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Abstract

We study the assignment of indivisible objects with quotas (houses, jobs, or offices) to a set of agents (students, job applicants, or professors). Each agent receives at most one object and monetary compensations are not possible. We characterize efficient priority rules by *efficiency*, *strategy-proofness*, and *reallocation-consistency*. Such a rule respects an acyclical priority structure and the allocations can be determined using the deferred acceptance algorithm.

Keywords : acyclical priority structures, indivisible objects

1 Introduction

We study a basic indivisible-objects model with a finite number of object types and a finite quota of available objects of each type. Examples are the determination of access to education, allocation of graduate housing, offices, or tasks. Agents have strict preferences over object types and remaining unassigned. An assignment is an allocation of the objects to the agents such that every agent receives at most one object and quotas are binding. A rule associates an assignment to each preference profile. When the quota of each object type is one, this problem is known as house allocation. A number of recent papers studied the house allocation problem (for example, Abdulkadiroğlu and Sönmez (1999), Svensson (1999), Pápai (2000), Ergin (2000), Bogomolnaia and Moulin (2001), Ehlers (2002), and Ehlers, Klaus, and Pápai (2002)).[1]

In education usually a ranking of the students is obtained through an objective test such as an entry exam at a university. Then students who achieved higher test scores than others have higher priority in that university. This situation can be recorded as a strict priority ranking of individuals for each object type where $i \succ_a j$ means "i has higher priority for object type a than j." A priority structure is a collection specifying for each object type a strict priority ranking. A rule violates the priority of agent i for object a if there is a preference profile under which i envies agent j who obtains a even though i has a higher priority for a than j. A rule respects a priority structure if it never violates the specified priorities. Gale and Shapley's (1962) deferred acceptance algorithm is a so-called best (efficient) rule respecting a given priority structure. This means that any assignment, which does not violate any priority of any agent, is Pareto-dominated by the assignment calculated by the deferred acceptance algorithm. Loosely speaking, Ergin's (2002, Theorem 1) main result demonstrates that for rules that respect a fixed priority structure, efficiency, group strategy-proofness[2], consistency[3], and the acyclicity[4] of the priority structure are all equivalent.

In Ergin (2002) a priority structure is exogenously given. We drop this assumption and allow for all rules. We say that a rule is a priority rule

[1] This list is not exhaustive.

[2] Group strategy-proofness means that no group of agents can profit by joint misrepresentation of their preferences such that all members of the group weakly gain and one member of the group strictly gains.

[3] We discuss consistency in Section 2. To be precise, in his characterization Ergin (2002, Theorem 1) requires consistency to hold for the so-called extended best rule.

[4] A formal definition of acyclicity is given in Section 3.

if there exists an endogenously given priority structure such that this rule chooses the same allocations that the deferred acceptance algorithm finds using that priority structure. Our main result is that a rule satisfies efficiency, strategy-proofness, and reallocation-consistency if and only if it is an efficient priority rule. In other words, any rule satisfying our combination of axioms is a best rule for an endogenously given acyclical priority structure.

Here the third property is a stability condition requiring that when a set of agents leaves with their allotments, then their assignments should remain unchanged when applying the same rule to the reallocation problem that consists of these agents and their allotments. For instance, in Germany medical students are assigned to universities through a centralized rule each year. Some students may wonder if they can improve their assignments by reallocation among themselves. However, if their positions are reallocated among themselves using the same rule, then by reallocation-consistency the assignments would not change.

The paper is organized as follows. In Section 2 we introduce the model and our axioms. In Section 3 we define priority rules and present the characterization of efficient priority rules. Section 4 contains some concluding remarks.

2 Object Allocation with Quotas

Let N denote the finite set of agents. Let A denote the finite set of indivisible object types. Given object type $a \in A$, let $q_a \geq 1$ denote the number of available objects, or quota, of type a. Let $q \equiv (q_a)_{a \in A}$. Let 0 represent the null object. Not receiving any object is called "receiving the null object." The null object does not belong to A and is available in any economy.

Each agent $i \in N$ is equipped with a strict preference relation R_i over $A \cup \{0\}$. In other words, R_i is a linear order[5] over $A \cup \{0\}$. Given $x, y \in A \cup \{0\}$, $x \, P_i \, y$ means that agent i strictly prefers x to y under R_i. Let $\mathcal{R}$ denote the set of all linear orders over $A \cup \{0\}$. Let $\mathcal{R}^N$ denote the set of all (preference) profiles $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. Given $R \in \mathcal{R}^N$ and $M \subseteq N$, let R_M denote the restriction of R to M. We also use the notation $R_{-i} = R_{N \setminus \{i\}}$. For example, $(\tilde{R}_i; R_{-i})$ denotes the profile obtained from R by replacing R_i by $\tilde{R}_i$.

An economy consists of a set of agents $N' \subseteq N$, their preferences $R' \in \mathcal{R}^{N'}$, and a vector of quotas $q' = (q'_a)_{a \in A}$ such that for all $a \in A$, $q_a \geq q'_a \geq 0$. We suppress the set of agents and denote this economy by $(R'; q')$.

