Centre interuniversitaire de recherche
en économie quantitative

CIREQ

Cahier 11-2006

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Ce cahier a également été publié par le Département de sciences économiques de l'Université de Montréal sous le numéro 2006-14.

This working paper was also published by the Department of Economics of the University of Montreal under number 2006-14.

Dépôt légal - Bibliothèque nationale du Canada, 2006, ISSN 0821-4441
Dépôt légal - Bibliothèque et Archives nationales du Québec, 2006
ISBN-13 : 978-2-89382-533-8; ISBN-10 : 2-89382-533-5

Single-peaked choice*

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September 2006

Abstract

Single-peaked preferences have played an important role in the literature ever since they were used by Black (1948) to formulate a domain restriction that is sufficient for the exclusion of cycles according to the majority rule. In this paper, we approach single-peakedness from a choice-theoretic perspective. We show that the well-known axiom independence of irrelevant alternatives (a form of contraction consistency) and a weak continuity requirement characterize a class of single-peaked choice functions. Moreover, we examine the rationalizability and the rationalizability-representability of these choice functions. *Journal of Economic Literature* Classification Nos.: D11, D71.

Keywords: Single-peakedness, choice functions, rationalizability, representability.

*We thank John Weymark for comments. Financial support from the Research Institute CIREQ of the Université de Montréal, the Research School METEOR of the University of Maastricht and the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged.

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1 Introduction

Single-peaked preferences have played an important role in the literature ever since they were used by Black (1948) to formulate a domain restriction that is sufficient for the exclusion of cycles according to the majority rule; see also Inada (1969) and Sen (1970) for early contributions that employ domain assumptions of that nature. An example for more recent applications is the area devoted to the study of *strategy-proofness,* where this restriction on preferences has proven to allow for several classes of possible collective choice rules in one-dimensional policy spaces; see, for example, Moulin (1980), Sprumont (1991), Barberà, Gul and Stacchetti (1993), Barberà and Jackson (1994), Ehlers and Storcken (2002), and Dutta, Peters and Sen (2003), to name but a few. Single-peakedness (or more specific notions such as *spatial* or *Euclidean* preferences) can be defined in higher dimensions as well; see, for instance, Le Breton and Weymark (2006) for a detailed discussion. They arise naturally in many economic models, e.g. by maximizing a strongly quasi-concave utility function on a linear budget set in consumer theory. Ballester and Haeringer (2006) provide a characterization of (one-dimensional) single-peaked preference *profiles.* They examine the question under what conditions there exists a single ranking of the alternatives such that all preferences within the profile are single-peaked with respect to this ranking.

In this paper, we approach single-peakedness from a choice-theoretic perspective. The universal set of alternatives is represented by a Euclidean space (of fixed but arbitrary dimension), and a choice function assigns a unique chosen alternative to each feasible set within the domain of this function. We assume that the domain consists of all non-empty, compact and convex subsets of the Euclidean space, an assumption that is standard in the context of choice in economic environments. Moreover, we examine the rationalizability and the rationalizability-representability of these choice functions.

Unlike Ballester and Haeringer (2006), we consider a single choice function of a single decision maker and we show that the conjunction of the well-known axiom independence of irrelevant alternatives (IIA: a form of contraction consistency introduced by Nash, 1950, in the context of bargaining problems) and a weak continuity requirement characterizes a class of single-peaked choice functions. Moulin (1984) examines social choice functions defined on a domain involving single-peaked preference profiles that satisfy suitably formulated versions of independence properties. As a by-product of his analysis he obtains a characterization of single-peaked choice from closed intervals within $[0, 1]$ based on IIA and continuity (see Remark 1 in Moulin, 1984). This is closely related to our results, applied to the one-dimensional case.

Our notion of single-peaked choice is based on the following idea. Consider two distinct points x and y and suppose x is the choice from some feasible set also containing y. Now consider a point w on the half-line starting at x and passing through y and a point v in the interval $[x, w]$. Single-peakedness demands that v is chosen from the interval $[v, w]$. This requirement is motivated by the observation that if a point x is chosen over another point y and all points on the line segment joining the two are feasible (which is implied by the convexity assumption), then the choice from an interval included in this line segment should be the point closest to x. Moreover (with some abuse of language) single-peakedness requires that there is at most one 'peak,' that is, a point that is always chosen when it is feasible and for all feasible sets excluding the peak, a boundary point must be chosen. This definition specializes to the rationalizability of the choice function by a single-peaked preference relation in the one-dimensional case.

