouchables of the finance world, went down stating inability to honor short term liabilities.<sup>1</sup> US government gave in and devised a \$700 billion scheme to rescue the financial system from total collapse. *Otherwise more banks were on the line.*

About a year later, many capital-injected, too-big-to-fail banks announced *solid* profits again, arguably due partly to the changes in the accounting standards used in the valuation of their assets and partly to the extremely low cost of emergency borrowing from the lenders of last resort<sup>2</sup>. The issue is that at times of great economic distress, credit markets freeze and the pricing mechanism stops working. Transactions are carried out at fire sale prices. The drought of *liquidity* in the system halts the well-functioning of the financial intermediaries and consequently that of the economic system.

This is, of course, not the first time we experience such turmoil. During the credit crunch of 1990, banks started cutting back on lending immensely. Limited bank capital relative to the loan demand contributed to restrictive bank lending during the recession of 1990/91.<sup>3</sup> We know by now that lender capital has a significant effect on lending and economic activity. As their capital ratios fall, banks become more conservative in their lending.

The situation is worse for relatively *small lenders* specializing on local industries and *small firms* that depend on small lenders in the form of backing lines of credit and other forms of external finance. Small firms do not enjoy the same market access and low interest rates as do big ones (see

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<sup>1</sup>On January 29, 2008, Lehman Brothers reported record revenues of nearly \$60 billion and record earnings in excess of \$4 billion for the fiscal year 2007. Its stock traded as high as \$65.73 per share and averaged in the high to mid-fifties. Less than eight months later, on September 12, its stock closed under \$4, a decline of nearly 95%. On September 15, the once fourth largest investment bank by market capitalization, sought Chapter 11 protection, in the largest bankruptcy proceeding ever filed (From the *Lehman Brothers Holdings Inc. Chapter 11 Proceedings Examiner's Report*, Jenner&Block, March 2010, New York and Chicago. Available publicly at <http://lehmanreport.jenner.com/>).

<sup>2</sup>Here is an excerpt from an article in the Financial Times, Markets Section, dated December 3, 2009, on US banking sector: "A year ago, few people thought the banking sector was the place to be. However, its performance in the stock market has been nothing short of stellar in 2009."

<sup>3</sup>See [Bernanke and Lown \(1991\)](#) on the '*Credit Crunch*'. They give anecdotal evidence on Richard Syron, then president of the Federal Reserve Bank of Boston, calling the crunch a '*Capital Crunch*'. Syron argued in a testimony before Congress that the credit crunch in New England was due to a shortage in bank capital. Banks in the region had to write down loans, forced by the real estate bubble, which led to the depletion of their equity capital. In order to meet regulatory requirements, they had to sell assets and scale down their lending.

e.g. Beck, Demirgüç-Kunt and Maksimovic, 2008). Moreover, for small borrowers, obtaining a line of credit is no guarantee that they can draw on their line when liquidity is needed. Most credit line arrangements come with restrictive covenants and contingencies and are more sensitive to lender and borrower conditions in the case of small borrowers than in the case of large borrowers (see e.g. James, 2009 and Sufi, 2009). Empirical evidence supports the view that dynamic bank relationships help borrowers (especially small ones loyal to the same lender) through implicit contracting (see Petersen and Rajan, 1995, Berger and Udell, 1994, and Hoshi, Kashyap and Scharfstein, 1990a,b, 1991).

The empirical literature on the link between lender capital and lending is well-developed (see Sharpe, 1995 for an extensive survey).<sup>4</sup> Theoretical literature on the issue suggests that higher lender capital tends to increase lending (see Besanko and Kanatas, 1996, Thakor, 1996, Holmström and Tirole, 1997, and Diamond and Rajan, 2000). The analyses in Thakor (1996) and Holmström and Tirole (1997) are most consistent with the findings of the empirical literature. Yet, a proper investigation of these issues necessitates a genuinely dynamic model with endogenous capital constraints on lenders.<sup>5</sup>

This paper aims at incorporating all these empirical facts into a theoretical model in order to study the nature of long-term liquidity provision between lenders and borrowers in the absence of perfect enforceability and when both parties are credit constrained. To this end, I build an infinite horizon model of long-term lending and borrowing and analyze in what ways liquidity shortages on both sides affect the evolution of the economy and investment activity in particular.

An infinitely-lived, risk neutral borrower (firm or entrepreneur) repeatedly has projects every period with some probability.<sup>6</sup> The projects require a lump-sum investment and have positive expected net present value. The borrower's periodic revenue is not sufficient hence he needs external financing. I assume limited liability on the part of the borrower, thus net payments are nonnegative. All these factored in generate a *demand for liquidity* for the borrower. This demand is not always matched by an associated supply of credit since the contracts are not perfectly enforceable: The

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<sup>4</sup>Most of these studies have been conducted for US data inquiring into whether implementation of the 1998 Basel accords' capital standards caused a 'credit crunch' in the US. Sharpe Sharpe (1995) finds that empirical evidence suggests that loan losses have a negative and bank profitability has a positive effect on loan growth.

<sup>5</sup>We quote (emphasis ours) from (Holmström and Tirole, 1997, p.690): "*Limited intermediary capital* is a necessary ingredient in the study of credit crunches and cyclical solvency ratios." They also mention that a proper investigation of these issues requires endogenous intermediary capital and an explicitly dynamic model.

<sup>6</sup>This paper is not concerned with consumption smoothing. We are focusing on the asset side of the lenders' balance sheets to study liquidity provision to borrowers for productive purposes. For a classical treatment of the liquidity provision role of banks for consumption smoothing purposes, see Diamond and Dybvig (1983). For an analysis of intertemporal smoothing by long-lived intermediaries, see Allen and Gale (1997).

borrower can renege on and run away with the return on investment. As a consequence, the lender offers incentive-compatible contracts and the borrower is credit constrained.

The lender is infinitely-lived, risk neutral, and provides the borrower with **liquidity**. Although, it might mean many different things depending on the environment (see e.g. Brunnermeier and Pedersen, 2009, Holmström and Tirole, 2011, Nikolau, 2009, and von Thadden, 2004 ), definition of liquidity for this paper is the availability of credit for productive purposes whenever there is demand for it. The lender can commit to contractual agreements and has a storage technology (interest rate on deposits) that makes it possible for the entrepreneur to accumulate wealth through the lender.<sup>7</sup> Contrary to the common practice, we assume that lenders are credit-constrained. This is to capture in a simple way the fact that lenders also face financial frictions in raising funds (see e.g. Kashyap and Stein (2000)): For example, they might face liquidity shortfalls due to other financial commitments. Financial collateral also plays an important role in our framework. When a borrower defaults, the lender can only recoup up to the collateral that the borrower keeps with the lender.

A **dynamic liquidity contract** is a mechanism that specifies transfers to and payments from the lender as a function of the entire history of the relationship. I first study, in Section 3, optimal contracts in the absence of enforceability problems. In the absence of default, optimal accumulation and investment decisions are independent of the surplus sharing rule. Optimal contracts trace a strictly decreasing Pareto frontier between the value to the borrower and the value to the lender. A reduction in the value to the lender is an equivalent increase in the value to the borrower; there is no loss of value because there is no possibility of default. However, it might happen that for some capital levels, the resource constraint for the lender binds, and projects will be passed up (although they have positive net present value); so there will be **credit rationing**. Nevertheless, if the agents are patient enough, the economy accumulates capital by saving and starts investing again in finite time, with probability one.

First-best savings are bounded from above: Since both parties are risk neutral, saving happens only for future productive possibilities. This is the first-order effect. There is also the second-order effect: A sequence of unlucky draws can lead to drying up of capital. Thus, agents would like to save to insure themselves against a capital depletion. However, as the capital levels increase, the utility cost of saving one more dollar outweighs the gain from insurance, leading to savings being bounded. Clearly, relatively more patient agents save more. Finally, first-best investment rule is

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<sup>7</sup>There is no other way for the borrower to save. This is not restrictive in the sense that it is the simplest way to capture the idea that the return on deposits with the lender is higher than the return on the borrower's storage technology (self-insurance).

monotonic in the level of capital.

A second-best contract is an incentive-compatible optimal contract. The worst punishment that can be inflicted upon the borrower in case of default is exclusion from the credit markets. When it happens, the borrower is left to consume what he expropriated plus his stream of future endowment. This is a rather standard assumption in the literature on dynamic contracts (see [Albuquerque and Hopenhayn \(2004\)](#), [Alvarez and Jermann \(2000\)](#), [Atkeson \(1991\)](#), [Kehoe and Levine \(1993\)](#), and [Thomas and Worrall \(1994\)](#)).<sup>8</sup>

I then characterize properties of second-best contracts. Enforcement problems and endogenous resource constraints severely reduce the possibility of financing projects. Investment and savings are functions both of the level of resources and the surplus sharing rule, in contrast to the first-best contracts. Investment is (weakly) under-provided. Second-best savings are (weakly) less than first-best savings capturing the intuition that it does not pay off to postpone consumption/investment to self-insure against a possible ‘**credit crunch**’. When agents are relatively patient, they exhibit investment cycles. For any initial capital level, the economy falls into a ‘credit crunch’ in finite time, almost surely. With patient agents, this is not an absorbing state. The economy recovers in the long-run, the capital levels are restored and investment activity goes back to normal, the lag between a phase of investment and a phase of no-investment gets longer. In economies with relatively impatient agents, the system gets stuck in that phase and we observe constant stagnation.

If the borrower collateral is too small, the lender might find it too costly to prevent default. Thus, although the projects have positive net present value, credit might be rationed. This does not lead to the collapse of the relationship though. As time evolves, the share of the borrower increases (on average) and credit is extended for the same level of lender capital where initially credit was rationed. That is because over time borrower rebuilds collateral and holds it with the lender as a guarantee.

The theoretical structure in the current paper is related to and/or builds upon the literature on dynamic and relational contracts. See for e.g., [Albuquerque and Hopenhayn \(2004\)](#), [Alvarez and Jermann \(2000\)](#), [Atkeson \(1991\)](#), [Cooley, Marimon, and Quadrini \(2004\)](#), [Hopenhayn and Werning \(2008\)](#), [Kovrijnykh \(2010\)](#), [Kehoe and Levine \(1993\)](#), [MacLeod and Malcomson \(1989\)](#), [Ray \(2002\)](#), and [Thomas and Worrall \(2010\)](#). The paper that comes closest to the current one is [Thomas and Worrall \(1994\)](#) although there are important distinctions. First of all, theoretically, the current one is a dynamic game whereas theirs is a repeated game. More importantly, I am interested in endogenous credit constraints on the part of the lender alongside the by-now more standard enforceability problems. To the best of my knowledge, the current paper is the first full-blown

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<sup>8</sup>For exceptions see e.g. [Cooley, Marimon, and Quadrini \(2004\)](#) and [Phelan Phelan \(1995\)](#).

infinite-horizon dynamic contracting model that allows for “shallow-pocketed” lenders.

[Bernanke and Gertler \(1989\)](#) study an OLG model of business cycle dynamics where borrowers’ balance sheet positions play an important role. They show that agency costs associated with the undertaking of physical investment are decreasing in the borrower’s net worth, and that this results in the emergence of *accelerator* effects on investment. Strengthened balance sheets of borrowers during good times in turn expand investment demand which tends to amplify the boom; weakened balance sheets during bad times work in the opposite direction. The same kind of accelerator effect is exhibited by the dynamics of the second-best contracts in the present paper. However, the present paper is a full-fledged infinite horizon model of borrowing and lending, internalizing the gains from long-term relationships. Moreover, in the current paper, unlike [Bernanke and Gertler \(1989\)](#), cycles can be efficient; i.e., we can observe investment cycles not because of agency costs but because lender’s capital is scarce and the system enters a ‘capital crunch’.

This paper also contributes a technical analysis of the existence and characterization of optimal dynamic contracts that are solutions to a non-convex dynamic programming problem. Indivisibility of the projects along with credit constraints make the set of feasible contracts a non-convex set. The resulting value functions exhibit discontinuities. The standard methods using concave programming and/or (super)differentiability of the value functions are not of help. I use a direct strategy and exploit monotonicity of the resulting operators. The problem of the value function entering the constraint set was introduced in [Thomas and Worrall \(1994\)](#). In our case, the problem is exacerbated by the fact that both the borrower and the lender have limited liability constraints and the continuation values should be nonnegative for both.<sup>9</sup> My task is further complicated since, in game theoretical terms, the problem in their paper is a repeated game whereas the current problem is a dynamic game with an unbounded state space.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 analyzes the efficient contracts in the absence of enforceability problems. Section 4 provides full characterization of second-best dynamic liquidity contracts and studies their dynamic properties. Section 5 presents a summary and conclusions along with possible future research. All proofs and technical results are collected in the Appendix.

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<sup>9</sup>[Thomas and Worrall \(1994\)](#) show that the resulting dynamic program is not a standard concave programming problem and the operator is not a contraction mapping in the supremum metric, despite the presence of strict discounting. The technical reason is the presence of the value function in the constraints.

## 2 The Model

Time is infinite and is indexed by  $t = 0, 1, \dots$ . There are two agents, a borrower (B) and a lender (L), both infinitely lived. The borrower receives a deterministic endowment of  $Y > 0$  units of the only consumption good in the economy, every period. Each period, with some probability  $p \in (0, 1)$ , he has a project that needs to be implemented within that period. Investment requires  $I > Y$  units of the consumption good and generates a verifiable, financial return of  $D > I$  units, in the same period, with probability  $q$ , and 0 units with probability  $(1 - q)$ . The net present value of the project is positive, i.e.,  $qD - I > 0$ . Let  $\theta_t$  be the random variable that takes the value 1 if the agent has a project at time  $t$  (*liquidity shock*), 0 otherwise. Similarly, let  $\mu_t$  be the random variable that takes the value (conditional on investment) 1 if the project is a success and 0 if it is a failure. Figure 1 depicts the timing of actions within period  $t$ .

Let  $H^t \equiv \{h^t = (\theta_0, \mu_0, \dots, \theta_t, \mu_t)\}$  be the set of  $t$ -period histories of past realizations of the stochastic process  $(\theta_t, \mu_t)$ , for  $t = 0, 1, \dots$ . This brings us to the following

**Definition 1** A *dynamic liquidity contract*  $\sigma = (\sigma_t) = (c_t, m_t, I_t, S_{t+1})$  is a vector of stochastic plans such that after any history  $h^{t-1}$ ,  $I_t(h^{t-1}, \theta_t)$  is the level of investment,  $c_t(h^{t-1}, \theta_t, \mu_t)$  is the borrower's suggested compensation,  $m_t(h^{t-1}, \theta_t, \mu_t)$  is net payments to the lender, and finally  $S_{t+1}(h^{t-1}, \theta_t, \mu_t)$  is the amount of capital transferred to the next period.