[5] A linear order is a complete, reflexive, transitive, and antisymmetric binary relation.

When allocating objects each agent either receives an object of type $a \in A$ or the null object. The null object can be assigned to several agents without any restriction, but for all other objects the associated quota is binding. Formally, given an economy (R', q') an allocation for (R', q') is a list $\alpha = (\alpha_i)_{i \in N'}$ such that for all $i \in N'$, $\alpha_i \in A \cup \{0\}$, and for all $a \in A$, $|\{i \in N' : \alpha_i = a\}| \leq q'_a$. Note that not all available objects need to be assigned. Given $i \in N'$, we call α_i the allotment of agent i at α. An unrestricted (allocation) rule is a function φ that assigns to each economy (R', q') an allocation $\varphi(R', q')$.

We are only interested in economies where all agents are present and all objects are available with quotas q and all economies that result as reallocation problems from those economies. Therefore, we restrict any unrestricted rule to these economies. The set of admissible economies with agent set N is $\mathcal{E}^N = \{(R, q) : R \in \mathcal{R}^N\}$.

Given an unrestricted rule φ, we consider situations where, departing from an economy in $\mathcal{E}^N$, some agents may want to reallocate the objects assigned to them under φ. Given $R \in \mathcal{R}^N$ and $N' \subsetneq N$, let $r^{\varphi}_{N'}(R, q)$ denote the reallocation problem that the agents N' face after having left the economy (R, q) with their allotments at $\varphi(R, q)$. Formally, $r^{\varphi}_{N'}(R, q)$ denotes the economy $(R_{N'}, q')$ where $q'_a = |\{i \in N' : \varphi_i(R, q) = a\}|$ for all $a \in A$. Note that in a reallocation problem there are at most as many objects available as agents are present. Given an unrestricted rule φ, the set of admissible economies (or reallocation problems) with agent set $N' \subsetneq N$ is $\mathcal{E}^{N'}_{\varphi} = \{r^{\varphi}_{N'}(R, q) : R \in \mathcal{R}^N\}$. Slightly abusing notation, we write $\mathcal{E}^N_{\varphi}$ instead of $\mathcal{E}^N$.

Starting from an unrestricted rule φ, we consider the restriction of φ to all its admissible economies. Again slightly abusing notation, we use the same symbols for the restricted and the unrestricted rule. An (allocation) rule is a function φ that assigns to all $N' \subseteq N$ and all admissible economies $(R', q') \in \mathcal{E}^{N'}_{\varphi}$ an allocation $\varphi(R', q')$. Note that different unrestricted rules could induce the same rule.

Next, we introduce our main properties for rules. First, a rule chooses only (Pareto) efficient allocations.

Efficiency: For all $R \in \mathcal{R}^N$, there is no allocation $\alpha = (\alpha_i)_{i \in N}$ for (R, q) such that for all $i \in N$, $\alpha_i \, R_i \, \varphi_i(R, q)$, and for some $j \in N$, $\alpha_j \, P_j \, \varphi_j(R, q)$.

Second, no agent ever benefits from misrepresenting his preference relation.

Strategy-Proofness: For all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\varphi_i(R;q) \; R_i \; \varphi_i((R'_i, R_{-i});q)$.

Note that as in Ergin (2002) we only require efficiency and strategy-proofness when all agents belonging to N are present and all objects are available with their maximal quotas in the economy.

Our last property is a stability condition for the allocations chosen by the rule when all agents are present. Suppose after the objects have been allocated, some agents decide to reallocate their allotments among themselves. What if the same rule is applied to the "reallocation problem"? The rule is "unstable" if its assignment to the agents in the reallocation problem differs from its original allotments to them. Here we are only interested in reallocation problems that are derived from economies in which all agents are present and all objects are available with quotas q.

Reallocation-Consistency: For all $R \in \mathcal{R}^N$, all $N' \subsetneq N$, and all $i \in N'$, $\varphi_i(R;q) = \varphi_i(r^{\varphi}_{N'}(R;q))$.

At first glance one may think that reallocation-consistency is equivalent to the "generic" consistency property for this model defined as follows: Suppose a group of agents leave with their allotments. Then the reduced economy consists of the remaining agents and the remaining resources (the allotments of the remaining agents and all unassigned objects). A rule is consistent if the allotments to the remaining agents do not change when the rule is applied to the reduced economy.[6] In a reduced economy there may be some unassigned objects in addition to the remaining agents' allotment – an incidence that cannot occur in a reallocation problem where agents can only reallocate their allotments among themselves. Ergin (2000) studies consistency for the house allocation problem in various combinations with efficiency, converse consistency, neutrality, and anonymity. Ehlers and Klaus (2002) analyze consistency in combination with efficiency and strategy-proofness. In models where always all resources are assigned, both properties are indeed the same.