Rationalizability means that the choice function can be represented by a transitive binary relation. By constructing a specific class of single-peaked choice functions we show that for any $n \geq 3$ there are choice functions with exactly one peak that have no revealed preference cycles of length n but that do have such cycles of length $n + 1$. The construction involves classical geometric arguments. The situation is somewhat better for single-peaked choice functions without a peak – i.e., with an 'infinite' peak. In the two-dimensional case without a peak, acyclicity of the revealed preference relation is implied by independence of irrelevant alternatives and our weak continuity assumption. Our results on representability state that under the assumptions of acyclicity and (strong) continuity a representing strongly quasi-concave utility function exists.

In the next section, we introduce our basic definitions. Section 3 contains our characterization result for single-peaked choice functions. In Section 4, we examine the rationalizability of these choice functions, and Section 5 provides results regarding the existence of real-valued representations of such rationalizations. A brief concluding section summarizes.

2 Preliminaries

We consider choices from sets of k-dimensional vectors, where $k \in \mathbb{N}$ is arbitrary but fixed. For a non-empty set $C \subseteq \mathbb{R}^k$, $\mathrm{conv}(C)$ denotes the convex hull of C, and $\mathrm{bd}(C)$ denotes the (topological) boundary of C. If $C = \{x, y\}$ for some $x, y \in \mathbb{R}^k$, we also write $[x, y]$ instead of $\mathrm{conv}(C)$ and refer to this set as an *interval*. The (relatively) open or half-open sets $(x, y]$, $[x, y)$, and (x, y) are defined in the obvious way. For distinct $x, y \in \mathbb{R}^k$, $[x, y, \rightarrow)$ denotes the half-line through y starting at x and $\ell(x, y)$ denotes the straight line through x and y. The convergence of a sequence of subsets of $\mathbb{R}^k$ is defined in terms of the Hausdorff metric.

For a (binary) relation $R \subseteq \mathbb{R}^k \times \mathbb{R}^k$, we use P to denote the *asymmetric factor* of R, that is, xPy if and only if xRy and $\neg yRx$ for all $x, y \in \mathbb{R}^k$. A relation R on $\mathbb{R}^k$ is: (i) *reflexive* if xRx for all $x \in \mathbb{R}^k$; (ii) *complete* if xRy or yRx for all $x, y \in \mathbb{R}^k$ such that $x \neq y$; (iii) *transitive* if $[xRy$ and $yRz]$ implies xRz for all $x, y, z \in \mathbb{R}^k$; (iv) *antisymmetric* if $[xRy$ and $yRx]$ implies $x = y$ for all $x, y \in \mathbb{R}^k$; (v) *acyclic* if there exist no $m \in \mathbb{N} \setminus \{1, 2\}$ and $x^1, \ldots, x^m$ such that $x^i P x^{i+1}$ for all $i \in \{1, \ldots, m\}$, where $x^{m+1} := x^1$.

The set of all non-empty, compact and convex subsets of $\mathbb{R}^k$ is denoted by $\mathcal{C}$. A *choice function* is a mapping $\varphi\colon \mathcal{C} \rightarrow \mathbb{R}^k$ such that $\varphi(C) \in C$ for all $C \in \mathcal{C}$. In particular, this means that φ is single-valued and is defined for *every* non-empty, compact and convex subset of $\mathcal{C}$, including single points. A choice function φ on $\mathcal{C}$ induces a relation R_φ on $\mathbb{R}^k$ defined by

$$xR_\varphi y :\Leftrightarrow \exists C \in \mathcal{C} \text{ such that } y \in C \text{ and } \varphi(C) = x$$

for all $x, y \in \mathbb{R}^k$. The relation R_φ is called the *direct revealed preference relation* corresponding to the choice function φ. Due to our domain assumption (in particular, because $\{x\} \in \mathcal{C}$ for all $x \in \mathbb{R}^k$), R_φ is reflexive. The (indirect) revealed preference relation $\overline{R_\varphi}$ corresponding to φ is the transitive closure of R_φ, that is,

$$\begin{aligned} x\overline{R_\varphi}y \quad :\Leftrightarrow \quad & \exists m \in \mathbb{N} \setminus \{1\} \text{ and } x^1, \ldots, x^m \in \mathbb{R}^k \text{ such that} \\ & x = x^1,\ x^i R x^{i+1}\ \forall i \in \{1, \ldots, m-1\} \text{ and } x^m = y \end{aligned}$$

for all $x, y \in \mathbb{R}^k$.

The following properties of a choice function $\varphi\colon \mathcal{C} \rightarrow \mathbb{R}^k$ are of importance in this paper.

Independence of Irrelevant Alternatives (IIA). For all $C, D \in \mathcal{C}$,

$$C \subseteq D \text{ and } \varphi(D) \in C \;\Rightarrow\; \varphi(C) = \varphi(D).$$

IIA is the standard contraction-independence property for single-valued choice functions; see Nash (1950) for its application in a bargaining framework.