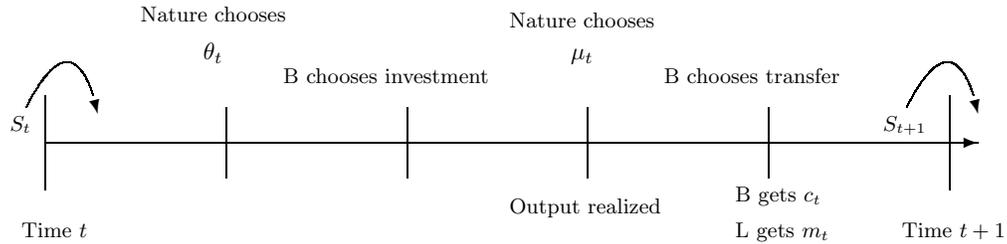


Figure 1: Timing of Decisions

The borrower is risk neutral and ranks allocations according to their consumption sequences,  $c = (c_t)$

$$U^B(\sigma) = (1 - \beta) \sum_{t=0}^{\infty} \beta^t c_t.$$

The lender is risk neutral and ranks allocations with respect to their sequences of net payments,

$$m = (m_t)$$

$$U^L(\sigma) = (1 - \beta) \sum_{t=0}^{\infty} \beta^t m_t.$$

and  $\beta \in (0, 1)$  is the common discount factor. I follow the common practice in the repeated games literature and normalize the utility levels to make them comparable to period utilities.

**Assumption 1 (One-sided Strategic Default)** *The lender honors his promises whereas the borrower might renege on the current contract at any time. If the borrower chooses to default, he is excluded from the credit markets forever.*

The borrower cannot store goods unless he saves by keeping an account with the lender, who has a storage technology that returns one unit next period for every unit stored in the current one.<sup>10</sup> The endowment stream of the borrower guarantees him  $Y$  every period. Thus, the autarkic level of a borrower who does not enter into a long-term contract is defined as  $v_{aut}^B = Y$ . The lender has an initial capital level of  $S_0 \geq 0$  units.

### 3 First-Best Liquidity Contracts

In this section, I solve for efficient contracts, without default. They constitute the benchmark case relative to which the welfare costs of allowing strategic default are evaluated. Assuming that the planner has the same information that the agents have and that there are no incentive problems, any feasible contract should satisfy  $\forall t, \forall h^t$

$$\begin{aligned} S_{t+1}(h^t) \leq & S_t(h^{t-1}) + Y - m_t(h^t) - c_t(h^t) \\ & + D \mathbf{1}_{\{\theta_t=1, \mu_t=1 \text{ and } I_t(h^{t-1}, \theta_t) \geq I\}} - I_t(h^{t-1}, \theta_t) \end{aligned} \quad (1)$$

which is an **aggregate feasibility constraint**.

The idea that liquidity might be limited is captured by the following two constraints: The **liquidity constraints** (or equivalently resource or capital constraints), i.e., for any period  $t$ , and for any history  $h^t$ ,

$$I_t(h^{t-1}, \theta_t) \leq S_t(h^{t-1}) + Y \quad (2)$$

and the **limited liability constraints** (nonnegative net payments)

$$c_t, m_t, I_t, S_{t+1} \geq 0 \quad (3)$$

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<sup>10</sup>This is just a normalization. As long as the rate that the bank pays for deposits is lower than the rate it can get for its funds at the market  $r$  where  $\beta = 1/1+r$ , results are not affected.

Technically, these last two constraints guarantee that the utility possibility set is bounded from below and above, for a given initial capital level of  $S_0$ . Figure 2 shows the possible contingencies that can be generated by a contract at period  $t$ , conditional on the availability of capital to implement a project. The possible scarcity of liquidity and the presence of limited liability are the reasons why

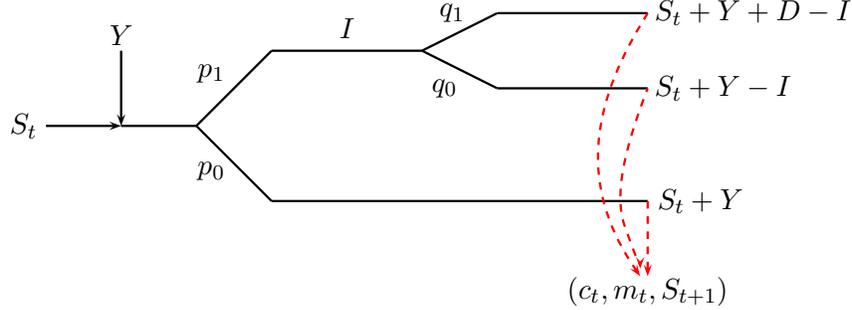


Figure 2: Decision Tree

we obtain interesting credit cycles. A dynamic liquidity contract is said to be **admissible** if it is an element of the following set with initial stock level  $S$ ,

**Definition 2** Let  $\Sigma^{FB}(S) = \{\sigma = (c_t, m_t, I_t, S_{t+1}) \text{ that satisfy (1)-(3), with } S_0 = S\}$

Note that  $\Sigma^{FB}(S)$  is not convex due to the non-convexity of the production technology. Technically, this is due to the presence of the indicator function on the right hand side of (1). Although this complicates the analysis, the utility possibility set for each agent is nevertheless a convex interval, which makes the problem manageable in the space of utilities.

With this machinery at hand, I can characterize the set of **efficient contracts** as solutions to the following parameterized family of problems

$$P(v, S) = \max_{\{\sigma \in \Sigma^{FB}(S)\}} \{EU^L(\sigma) \mid EU^B(\sigma) \geq v\} \quad (4)$$

where  $v$  is feasible in the sense that there exists a feasible contract  $\sigma$  that gives the borrower an ex-ante present discounted utility of at least  $v$ . An efficient contract has to be efficient after any history. Otherwise, it would be possible to replace the continuation contract after some history with a Pareto-improving one. This would make the original contract inefficient at time  $t = 0$ , since all histories are reached with positive probability. This property of an efficient contract permits me

to write (4) as a **Recursive First-Best** problem (RFB). It follows that

$$\text{(RFB)} \quad P(v, S) = \max_{(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_{\theta}, v_{\theta\mu}) \in \mathbb{R}_+^{18}} \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)m_{\theta\mu} + \beta P(v_{\theta\mu}, S_{\theta\mu})] \quad (5)$$

$$\text{s.t.} \quad \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu}] \geq v \quad (6)$$

$$\text{and } \forall \theta, \forall \mu : \quad S_{\theta\mu} \leq S + Y + D1_{\{\theta=1, \mu=1, I_{\theta} \geq I\}} - I_{\theta} - m_{\theta\mu} - c_{\theta\mu} \quad (7)$$

$$I_{\theta} \leq S + Y \quad (8)$$

$$v_{\theta\mu} \in [0, \bar{v}_{S_{\theta\mu}}]. \quad (9)$$

where we use  $p_1 = 1 - p_0 = p$  and  $q_1 = 1 - q_0 = q$ . The random variables  $\theta$  and  $\mu$  have the same support and distribution as  $\theta_t$  and  $\mu_t$ , respectively, for any  $t$ . The constraint in (6) is a *promise-keeping constraint* that guarantees the borrower utility  $v$  (*average cash flow*), on average. The one in (9) makes sure that the continuation values offered to the borrower are feasible. (7) and (8) are the recursive versions of (1) and (2), respectively.<sup>11</sup>

**Proposition 1 (Existence, Independence and Continuity of Efficient Contracts)** *An optimal first-best dynamic liquidity contract exists and has the following properties*

- (i) *Given any initial capital level  $S \geq 0$ , the optimal investment and saving policies  $(v, S) \rightarrow I_{\theta}(v, S)$  and  $(v, S) \rightarrow S_{\theta\mu}(v, S)$  are **independent of  $v$** , i.e.,  $\forall \theta, \forall \mu$ , and  $\forall v, v'$  feasible,  $I_{\theta}(v, S) = I_{\theta}(v', S)$ ; similarly for  $S_{\theta\mu}$ .*
- (ii) *The Pareto Frontier is characterized by  $P(v, S) = \bar{v}_S - v$  with  $v \in [0, \bar{v}_S]$ , where  $\bar{v}_S$  is the highest possible surplus in a feasible contract, given  $S \geq 0$ . Moreover,  $P$  is **strictly increasing in  $S$**  and **strictly decreasing in  $v$** .*

Note that the existence result in Proposition 1 does not mention uniqueness. This is because the time path of transfers is not uniquely determined as both parties are risk neutral and have the same discount factor. First part of Proposition 1 states that in the absence of default, the way the surplus is shared does not affect investment and saving plans. The intuition is straightforward: If no party has the power to renege on the contract, what matters is to maximize the surplus to be shared first, then to split it according to a predetermined sharing rule,  $v$ . This brings us to the second part of Proposition 1: A one unit reduction in  $v$  leads to a one unit reduction in  $P$ , since

<sup>11</sup>We show in the Appendix the equivalence of these two programs and the existence of the value function as stated in the following proposition, along with some characterization results.

the optimal investment and saving policies are not affected from this change and that both agents are risk neutral. We characterize next the recursive behavior of efficient saving and investment policies.

**Proposition 2 (Uniform Bounds on Efficient Savings and Investment)** *Given the values for  $Y, I, D, p, q, \beta$ , there exist two threshold levels of capital,  $\underline{S}$  ( $\geq I - Y$ ) and  $\bar{S}$ , such that*

(i)  $S_{\theta\mu}(v, S) \leq \bar{S}$  for all feasible  $(v, S)$  and all  $(\theta, \mu)$ .

(ii) The optimal investment rule is to ‘invest’ if and only if  $S \geq \underline{S}$  and  $\theta = 1$ .

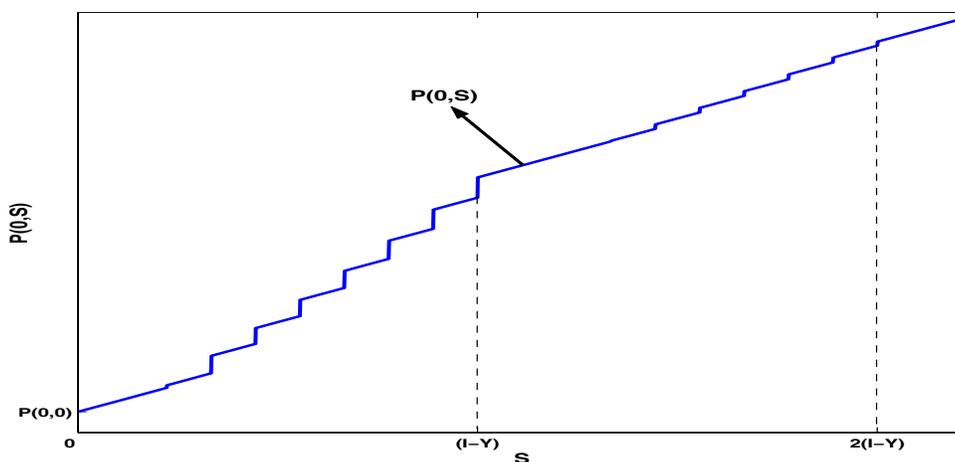


Figure 3: Total Surplus as a Function of Capital

First part of proposition 2 says that, efficient savings are uniformly bounded from above independently of the current state of the relationship. That is because saving is costly ( $\beta^{-1} < 1$ ) and it makes sense to save only if it increases the future value of the partnership significantly (manifested as a jump in the value function; see Figure 3). The intensity of those jumps decreases as  $S$  increases, since they become higher-order. As  $S$  gets larger, saving eventually ceases to be worthwhile since the per unit gain from saving does not cover per unit cost of foregone current consumption (think dividends) anymore. The second part states that the optimal investment rule is a threshold strategy. If the contract dictates investment for some level of capital, it does so for higher levels of capital too. Remember that the first-best optimal policies are independent of  $v$ , hence the results in the proposition are independent of it too.

The next Proposition summarizes our initial idea of investment cycles generated by the liquidity constraints on the lender. These are efficient cycles in the sense that they cannot be Pareto improved upon by budget-balanced interventions.

**Proposition 3 (Optimal Cycles)** *Given any economy, **investment cycles** are observed almost surely, for economies with relatively patient agents. For low discount rates, the economy gets stuck in a ‘**credit crunch**’ region with probability one, in finite time. Conditional on productive investment being undertaken, the expected number of periods it takes the economy to move into a no-investment state is an increasing function of the discount factor  $\beta$  and of the probability of productive investment  $q$ .*

The economic intuition is clear: for relatively patient parties, even if the joint capital level is not sufficient to undertake positive NPV projects, the relationship does not collapse. The value of the possibility of undertaking projects in the future is high enough to make the parties abide by the contractual terms, build up capital to invest in the future. For relatively impatient parties, that all works in the opposite direction. Once the economy gets into the ‘credit crunch’ region, it stagnates there forever. There is no production in the economy anymore because the time value of continuation by accumulating resources is too low compared to immediate consumption.

This result is important because the cycle argument does not necessitate any sort of agency costs and/or inefficiency. These cycles are *efficient cycles*. There is no room for an authority to intervene to improve upon the current allocation. We will see in section 4 that the likelihood of these cycles increases with the introduction of agency costs due to the imperfect enforceability of contracts.

### 3.1 Examples of Efficient Dynamics

The following will be our working example in this and the next section. The economy considered is a special case of the general economy outlined above. Propositions 1 and 4 apply and I will give a more explicit characterization of the optimal contract and use that to stress the important aspects stemming from the incentive compatibility and the resource constraint. I first present the full characterization of the first-best contracts for two classes of economies with relatively low discount factors.

**Example 1** *Given the values for  $Y, I, D, p, q$ , there exist two threshold levels of the discount factor,  $\beta_1$  and  $\beta_2$ , where  $0 < \beta_1 < \beta_2 < 1$  such that the first-best contracts have the following properties*

1. *For any  $\beta \in [0, \beta_1)$ , a first-best contract exists and is characterized by:*

- $I_1(v, S) = I$  iff  $S \geq (I - Y)$ .
- Do never save.

2. For any  $\beta \in [\beta_1, \beta_2)$ , a first-best contract exists and is characterized by:

- $I_1(v, S) = I$  iff  $S \geq (I - Y)$ .
- Save  $(I - Y)$  if feasible, otherwise save 0

Notice that  $v$  is not part of the characterization since from Proposition 1, the optimal investment and saving decisions are independent of the promise level. Moreover, consumption paths are not part of the characterization either, due to the risk neutrality and the same time preferences of the agents.