Note that when considering efficiency, strategy-proofness, and reallocation-consistency in their present form, we can only derive conclusions for economies with agent set N and full quotas q and for all reallocation

[6] **Consistency:** For all $R \in \mathcal{R}^N$, all $N' \subsetneq N$, and all $i \in N'$, $\varphi_i(R;q) = \varphi_i(R_{N'};\tilde{q})$ where $\tilde{q}_a = q_a - |\{j \in N \setminus N' : \varphi_j(R;q) = a\}|$ for all $a \in A$.

problems that are induced from these economies by the unrestricted rule. We do not require strategy-proofness for reallocation problems since agents revealed their preferences before reallocation and it is not possible for them to change them. For instance, our axioms do not impose any requirements on any economy in which not all agents are present and more objects are available than agents. This is why we restricted the domain of a rule to all its admissible economies. As in Ergin (2002) we require that when all agents are present, then all objects are available with quotas q. For example, for a serial dictatorship where agent 1 is the first dictator, it is not meaningful to consider (sub)economies as reallocation problems in which agent 1 is present but the quota of his favorite object type is 0. Such economies simply do not arise as reallocation problems for this rule (and our axioms do not impose any requirements on them).

3 Priority Rules

Given $a \in A$, let $\succ_a$ denote a linear order over N. We call $\succ_a$ a priority ordering for object type a. A priority structure is a profile $\succ = (\succ_a)_{a \in A}$ specifying for each object type a priority ordering. Given $N' \subseteq N$, an economy $(R'; q')$, $i \in N'$, $a \in A$, and a priority structure $\succ$, an allocation α for $(R'; q')$ violates the priority of i for a if there exists $j \in N'$ such that $\alpha_j = a$, $i \succ_a j$, and $a P_i \alpha_i$ (i.e., i has higher priority for object type a than j but j receives a and i envies j). A rule φ respects a priority structure $\succ$ if for all $N' \subseteq N$ and all $(R'; q') \in \mathcal{E}^{N'}$, $\varphi(R'; q')$ does not violate the priority of any agent for any object type.[7]

Given a priority structure $\succ$ and $R \in \mathcal{R}^N$, Balinski and Sönmez (1999, Theorem 2) show that the deferred acceptance (DA) algorithm applied to $\succ$ and $(R; q)$ yields the best allocation among all allocations which do not violate the priority of any agent for any object type. In other words, if an allocation respects $\succ$ at the economy $(R; q)$, then it is Pareto-dominated by the allocation calculated by the DA-algorithm. Let $f^{\succ}$ denote the deferred acceptance rule with priority structure $\succ$. Note that under $f^{\succ}$ the agents propose to object types and, using $\succ_a$, object type a rejects agents once the quota is filled. Formally, given $N' \subseteq N$ and an economy $(R'; q')$ with agent set N', the allocation $f^{\succ}(R'; q')$ is determined as follows:

- At the first step every agent in N' "proposes" to his favorite object type in $A \cup \{0\}$. For each object type a, the q'_a applicants who have

[7] Ergin (2002) uses the expression "a rule adapts to a priority structure" instead of "a rule respects a priority structure".

the highest priority under $\succ_a$ (all if there are fewer than q'_a) are placed on the waiting list of a, and the others are rejected.

- At the lth step every newly rejected agent proposes to his next best object type in $A \cup \{0\}$. For each object type a, the q'_a applicants who have the highest priority under $\succ_a$ (all if there are fewer than q'_a) among the new applicants and those on the waiting list are placed on the new waiting list and the others are rejected.

- The algorithm terminates when every agent belongs to a waiting list. Then object $a \in A$ is assigned to the agents on the waiting list of a.

Note that any agent who proposes to the null object is immediately accepted. Although the DA-algorithm calculates for each economy the best allocation among the allocations that respect the priority structure, the deferred acceptance rule may not be efficient.[8] Ergin (2002) identifies a necessary and sufficient condition on a priority structure such that the deferred acceptance rule yields an efficient allocation for all economies with agent set N.

Given a priority structure $\succ$, a cycle is constituted of distinct $a, b \in A$ and $i, j, k \in N$ such that the following are satisfied

(C) Cycle condition: $i \succ_a j \succ_a k$ and $k \succ_b i$ and

(S) Scarcity condition: there exist (possibly empty) disjoint sets $N_a, N_b \subseteq N \setminus \{i, j, k\}$ such that $N_a \subseteq \{l \in N : l \succ_a j\}$, $N_b \subseteq \{l \in N : l \succ_b i\}$, $|N_a| = q_a - 1$, and $|N_b| = q_b - 1$.

A priority structure is acyclical if no cycles exist.

If quotas are all equal to 1, then the scarcity condition is automatically satisfied. For other quotas, the scarcity condition limits the definition of a cycle to cases where there indeed exist economies in E^N such that agents i, j, and k actually compete for objects a and b (in the absence of this competition, e.g., because the quotas in fact do not limit the access of the agents to objects a and b, a cycle will not lead to the violation of either efficiency or the given priorities – see Ergin (2002) for further discussion).

Proposition 1 (Ergin, 2002, Theorem 1). *Let $\succ$ be a priority structure. Then $f^{\succ}$ is efficient if and only if $\succ$ is acyclical.*

[8] See Roth and Sotomayor (1990, Example 2.31).