Continuity (CON). For all $C \in \mathcal{C}$ and for all sequences $\langle C^i \rangle_{i \in \mathbb{N}}$ with $C^i \in \mathcal{C}$ for all $i \in \mathbb{N}$,

$$\lim_{i \to \infty} C^i = C \;\Rightarrow\; \lim_{i \to \infty} \varphi(C^i) = \varphi(C).$$

For our purposes, the following weaker version of continuity is of interest. In this weakening the continuity axiom applies to intervals along the same line only.

Collinear Interval Continuity (CIC). For all distinct $x, y \in \mathbb{R}^k$ and for all sequences $\langle x^i \rangle_{i \in \mathbb{N}}$ and $\langle y^i \rangle_{i \in \mathbb{N}}$ with $x^i, y^i \in \ell(x, y)$ for all $i \in \mathbb{N}$,

$$\lim_{i \to \infty} x^i = x \text{ and } \lim_{i \to \infty} y^i = y \;\Rightarrow\; \lim_{i \to \infty} \varphi([x^i, y^i]) = \varphi([x, y]).$$

If φ satisfies IIA, then $xR_\varphi y$ implies $\neg yR_\varphi x$ for all $x, y \in \mathbb{R}^k$ such that $x \neq y$: this is so since, under IIA, $xR_\varphi y$ is equivalent to $\varphi([x, y]) = x$. Thus, if φ satisfies IIA, then the direct revealed preference relation R_φ is antisymmetric.

We conclude this section with definitions of the well-known notions of *rationalizability* and *representability* of a choice function, formulated for our specific environment. A choice function $\varphi\colon \mathcal{C} \to \mathbb{R}^k$ is rationalizable if there exists a transitive relation R on $\mathbb{R}^k$ such that

$$\{\varphi(C)\} = \{x \in C \mid xRy \;\forall y \in C\}$$

for all $C \in \mathcal{C}$.[1] A choice function φ is rationalizable-representable if there exist a transitive relation R on $\mathbb{R}^k$ and a function $u\colon \mathbb{R}^k \to \mathbb{R}$ such that R rationalizes φ and

$$[xRy \;\Rightarrow\; u(x) \geq u(y)] \text{ and } [xPy \;\Rightarrow\; u(x) > u(y)]$$

for all $x, y \in \mathbb{R}^k$. A function $u\colon \mathbb{R}^k \to \mathbb{R}$ is *strongly quasi-concave* if, for all $x, z, z' \in \mathbb{R}^k$ with $z \neq z'$ and $u(z), u(z') \geq u(x)$ and for all $0 < \alpha < 1$, $\alpha z + (1 - \alpha) z'$ is an interior point of the set $\{y \in \mathbb{R}^k \mid u(y) \geq u(x)\}$. In other words, u is strongly quasi-concave if its so-called upper contour sets are strictly convex.

[1] Requiring the rationalizing relation to be reflexive, complete and transitive leads to an equivalent formulation of rationalizability; see Richter (1966, 1971).

3 Single-peaked choice functions

A choice function φ is *single-peaked* if

(I) for all $x, y \in \mathbb{R}^k$ such that $x \neq y$ and $xR_\varphi y$, for all $w \in [x, y, \rightarrow)$ and for all $v \in [x, w]$, $\varphi([v, w]) = v$

and either

(II.a) there exists $p \in \mathbb{R}^k$ such that $\varphi(C) = p$ for all $C \in \mathcal{C}$ with $p \in C$ and $\varphi(C) \in \mathrm{bd}(C)$ for all $C \in \mathcal{C}$ with $p \notin C$

or

(II.b) $\varphi(C) \in \mathrm{bd}(C)$ for all $C \in \mathcal{C}$.

If case (II.a) applies, the point p is called a *peak* of φ. Clearly, a single-peaked choice function has either one peak or none.

For $k = 1$, this definition reduces to rationalizability by single-peaked preferences: if there is a peak $p \in \mathbb{R}$, define the relation R by letting

$$xRy :\Leftrightarrow \; |x - p| \leq |y - p|$$

for all $x, y \in \mathbb{R}$; if there is no peak, let either

$$xRy :\Leftrightarrow \; x \leq y$$

for all $x, y \in \mathbb{R}$ or

$$xRy :\Leftrightarrow \; x \geq y$$

for all $x, y \in \mathbb{R}$, whichever case applies. Moreover, rationalizability-representability is guaranteed: for the three possibilities illustrated above, the corresponding utility function $u \colon \mathbb{R} \rightarrow \mathbb{R}$ can be defined by

$$u(x) = -|x - p|$$

for all $x \in \mathbb{R}$ or

$$u(x) = -x$$

for all $x \in \mathbb{R}$ or

$$u(x) = x$$

for all $x \in \mathbb{R}$, respectively.[2]

Single-peaked choice functions are characterized by IIA and CIC, as established in the following theorem.