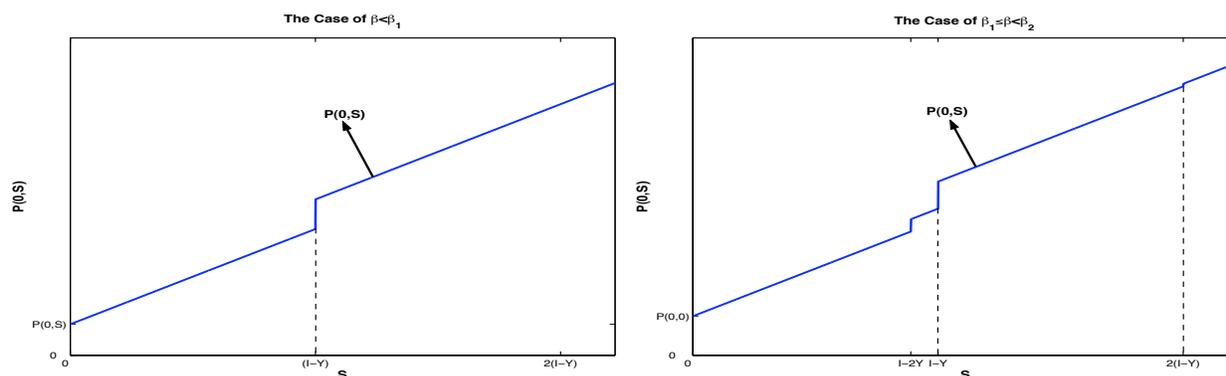


Figure 4: Maximum Surplus for Two Classes of Discount Factors

First part of the example is for economies with extremely impatient agents. The continuation value of the partnership is too small for these agents to save at all. The threshold discount factor  $\beta_1 = \frac{(I - Y)}{(I - Y) + p_1(q_1 D - I)}$ , as we show it in the proof, is the one that makes the agents indifferent between saving and not saving. The problem, de facto, becomes one of one-period project funding. Since investing is socially optimal, barring any incentive problems, investment is undertaken whenever it is feasible to do so. A close look at the threshold discount factor provides further insights.  $\beta_1$  is an increasing function of  $I$  and a decreasing function of  $p_1, q_1, D$  and  $Y$ . The intuition is clear: as the probability of project arrival and the probability of success of investments increase, the marginal types start saving. That's because the continuation value of the partnership increases. Similarly for  $D$ . If  $Y$  increases, the number of periods that the agents should accumulate before they start

investing decreases, which makes it worthwhile for some non-savers to start saving. A decrease in the value of  $I$  works exactly in the same direction. Hence, the set of types (discount rates) who will save becomes larger.

Second part is for agents who are “just patient” enough to save for the next period the amount  $(I - Y)$  that will make it feasible to invest in case of a productive shock. The added value of saving more than this amount to self-insure for more than one period, is not enough to compensate for the utility cost incurred. Similarly, it is not worth building the necessary stock to be able to invest in case of a liquidity shock in the future, if the initial resources are too small. The same comparative statics exercise that we did above for  $\beta_1$  can be undertaken for  $\beta_2$  and shown that  $\beta_2$  behaves the same way.<sup>12</sup>

We now ask what happens in the two classes of economies in the longer run. For that purpose, let the following be the set of states for the Markov aggregate system of our economy, generated by the optimal investment and saving rules.

$$\mathcal{S} \equiv \{S^* \mid S^* \text{ is the optimal saving level for some level of end-of-period wealth}\}$$

Optimal savings are at the discontinuity points of the value function  $P$ . This is because saving is relatively costly (or rate of return on deposits at the bank is smaller than the market interest rate) and if savings are at a continuity point of  $P$ , there is always the temptation of cutting them down since the gain from saving one unit less is  $(1 - \beta) > \beta(1 - \beta)$ , the cost of continuing with one unit less ( $P$  has constant slope  $(1 - \beta)$  at continuity points). Hence, in the case of the first class of economies,  $\mathcal{S} = \{0\}$ , since saving zero is the optimal strategy for any  $S$ . This is an ‘absorbing state’ and the economy will be in that state forever at the period-ends, from second period on. From the second period on, no investment projects will be undertaken. Capital will be depleted and the economy will be in a constant state of stagnation.

For the second class of economies, the transitions are a bit more interesting. In this case,  $\mathcal{S} = \{0, (I - Y)\}$ . The Markov transition matrix,  $R$ , can be computed easily, by referring to Figure 2.

$$R \equiv \begin{bmatrix} 1 & 0 \\ p_1 q_0 & 1 - p_1 q_0 \end{bmatrix}$$

Let the first row denote state 0 and the second row represent state  $(I - Y)$ . For example, the probability of moving from state  $(I - Y)$  to state 0 is given by  $R_{21} = p_1 q_0$ . The probability of

<sup>12</sup>Although we don’t have a clear explicit form for  $\beta_2$ , a look at Figure 7 (in the Appendix) reveals that an increase in  $p_1, q_1, D$  and  $Y$  pulls up the intersection of the function with the vertical axis,  $C$ , whereas a decrease in  $I$  takes the function down by taking its value at  $\beta_1 = -F$ , down. Both these movements make the intersection of the function with the horizontal axis,  $\beta_2$ , move to the left.

ending up in the absorbing state in finite time is

$$1 - \lim_{n \rightarrow \infty} (p_1 q_0)^n = 1$$

since it is simply the complement of the event ‘always in state  $(I - Y)$ ’. Once again, the dynamics are simple. From the second period on, if the capital level is sufficient to implement projects, the economy stays in state  $(I - Y)$ , for some time, with positive probability. This is a very fragile state since one bad shock (project failure) is sufficient to move the economy into the state of capital crunch where it stays forever. The lender’s resource constraint binds; lender cannot provide the borrower with any liquidity since he has no funds available. Positive net present value projects are passed up. The interesting feature is that these are ‘efficient dynamics’.

## 4 Optimal Dynamic Liquidity Contracts with Strategic Default

In this section, I study equilibrium contracts when the borrower can no longer commit. He has the opportunity to renege on the agreement after the investment is undertaken and can run away with the return on investment that he confiscates. Remember that the borrower is excluded from the credit markets forever in case he defaults. The lender commits to the terms of the contract as long as his participation constraint is satisfied at  $t = 0$ . A second-best liquidity contract, then, analogously to the first-best, is said to be **feasible** if it is an element of the following set with initial capital level  $S$ ,<sup>13</sup>

**Definition 3** Let  $\Sigma^{SB}(S) = \{\sigma = (c_t, m_t, I_t, S_{t+1})$  that satisfy (1)-(3), with  $S_0 = S$  and an *incentive compatibility constraint (IC)*, i.e.,  $\forall t, \forall h^t$ ,

$$(1 - \beta) c_t(h^t) + \beta EU^B(\sigma | h^t) \geq (1 - \beta) D 1_{\{\theta_t=1, \mu_t=1 \text{ and } I_t(h^{t-1}, \theta_t) \geq I\}} + \beta Y$$

Taking these constraints into account, second-best contracts will be the solutions to the following program

$$Q(v, S) = \max_{\sigma \in \Sigma^{SB}(S)} \{EU^L(\sigma) | EU^B(\sigma) \geq v\} \quad (10)$$

where  $v$  is feasible in the sense that there exists a feasible contract  $\sigma$  that yields the borrower at least  $v$  ex-ante. Notice that the above constraints are “best deviation” constraints for the borrower.

<sup>13</sup>Alternatively, one can divide the joint capital level  $S$  into  $S^L$  and  $S^B$  and track them separately. This complicates the computations and the presentation with no apparent contribution to economic insight. In such a treatment, the borrower would keep  $S^B$  with the bank since he has no other way of storing and the bank can confiscate  $S^B$  in case the borrower defaults. This division argument might yield different results in the presence of competing multiple lenders and access to credit markets, which are outside the scope of the current paper.

In general, the borrower might deviate from what the allocation prescribes in many different ways; none of these yields him a higher payoff than does the best deviation strategy. Thus, if a contract satisfies the best deviation constraints, it will be incentive compatible.

The program above can be written recursively, using  $v$  and  $S$  as state variables. These two are “sufficient statistics” providing the necessary information required to solve for the second-best contract. Hence, the recursive second-best program (RSB) is

$$\begin{aligned}
\text{(RSB)} \quad Q(v, S) &= \max_{(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_{\theta}, v_{\theta\mu}) \in \mathbb{R}_+^{18}} \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)m_{\theta\mu} + \beta Q(v_{\theta\mu}, S_{\theta\mu})] \\
\text{s.t.} \quad &\sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu}] \geq v & (11) \\
&S_{\theta\mu} \leq S + Y + D \mathbf{1}_{\{\theta=1, \mu=1, I_{\theta} \geq I\}} - I_{\theta} - m_{\theta\mu} - c_{\theta\mu} & (12) \\
&I_{\theta} \leq S + Y & (13) \\
&(1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \geq (1 - \beta)D \mathbf{1}_{\{\theta=1, \mu=1, I_{\theta} \geq I\}} + \beta Y & (14) \\
&v_{\theta\mu} \in [Y, \bar{v}_{S_{\theta\mu}}]. & (15)
\end{aligned}$$

The extra constraint (14) guarantees that the borrower gets a utility level at least as high as what he gets by defaulting, on the paths where investment is undertaken and it succeeds. On other paths, (14) is implied by (15). The next result is at the core of our analysis. It fully characterizes optimal liquidity contracts while at the same time making sure that at least one exists.

**Proposition 4** (i) *An optimal second-best contract exists.*

(ii) *There exists an  $\underline{S} \geq (I - Y)$  such that for all  $S \geq \underline{S}$ , there are two promise values  $0 \leq v_*(S) \leq v^*(S) \leq \bar{v}_S$  with*

(a)  $I_1(v, S) = 0$ , for  $v \in [Y, v_*(S)]$ ,

(b)  $I_1(v, S) = I$ , for  $v \in [v_*(S), \bar{v}_S]$

(c) *The value function in (RSB) is given by*

$$Q(v, S) = \begin{cases} \bar{v}_S - v & \text{if } v \geq v^*(S) \\ \bar{v}_S - v^*(S) & \text{if } v \in [v_*(S), v^*(S)] \\ \bar{v}_S - [v^*(S) - v_*(S)] - v & \text{if } v \in [Y, v_*(S)] \end{cases}$$

(d)  $v^*(S)$  is nondecreasing in  $S$ .

The statements in Proposition 4, which are also depicted in Figure 5, carry a nice economic intuition. For each level of  $v$  and  $S$ , the number of ways the incentives can affect the first-best utility levels is two. What distinguishes the graph on the left from the one on the right is the fact that, first-best optimal policies are also second-best optimal on the left. On the right, below a level of  $v$ , investment is not undertaken under the second-best rule although it is under the first-best rule. Notice that, both these cases refer to the first-best optimal ‘Invest’ regime. When the first-best rule is not to invest in the current period, there is no distributional conflicts arising from incentive compatibility; hence first-best and second-best utility levels coincide, i.e.,  $Q(v, S) = P(v, S)$  (see Figure 6).

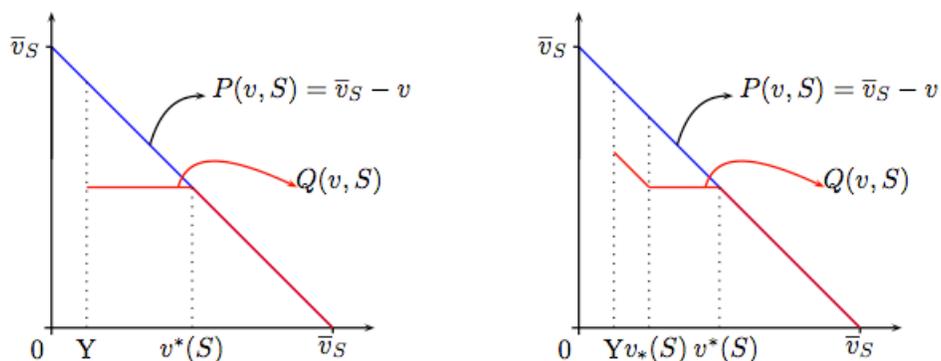


Figure 5: Second-Best Frontier for  $S \geq \underline{S}$ .

**Proposition 5** (i) *Second-best savings are less than or equal to the first-best savings.*  
(ii) *The optimal investment rule,  $(v, S) \rightarrow I_\theta(v, S)$ , is a nondecreasing function of  $v$ .*

A few remarks are in order. Proposition 5-(i) captures the intuition that it does not pay off to over-save relative to the first-best level to self-insure against the possibility of ending up in a “**credit crunch**” regime, where there is not enough capital to undertake projects in case of a liquidity shock. Combined with Proposition 2-(i), Proposition 5 implies that the optimal second-best savings also are bounded from above.

For each  $S$ ,  $v^*(S)$  is the minimum promise level below which (14) holds with strict inequality. The reason is first that the contract has to make sure that the borrower doesn’t default. Hence it needs to provide him with at least the default utility, on the success realization path. Second, the

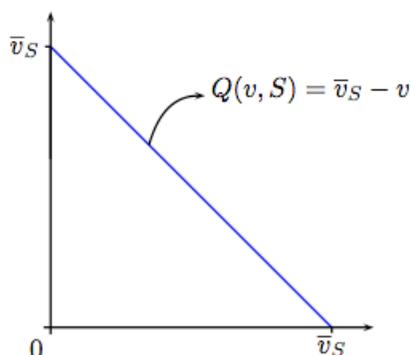


Figure 6: Second-Best Frontier for  $S < \underline{S}$ .

continuation promise levels must satisfy the borrower’s individual rationality constraint. Hence, even if the initial level of  $v$  is below  $v^*(S)$ , the de facto payment, on average is equal to  $v^*(S)$ . For some parameterizations, there is another threshold level  $v_*(S)$  below which “**rationing the credit**” is optimal although it is socially optimal to undertake the investment in the absence of incentive problems. The idea is that, if investment is undertaken, the cost of making sure that the borrower does not default is so high that the lender prefers not extending the credit although first-best requires him to do so.

The fact that first-best and second-best investment and saving decisions might not agree lead to the following important conclusion: *investment cycles might become further skewed toward states of liquidity drought*. The reason is straightforward. The only reason agents save in this economy is to make sure that productive investment opportunities do not go unimplemented. The lack of enforcement diminishes the future value of the partnership hence making it less appealing to save for the future. Consequently, it becomes more likely that the system ends up in the ‘liquidity drought’ region. In a similar fashion, low collateral (proxied by  $v$ ) on the part of the borrower can make it extremely costly to keep the borrower from renegeing, which in turn leads to credit being rationed; this is another channel through which the flow of new capital into the system is interrupted which in turn makes the possibility of liquidity drought more likely.

As we did for the first-best contracts, we now turn to two classes of economies where we can solve for the contracts in closed form and shed light on the second-best dynamics.

## 4.1 Second-Best Dynamics

Here, we take it from where we left in the previous section's Example 1 and analyze the behavior of the second-best contracts and compare it to the benchmark case of first-best contracts. The explicit characterization of the former for both economies makes it clear what kind of distortions the enforceability problems cause to the socially optimal allocations and utility levels.