We say that a rule φ is a priority rule if there exists a priority structure $\succ$ such that $\varphi = f^{\succ}$. We call a rule φ an efficient priority rule if there exists an acyclical priority structure $\succ$ such that $\varphi = f^{\succ}$.

While Ergin (2002) focuses on the class of rules that respect an exogenously given priority structure, we consider all rules. Our main result shows that if a rule satisfies efficiency, strategy-proofness, and reallocation-consistency, then it is a best rule for an endogenously given acyclical priority structure.

Theorem 1. *Efficient priority rules are the only rules satisfying efficiency, strategy-proofness, and reallocation-consistency.*

Before proving Theorem 1 we establish the independence of the axioms. The rule that assigns the null object to all agents for all admissible economies satisfies strategy-proofness and reallocation-consistency, but not efficiency.

Let $\succ$ denote the priority structure such that for all $a \in A$, $1 \succ_a 2 \succ_a \cdots \succ_a |N|$. Let $\succ'$ denote the priority structure such that for all $a \in A$, $2 \succ'_a 3 \succ'_a \cdots \succ'_a |N| \succ'_a 1$. Given $b \in A$, let φ^b be the rule such that for all $N' \subseteq N$ and all $(R', q') \in \mathcal{E}^{N'}_{\varphi^b}$, (i) if $1 \in N'$ and $b \, P'_1 \, 0$, then $\varphi^b(R', q') \equiv f^{\succ}(R', q')$ and (ii) otherwise $\varphi^b(R', q') \equiv f^{\succ'}(R', q')$. Then φ^b satisfies efficiency and reallocation-consistency, but not strategy-proofness.

Given $\succ$ and $\succ'$ as above, define φ as follows: (i) for all $R \in \mathcal{R}^N$, $\varphi(R, q) \equiv f^{\succ}(R, q)$ and (ii) for all $N' \subsetneq N$ and all $(R', q') \in \mathcal{E}^{N'}_{\varphi}$, $\varphi(R', q') \equiv f^{\succ'}(R', q')$. Then φ satisfies efficiency and strategy-proofness, but not reallocation-consistency.

Proof of Theorem 1

Let φ be an efficient priority rule. Then there exists an acyclical priority structure $\succ$ such that $\varphi = f^{\succ}$. Since any deferred acceptance rule is strategy-proof it follows that φ is strategy-proof. To show reallocation-consistency, let $R \in \mathcal{R}^N$. Because $\succ$ is acyclical, $f^{\succ}$ is efficient. Thus, for all $a \in A$, if $|\{i \in N : f^{\succ}_i(R, q) = a\}| < q_a$, then for all $i \in N$, $\varphi_i(R, q) \, R_i \, a$. For all $a \in A$, let $q'_a \equiv |\{i \in N : \varphi_i(R, q) = a\}|$. When calculating $f^{\succ}(R, q)$ the waiting list of any object type contains at any step at most q'_a applicants. Thus, applying the DA-algorithm to (R, q') yields $f^{\succ}(R, q)$. Hence,

$$f^{\succ}(R, q') = f^{\succ}(R, q). \qquad (1)$$

By definition of q', at $f^{\succ}(R, q')$ all objects are assigned. Because $\succ$ is acyclical, $f^{\succ}$ is consistent (Ergin (2002), Theorem 1). Now for all $N' \subsetneq N$ and all $i \in N'$,

$$\varphi_i(R;q) = f_i^{\succ}(R;q) = f_i^{\succ}(R;q') = f_i^{\succ}(r^{\varphi}_{N'}(R;q)) = \varphi_i(r^{\varphi}_{N'}(R;q));$$

where the first and the last equality follow from $\varphi = f^{\succ}$, the second from (1), and the third from the facts that at $f^{\succ}(R;q')$ all objects are assigned and $f^{\succ}$ is consistent.[9] Hence, φ satisfies reallocation-consistency.

Conversely, let φ be a rule satisfying efficiency, strategy-proofness, and reallocation-consistency. First we construct for each object type a priority ordering. Second we show that the constructed priority structure is acyclical. Third we prove that φ and the best rule respecting the constructed priority structure coincide.

Given $x \in A \cup \{0\}$, fix $R^x \in \mathcal{R}^N$ such that for all $i \in N$ and all $y \in (A \cup \{0\}) \setminus \{x\}$, $x R_i^x 0 R_i^x y$. Given $a \in A$, we define $\succ_a$ inductively as follows:

Step 1: For all $i, j \in N$, (a) if $\varphi_i(R^a;q) = a \neq \varphi_j(R^a;q)$, then $i \succ_a j$, and (b) if $\varphi_i(R^a;q) = a = \varphi_j(R^a;q)$ and $i < j$, then $i \succ_a j$.

If $q_a \geq |N| - 1$, then for all distinct $i, j \in N$ we have $i \succ_a j$ or $j \succ_a i$ and $\succ_a$ is completely defined. If $q_a < |N| - 1$, then it is possible that for distinct $i, j \in N$, $\varphi_i(R^a;q) = 0 = \varphi_j(R^a;q)$. To define $\succ_a$ in these cases, we extend the definition inductively.