[2] As mentioned in the Introduction, a similar result was derived in Moulin (1984) for choice functions defined on subsets of a closed interval. See in particular his Remark 1.

Theorem 3.1 *Let $k \in \mathbb{N}$. A choice function $\varphi\colon \mathcal{C} \to \mathbb{R}^k$ satisfies IIA and CIC if and only if φ is single-peaked.*

Proof. 'If.' Suppose φ is single-peaked.

To establish IIA, suppose that $C, D \in \mathcal{C}$ with $C \subseteq D$ and $\varphi(D) \in C$.

Case (i): There exists a peak p and $p \in C$. This implies $p \in D$ because $C \subseteq D$. By the definition of single-peaked choice, it follows that $\varphi(C) = \varphi(D) = p$.

Case (ii): There exists no peak or there exists a peak p and $p \notin C$. If there exists a peak p and $p \in D$, it follows that $p = \varphi(D) \in C$, a contradiction. Thus, if a peak p exists, it cannot be in D either. Suppose $\varphi(C) \neq \varphi(D)$. Because $C \subseteq D$, it follows that $\varphi(C) \in D$ and thus $\varphi(D) R_\varphi \varphi(C)$. Letting $x = v = \varphi(D)$ and $y = w = \varphi(C)$ in part (I) of the definition of single-peakedness, we obtain

$$\varphi([\varphi(C), \varphi(D)]) = \varphi(D). \tag{1}$$

Moreover, because $\varphi(D) \in C$ by assumption, we have $\varphi(C) R_\varphi \varphi(D)$, and interchanging the roles of $\varphi(C)$ and $\varphi(D)$ in the previous argument yields $\varphi([\varphi(C), \varphi(D)]) = \varphi(C)$. Together with (1), this contradicts our hypothesis $\varphi(C) \neq \varphi(D)$.

To prove that CIC is satisfied, let x, y be distinct elements of $\mathbb{R}^k$ and consider sequences $\langle x^i \rangle_{i \in \mathbb{N}}$ and $\langle y^i \rangle_{i \in \mathbb{N}}$ with $x^i, y^i \in \ell(x, y)$ for all $i \in \mathbb{N}$, and with $\lim_{i \to \infty} x^i = x$ and $\lim_{i \to \infty} y^i = y$. Without loss of generality, $\lim_{i \to \infty} \varphi([x^i, y^i])$ exists, and we have to prove that it is equal to $\varphi([x, y])$. To the contrary, suppose that $v := \varphi([x, y]) \neq \lim_{i \to \infty} \varphi([x^i, y^i]) =: w \in [x, y]$. By (I), $v R_\varphi w$ and in particular (without loss of generality) $\varphi([x^i, y^i]) \neq v$ and $v R_\varphi \varphi([x^i, y^i])$ for all i. If $w \in (x, y)$ then we can choose i such that $\varphi([x^i, y^i]) \in (x^i, y^i)$, so that $\varphi([x^i, y^i]) R_\varphi x^i$ and $\varphi([x^i, y^i]) R_\varphi y^i$, and hence by (I), we obtain $\varphi([x^i, y^i]) R_\varphi v$, a contradiction. If $w \notin (x, y)$ then without loss of generality $w = y$; then $v \in [x, y)$ and we can choose i such that $\varphi([x^i, y^i]) \in (v, y^i]$, so that $\varphi([x^i, y^i]) R_\varphi v$ by (I), again a contradiction.

'Only if.' Suppose φ satisfies IIA and CIC. To establish part (I) of single-peakedness, let $x R_\varphi y$ (hence $\varphi([x, y]) = x$ by IIA), $x \neq y$, $w \in [x, y, \to)$ and $v \in [x, w]$. Suppose, contrary to what we wish to show, that $\varphi([v, w]) \in (v, w]$. We distinguish two cases.

Case (i): $w \in [x, y]$. In this case, let $\bar{v}$ be the point in $[x, v]$ closest to x for which $\varphi([\bar{v}, w]) = \varphi([v, w])$; this point exists because of CIC. Suppose $\bar{v} \neq x$. Then, for $\bar{\bar{v}} \in [x, \bar{v})$ close enough to $\bar{v}$, we have $\varphi([\bar{\bar{v}}, w]) = \varphi([v, w])$

by CIC and IIA, contradicting the definition of $\bar{v}$. Hence, $\bar{v} = x$. This implies $\varphi([x, w]) = \varphi([\bar{v}, w]) = \varphi([v, w]) \neq x$, a contradiction to IIA.

Case (ii): $w \notin [x, y]$. In this case, by an argument similar to the first one in case (i), we have $\varphi([x, w]) = x$. Since $v \in [x, w]$, we are back in case (i) by assuming $y = w$ there.