**Example 2** 1. For any  $\beta \in (0, \beta_1)$ , a stationary second-best contract exists and is characterized by:

(a) For  $S \geq I - Y$ , there exist  $0 \leq v_*(S) \leq v^*(S) \leq \bar{v}_S$  such that

i.  $I_1(v, S) = 0$ , for  $v \in [Y, v_*(S)]$ ,

ii.  $I_1(v, S) = I$ , for  $v \in [v_*(S), \bar{v}_S]$

(b) Do never save

(c) We have the figure on the right iff  $p_1 > \frac{Y}{I}$ .

2. For any  $\beta \in [\beta_1, \beta_2)$ , a stationary second-best contract exists and is characterized by:

(a) For  $S \geq I - Y$ , there exist  $0 \leq v_*(S) \leq v^*(S) \leq \bar{v}_S$  such that

i.  $I_1(v, S) = 0$ , for  $v \in [Y, v_*(S)]$ ,

ii.  $I_1(v, S) = I$ , for  $v \in [v_*(S), \bar{v}_S]$

(b) Save  $(I - Y)$  if feasible, otherwise save 0

One interesting feature of the first class of economies is the fact that we have the second-best frontier on the right hand side of Figure 5 if  $p_1 > \frac{Y}{I}$ . So, if the probability of a liquidity shock is too high, the lender does not extend credit to borrowers with little collateral,  $v \in [Y, v_*(S))$ , because it is too costly to ensure no-default in case investment is undertaken; an increase in  $p_1$  raises the weight of that state in the expected utility computation.

## 5 Conclusion

In this paper, I studied the nature of long-term liquidity provision between lenders and borrowers in the absence of perfect enforceability and when both parties are financially constrained. To this end, I built a tractable infinite horizon model of long-term lending and borrowing and analyzed in what ways liquidity shortages on both parties affect the evolution of the economy and investment activity in particular.

I show that enforcement problems and endogenous resource constraints can severely reduce the possibility of financing projects. Investment and saving decisions depend not only on the level of capital but also on the surplus sharing rule, in contrast to the first-best contracts. Investment is (weakly) under-provided. Second-best savings are (weakly) less than first-best savings capturing the intuition that it does not pay off to postpone consumption/investment to self-insure against a possible ‘credit crunch’.

I also show that the economy exhibits investment cycles of two different natures. First type of cycles happen because the lenders are credit constrained; these are efficient cycles. The second type of cycles are due to incentive compatibility. The punchline is: credit is rationed if either the lender has too little capital or the borrower has too little financial collateral.

The present work’s technical contribution is to show the existence and characterization of financial contracts that are solutions to a non-convex dynamic programming problem.

As in any model, I left out many things. The first next step in this research program should be to build and analyze a model in which the opportunity cost (capital constraint) of lenders is endogenized by studying explicitly the credit markets as a dynamic game between lenders for loanable funds. The introduction of competition among lenders would not only enrich the story that I am telling, but also would make it possible to think in a realistic way about renegotiation issues, reputation building on the part of the lender/borrower, and endogenous opportunity costs and credit rationing due to locking-up with a current set of borrowers (or lenders).

There is a couple of other points that are of importance for future research: One should look at economies where there are both aggregate and micro level liquidity shocks and consider heterogeneity among lenders and borrowers. Heterogeneity and aggregate uncertainty introduces the possibility of intra-institutional arrangements in a dynamic setting, that the present model cannot capture. Of course, all of this is for future work.

## 6 Appendix

### *Proof of Proposition 1*

Period utility functions and the space of feasible resources are unbounded. I first show that at the optimum, the set  $\{EU^L(\sigma) \mid EU^B(\sigma) \geq v, \sigma \in \Sigma^{FB}(S)\}$  is bounded from above by the value of a relaxed program for which the supremum exists and is finite-valued. This latter defines a set,  $F$ , of feasible  $(v, S)$  pairs. I define a functional space,  $B(F)$  and an operator,  $T$ , on that functional space, which is associated with the (RFB) problem in (5). The ‘limited liability’ assumption implies that

the value function be non-negative for any feasible value of  $v$  in the first-best program. So, the value function itself enters the constraint set of the problem. This brings up the question ‘*what is the set of feasible values of  $v$* ’ for any given  $S$ . I will first prove a series of 3 Lemmas which, put together, will deliver the results of interest.

Let  $P^*(v, S)$  be the value of the supremum in the first-best problem in (4) parameterized by  $S_0 = S \geq 0$  and a feasible  $v$ . I first show that the supremum of the sequence problem exists and is attained by a contract. Then, I proceed to show that the unique fixed point of the operator  $T$ , associated with the (RFB) problem, defined on the ‘right’ space of candidate value functions, is actually  $P^*$ . Finally, I characterize the first-best frontier and first-best contracts.

**Lemma 1**  *$P^*(v, S)$  exists and is attained by an optimal contract such that  $P^*(v, S) \leq (1 - \beta)S + Y + p_1q_1(D - I)$ . In particular,  $(1 - \beta)S + Y \leq P^*(0, S) \leq (1 - \beta)S + Y + p_1q_1(D - I)$ .*

*Proof:* The following program is a relaxed version of the first-best problem in (4) with  $v = 0$ , where the economy generates  $Y$  with certainty and  $(D - I)$  extra with probability  $p_1q_1$ , every period.

$$\begin{aligned} & \sup_{(m_t, S_{t+1})_{t=0}^{\infty}} E(1 - \beta) \sum_{t=0}^{\infty} \beta^t m_t \\ \text{s.t. } \forall t, \forall h^t & \quad S_{t+1}(h^t) \leq S_t(h^{t-1}) - m_t(h^t) + Y + (D - I)1_{\{\theta_t=1, \mu_t=1\}} \\ & \quad m_t, S_{t+1} \geq 0 \text{ and } S_0 = S \geq 0 \text{ is given.} \end{aligned} \quad (16)$$

The objective is linear and the constraint set is convex. So, this is a concave programming problem. Hence, the first-order conditions are necessary and sufficient for a maximum. Partially differentiating the objective with respect to  $S_{t+1}(h^t)$  gives

$$-\text{Prob}(h^t)\beta^t(1 - \beta) + \text{Prob}(h^t)\beta^{t+1}(1 - \beta) < 0$$

which implies that  $S_{t+1}(h^t) = 0$  for all  $h^t$ . Hence, the agent consumes all his wealth at the end of each period. The supremum is achieved and by simple algebra, is equal to

$$(1 - \beta)S + Y + p_1q_1(D - I)$$

Clearly, any contract that is feasible for the original problem is also feasible for the relaxed problem. Consequently, the supremum of the original problem exists and  $P^*(0, S) \leq (1 - \beta)S + Y + p_1q_1(D - I)$ . Since the constraint set of the problem with  $v \neq 0$  is smaller than that of the problem with  $v = 0$ , we also have  $P^*(v, S) \leq P^*(0, S)$  for any  $(v, S)$  feasible. Supremum in (16) is attained by an

optimal contract because the constraint set defined by a sequence of weak inequalities is closed. Hence, the sup operator in the statement of the problem can be replaced by a max operator and an optimal contract can be computed for each  $S$  and  $v$ . One feasible strategy in the original problem is to ‘never invest’ after any history. Conditioning on ‘never investing’, the optimal strategy solves

$$\begin{aligned} & \max_{(m_t, S_{t+1})_{t=0}^{\infty}} E(1 - \beta) \sum_{t=0}^{\infty} \beta^t m_t \\ \text{s.t. } \forall t, \forall h^t & \quad S_{t+1}(h^t) \leq S_t(h^{t-1}) - m_t(h^t) + Y \\ & \quad m_t, S_{t+1} \geq 0 \text{ and } S_0 = S \geq 0 \text{ is given.} \end{aligned}$$

By the same token as above, the maximum is achieved and is equal to  $(1 - \beta)S + Y$ . Since this is a possibly suboptimal strategy, we have  $(1 - \beta)S + Y \leq P^*(0, S)$ . Gathering both sides of the inequalities together, we obtain the stated result. Q.E.D.

Let  $F$  be the set of  $(v, S)$  pairs for which a feasible contract exists; namely

$$F \equiv \{(v, S) \in \mathbb{R}_+^2 \mid v \leq P^*(0, S)\}$$

Naturally, the ‘right’ space for our problem is the space of non-negative valued functions on  $F$ , that are non-increasing in  $v$ , non-decreasing in  $S$  and bounded above by the function  $(1 - \beta)S + Y + p_1 q_1 (D - I)$ . Namely,

$$\begin{aligned} B(F) & := \{f : F \rightarrow \mathbb{R}_+ \mid \text{s.t. (i) - (iii) hold where}\} \\ & \quad \text{(i) } f(v', S) \leq f(v, S) \text{ for } v' > v \\ & \quad \text{(ii) } f(v, S) \leq f(v, S') \text{ for } S' > S \\ & \quad \text{(iii) } 0 \leq f(v, S) \leq (1 - \beta)S + Y + p_1 q_1 (D - I) \end{aligned}$$

Let  $T$  be an operator defined on  $B(F)$  such that, for any  $(v, S) \in F$  and any  $P \in B(F)$ :

$$\begin{aligned} (TP)(v, S) & = \max_{(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_{\theta}, v_{\theta\mu}) \in \mathbb{R}_+^{18}} \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)m_{\theta\mu} + \beta P(v_{\theta\mu}, S_{\theta\mu})] \\ \text{s.t. } \forall \theta, \forall \mu & \quad (v_{\theta\mu}, S_{\theta\mu}) \in F \\ & \quad S_{\theta\mu} \leq S + Y + D 1_{\{\theta=1, \mu=1, I_{\theta} \geq I\}} - I_{\theta} - m_{\theta\mu} - c_{\theta\mu} \\ & \quad I_{\theta} \leq S + Y \\ & \quad \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu}] \geq v \end{aligned} \tag{17}$$

**Lemma 2**  $T : B(F) \rightarrow B(F)$  and  $P^*$  is its unique fixed point.

*Proof:*  $(TP)(v, S) \geq 0$  for any  $(v, S) \in F$  and  $P \in B(F)$  due to limited liability and the non-negativity of  $P$ . Let  $\bar{P}(v, S) \equiv (1 - \beta)S + Y + p_1 q_1 (D - I)$  be the upper bound function for elements of  $B(F)$ .  $\bar{P}$  clearly is an element of  $B(F)$ . With  $\bar{P}$  being the continuation function, the optimal consumption/saving decision is to consume everything at the end of the period. Conditioning on that, the optimal investment strategy is to invest whenever it is feasible. But then,

$$\begin{aligned} (T\bar{P})(v, S) \leq (T\bar{P})(0, S) &= \begin{cases} (1 - \beta)S + Y + p_1 q_1 D - (1 - \beta q_0) p_1 I, & \text{if } S \geq I - Y \\ (1 - \beta)S + Y + \beta p_1 q_1 (D - I), & \text{if } S < I - Y. \end{cases} \\ &\leq \bar{P}(v, S) \end{aligned}$$

Clearly,  $T$  is monotonic in  $P$  and  $(T\bar{P})$  is non-increasing in  $v$  and nondecreasing in  $S$  which makes the latter an element of  $B(F)$ . Hence,  $0 \leq (TP)(v, S) \leq (T\bar{P})(v, S) \leq \bar{P}(v, S)$  for any  $(v, S) \in F$  and  $P \in B(F)$ . This establishes that  $T$  is a self-map.

$P^* \in B(F)$  and by standard arguments (see [Stokey and Lucas \(1989\)](#), Theorem 4.2),  $TP^* = P^*$ . The next step is to show that this is the only fixed point of the operator  $T$ . What is required is a boundedness condition,  $\lim_{n \rightarrow \infty} \beta^n P(v_n, S_n) = 0$ , where  $(v_n, S_n)_{n=0}^{\infty}$  is a particular realization path of a feasible contract from  $(S_0, v_0)$ . This condition requires the value function to be bounded on any realized path from  $(v_0, S_0)$  for a feasible contract, and is sufficient to guarantee that any  $P$  that satisfies this condition is actually the supremum function. We will show that  $\lim_{n \rightarrow \infty} \beta^n P(v_n, S_n) = 0$  for any fixed point  $P$  of  $T$ . Since there is a finite number of states of nature each period, Theorem 4.3 in [Stokey and Lucas \(1989\)](#) can be modified to argue that any fixed point  $P$  of (17) that satisfies the boundedness condition is actually the supremum function  $P^*$ .

The fastest accumulation path for the problem in (16) is  $S_1 = S_0 + Y + p_1 q_1 (D - I)$ ,  $S_2 = S_1 + Y + p_1 q_1 (D - I) = S_0 + 2(Y + p_1 q_1 (D - I))$ ,  $\dots$ ,  $S_n = S_0 + n(Y + p_1 q_1 (D - I))$ ,  $\dots$ . Clearly, it is so for the original problem, too. Hence for any realized feasible path  $(v_n^*, S_n^*)_{n=0}^{\infty}$  of the original problem, we have

$$P(v_n^*, S_n^*) \leq P(0, S_n^*) \leq \bar{P}(v_n^*, S_n^*)$$

The discounted versions respect the same ordering

$$\beta^n P(v_n^*, S_n^*) \leq \beta^n P(0, S_n^*) \leq \beta^n \bar{P}(v_n^*, S_n^*)$$

Substituting for  $\bar{P}(v_n^*, S_n^*)$ , we have

$$\beta^n \bar{P}(v_n^*, S_n^*) = \beta^n [(1 - \beta)(S_0 + n(Y + p_1 q_1 (D - I))) + Y + p_1 q_1 (D - I)]$$

which goes to zero as  $n$  goes to infinity. Therefore, so does  $\beta^n P(v_n^*, S_n^*)$ . By the argument in the previous paragraph,  $P = P^*$ . Q.E.D.

Let  $P$  be the supremum function which is also the unique fixed point of (17) by Lemma 2. Then,  $P(0, S)$  is the maximum possible surplus with initial capital level  $S$ , due to the following Lemma.