Step 2: Suppose $q_a < |N| - 1$. Because $q_a \geq 1$, there exists $l_1 \in N$ such that for all $i \in N \setminus \{l_1\}$, $l_1 \succ_a i$. Then for all $i, j \in N \setminus \{l_1\}$, if $\varphi_i((R^0_{l_1}, R^a_{-l_1});q) = a \neq \varphi_j((R^0_{l_1}, R^a_{-l_1});q)$, then $i \succ_a j$. If $q_a \geq |N| - 2$, then for all distinct $i, j \in N$ we have $i \succ_a j$ or $j \succ_a i$.

Step 3: Suppose $q_a < |N| - 2$. Because $q_a \geq 1$, there exists $l_2 \in N \setminus \{l_1\}$ such that for all $i \in N \setminus \{l_1, l_2\}$, $l_2 \succ_a i$. Then for all $i, j \in N \setminus \{l_1, l_2\}$, if $\varphi_i((R^0_{\{l_1,l_2\}}, R^a_{N \setminus \{l_1,l_2\}});q) = a \neq \varphi_j((R^0_{\{l_1,l_2\}}, R^a_{N \setminus \{l_1,l_2\}});q)$, then $i \succ_a j$; etc.

After at most $n - 1$ inductive steps (if $q_a = 1$), $\succ_a$ is completely defined, i.e., for any distinct $i, j \in N$ we have $i \succ_a j$ or $j \succ_a i$.

Lemma 1. $\succ_a$ is a well-defined linear order.

Proof. First we show that $\succ_a$ is well-defined. Suppose that for some $i, j \in N$ we have both $i \succ_a j$ and $j \succ_a i$. Obviously, $i \succ_a j$ and $j \succ_a i$ cannot be defined in the same inductive step. Thus, in particular, $q_a < |N| - 1$. Without loss of generality, let $i \succ_a j$ be defined first.

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[9] By consistency, for all $i \in N'$, $f_i^{\succ}(R;q') = f_i^{\succ}(R_{N'};\tilde{q})$ where $\tilde{q}_a = q'_a - |\{j \in N \setminus N' : f_j^{\succ}(R;q') = a\}|$ for all $a \in A$. Note that $r^{\varphi}_{N'}(R;q) = (R_{N'};\hat{q})$ where $\hat{q}_a = |\{i \in N' : f_i^{\succ}(R;q) = a\}|$ for all $a \in A$. So, by construction, for all $a \in A$, $\tilde{q}_a = \hat{q}_a$. Thus, $(R_{N'};\tilde{q}) = r^{\varphi}_{N'}(R;q)$ and $f_i^{\succ}(R;q') = f_i^{\succ}(r^{\varphi}_{N'}(R;q))$.

Because $j \succ_a i$ there is some $t \in \{1, \ldots, |N| - 1\}$ such that for $L_t = \{l_1, \ldots, l_t\}$ we have $i, j \in N \setminus L_t$ and $\varphi_j((R^0_{L_t}, R^a_{N \setminus L_t}), q) = a \neq \varphi_i((R^0_{L_t}, R^a_{N \setminus L_t}), q)$. By efficiency, $\varphi_i((R^0_{L_t}, R^a_{N \setminus L_t}), q) = 0$. Let q^a denote the profile of quotas such that $q^a_a = 1$ and for all $b \in A \setminus \{a\}$, $q^a_b = 0$. Then $r^{\varphi}_{\{i,j\}}((R^0_{L_t}, R^a_{N \setminus L_t}), q) = (R^a_{\{i,j\}}, q^a)$. By reallocation-consistency,

$$\varphi_j(R^a_{\{i,j\}}, q^a) = a. \tag{2}$$

Because $i \succ_a j$ is defined before $j \succ_a i$, either

(a) there exists $L \subsetneq L_t$ such that $i, j \in N \setminus L$ and $\varphi_i((R^0_L, R^a_{N \setminus L}), q) = a \neq \varphi_j((R^0_L, R^a_{N \setminus L}), q)$ or

(b) $\varphi_i(R^a, q) = a = \varphi_j(R^a, q)$ and $i < j$ ((b) in Step 1).

If (a), then by efficiency, $\varphi_j((R^0_L, R^a_{N \setminus L}), q) = 0$. Then $r^{\varphi}_{\{i,j\}}((R^0_L, R^a_{N \setminus L}), q) = (R^a_{\{i,j\}}, q^a)$. By reallocation-consistency,

$$\varphi_i(R^a_{\{i,j\}}, q^a) = a. \tag{3}$$

By (2) and (3),

$$|\{k \in \{i,j\} : \varphi_k(R^a_{\{i,j\}}, q^a) = a\}| = |\{i,j\}| = 2 > 1 = q^a_a,$$

which contradicts the fact that $\varphi(R^a_{\{i,j\}}, q^a)$ is an allocation for $(R^a_{\{i,j\}}, q^a)$.