To establish part (II) of single-peakedness, clearly, (II.a) and (II.b) cannot both be true. Suppose (II.b) is not true. Then there is a $C \in \mathcal{C}$ such that $\varphi(C)$ is an interior point of C, say p. So $pR_\varphi x$ for all $x \in C$. Let $y \in \mathbb{R}^k \setminus C$. Since p is an interior point of C, there is an $x \in C \setminus \{p\}$ such that $y \in [p, x, \rightarrow)$. By part (I), $pR_\varphi y$. By IIA, $\varphi(D) = p$ whenever $p \in D$. $\square$

Single-peaked choice functions do not necessarily satisfy full continuity CON. For instance, the single-peaked choice function picking the lexicographic maximum of a choice set in $\mathbb{R}^2$ does not satisfy CON (cf. also Example 5.1).

4 Rationalizability

If $k = 1$, IIA and CIC together guarantee rationalizability, as mentioned earlier. However, matters are more complex in higher dimensions, even if CIC is strengthened to CON.

Suppose $k \geq 2$. As noted before, IIA implies that R_φ is antisymmetric and, thus, there are no cycles of length two in R_φ. Conversely, the absence of cycles of length two in R_φ implies IIA. Thus, these two conditions are equivalent. Although necessary, IIA (or, equivalently, the absence of cycles of length two) is not sufficient for rationalizability unless specific domain assumptions are made. What *is* sufficient on any domain is the *strong axiom of revealed preference* which, in our setting, is equivalent to the requirement that the revealed preference relation R_φ be acyclic (or, equivalently, that the revealed preference relation $\overline{R_\varphi}$ be antisymmetric). This raises the question whether IIA (possibly together with one of our continuity properties) is sufficient to rule out cycles of arbitrary length on our domain, thus guaranteeing rationalizability. More generally, we investigate whether ruling out cycles of length $n \in \mathbb{N}$ or less is sufficient to rule out longer cycles for arbitrary $n \geq 2$. In answering this question, it turns out that we have to distinguish between the two possibilities (II.a) and (II.b) in the definition of single-peakedness.

4.1 Choice functions with a peak

In the following example we show that, for any $n \in \mathbb{N}$ with $n \geq 3$, there is a choice function with a peak (in the example the point $\mathbf{0}$), satisfying IIA and CON, such that there are no cycles of length n (or smaller) but there are cycles of length $n+1$ (or larger). This implies that, without further assumptions, there is no n such that the exclusion of n-cycles implies the exclusion of all cycles, for a choice function satisfying IIA and CON.

Example 4.1 Let $k = 2$. Let $\alpha \in [0, \pi/2]$. We are going to construct a choice function φ^α. For $C \in \mathcal{C}$ with $\mathbf{0} \in C$ let $\varphi^\alpha(C) := \mathbf{0}$. Now let $C \in \mathcal{C}$ with $\mathbf{0} \notin C$.

For every $\beta \in [0, 2\pi)$ let ℓ_β be the half-line starting from the origin and forming an angle of β radians with the positive horizontal axis. Then the set $B(C) := \{\beta \in [0, 2\pi) \mid \ell_\beta \cap C \neq \emptyset\}$ is either of the form $[\beta_1, \beta_2]$ with $0 \leq \beta_1 \leq \beta_2 < 2\pi$ or of the form $[\beta_1, 2\pi) \cup [0, \beta_2]$ with $0 \leq \beta_2 < \beta_1 < 2\pi$. For every $\beta \in B(C)$ let x_β be the point in $C \cap \ell_\beta$ closest to the origin. We define the correspondence $c\colon B(C) \to [0, \pi/2]$ as follows: for every $\beta \in B(C)$, $c(\beta)$ is the interval of the non-obtuse angles (in radians) between ℓ_β and those supporting lines of the set C at the point x_β that (weakly) separate C from the origin. Then $0 \in c(\beta_1)$ and $0 \in c(\beta_2)$. There is a unique value of β with $\pi/2 \in c(\beta)$, namely the value of β such that x_β is the point of C with minimal Euclidean distance to the origin. The correspondence c strictly increases from β_1 to this value, and then strictly decreases again to β_2. In particular, there are at most two different values of β such that $\alpha \in c(\beta)$. Let these values be β' and β'' such that $\beta' \bmod \beta_1 \leq \beta'' \bmod \beta_1$, then $\varphi^\alpha(C) := x_{\beta'}$.

In other words, if C contains the origin then φ^α chooses the origin. Otherwise, φ^α chooses a point from the boundary of C such that there is a supporting line of C at this point that forms an angle of α radians with the line through this point and the origin. The chosen point is always the point on the latter line closest to the origin, and in case there are two such points, on different lines through the origin, then φ^α takes the first one going counter clockwise. For instance, for $\alpha = \pi/2$, $\varphi^\alpha(C)$ is the point of C closest to the origin. For $\alpha = 0$, it takes the point of C closest to the origin on the first supporting line of C passing through the origin when going counter clockwise. See Fig. 1.