**Lemma 3** *The optimal investment and saving strategies  $(I_t, S_{t+1})_{t=0}^\infty$  depend only on  $S$ , not on  $v$ . Moreover,  $P(v, S) = P(0, S) - v$  for  $v \in [0, P(0, S)]$ .*

*Proof:* Let  $(m_t^*, I_t^*, S_{t+1}^*)_{t=0}^\infty$  be the optimal contract that achieves the supremum in the first-best with  $v = 0$  and  $S_0 = S$ . Let  $v \in [0, P(0, S)]$ . We will first construct  $(c_t^{**}, m_t^{**}, I_t^{**}, S_{t+1}^{**})_{t=0}^\infty$  where  $c_t^{**} = \alpha m_t^*$  and  $m_t^{**} = (1 - \alpha)m_t^*$ , where  $\alpha \in (0, 1)$  is s.t.

$$v = E(1 - \beta) \sum_{t=0}^{\infty} \beta^t c_t^{**} = \alpha E(1 - \beta) \sum_{t=0}^{\infty} \beta^t m_t^*$$

and  $S_{t+1}^{**} = S_{t+1}^*$ ,  $I_t^{**} = I_t^*$  as before. Clearly, this new contract is feasible and achieves the utility level  $v$  for the borrower. Now, suppose for a contradiction that the new contract is Pareto dominated by  $(c'_t, m'_t, I'_t, S'_{t+1})_{t=0}^\infty$ , the new optimizer. Therefore,

$$v = E(1 - \beta) \sum_{t=0}^{\infty} \beta^t c'_t$$

since the IR constraint for the borrower binds necessarily at the optimum

$$P(v, S) = E(1 - \beta) \sum_{t=0}^{\infty} \beta^t m'_t > E(1 - \beta) \sum_{t=0}^{\infty} \beta^t m_t^{**} = E(1 - \beta) \sum_{t=0}^{\infty} \beta^t m_t^* (1 - \alpha)$$

But this is going to imply that the original contract for  $v = 0$  could not have been optimal. Let's define the contract  $(c''_t, m''_t, I''_t, S''_{t+1})_{t=0}^\infty$  by  $c''_t(h^t) = 0$ ,  $m''_t(h^t) = (m'_t(h^t) + c'_t(h^t))$  with  $I''_t = I'_t$  and  $S''_{t+1} = S'_{t+1}$ . This is clearly feasible for the problem with  $v = 0$  and

$$\begin{aligned} P(0, S) &\geq E(1 - \beta) \sum_{t=0}^{\infty} \beta^t m''_t &= E(1 - \beta) \sum_{t=0}^{\infty} \beta^t (c'_t + m'_t) \\ &> E(1 - \beta) \sum_{t=0}^{\infty} \beta^t c'_t + E(1 - \beta) \sum_{t=0}^{\infty} \beta^t m_t^* (1 - \alpha) \\ &= \alpha P(0, S) + (1 - \alpha) P(0, S) \\ &= P(0, S) \end{aligned}$$

a contradiction. As to the second claim, the contract  $(c_t^{**}, m_t^{**}, I_t^{**}, S_{t+1}^{**})_{t=0}^{\infty}$  is shown to be feasible and achieve the supremum for the first-best program with  $v \in (0, P(0, S))$ . Hence,

$$\begin{aligned}
P(v, S) &= E(1 - \beta) \sum_{t=0}^{\infty} \beta^t m_t^{**} \\
&= E(1 - \beta) \sum_{t=0}^{\infty} \beta^t (1 - \alpha) m_t^* \\
&= (1 - \alpha) E(1 - \beta) \sum_{t=0}^{\infty} \beta^t m_t^* \\
&= P(0, S) - v
\end{aligned}$$

from the definition of  $\alpha$ .

*Q.E.D.*

Lemmas (1)-(3) put together prove Proposition 1.

*Q.E.D.*

### ***Proof of Proposition 2***

**Part (i):** Suppose for a contradiction that for any  $\bar{S}$ , there is an  $S'$  such that for some  $(v, S)$  and  $(\theta, \mu)$ ,  $S' = S_{\theta\mu}(v, S) > \bar{S}$ . Then, we can construct a nondecreasing sequence  $(S'_n)$ :

$$S'_n = S_{\theta_n \mu_n}(v_n, S_n) > n \quad , \text{ for } n \in \mathbb{N}$$

for some  $(v_n, S_n)$  feasible, and  $(\theta_n, \mu_n)$  for each  $n$ . Let  $W_n$  be the end-of-period wealth on the realized path corresponding to  $(v_n, S_n)$  and  $(\theta_n, \mu_n)$ . Clearly, this sequence is unbounded and for each consecutive terms, we have (since first-best decisions are independent of  $v$  due to Proposition 1)

$$(1 - \beta)[W_{n+1} - S'_n] + \beta P(0, S'_n) \leq (1 - \beta)[W_{n+1} - S'_{n+1}] + \beta P(0, S'_{n+1})$$

due to optimality, with at least one strict inequality. But, then this ordering is independent of the particular wealth level since we can just get rid of those from both sides of the inequality. Hence,

$$(1 - \beta)[-S_1] + \beta P(0, S_1) < (1 - \beta)[-S_{n+1}] + \beta P(0, S_{n+1})$$

for large  $n$ . This in turn implies that

$$\frac{P(0, S_{n+1}) - P(0, S_1)}{S_{n+1} - S_1} > \frac{(1 - \beta)}{\beta} > (1 - \beta)$$

Hence,  $P$  should increase, on average, with a slope larger than  $(1 - \beta)/\beta$ , greater than  $(1 - \beta)$ . This would imply that  $P$  should intersect the line  $(1 - \beta)S + Y + p_1q_1(D - I)$  eventually, which is a contradiction since that is an upper bound for  $P$  from Lemma 1. Moreover, this level  $\bar{S}$  is achieved by the same token above. For large levels of  $W$ ,  $\bar{S}$  and everything less than that will be available. Since  $P$  is bounded above and  $P$  is clearly right-continuous, there is a threshold level of end-of-period wealth after which optimal savings are  $\bar{S}$ .

**Part (ii):** The existence of  $\underline{S}$  is implied by the arguments in Part (i) above. Since savings are bounded by  $\bar{S}$ , for levels of  $S \gg \bar{S}$ , savings are identical and equal to  $\bar{S}$ , at the period ends. But, then simple algebra shows that it is optimal to invest for  $S \geq \bar{S}$ . For the rest of the proof, there are two steps involved: (1) We first show that finitely truncated dynamic programs have monotone optimal saving/investment rules. (2) Then we show that the finite truncations converge to the infinite limit which has a monotone optimal investment policy.

1. The set of feasible optimal saving levels is given by  $[0, \bar{S}]$  due to Part (i), which is a compact sublattice of  $\mathbb{R}$ . For any  $n$ , the  $n$ -period truncated version of the first-best problem would be

$$\begin{aligned} \max_{(I_t, S_{t+1})_{t=0}^n} & E(1 - \beta) \sum_{t=0}^n \beta^t [S_t(h^{t-1}) + Y + D1_{\{\theta_t=1, \mu_t=1, I_t(h^{t-1}, \theta_t) \geq I\}} \\ & - I_t(h^{t-1}, \theta_t) - S_{t+1}(h^t)] \\ & I_t, S_{t+1} \geq 0 \text{ and } S_0 = S \geq 0 \text{ is given.} \end{aligned}$$

The truncated contracts  $(I_t, S_{t+1})_{t=0}^n$  live in a  $R^{6n}$  dimensional finite lattice. The period utility function is linear in the contractual terms, hence is trivially supermodular in  $(I_t, S_{t+1})_{t=0}^n$  for each  $S$ ; and satisfies increasing differences in  $((I_t, S_{t+1})_{t=0}^n, S)$ . But then Theorem 10.7, on p.259 of Sundaram (1996) guarantees that the optimal investment rule should be nondecreasing for each  $n$ , which concludes the first part of the proof.

2. Let  $(P_n)$  be a sequence of the values of the  $n$ -period truncated problems above. Clearly, using the same operator  $T$ , as before,  $P_{n+1} = TP_n$ . Let  $(\sigma_n)$  be a sequence of optimal policies of the truncated dynamic program with the operator  $T$ . Where the optimal saving rule is not unique, pick the highest one, wlog. Standard arguments as in Theorem 3.8, on p. 64 of Stokey and Lucas (1989) show that  $P_n \rightarrow P$  and  $\sigma_n \rightarrow \sigma$  pointwise where  $\sigma$  is the optimal stationary policy function of the original dynamic program. Hence, if  $(I_{\theta_n})$  are all monotone, the limit of the sequence,  $I_\theta$  should be a monotone function, which concludes the proof.

*Q.E.D.*

***Proof of Proposition 3***

Let the following be the set of states for the Markov aggregate system of a given economy, generated by the optimal investment and saving rules.

$$\mathcal{S} \equiv \{S^* \mid S^* \text{ is the optimal saving level for some level of end-of-period resources}\}$$

Optimal savings are at the discontinuity points of the value function  $P$  as we point out in the Proof of Example 1. Moreover, we know from Proposition 2 that this set is bounded. Moreover, it is a tedious but straightforward exercise to show (as in the proof of Example 1) that the jumps happen at intervals of  $I$  and thus they are finite in total. Hence,  $|\mathcal{S}| < \infty$ . The optimal saving policy induces a Markov transition matrix  $R$ , for our aggregate system. Clearly, the savings policy is monotonically nondecreasing (when more than one best response is possible we use the convention of picking the smallest one) in the level of capital. Let the states in  $\mathcal{S}$  be ordered in an increasing fashion, i.e.,  $\mathcal{S} = \{S_1 < S_2 < \dots < S_N = \bar{S}\}$  where  $N = |\mathcal{S}|$ . Hence our economy is going to be in one of these states at the end of each period with some probability. For small discount factors,  $\beta \in [0, \beta_2)$ , we show in Section 3.1 that the economy gets stuck in the absorbing state  $S_1 = 0$ , in finite time. We can observe implementation of projects before the system gets absorbed by that state, if the initial resources are large.

For larger discount factors, there is no absorbing state: As  $\beta \rightarrow 1$ , the utility from a unit current consumption becomes negligible and the utility of future consumption approaches 1. At the limit  $\beta = 1$ , an agent only cares about future consumption. From the continuity of the problem in  $\beta$ , the same must be true when  $\beta$  is sufficiently near 1. Even when capital level is low, then, the contract asks the agents to save consecutively until the capital level is sufficient to invest. So, for any  $n$ , with probability  $1 - p_1 q_0$ , the system moves to a state  $S' \geq S_n$  ( $S' > S_n$ , with probability at least  $p_1 q_1$ , for  $n < N$ ) and to a lower state with probability  $p_1 q_0$ . Consequently, the probability of going from one state to the other is always positive (irreducible Markov chain) which guarantees that the system will hit each state with probability one. Hence, investment cycles are observed almost surely.

As the discount factor increases, the set of states becomes finer which makes it harder to fall down into the region where resource constraint of the lender binds. Second, trivially, the probability of going down to a lower state is  $p_1 q_0$  from second period on. This probability decreases if  $q_1$  goes up ( $q_0$  goes down), which increases the expected number of periods that takes the system to go down to the ‘capital crunch’ region.

*Q.E.D.*

**Proof of Proposition 4**

The existence of the second-best recursive contract is a little involved. The reason is that the value function has to be non-negative for any feasible value of  $v$  in the second-best program. So, the value function itself enters the constraint set of the problem which makes it a nonstandard dynamic programming problem. The method of proof for the existence and computation of the second-best value function in the first part of the proof is the same as in [Thomas and Worrall \(1994\)](#).

1. Let  $F$  and  $B(F)$  be defined as in the proof of Proposition 1. Let the operator  $T$  be defined on  $B(F)$  for any  $(v, S) \in F$  and any  $Q \in B(F)$  by:

$$\begin{aligned}
(TQ)(v, S) &= \max_{(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_{\theta}, v_{\theta\mu}) \in \mathbb{R}_+^{18}} \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)m_{\theta\mu} + \beta Q(v_{\theta\mu}, S_{\theta\mu})] \\
\text{s.t. } \forall \theta, \forall \mu & \quad Q(v_{\theta\mu}, S_{\theta\mu}) \geq 0 \\
& \quad S_{\theta\mu} \leq S + Y + D1_{\{\theta=1, \mu=1, I_{\theta} \geq I\}} - I_{\theta} - m_{\theta\mu} - c_{\theta\mu} \\
& \quad I_{\theta} \leq S + Y \\
& \quad \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu}] \geq v \\
& \quad (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \geq (1 - \beta)D1_{\{\theta=1, \mu=1, I_{\theta} \geq I\}} + \beta Y
\end{aligned} \tag{18}$$

We know from before that the first-best value function,  $P \in B(F)$ . With one extra constraint to consider, the feasible set for the above problem is smaller than that of the first-best problem; hence  $TP \leq P$ . Clearly,  $T$  is a monotone operator which implies that  $T^n P \leq T^{n-1} P$ , all  $n$ , by a simple induction argument. Hence, for each  $(v, S)$ ,  $((T^n P)(v, S))$  is a decreasing sequence which is bounded from below by 0. So, it should converge pointwise to a limit value, say, to  $P_{\infty}(v, S)$ .

- We need to show that  $P_{\infty}$  is a fixed point of the operator  $T$ . For any given  $(v, S)$ , consider the sequence of optimal actions taken at each iteration of the operator  $T$ ,  $(c_{\theta\mu}^n, m_{\theta\mu}^n, S_{\theta\mu}^n, I_{\theta}^n, v_{\theta\mu}^n)$ . Since  $(T^n(P)(v, S))$  is a decreasing sequence, the constraint in (18) is not going to relax as  $n$  increases, which makes the feasible set of values that the sequence of optimal actions live in, a compact set. Hence,  $(c_{\theta\mu}^n, m_{\theta\mu}^n, S_{\theta\mu}^n, I_{\theta}^n, v_{\theta\mu}^n)$  has a convergent subsequence, converging to  $(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_{\theta}, v_{\theta\mu})$ . The sequence satisfies all the constraints of the problem, for each  $n$ ; so, the limit contract should do so too since these are all inequality constraints. In particular,  $T^n(P)(v, S) \geq 0$ , for all  $n$  and  $T^n(P)(v, S) \rightarrow P_{\infty}(v, S)$  hence  $P_{\infty}(v, S) \geq 0$ , too. So, this limit contract is feasible and provides the borrower with an ex-ante discounted utility of at least  $v$ . The actual optimal contract should do at least as good which implies that

$(TP_\infty)(v, S) \geq P_\infty(v, S)$ . We also know, from the monotonicity of the operator  $T$ , that, for all  $n$ ,  $(T^n P)(v, S) \leq (T^{n-1} P)(v, S)$  hence  $(T^{n-1} P)(v, S) \geq P_\infty(v, S)$ . Therefore,  $(T^n P)(v, S) \geq (TP_\infty)(v, S)$  for all  $n$  which implies that the limit of that sequence admits the same ordering:  $(T^n P)(v, S) \rightarrow P_\infty(v, S) \geq (TP_\infty)(v, S)$ . We showed that  $P_\infty(v, S) \geq (TP_\infty)(v, S)$  and  $P_\infty(v, S) \leq (TP_\infty)(v, S)$  which implies that  $P_\infty(v, S) = (TP_\infty)(v, S)$  hence  $P_\infty$  is a fixed point of the operator  $T$ .