If (b), then by efficiency, there exists $k \in N$ such that $\varphi_k(R^a, q) = 0$ and $\varphi_k((R^0_{L_t}, R^a_{N \setminus L_t}), q) = a$. Hence, by reallocation-consistency and by similar arguments as for (a), $\varphi_i(R^a_{\{i,k\}}, q^a) = a$ and $\varphi_k(R^a_{\{i,k\}}, q^a) = a$. Similarly as in (a) this yields a contradiction.

Completeness and transitivity of $\succ_a$ follow straightforwardly from the inductive definition. $\square$

Lemma 2. The priority structure $\succ \equiv (\succ_a)_{a \in A}$ is acyclical.

Proof. Suppose that $\succ$ contains a cycle. Then there are $a, b \in A$ and $i, j, k \in N$ such that (C) $i \succ_a j \succ_a k$ and $k \succ_b i$ and (S) there exist (possibly empty) disjoint sets $N_a, N_b \subseteq N \setminus \{i, j, k\}$ such that $N_a \subseteq \{l \in N : l \succ_a j\}$, $N_b \subseteq \{l \in N : l \succ_b i\}$, $|N_a| = q_a - 1$, and $|N_b| = q_b - 1$.

Let $R \in \mathcal{R}^N$ be such that

- for all $l \in N_a$, $R_l = R^a_l$,
- for all $l \in N_b$, $R_l = R^b_l$,

- for all $l \in N \setminus (N_a \cup N_b \cup \{i, j, k\})$, $R_l = R_l^0$,
- $R_j = R_j^a$ and for all $c \in A \setminus \{a, b\}$, $b \, P_i \, a \, P_i \, 0 \, P_i \, c$ and $a \, P_k \, b \, P_k \, 0 \, P_k \, c$.

We now calculate $\varphi(R; q)$. By efficiency, $|\{l \in N : \varphi_l(R; q) = a\}| = q_a$ and $|\{l \in N : \varphi_l(R; q) = b\}| = q_b$. Because $|N_a \cup N_b \cup \{i; j; k\}| = q_a + q_b + 1$, there exists $\hat{l} \in N_a \cup N_b \cup \{i; j; k\}$ such that $\varphi_{\hat{l}}(R; q) = 0$.

If $\hat{l} \in N_a \cup \{j\}$, then by efficiency, $\varphi_k(R; q) = a$. Thus, by strategy-proofness, $\varphi_k((R_k^a; R_{-k}); q) = a$ and for some $\tilde{l} \in N_a \cup \{j\}$, $\varphi_{\tilde{l}}((R_k^a; R_{-k}); q) = 0$. By reallocation-consistency and $r^{\varphi}_{\{k; \tilde{l}\}}((R_k^a; R_{-k}); q) = ((R_k^a; R_{\tilde{l}}^a); q^a)$,

$$\varphi_k((R_k^a; R_{\tilde{l}}^a); q^a) = a \text{ and } \varphi_{\tilde{l}}((R_k^a; R_{\tilde{l}}^a); q^a) = 0: \quad (4)$$

On the other hand, since by (C) $j \succ_a k$, $N_a \cup \{j\} \subseteq \{l \in N : l \succ_a k\}$. Thus, $\tilde{l} \succ_a k$ and by definition of $\succ_a$ either

(a) there exists $L \subseteq N$ such that $\tilde{l}; k \in N \setminus L$, $\varphi_{\tilde{l}}((R_L^0; R_{N \setminus L}^a); q) = a \neq \varphi_k((R_L^0; R_{N \setminus L}^a); q)$ or

(b) $\varphi_{\tilde{l}}(R^a; q) = a = \varphi_k(R^a; q)$ and $\tilde{l} < k$ ((b) in Step 1).

If (a), then by reallocation-consistency and $r^{\varphi}_{\{k; \tilde{l}\}}((R_L^0; R_{N \setminus L}^a); q) = ((R_k^a; R_{\tilde{l}}^a); q^a)$,

$$\varphi_k((R_k^a; R_{\tilde{l}}^a); q^a) = 0 \text{ and } \varphi_{\tilde{l}}((R_k^a; R_{\tilde{l}}^a); q^a) = a: \quad (5)$$

Now (4) and (5) contradict the fact that $q_a^a = 1$.

If (b), then, because $|N_a \cup \{j; k\}| = q_a + 1$, there exists $l' \in (N_a \cup \{j\}) \setminus \{\tilde{l}\}$ such that $\varphi_{l'}(R^a; q) = 0$. Thus, by the definition of $\succ_a$, $k \succ_a l'$. This contradicts $l' \in N_a \cup \{j\} \subseteq \{l \in N : l \succ_a k\}$.