For every $\alpha \in [0, \pi/2]$, the choice function φ^α is well-defined and satisfies IIA. For every $\alpha \in (0, \pi/2]$, φ^α satisfies CON, but for $\alpha = 0$ it does not. $\square$

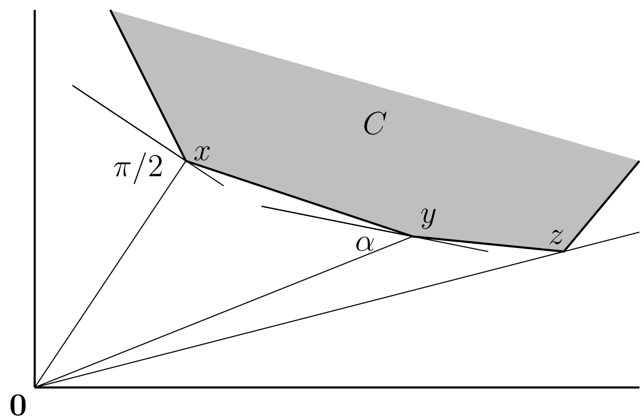

Figure 1: Illustration of the set of choice functions defined in Example 4.1. The choice set is the shaded set C, and $x = \varphi^{\pi/2}(C)$, $y = \varphi^{\alpha}(C)$ for some value α between 0 and $\pi/2$, and $z = \varphi^{0}(C)$.

We will prove a theorem which implies our earlier claim: that, for every $n \geq 3$, there is a choice function satisfying IIA and CON which has no cycles of length n or smaller, but which has cycles of length $n+1$ or larger. In the proof, we make use of the following result which is proven in Appendix A. For every $n \in \mathbb{N}$ with $n \geq 2$, define

$$A(n) = \frac{n-2}{2n}\pi \ .$$

The number $A(n)$ is equal to half the angle at a vertex in a regular n-polygon.

Theorem 4.2 *Let $k = 2$, let $\mathcal{P}$ be a convex n-polygon with vertices $x^1, \ldots, x^n$ and let $\hat{x}$ be a point of $\mathcal{P}$ such that $\measuredangle \hat{x}x^1x^2 = \measuredangle \hat{x}x^2x^3 = \ldots = \measuredangle \hat{x}x^{n-1}x^n =: \alpha < \pi/2$ and $\measuredangle \hat{x}x^nx^1 \geq \alpha$. Then $\alpha \leq A(n)$.*

The main result of this subsection is the following theorem.

Theorem 4.3 *Let $k = 2$, let $\alpha \in (0, \pi/2)$ and let $n \in \mathbb{N} \setminus \{1\}$ be such that $A(n) < \alpha \leq A(n+1)$. Then R_{φ^α} has no cycle of length n but it does have a cycle of length $n+1$.*

Proof. We first exhibit an $(n+1)$-cycle for R_{φ^α}, and then show that there are no smaller cycles.

To exhibit an $(n+1)$-cycle, take a regular $(n+1)$-polygon with $\mathbf{0}$ as center. Let $x^1, \ldots, x^{n+1}$ be the successive vertices of this polygon. Since $\measuredangle \mathbf{0}x^1x^2 = A(n+1) \geq \alpha$, it follows that $\varphi^\alpha([x^1, x^2]) = x^1$ by definition of φ^α, hence $x^1 R_{\varphi^\alpha} x^2$. This argument can be repeated for any successive pair of vertices, so that $x^1 R_{\varphi^\alpha} x^2 R_{\varphi^\alpha} \ldots R_{\varphi^\alpha} x^{n+1} R_{\varphi^\alpha} x^1$. So R_{φ^α} has an $(n+1)$-cycle.

Next, we show that there are no smaller cycles. It is sufficient to show that there are no cycles of length n. Suppose, to the contrary, that there is such a cycle

$$x^1 R_{\varphi^\alpha} x^2 R_{\varphi^\alpha} \ldots R_{\varphi^\alpha} x^n R_{\varphi^\alpha} x^1 \ .$$

Obviously, $\mathbf{0}$ is not an element of this cycle since it is the peak of φ^α. Furthermore, we may assume that for all $i, j \in \{1, \ldots, n\}$, $x^i R_{\varphi^\alpha} x^j$ implies that either $i \leq n$ and $j = i+1$ or $i = n$ and $j = n+1$ since otherwise we would have an even shorter cycle with $\ldots R_{\varphi^\alpha} x^i R_{\varphi^\alpha} x^j R_{\varphi^\alpha} \ldots$ by leaving out the points between x^i and x^j in the original cycle.