- Clearly, each fixed point  $Q$  of the operator  $T$  corresponds to a second-best contract. It is the standard unravelling idea. Start with an initial  $(v, S)$ ; the optimal actions in period 1 are given by  $(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_\theta, v_{\theta\mu})$ . Let  $\sigma_1$  be defined as  $c_1(\theta, \mu) = c_{\theta\mu}$ ,  $m_1(\theta, \mu) = m_{\theta\mu}$ ,  $S_2(\theta, \mu) = S_{\theta\mu}$  and  $I_1(\theta) = I_\theta$ . The second period contract conditional on the realization of  $\theta, \mu$  in the first period is the optimal action vector starting with an initial  $(v_{\theta\mu}, S_2(\theta, \mu))$ , and so on by repeatedly applying the operator  $T$ . The contract  $\sigma$  constructed this way satisfies all the constraints of the original second-best problem and delivers the borrower and the lender the utility levels  $v$  and  $Q(v, S)$ , respectively. Moreover, it is optimal from each history on; hence it is a second-best contract.
- We know that  $P \geq Q$  where  $Q$  is a fixed point of  $T$ , from above, which leads to  $T^n P \geq T^n Q = Q$ . But then, the latter also holds in the limit:  $P_\infty \geq T^n Q = Q$ . Since every fixed point of  $T$  corresponds to a second-best contract and by the optimality of  $Q$ , we also have  $P_\infty \leq Q$ . Hence  $P_\infty = Q$ .

In summary, we showed that a fixed point,  $Q$ , of the operator  $T$  exists and can be computed by an iterative application of  $T$ , starting initially with the first-best value function  $P$ . In addition, a second-best contract exists that is associated with that value function which delivers the borrower and the lender the utility levels  $v$  and  $Q(v, S)$ , respectively.

2. The existence of  $\underline{S}$  follows from Proposition 2-(ii). The idea is that if  $v = \bar{v}_S$ , i.e., the whole surplus goes to the borrower, then the first-best rules are implemented. So, for  $v = \bar{v}_S$  and  $S \geq \underline{S}$ , ‘invest’ is the optimal strategy. Now, the sketch of the proof is as follows: as we know from the existence proof, starting from the first-best value function  $P$ , repeated application of the operator  $T$  on  $P$  leads to the second-best value function  $Q$ . At each iteration, we will show that the optimal policy rules are of the monotonic nature and that in the limit, they converge to the stated form.

So, initially, we assume the continuations are given by  $P$  and solve

$$\begin{aligned}
(TP)(v, S) &= \max_{(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_{\theta}, v_{\theta\mu}) \in \mathbb{R}_+^{18}} \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)m_{\theta\mu} + \beta P(v_{\theta\mu}, S_{\theta\mu})] \\
\text{s.t. } \forall \theta, \forall \mu & \quad P(v_{\theta\mu}, S_{\theta\mu}) \geq 0 \\
& \quad S_{\theta\mu} \leq S + Y + D1_{\{\theta=1, \mu=1, I_{\theta} \geq I\}} - I_{\theta} - m_{\theta\mu} - c_{\theta\mu} \\
& \quad I_{\theta} \leq S + Y \\
& \quad \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu}] \geq v \\
& \quad (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \geq (1 - \beta)D1_{\{\theta=1, \mu=1, I_{\theta} \geq I\}} + \beta Y
\end{aligned} \tag{19}$$

where  $P(v, S) = \bar{v}_S - v$ . So, the above program can be written as

$$\begin{aligned}
(TP)(v, S) &= \max_{(c_{\theta\mu}, m_{\theta\mu}, S_{\theta\mu}, I_{\theta}, v_{\theta\mu}) \in \mathbb{R}_+^{18}} \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta) (S + Y + D1_{\{\theta=1, \mu=1, I_{\theta} \geq I\}} - I_{\theta} - S_{\theta\mu}) \\
& \quad + \beta \bar{v}_S - (1 - \beta)c_{\theta\mu} - \beta v_{\theta\mu}] \\
\text{s.t. } \forall \theta, \forall \mu & \quad P(v_{\theta\mu}, S_{\theta\mu}) \geq 0 \\
& \quad I_{\theta} \leq S + Y \\
& \quad \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu}] \geq v \\
& \quad (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \geq (1 - \beta)D1_{\{\theta=1, \mu=1, I_{\theta} \geq I\}} + \beta Y
\end{aligned} \tag{20}$$

*Savings:* The first-best saving rule maximizes the first part of the lender's objective and is independent of the optimal choice of  $(c_{\theta\mu}, v_{\theta\mu})$ .

*Investment:* There are two different regimes. Let  $v^{*1}(S) := (1 - \beta)p_1q_1D + \beta Y$ .

1. If  $v \geq v^{*1}(S)$ : The (IR) constraint (20) binds. The reason is that, if the investment is undertaken, this is the minimum amount that the borrower needs to be provided with, ex-ante, in order to prevent default. If it does not bind, you can always lower some continuation values without violating any of the incentive compatibility constraints. So,  $EU^B = v$ . The alternative is not to invest. But the comparison is exactly that of the first-best investment decision whose answer is to 'invest'.
2. If  $v < v^{*1}(S)$ : The constraint (20) holds with strict inequality.  $EU^B = v^{*1}(S)$ . If there is no investment,  $EU^E = v$ . So, the comparison is between

$$P(v, S) - v^{*1}(S)$$

which is the first-best level of surplus minus the average ex-ante payment to B, in case of investment and

$$(1 - \beta)(S + Y - S'_{FB}) + \beta \bar{v}_{S'_{FB}} - v$$

in case of ‘not investing’, where  $S'_{FB}$  is the first-best level of savings. Hence, there is a level  $v_*^1(S) > Y$  ( $v_*^1(S) = Y$  if it is optimal to invest always) such that it is optimal to invest for  $v \geq v_*^1(S)$  and not to invest for  $v \leq v_*^1(S)$ .<sup>14</sup>

So, we have a full characterization of  $TP$ , i.e.,

1. Do not invest for  $v \in [Y, v_*^1(S)]$ ; Invest for  $v \in [v_*^1(S), \bar{v}_S]$
2. Save according to the first-best rule.
3.  $TP$  is given by

$$TP(v, S) = \begin{cases} \bar{v}_S - v & \text{if } v \geq v_*^1(S) \\ \bar{v}_S - v_*^1(S) & \text{if } v \in [v_*^1(S), v_*^1(S)] \\ \bar{v}_S - [v_*^1(S) - v_*^1(S)] - v & \text{if } v \in [Y, v_*^1(S)] \end{cases}$$

We know that if the whole surplus goes to the borrower, i.e.,  $v = \bar{v}_S$ , the first-best rule is implemented. In the second iteration of the above problem ( $T^2P$ ), for the same  $S$ , there is a nonempty interval of values of  $v$  (including  $\bar{v}_S$ ) for which the optimal rule is to invest. That is because of the continuity of the problem w.r.t.  $v$  and the fact that  $P$  and  $TP$  coincide for high values of  $v$ . Similar reasoning guarantees that if there is an interval of values for which the optimal rule is ‘not to invest’ for the first iteration, there is such an interval for the second iteration, since the continuation value is lower than the original continuation value ( $TP \leq P$ ). Then, we need to show two things: (i)  $T^n P$  has the same shape as  $TP$  and (ii)  $v^*(S)$  is nondecreasing. These two combined will deliver the result. For  $n = 1$ , it is trivially true. For  $n > 1$ , assume that it is true for  $n - 1$ . We know that the optimal rule is to invest for  $v \geq v^{*n-1}(S)$  since the first-best and second-best values coincide for that interval.  $T^n P \leq T^{n-1} P$  from the existence part, hence  $v^{*n}(S) \geq v^{*n-1}(S)$  since otherwise  $T^n P(v^{*n}(S), S) > T^{n-1} P(v^{*n}(S), S)$ , a contradiction. For  $v < v^{*n-1}(S)$ ,  $EU^B = v^{*n}(S)$  from the same token as above. We know from above that  $v_*^{n-1}(S)$  is such that

$$(1 - \beta)(S + Y - S'_{FB}) + \beta \bar{v}_{S'_{FB}} - v_*^{n-1}(S) = P(v, S) - v^{*n-1}(S)$$

Since  $v^{*n}(S) \geq v^{*n-1}(S)$ , there is a  $v_*^n(S) \geq v_*^{n-1}(S)$  (where  $v_*^n(S) - v_*^{n-1}(S) = v^{*n}(S) - v^{*n-1}(S)$ ) such that the equality still holds and it is optimal to invest for  $v \geq v_*^n(S)$  and not to invest for

<sup>14</sup>It is not always the case that investing is the optimal decision no matter what  $v$  is, as Example 2 shows.

$v \leq v_*^n(S)$ . Hence,  $T^n P$  has the same shape as  $TP$ . Now,  $(v_*^n(S))$  forms a monotonically non-decreasing sequence of thresholds, bounded from above by  $\bar{v}_S$ . So, it should converge to  $v^*(S)$  as  $n \rightarrow \infty$ . But, trivially then,  $(v_*^n(S)) \rightarrow v_*(S)$ .

Let  $S' > S$ . So,  $\bar{v}_{S'} > \bar{v}_S$  since  $P$  is strictly increasing from Proposition 1.3. The fact that the individual rationality constraint is tight means that, conditioning on investing today, a constraint will be violated if at least  $v^*(S)$  is not delivered to the borrower. For  $S' > S$ , the same contract will violate that same constraint, since the constraints are stationary. Hence,  $v^*(S') \geq v^*(S)$ . That concludes the proof. Q.E.D.

### **Proof of Proposition 5**

1. Suppose for a contradiction that  $S_{\theta\mu} > S_{FB,\theta\mu}$ . Let  $(c_{\theta\mu}, v_{\theta\mu}, S_{\theta\mu})$  be the optimal second-best contract given  $(v, S)$  is the state variable. On a realized path  $(\theta\mu)$ , utility to L is

$$(1 - \beta) [W - c_{\theta\mu} - S_{\theta\mu}] + \beta Q(v_{\theta\mu}, S_{\theta\mu})$$

which can be written as

$$(1 - \beta) [W - c_{\theta\mu} - S_{\theta\mu}] + \beta \begin{cases} \bar{v}_{S_{\theta\mu}} - v_{\theta\mu} & \text{if } v_{\theta\mu} \geq v^*(S_{\theta\mu}) \\ \bar{v}_{S_{\theta\mu}} - v^*(S_{\theta\mu}) & \text{if } v_{\theta\mu} \in [v_*(S_{\theta\mu}), v^*(S_{\theta\mu})] \\ \bar{v}_{S_{\theta\mu}} - [v^*(S_{\theta\mu}) - v_*(S_{\theta\mu})] - v_{\theta\mu} & \text{if } v_{\theta\mu} \in [Y, v_*(S_{\theta\mu})] \end{cases}$$

which is equivalent to

$$(1 - \beta) [W - S_{\theta\mu}] + \beta \begin{cases} \bar{v}_{S_{\theta\mu}} \\ \bar{v}_{S_{\theta\mu}} \\ \bar{v}_{S_{\theta\mu}} - [v^*(S_{\theta\mu}) - v_*(S_{\theta\mu})] \end{cases} - \begin{cases} (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \\ (1 - \beta)c_{\theta\mu} + \beta v^*(S_{\theta\mu}) \\ (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \end{cases} \quad (22)$$

Necessary condition for optimality requires that either

$$v_{\theta\mu} \geq v^*(S_{\theta\mu}) \geq v^*(S_{FB,\theta\mu})$$

or

$$v_{\theta\mu} \leq v_*(S_{FB,\theta\mu}) \leq v^*(S_{FB,\theta\mu}) \leq v^*(S_{\theta\mu})$$

which implies that  $(c_{\theta\mu}, v_{\theta\mu})$  is feasible also under the first-best saving rule  $S_{FB,\theta\mu}$  since  $S_{\theta\mu} > S_{FB,\theta\mu}$ . So, the payoff to the lender on the realized path from saving according to the first-best rule and paying the borrower  $(c_{\theta\mu}, v_{\theta\mu})$ , is better than the original scheme since the LHS of (22) is

maximized at  $S_{\theta\mu} = S_{FB,\theta\mu}$ , i.e.,

$$(1 - \beta) [W - S_{\theta\mu}] + \beta \begin{cases} \bar{v}_{S_{\theta\mu}} \\ \bar{v}_{S_{\theta\mu}} \\ \bar{v}_{S_{\theta\mu}} - [v^*(S_{\theta\mu}) - v_*(S_{\theta\mu})] \end{cases} - \begin{cases} (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \\ (1 - \beta)c_{\theta\mu} + \beta v^*(S_{\theta\mu}) \\ (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \end{cases} \\ < (1 - \beta) [W - S_{FB,\theta\mu}] + \beta \begin{cases} \bar{v}_{S_{FB,\theta\mu}} \\ \bar{v}_{S_{FB,\theta\mu}} \\ \bar{v}_{S_{FB,\theta\mu}} - [v^*(S_{FB,\theta\mu}) - v_*(S_{FB,\theta\mu})] \end{cases} - \begin{cases} (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \\ (1 - \beta)c_{\theta\mu} + \beta v^*(S_{FB,\theta\mu}) \\ (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \end{cases}$$

if  $v_{\theta\mu} \leq v_*(S_{\theta\mu})$ , or

$$(1 - \beta) [W - S_{\theta\mu}] + \beta \begin{cases} \bar{v}_{S_{\theta\mu}} \\ \bar{v}_{S_{\theta\mu}} \\ \bar{v}_{S_{\theta\mu}} - [v^*(S_{\theta\mu}) - v_*(S_{\theta\mu})] \end{cases} - \begin{cases} (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \\ (1 - \beta)c_{\theta\mu} + \beta v^*(S_{\theta\mu}) \\ (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \end{cases} \\ < (1 - \beta) [W - S_{FB,\theta\mu}] + \beta \begin{cases} \bar{v}_{S_{FB,\theta\mu}} \\ \bar{v}_{S_{FB,\theta\mu}} \\ \bar{v}_{S_{FB,\theta\mu}} - [v^*(S_{FB,\theta\mu}) - v_*(S_{FB,\theta\mu})] \end{cases} - \begin{cases} (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \\ (1 - \beta)c_{\theta\mu} + \beta v^*(S_{FB,\theta\mu}) \\ (1 - \beta)c_{\theta\mu} + \beta v^*(S_{FB,\theta\mu}) \end{cases}$$

if  $v_{\theta\mu} > v_*(S_{\theta\mu})$  since  $Q$  is strictly increasing in  $S$ . Therefore, saving more than the first-best rule makes L strictly worse off, which concludes the proof.

2. This is implied by Proposition 4-(ii). Assuming the latter is true, pick a feasible  $(v, S)$  pair.  $I_0(v, S) = 0$  trivially for any feasible  $(v, S)$  since it does not pay off to allocate resources to investment when there is no project. For a given  $S$ ,  $I_1$  is trivially monotonically increasing in  $v$ , since it is a step function from Proposition 4.