Recall that so far we have assumed that $\varphi_{\hat{l}}(R; q) = 0$ for $\hat{l} \in N_a \cup \{j\}$. If $\hat{l} \in N_b \cup \{k\}$, then by $\hat{l} \notin N_a \cup \{j\}$ we have for all $l \in N_a \cup \{j\}$, $\varphi_l(R; q) = a$. Thus, by efficiency, $\varphi_i(R; q) = b$. Then, by strategy-proofness, $\varphi_i((R_i^b; R_{-i}); q) = b$ and for some $\tilde{l} \in N \setminus \{i\}$, $\varphi_{\tilde{l}}((R_i^b; R_{-i}); q) = 0$. We have already shown that $\tilde{l} \in N_a \cup \{j\}$ yields a contradiction. Hence, $\tilde{l} \in N_b \cup \{k\}$. By reallocation-consistency and $r^{\varphi}_{\{i; \tilde{l}\}}((R_i^b; R_{-i}); q) = ((R_i^b; R_{\tilde{l}}^b); q^b)$,

$$\varphi_i((R_i^b; R_{\tilde{l}}^b); q^b) = b \text{ and } \varphi_{\tilde{l}}((R_i^b; R_{\tilde{l}}^b); q^b) = 0: \quad (6)$$

On the other hand, since by (C) $k \succ_b i$, $N_b \cup \{k\} \subseteq \{l \in N : l \succ_b i\}$. Thus, $\tilde{l} \succ_b i$. Now, similarly as before, we derive a contradiction using (6) and the definition of $\succ_b$.

Finally, if $\hat{\imath} = i$, then for all $l \in N_a \cup \{j\}$, $\varphi_l(R;q) = a$. In particular, $\varphi_j(R;q) = a$. By strategy-proofness and efficiency, $\varphi_i((R_i^a; R_{-i}); q) = 0$ and $\varphi_j((R_i^a; R_{-i}); q) = a$. By reallocation-consistency and $r'_{\{i,j\}}((R_i^a; R_{-i}); q) = ((R_i^a; R_j^a); q^a)$,

$$\varphi_j((R_i^a; R_j^a); q^a) = a \text{ and } \varphi_i((R_i^a; R_j^a); q^a) = 0. \tag{7}$$

On the other hand (C) $i \succ_a j$. Now, similarly as before, we derive a contradiction using (7) and the definition of $\succ_a$. ¤

Lemma 3. $\varphi = f^{\succ}$.

Proof. By Lemma 2, $\succ$ is acyclical. Thus, $f^{\succ}$ is reallocation-consistent. Because φ is reallocation-consistent, in showing $\varphi = f^{\succ}$ it suffices to show that for all $R \in \mathcal{R}^N$, $\varphi(R;q) = f^{\succ}(R;q)$. First we show the following claim.

Claim: If $\varphi \neq f^{\succ}$, then there exists $R \in \mathcal{R}^N$ such that $\varphi(R;q) \neq f^{\succ}(R;q)$ and for all $i \in N$, $\varphi_i(R;q) \neq f_i^{\succ}(R;q)$ implies $R_i \in \{R_i^x : x \in A\}$.

Proof of Claim: Suppose $\varphi \neq f^{\succ}$. Let $R \in \mathcal{R}^N$ be such that $\varphi(R;q) \neq f^{\succ}(R;q)$. Let $i \in N$ be such that $\varphi_i(R;q) \neq f_i^{\succ}(R;q)$ and $R_i \notin \{R_i^x : x \in A\}$. Without loss of generality, suppose $\varphi_i(R;q)\, P_i\, f_i^{\succ}(R;q)$. By efficiency, $\varphi_i(R;q) \in A$. Let $\varphi_i(R;q) = a$. Because both φ and $f^{\succ}$ are strategy-proof, we have $\varphi_i((R_i^a; R_{-i}); q) = a$ and $f_i^{\succ}((R_i^a; R_{-i}); q) = 0$. Thus, $\varphi((R_i^a; R_{-i}); q) \neq f^{\succ}((R_i^a; R_{-i}); q)$ and i's preference relation belongs to $\{R_i^x : x \in A\}$. Continuing this procedure yields the desired profile specified in the Claim.

Suppose $\varphi \neq f^{\succ}$. Then by the Claim there exists $R \in \mathcal{R}^N$ such that $\varphi(R;q) \neq f^{\succ}(R;q)$ and for all $i \in N$, $\varphi_i(R;q) \neq f_i^{\succ}(R;q)$ implies $R_i \in \{R_i^x : x \in A\}$. Because $\succ$ is acyclical, $f^{\succ}$ is efficient. Thus, by $\varphi(R;q) \neq f^{\succ}(R;q)$ and efficiency of φ, there exists $j \in N$ such that $f_j^{\succ}(R;q)\, P_j\, \varphi_j(R;q)$. By efficiency, $f_j^{\succ}(R;q) \in A$. Let $f_j^{\succ}(R;q) = a$. By our choice of R and $\varphi_j(R;q) \neq f_j^{\succ}(R;q)$, we have $R_j = R_j^a$. Hence, $\varphi_j(R;q) = 0$ and by efficiency, $|\{i \in N : \varphi_i(R;q) = a\}| = q_a$. Thus, there exists $k \in N$ such that $\varphi_k(R;q) = a \neq f_k^{\succ}(R;q)$. But then again by our choice of R we have $R_k = R_k^a$ and $f_k^{\succ}(R;q) = 0$. Thus, $r'_{\{j,k\}}(R;q) = ((R_j^a; R_k^a); q^a)$. By $\varphi_k(R;q) = a$ and reallocation-consistency,

$$\varphi_k((R_j^a; R_k^a); q^a) = a. \tag{8}$$

On the other hand $f^{\succ}$ respects $\succ$. Thus, by $f_j^{\succ}(R;q) = a$, $f_k^{\succ}(R;q) = 0$, and $a\, P_k\, 0$, we have $j \succ_a k$. Hence, by definition of $\succ_a$ either