Consider x^1 and x^2. Since $x^1 R_{\varphi^\alpha} x^2$, x^2 is a point separated from $\mathbf{0}$ by the line ℓ_1 passing through x^1 and forming an angle of α radians with the line through $\mathbf{0}$ and x^1. Suppose x^2 is not on ℓ_1. Since x^3 must be on the same side of ℓ_1 as $\mathbf{0}$, we can replace x^2 by the unique point x' in $[x^2, x^3] \cap \ell_1$ and still have $x^1 R_{\varphi^\alpha} x' R_{\varphi^\alpha} x^3$. Hence, we can without loss of generality assume that x^2 is on ℓ_1. There are two cases to consider, namely (i) $\measuredangle \mathbf{0}x^1x^2 = \alpha$; and (ii) $\measuredangle \mathbf{0}x^1x^2 = \pi - \alpha$.

Case (i): Assume $\measuredangle \mathbf{0}x^1x^2 = \alpha$ and now consider x^2 and x^3. By repeating the argument of the previous paragraph, we may assume that x^3 is a point on the line ℓ_2 passing through x^2 and forming an angle of α radians with the line through $\mathbf{0}$ and x^2. We must have $\measuredangle \mathbf{0}x^2x^3 = \alpha$, since $\measuredangle \mathbf{0}x^2x^3 = \pi - \alpha$ would imply that x^3 would be separated from $\mathbf{0}$ by ℓ_1 and, thus, $x^1 R_{\varphi^\alpha} x^3$, a contradiction. See Fig. 2(i) for the construction so far.

Repeating this argument up to and including the pair x^{n-1} and x^n, we have $n-1$ lines $\ell_1, \ldots, \ell_{n-1}$, with, for each $i = 1, \ldots, n-1$, ℓ_i passing through x^i and x^{i+1} and such that each x_j ($j \in \{1, \ldots, n\} \setminus \{i, i+1\}$) and $\mathbf{0}$ are on the same side of ℓ_i but not on ℓ_i. Moreover, $\measuredangle \mathbf{0}x^ix^{i+1} = \alpha$ for each $i = 1, \ldots, n-1$. Hence, $\mathcal{P} := \mathrm{conv}\{x^1, \ldots, x^n\}$ is a convex n-polygon with vertices $x^1, \ldots, x^n$, containing $\mathbf{0}$ as an interior point, such that $\measuredangle \mathbf{0}x^ix^{i+1} = \alpha$ for each $i = 1, \ldots, n-1$ and $\measuredangle \mathbf{0}x^nx^1 \geq \alpha$. The last inequality follows since $x^n R_{\varphi^\alpha} x^1$. (Observe that we cannot assume equality here: this might involve having to replace x^1 by a different point x' but then $\measuredangle \mathbf{0}x'x^2 > \alpha$.) By applying Theorem 4.2 to $\mathcal{P}$, it follows that $\alpha \leq A(n)$, a contradiction.

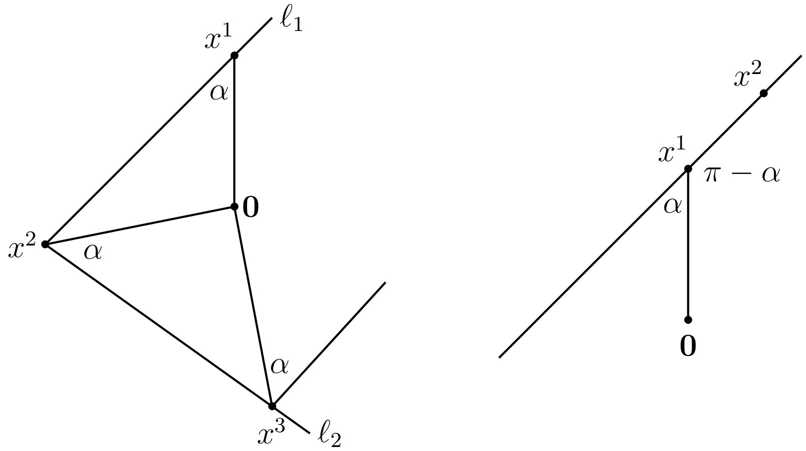

Figure 2: Illustrations for the proof of Theorem 4.3

Case (ii): Assume $\measuredangle \mathbf{0}x^1x^2 = \pi - \alpha$; see Fig. 2(ii). Since $\alpha < \pi/2$, this implies $||x^2|| > ||x^1||$ (where $||\cdot||$ denotes the Euclidean norm). Since also $\measuredangle \mathbf{0}x^2x^3 \geq \pi - \alpha$ (otherwise we would have $x^1 R_{\varphi^\alpha} x^3$, a contradiction), we have $||x^3|| > ||x^2||$. Continuing this argument, we obtain $||x^1|| < ||x^2|| < \ldots < ||x^n|| < ||x^1||$. This contradiction takes care of case (ii) and completes the proof of the theorem. $\square$

Intuitively it is clear that acyclicity of the revealed preference relation is even harder to obtain for higher dimensions. One way to extend Example 4.1 and Theorem 4.3 to $k > 2$ may be to 'embed' the choice function φ^α in higher dimensions, similar to the construction in Peters and Wakker (1994).