*Q.E.D.*

### ***Proof of Example 1***

We know from Proposition 1 that  $P$  exists, that there exists  $\sigma = (c_{\theta\mu}, m_{\theta\mu}, I_{\theta\mu}, S_{\theta\mu})$ , an optimal stationary contract and the optimal saving and investment strategies are independent of  $v$ , meaning that  $P(v, S) = P(0, S) - v$ . Let  $W$  denote end-of-period wealth.

1. We first conjecture that for any  $\beta \in (0, \beta_1)$ , where  $\beta_1 \equiv \frac{(I-Y)}{(I-Y)+p_1(q_1D-I)}$ , we have

$$P(0, S) = \begin{cases} (1 - \beta) S + Y & , \text{ if } S < I - Y \\ (1 - \beta) [S + p_1(q_1D - I)] + Y & , \text{ if } S \geq I - Y \end{cases}$$

It will be shown, next, that the conjectured function  $P$  is indeed the value function and the optimal policy functions are as specified in the proposition. The proof proceeds in two steps: First, we check if deviations from the savings strategy pay off; second, we look for an improved investment policy.

*Savings:* A simple arbitrage argument shows that the optimal savings need to be at the points of discontinuity: At a point of differentiability, the derivative of

$$\max_{0 \leq S' \leq S} (1 - \beta)(W - S') + \beta P(0, S')$$

with respect to  $S'$  is  $-(1 - \beta) + \beta(1 - \beta) < 0$ , which implies that it pays-off to decrease  $S'$ , if possible. This means that the only possible saving strategy would be to save  $I - Y$  when feasible. We claim that saving 0 is optimal. To this end, we look at the difference of the maximum surpluses from saving  $I - Y$  and 0, in that order

$$(1 - \beta) [W - (I - Y)] + \beta P(0, I - Y) - (1 - \beta)W - \beta P(0, 0)$$

which is equivalent to

$$\begin{aligned} &= -(1 - \beta)(I - Y) + \beta [P(0, I - Y) - P(0, 0)] \\ &= -(1 - \beta)(I - Y) + \beta(1 - \beta) [(I - Y) + p_1(q_1 D - I)] \end{aligned} \quad (23)$$

$$\begin{aligned} &= (1 - \beta) [-(I - Y)(1 - \beta) + \beta p_1(q_1 D - I)] < 0 \\ \iff \beta < \beta_1 &= \frac{(I - Y)}{(I - Y) + p_1(q_1 D - I)} \end{aligned} \quad (24)$$

which is true by hypothesis. All we needed was to show that there was no one-shot profitable deviation from the conjectured saving strategy, which we did.

*Investment:* Conditioning on the fact that  $\theta = 1$  (productive shock), we need to show that  $\forall S \geq (I - Y)$ , investing gives a higher payoff than not investing does. Given the optimal saving policy, investing brings:

$$(1 - \beta) (S + Y + q_1 D - I) + \beta P(0, 0)$$

where not investing brings:

$$(1 - \beta) (S + Y) + \beta P(0, 0)$$

whose difference is

$$(1 - \beta)(q_1 D - I) > 0$$

Hence, investing is optimal. It is easy to see that, this saving/investment strategy yields the conjectured maximum surplus function. Then, Proposition 1 implies that any first-best optimal contract,

independent of  $v$ , will necessarily have this saving/investment strategy pair, as part of it.

2. We first conjecture that for any  $\beta \in [\beta_1, \beta_2)$ , where  $\beta_2$  will be computed below,  $P(0, S)$  is given by<sup>15</sup>

$$P(0, S) = \begin{cases} (1 - \beta)S + Y & \text{if } S < I - 2Y \\ (1 - \beta)[S - (I - 2Y)] + \beta P(0, I - Y) & \text{if } I - 2Y \leq S < I - Y \\ (1 - \beta)\Delta [(I - Y)p_1q_0 + p_1(q_1D - I)] + Y & \text{if } I - Y \leq S < 2(I - Y) \\ (1 - \beta)[S + Y - (I - Y) + p_1(q_1D - I)] + \beta P(0, I - Y) & \text{if } S \geq 2(I - Y) \end{cases}$$

with  $\Delta \equiv [1 - \beta(1 - p_1q_0)]^{-1}$ . Once again, we need to show that the conjectured value function is indeed the correct one. To that effect, we show that the stated saving and investment rules are the optimal ones given  $P$ .

*Savings:* We know that savings need to be at the discontinuity points of the value function, by the arbitrage argument that we provided before. Hence, the candidates are:  $I - 2Y$ ,  $I - Y$ ,  $2(I - Y)$  and not saving at all. We just need to check that saving  $I - Y$  does better than all other possibilities:

- $\forall W \geq I - Y$ , saving 0 is not a better policy which translates into

$$(1 - \beta)[W - (I - Y)] + \beta P(0, I - Y) - (1 - \beta)W - \beta P(0, 0) \geq 0$$

the left hand side of which is equivalent to

$$\begin{aligned} &= -(1 - \beta)(I - Y) + \beta\Delta(1 - \beta)[(I - Y)p_1q_0 + p_1(q_1D - I)] \\ &= (1 - \beta)[-(I - Y)(1 - \beta\Delta p_1q_0) + \beta\Delta p_1(q_1D - I)] \end{aligned} \quad (25)$$

By algebra,  $(1 - \beta\Delta p_1q_0) = (1 - \beta)\Delta$ , which implies that the last line can be written as:

$$\begin{aligned} &= (1 - \beta)\Delta[-(I - Y)(1 - \beta) + \beta p_1(q_1D - I)] \geq 0 \\ \iff &-(I - Y)(1 - \beta) + \beta p_1(q_1D - I) \geq 0 \\ \iff &\beta \geq \beta_1 \end{aligned}$$

which is the case by hypothesis.

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<sup>15</sup>This is the proof for the case  $0 < Y < I - Y$ , which is more interesting since it is harder to self-finance. The proof for the case of  $Y \geq I - Y$  is similar.

- $\forall W \geq I - Y$ , *saving  $I - 2Y$  is not a better policy.* Once again, difference of surpluses yields
 
$$\begin{aligned}
 &= -(1 - \beta)Y + \beta[P(0, I - Y) - P(0, (I - 2Y))] \\
 &= -(1 - \beta)Y + \beta(1 - \beta)P(0, I - Y) \\
 &= (1 - \beta)[-(1 - \beta)Y + \beta\Delta(1 - \beta)[(I - Y)p_1q_0 + p_1(q_1D - I)]] \\
 &> (1 - \beta)[-(1 - \beta)(I - Y) + \beta\Delta(1 - \beta)[(I - Y)p_1q_0 + p_1(q_1D - I)]] \\
 &\geq 0
 \end{aligned}$$

where the last inequality follows from (25).

- $\forall W \geq 2(I - Y)$ , *saving  $2(I - Y)$  is not a better policy.* Difference of surpluses yields
 
$$\begin{aligned}
 &= (1 - \beta)(I - Y) + \beta[P(0, I - Y) - P(0, 2(I - Y))] \\
 &= (1 - \beta)(I - Y) + \beta[P(0, I - Y) - (1 - \beta)[(I - Y) + p_1(q_1D - I) + Y] - \beta P(0, I - Y)] \\
 &= (1 - \beta)^2(I - Y) + \beta(1 - \beta)[P(0, I - Y) - Y - p_1(q_1D - I)] \\
 &= (1 - \beta)^2(I - Y) + \beta(1 - \beta)[[(I - Y)p_1q_0 + p_1(q_1D - I)](1 - \beta)\Delta - p_1(q_1D - I)] \\
 &= (I - Y)(1 - \beta)^2[1 + \beta p_1q_0\Delta] - \beta(1 - \beta)[1 - \Delta(1 - \beta)]p_1(q_1D - I) \\
 &= (I - Y)(1 - \beta)^2[1 + \beta p_1q_0\Delta] - \beta(1 - \beta)\beta p_1q_0\Delta p_1(q_1D - I) \\
 &= (1 - \beta)[(I - Y)(1 - \beta)(1 + \beta p_1q_0\Delta) - \beta^2 p_1q_0\Delta p_1(q_1D - I)]
 \end{aligned}$$

Now, let

$$\begin{aligned}
 r &:= p_1q_0 \\
 C &:= I - Y > 0 \\
 F &:= r p_1(q_1D - I) > 0
 \end{aligned}$$

which makes the last line into

$$\begin{aligned}
 &= (1 - \beta) \left[ C(1 - \beta) \left( 1 + \frac{\beta r}{1 - \beta(1 - r)} \right) - \frac{\beta^2 F}{1 - \beta(1 - r)} \right] \\
 &= \frac{(1 - \beta)}{1 - \beta(1 - r)} [C(1 - \beta)(1 - \beta + 2\beta r) - \beta^2 F] \\
 &= \frac{(1 - \beta)}{1 - \beta(1 - r)} [\beta^2 (C(1 - 2r) - F) - 2\beta C(1 - r) + C]
 \end{aligned}$$

Set

$$\begin{aligned}
 A &:= \frac{(1 - \beta)}{1 - \beta(1 - r)} [\beta^2 (C(1 - 2r) - F) - 2\beta C(1 - r) + C] \\
 B &:= \beta^2 (C(1 - 2r) - F) - 2\beta C(1 - r) + C
 \end{aligned}$$

We have  $B$ , a quadratic function of  $\beta$ , whose determinant is

$$\begin{aligned} & [2C(1-r)]^2 - 4C[C(1-2r) - F] \\ &= 4C^2r^2 + 4CF > 0 \end{aligned}$$

Hence the equation has two real roots. Call them  $x_1$  and  $x_2$  and assume wlog that  $x_1 < x_2$ . Here are some facts that we use:

1.  $B$  evaluated at  $\beta = 1$  is

$$B |_{\beta=1} = C(1-2r) - F - 2C(1-r) + C = -F < 0$$

2.  $A$  evaluated at the first threshold  $\beta_1$  is

$$\begin{aligned} A |_{\beta=\beta_1} &= (1-\beta_1) [(I-Y)(1-\beta_1)(1+\beta_1r\Delta) - \beta_1^2r\Delta p_1(q_1D-I)] \\ &> (1-\beta_1) [(I-Y)(1-\beta_1) - \beta_1p_1(q_1D-I)] \\ &= 0 \end{aligned}$$

the inequality due to the fact that  $(1+\beta_1r\Delta) > \beta_1r\Delta$ ; the last equality from the definition of the threshold  $\beta_1$ .

3.  $A$  and  $B$  evaluated at  $\beta = 0$  give

$$A |_{\beta=0} = C > 0 = B |_{\beta=0}$$

4. The sign of  $A$  is determined by  $B$  since

$$\frac{\partial}{\partial \beta} \left( \frac{1-\beta}{1-\beta(1-r)} \right) < 0$$

and

$$\frac{1-\beta}{1-\beta(1-r)} |_{\beta=0} = 1 \quad \text{and} \quad \frac{1-\beta}{1-\beta(1-r)} |_{\beta=1} = 0$$

We have three cases to consider, depending on the coefficient of the highest exponent. The task in each of them is to show that  $0 < \beta_1 < \beta_2 < 1$ . The listed facts combined with the restriction on the coefficient of the highest order term in each case delivers the result.

- Case 1:  $C(1-2r) - F > 0$ . This is the case where the product of the roots  $x_1x_2 = \frac{C}{C(1-2r)-F} > 0$ , the sum of the roots  $x_1 + x_2 = \frac{2C(1-r)}{C(1-2r)-F} > 0$  and the quadratic troughs at  $\beta = \frac{2C(1-r)}{2[C(1-2r)-F]} > 1$ . Hence, we have two positive roots,  $0 < x_1 < 1 < x_2$ .

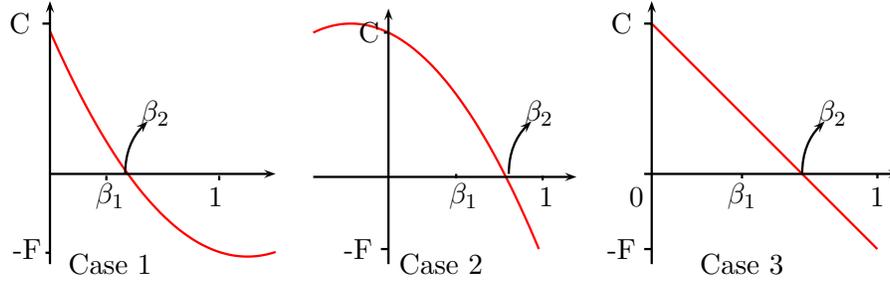


Figure 7: Three Different Cases for  $\beta_2$

- Case 2:  $C(1 - 2r) - F < 0$ . The product of the roots  $x_1x_2 < 0$ , the sum of the roots  $x_1 + x_2 < 0$  and the function peaks at  $\beta = \frac{2C(1-r)}{2[C(1-2r)-F]} < 0$ . So, we have 1 positive and 1 negative root,  $x_1 < 0 < x_2 < 1$ .
- Case 3:  $C(1 - 2r) - F = 0$ . This is the linear case.  $B$  becomes  $-\beta 2C(1 - r) + C$  which assumes the value  $-C(1 - 2r) = -F < 0$  at  $\beta = 1$ . Hence the intersection of the line with the horizontal axis happens in the interval  $(0, 1)$ .

As depicted in Figure 7,  $B$  is positive for  $\beta \in (0, \beta_2)$  where  $0 < \beta_1 < \beta_2$ . This implies that the original difference between maximum surpluses from saving  $I - Y$  and saving  $2(I - Y)$  is positive. Hence, saving  $I - Y$  pays more than the latter.

We showed that saving  $I - Y$  dominates all of the other saving policies, hence it is optimal.

*Investment:* There are two different regimes to consider:

- $\forall S \geq 2(I - Y)$ , Investing brings

$$q_1 [(1 - \beta)(S + Y + D - I - (I - Y)) + \beta P(0, I - Y)] + \\ q_0 [(1 - \beta)(S + Y - I - (I - Y)) + \beta P(0, I - Y)]$$

where not investing brings:

$$(1 - \beta)(S + Y - (I - Y)) + \beta P(0, I - Y)$$

whose difference is

$$(1 - \beta)(q_1 D - I) > 0$$

Hence, investing is the optimal strategy in that range, given the optimal saving strategy.