(a) there exists $L \subseteq N$ such that $j, k \in N \setminus L$, $\varphi_j((R^0_L, R^a_{N\setminus L}), q) = a \neq \varphi_k((R^0_L, R^a_{N\setminus L}), q)$ or

(b) $\varphi_j(R^a, q) = a = \varphi_k(R^a, q)$ and $j < k$ ((b) in Step 1).

If (a), then by reallocation-consistency and $r^{\varphi}_{\{j,k\}}((R^0_L, R^a_{N\setminus L}), q) = ((R^a_j, R^a_k), q^a)$,

$$\varphi_j((R^a_j, R^a_k), q^a) = a. \tag{9}$$

Now (8) and (9) contradict the fact that $q^a_a = 1$.

If (b), then by efficiency, there must exist $l \in N$ such that $\varphi_l(R^a, q) = 0$ and $\varphi_l(R, q) = a$. Thus, $j \succ_a k \succ_a l$. If $f^{\succ}_l(R, q) = a$, then by $f^{\succ}_k(R, q) = 0$, $k \succ_a l$, and $R_k = R^a_k$, $f^{\succ}(R, q)$ does not respect $\succ$, a contradiction. Hence, $f^{\succ}_l(R, q) \neq \varphi_l(R, q)$ and by construction, $R_l = R^a_l$ and $f^{\succ}_l(R, q) = 0$. Thus, by reallocation-consistency and $r^{\varphi}_{\{j,l\}}(R, q) = ((R^a_j, R^a_l), q^a)$, $\varphi_l((R^a_j, R^a_l), q^a) = a$. Since $\varphi_l(R^a, q) = 0$ and $\varphi_j(R^a, q) = a$, $r^{\varphi}_{\{j,l\}}(R^a, q) = ((R^a_j, R^a_l), q^a)$. Thus, by reallocation-consistency for $r^{\varphi}_{\{j,l\}}(R^a, q)$, $\varphi_j((R^a_j, R^a_l), q^a) = a$. This and $\varphi_l((R^a_j, R^a_l), q^a) = a$ contradict the fact that $q^a_a = 1$. This finishes the proof. □

4 Concluding Remarks

We have shown that any rule satisfying efficiency, strategy-proofness, and reallocation-consistency is an efficient priority rule. For a designer who wishes to implement a rule satisfying these properties this means that he must choose an acyclic priority structure, which he then can use to calculate outcomes according to the deferred acceptance algorithm.

Our formulation of the axioms is identical with the one by Ergin (2002)—efficiency and strategy-proofness are only required for all economies with agent set N and quotas q and reallocation-consistency only needs to hold for all reallocation problems arising from such economies. If we defined our axioms for all economies, then the characterized (unrestricted) rules are priority rules such that the associated priority structure does not satisfy the cycle condition (C). In other words the scarcity condition (S) becomes redundant. This is because when considering the full domain, economies are admissible in which each object type is available with quota one or zero. The same is true for Ergin's (2002, Theorem 1) main result. If all economies are considered, then for best rules, efficiency, group strategy-proofness, consistency, and not satisfying (C) are all equivalent. When the quota of each object is one, then the class of efficient priority rules and the rules characterized by Ehlers, Klaus, and Pápai (2002) coincide.

Finally we remark that our result does not remain true if we restrict the domain of preferences to be the domain of preferences where any object type is strictly preferred to the null object. The following example demonstrates that then there are other rules satisfying our axioms.

Example 1. Let $N = \{1, 2, 3\}$, $A = \{a, b, c\}$, and $q = (1, 1, 1)$. Let $\mathcal{R}_0$ denote the domain of all preference relations $R_i \in \mathcal{R}$ such that for all $x \in A$, xP_i0. The set of all admissible economies with agent set N is $\mathcal{E}_0^N = \{(R, q) : R \in \mathcal{R}_0^N\} (= \mathcal{E}^N$ in this example). Let $\succ$ be the priority structure such that for all $x \in A$, $1 \succ_x 2 \succ_x 3$. Let $\succ'$ be the priority structure such that for all $x \in A$, $1 \succ'_x 3 \succ'_x 2$. For all $N' \subseteq N$ and all $(R', q') \in \mathcal{E}^{N'}$, (i) if ($1 \in N'$ and $f_1^{\succ}(R', q') = b$) or ($1 \notin N'$ and $q'_b = 0$), then $\varphi(R', q') \equiv f^{\succ}(R', q')$, and (ii) otherwise $\varphi(R', q') \equiv f^{\succ'}(R', q')$. Then φ satisfies efficiency, strategy-proofness, and reallocation-consistency, but φ is not an efficient priority rule. □

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