- $\forall S \in [I - Y, 2(I - Y)]$ , investing gives

$$\begin{aligned} & q_1 [(1 - \beta)(S + Y + D - I - (I - Y)) + \beta P(0, I - Y)] + \\ & q_0 [(1 - \beta)(S + Y - I) + \beta P(0, 0)] \end{aligned}$$

where not investing brings:

$$(1 - \beta)(S + Y - (I - Y)) + \beta P(0, I - Y)$$

whose difference is

$$\begin{aligned} & (1 - \beta)[(q_1 D - I) - \beta q_0 \Delta p_1 (q_1 D - I) + \\ & (I - Y)q_0 - \beta \Delta p_1 q_0^2 (I - Y)] > 0 \end{aligned}$$

Hence, investing is optimal in this range, too.

Therefore, investment policy is the threshold rule stated in the proposition.

*Q.E.D.*

### ***Proof of Example 2***

We know that both  $P$  and  $Q$  exist and that  $0 \leq Q(v, S) \leq P(v, S)$  for any feasible  $(v, S)$ .

1. I will first show that the saving strategy is as in the first-best case. Then, I will fully characterize the optimal threshold-investment behaviour.

*Savings:* The ‘trick’ here is to solve the alternate program where the continuation is replaced by  $P$ , then go back and show that one achieves the same utility level with that solution in the original problem, as well. On a realized path, optimal contract, on this path, should solve

$$\begin{aligned} & \max_{(c_{\theta\mu}, v_{\theta\mu}, S_{\theta\mu})} (1 - \beta) [W - c_{\theta\mu} - S_{\theta\mu}] + \beta Q(v_{\theta\mu}, S_{\theta\mu}) \\ \text{s.t.} \quad & (1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu} \geq (1 - \beta)D1_{\{\theta=1, \mu=1, I_\theta \geq Y\}} + \beta Y \quad (*) \\ & \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)c_{\theta\mu} + \beta v_{\theta\mu}] \geq v \in [Y, \bar{v}_S] \quad (**) \end{aligned}$$

given the optimal choices for the other paths. Let's first consider the alternate program where we continue with the first-best value function,  $P$ , given that both (\*) and (\*\*) are satisfied

$$\begin{aligned} & \max(1 - \beta) [W - c_{\theta\mu} - S_{\theta\mu}] + \beta P(v_{\theta\mu}, S_{\theta\mu}) \\ = & \max(1 - \beta) [W - c_{\theta\mu} - S_{\theta\mu}] + \beta [P(0, S_{\theta\mu}) - v_{\theta\mu}] \\ = & \max(1 - \beta) [W - S_{\theta\mu}] + \beta P(0, S_{\theta\mu}) - (1 - \beta)c_{\theta\mu} - \beta v_{\theta\mu} \end{aligned}$$

LHS of this problem is maximized at  $S_{\theta\mu} = 0$ , independently of the RHS. That's because any utility level arising from an optimal  $(c_{\theta\mu}, v_{\theta\mu})$  pair that is feasible under an alternative saving strategy can be replicated by a corresponding contract  $(c', v')$  under the surplus-maximizing saving strategy. Since there is no saving, the only incentive compatible continuation value is  $v_{\theta\mu} = Y$  ( $P(0, 0) = Y$  from Proposition 1). This scheme is feasible under the original program too. Since

$$0 \leq Q(Y, 0) \leq P(Y, 0) = 0$$

the first inequality from the definition of second-best and the second from Proposition 1, we achieve the same utility under the original second-best program that we do under the alternate program. Hence, the necessary condition for optimality for a second-best contract is not saving at all as in the first-best case.

*Investment:* Given the optimal saving rule, there are two possible investment strategies:

1. If there is no investment ( $I_1(v, S) = 0$ ), second-best problem solves

$$\begin{aligned} & \max_{v, c} (1 - \beta) [S + Y - c] \\ \text{s.t.} & (1 - \beta)c + \beta Y \geq v \in [Y, \bar{v}_S] \end{aligned}$$

In the second-best optimum, the constraint should bind and

$$c = \left( \frac{v - \beta Y}{1 - \beta} \right) \leq \frac{\bar{v}_S - \beta Y}{1 - \beta} \leq \frac{\bar{v}_S}{1 - \beta} = \frac{(1 - \beta)[S + Y]}{1 - \beta} = S + Y$$

Moreover,  $c \geq 0$  clearly, which makes it feasible. Notice that we are assuming that  $c_{\theta\mu} = c$  here. This is just one of the solutions because of the linearity of the problem. However, all solutions leave L and B with the same utility levels. Hence,

$$EU^L = (1 - \beta) \left[ S + Y - \frac{v - \beta Y}{1 - \beta} \right] = (1 - \beta)S + (Y - v)$$

2. If there is investment ( $I_1(v, S) = I \leq S + Y$ ), since the optimal saving rule implies  $S_{\theta\mu} = 0$  and  $v_{\theta\mu} = Y$ , L's program becomes

$$\begin{aligned} & \max_{\theta\mu} \sum_{\theta\mu} p_{\theta} q_{\mu} (1 - \beta) (S + Y + D1_{\{\theta=1, \mu=1, I_{\theta} \geq I\}} - I_{\theta} - c_{\theta\mu}) \\ \text{s.t.} \quad & \sum_{\theta\mu} p_{\theta} q_{\mu} [(1 - \beta)c_{\theta\mu} + \beta Y] \geq v \end{aligned} \quad (26)$$

$$c_{\theta\mu} \geq D1_{\{\theta=1, \mu=1, I_{\theta} \geq I\}} \quad (27)$$

Let  $\lambda$  and  $\gamma$  be the Lagrange multipliers for the IR constraint, in (26), and the IC constraint  $c_{11} \geq D$ , in (27), respectively. First order conditions for  $c_{11}$ ,  $c_{\theta\mu}$  for  $\theta\mu \neq 1$ , are, in that order

$$-1 + \lambda + \frac{\gamma}{(1 - \beta)p_1 q_1} \leq 0, \quad c_{11} \geq 0, \quad \left[ -1 + \lambda + \frac{\gamma}{(1 - \beta)p_1 q_1} \right] c_{11} = 0 \quad (28)$$

$$-1 + \lambda \leq 0, \quad c_{\theta\mu} \geq 0, \quad [-1 + \lambda] c_{\theta\mu} = 0 \quad (29)$$

Let  $v^* \equiv \max\{Y, (1 - \beta)p_1 q_1 D + \beta Y\}$ . This quantity is crucial in determining whether (26) binds or not.

- (a) If  $v \in [Y, v^*]$ , (26) is an inequality, which implies that  $\lambda = 0$ . Then, (29) implies that  $c_{\theta\mu} = 0$  for  $\theta\mu \neq 1$ . Finally,  $c_{11} = D$  since otherwise, decreasing  $c_{11}$  would increase the objective without violating any constraints, which would be a contradiction to optimality. Therefore, the expected utility to L from investing is

$$EU_I^L = (1 - \beta) [S + p_1(q_1 D - I)] + Y - v^*$$

where the corresponding level from non-investing is

$$EU_{NI}^L = (1 - \beta)S + Y - v.$$

Now,  $EU_I^L - EU_{NI}^L \rightarrow (1 - \beta)p_1(q_1 D - I) > 0$  as  $v \rightarrow v^*$ . As  $v \rightarrow Y$ ,  $EU_I^L - EU_{NI}^L \rightarrow (1 - \beta)(Y - p_1 I)$ . If the latter is negative ( $p_1 > \frac{Y}{I}$ ), by the Intermediate Value Theorem, there exists a  $v_* = (1 - \beta)p_1 I + \beta Y \in [Y, v^*)$  such that  $EU_I^L - EU_{NI}^L < 0$  for  $v \in [Y, v_*)$ . If it is nonnegative ( $p_1 \leq \frac{Y}{I}$ ), it means that investing is optimal for all  $v \in [Y, v^*)$ , i.e.,  $v_* = Y$ .

- (b) If  $v \in (v^*, \bar{v}_S]$ , (26) binds.  $\lambda = 1$ , hence the slope of the second-best frontier is -1. If  $\lambda < 1$ , (28) and (29) imply that  $c_{\theta\mu} = 0$  for  $\theta\mu \neq 1$  and  $c_{11} = D$ , which would imply, in

turn, that  $v = v^*$ . But, this latter is a contradiction. Therefore, the expected utility to L from investing is

$$EU_I^L = (1 - \beta) [S + p_1(q_1D - I)] + Y - v$$

where the corresponding level from non-investing is

$$EU_{NI}^L = (1 - \beta)S + Y - v.$$

So,  $EU_I^L - EU_{NI}^L = (1 - \beta)p_1(q_1D - I) > 0$  which makes investing the optimal decision for all  $v \in (v^*, \bar{v}_S]$

2. The proof of the second part follows the same reasoning and machinery that the first one does. For that reason, it will be omitted.

*Q.E.D.*

## References

- ALBUQUERQUE, R. AND HOPENHAYN, H. A. (2004), "Optimal Lending Contracts and Firm Dynamics", *Review of Economic Studies*, **71**, 285-315.
- ALLEN, F. AND GALE, D. (1997), "Financial Markets, Intermediaries, and Intertemporal Smoothing", *Journal of Political Economy*, **105**, 523-546.
- ALVAREZ, F. AND JERMANN, U. J. (2000), "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica*, **59**, 1069-1089.
- ATKESON, A. (1991), "International Lending with Moral Hazard and Repudiation", *Econometrica*, **59**, 1069-1089.
- BECK, T., DEMIRGÜÇ-KUNT, A. AND MAKSIMOVIC, V. (2008), "Financing Patterns Around the World: Are Small Firms Different?", *Journal of Financial Economics*, **89**, 467-487.
- BERGER A. N. AND UDELL, G. (1994), "Did Risk-Based Capital Allocate Bank Credit and Cause a Credit Crunch in the U.S.?", *Journal of Money, Credit and Banking*, **26**, 585-628.
- BERNANKE, B. S. AND LOWN, C. S. (1991), "The Credit Crunch," *Brookings Papers on Economic Activity*, **2**, 205-247.
- BERNANKE, B. S. AND GERTLER, M. (1989), "Agency Cost, Net Worth and Business Fluctuations," *American Economic Review*, **79**, 14-31.
- BESANKO, D. AND KANATAS, G. (1996), "The Regulation of Bank Capital: Do Capital Standards Promote Bank Safety?" *Journal of Financial Intermediation*, **5** (2), 160-183.
- BRUNNERMEIER M. AND L. H. PEDERSEN (2009), "Market Liquidity and Funding Liquidity", *Review of Financial Studies*, **22** (6), 2201-2238.
- COOLEY, T., MARIMON R. AND QUADRINI V. (2004), "Aggregate Consequences of Limited Contract Enforceability", *Journal of Political Economy*, **112**, 817-847.
- DIAMOND, D. AND DYBVIK, P. (1983), "Bank Runs, Liquidity, and Deposit Insurance", *Journal of Political Economy*, **91**, 401-419.
- DIAMOND, D. AND RAJAN, R. (2000), "A Theory of Bank Capital," *Journal of Finance*, **55**, 2431-2465.

- HOLMSTRÖM, B. AND TIROLE, J. (1997), "Financial Intermediation, Loanable Funds, and the Real Sector," *Quarterly Journal of Economics*, **112**, 663-692.
- HOLMSTRÖM, B. AND TIROLE, J. (2011), *Inside and Outside Liquidity*, Cambridge, MA: The MIT Press.
- HOPENHAYN, H. AND WERNING, I. (2008), "Equilibrium Default," Mimeo, University of California at Los Angeles.
- HOSHI, T., KASHYAP, A. AND SCHARFSTEIN, D. (1990a), "Bank Monitoring and Investment: Evidence from the Changing Structure of Japanese Corporate Banking Relationships," in R. Glenn Hubbard, editor, *Asymmetric Information, Corporate Finance and Investment* (University of Chicago Press, Chicago, Illinois).
- HOSHI, T., KASHYAP, A. AND SCHARFSTEIN, D. (1990b), "The Role of Banks in Reducing the Costs of Financial Distress in Japan," *Journal of Financial Economics*, **27**, 67-88.
- HOSHI, T., KASHYAP, A. AND SCHARFSTEIN, D. (1991), "Corporate Structure, Liquidity and Investment: Evidence from Japanese Industrial Groups," *Quarterly Journal of Economics*, **106**, 33-60.
- JAMES, C. M. (2009), "Credit Market Conditions and the Use of Bank Lines of Credit", *FRBSF Economic Letter*, 2009-27.
- KASHYAP, A. AND STEIN, J. (2000), "What do a Million Observations on Banks Say about the Transmission of Monetary Policy?" *American Economic Review*, **90** (3), 407-428.
- KEHOE, T. J. AND LEVINE, D. K. (1993), "Debt-Constrained Asset Markets," *Review of Economic Studies*, **60**, 865-888.
- KOVRIJNYKH, N. (2010), "Debt Contracts with Partial Commitment", mimeo, Arizona State University.
- MACLEOD, W. B. AND MALCOMSON, J. (1989), "Implicit Contracts, Incentive Compatibility and Involuntary Unemployment," *Econometrica*, **57**, 447-480.
- NIKOLAU, K. (2009), "Liquidity (Risk) Concepts: Definitions and Interactions", European Central Bank Working Paper 1008.

- PETERSEN, M. AND RAJAN, R. (1995), "The Effect of Credit Market Competition on Lending Relationships," *Quarterly Journal of Economics*, **110**, 407-443.
- PHELAN, C. (1995), "Repeated Moral Hazard and One-Sided Commitment", *Journal of Economic Theory*, **66**, 488-506.
- RAY, D. (2002), "The Time Structure of Self-Enforcing Agreements", *Econometrica*, **70**, 547-582.
- SHARPE, S. (1995), "Bank Capitalization, Regulation, and the Credit Crunch: A Critical Review of the Research Findings." Finance and Economics Discussion Series paper No. 95/20, Board of Governors of the Federal Reserve System.
- STOKEY, N. L. AND LUCAS, R. E. (WITH PRESCOTT, E. C.) (1989), *Recursive Methods in Economic Dynamics*, Cambridge, MA: Harvard University Press.
- SUFI, A. (2009), "Bank Lines of Credit in Corporate Finance: An Empirical Analysis", *Review of Financial Studies*, **22** (3), 1057-1088.
- SUNDARAM, R. K. (1996), *A First Course in Optimization Theory*, Cambridge, Cambridge University Press.
- THAKOR, A. (1996), "Capital Requirements, Monetary Policy, and Aggregate Bank Lending: Theory and Empirical Evidence," *Journal of Finance*, **51** (1), 279-324.
- THOMAS, J. AND WORRALL, T. (1994), "Foreign Direct Investment and the Risk of Expropriation", *Review of Economic Studies*, **61**, 81-108.
- THOMAS, J. AND WORRALL, T. (2010), "Dynamic Relational Contracts with Credit Constraints", mimeo, University of Manchester.
- VON THADDEN, E-L. (2004), "Liquidity", in S. Bhattacharya, A. Boot, A. Thakor (eds.): *Credit, Intermediation, and the Macroeconomy*, Oxford University Press, 249-261.