4.2 Choice functions without a peak

If φ is a choice function such that $\varphi(C)$ is on the boundary of C for every feasible set C (as in case (II.b) in the definition of single-peakedness), the situation is different. The first observation is that, in this case, cycles of length three cannot occur if $k = 2$. Moreover, this observation is a consequence of IIA alone – no continuity requirement is needed.

Lemma 4.4 *Let $k = 2$. If a choice function $\varphi: \mathcal{C} \to \mathbb{R}^k$ satisfies IIA and $\varphi(C) \in bd(C)$ for all $C \in \mathcal{C}$, then R_φ has no cycles of length three.*

Proof. Suppose, to the contrary, that $x^1, x^2, x^3 \in \mathbb{R}^2$ are three different points and $x^1 R_\varphi x^2 R_\varphi x^3 R_\varphi x^1$. If these three points are not on the same line, then $\varphi(\text{conv}(\{x^1, x^2, x^3\}))$ must be a point on the boundary, i.e., in $[x^1, x^2]$ or in $[x^2, x^3]$ or in $[x^3, x^1]$. In the first case, $\varphi(\text{conv}(\{x^1, x^2, x^3\})) = x^1$ by IIA, but then $x^1 R_\varphi x^3$, a contradiction. The other two cases lead to similar contradictions. If the three points are on the same line then an analogous argument applies. $\square$

The following theorem shows that adding CIC allows us to extend this result to cycles of arbitrary length.

Theorem 4.5 *Let $k = 2$. If a choice function $\varphi: \mathcal{C} \to \mathbb{R}^k$ satisfies IIA and CIC and $\varphi(C) \in bd(C)$ for all $C \in \mathcal{C}$, then R_φ is acyclic.*

Proof. The proof proceeds by induction on the cycle length. By IIA, there are no cycles of length two and by Lemma 4.4 there are no cycles of length three. Let $m \geq 4$ and assume as induction hypothesis that there are no cycles of length smaller than m. Suppose that $x^1, \ldots, x^m$ are m different points such that $x^1 R_\varphi x^2 R_\varphi \ldots R_\varphi x^m R_\varphi x^1$. Let C be the convex hull of these m points, and let $x := \varphi(C)$. Then $x \in \text{bd}(C)$, and by IIA, there are x^i and x^k with $i, k \in \{1, \ldots, m\}$ and $k \notin \{(i-1) \bmod m, i, (i+1) \bmod m\}$ such that $x \in (x^i, x^k)$ and $(x^i, x^k) \cap \{x^1, \ldots, x^m\} = \emptyset$. We now consider four cases concerning the location of x^{i+1}. (For simplicity of notation we write $i+1$ and $i-1$ instead of $(i+1) \bmod m$ and $(i-1) \bmod m$.) See Fig. 3, where these four cases are illustrated.

Case (i): x^{i+1} is a point on the line through x^i and x^k. In this case $x^{i+1} \in [x, x^i, \rightarrow)$, and in particular $x^i \in [x, x^{i+1}]$, since otherwise $x^k R_\varphi x^{i+1}$ by Theorem 3.1, a contradiction. Then for every point $y \in [x^{i-1}, x^{i+1}] \cup [x^{i+1}, x^i]$ we have $[x, y] \cap [x^{i-1}, x^i] \neq \emptyset$, hence, by Theorem 3.1, if $\hat{y}$ is the point of intersection then $\hat{y} R_\varphi y$. This implies that $\varphi(\text{conv}(\{x^{i-1}, x^i, x^{i+1}\}))$ must be a point of $[x^{i-1}, x^i]$, and hence, by IIA, $\varphi(\text{conv}(\{x^{i-1}, x^i, x^{i+1}\})) = x^{i-1}$. This, however, implies $x^{i-1} R_\varphi x^{i+1}$, so that we obtain a cycle of length $(m-1)$ by dropping the point x^i from the original cycle of length m. This contradicts the induction hypothesis.

Case (ii): x^{i+1} is not a point on the line through x^i and x^k, and the half-line $[x^i, x^{i+1}, \rightarrow)$ is in the convex hull of the half-lines $[x^i, x, \rightarrow)$ and $[x^i, x^{i-1}, \rightarrow)$. (Observe that $x^i \notin [x, x^{i-1}]$ since otherwise $x^i R_\varphi x^{i-1}$ by Theorem 3.1, a contradiction.) In this case, consider a half-line ℓ starting from x

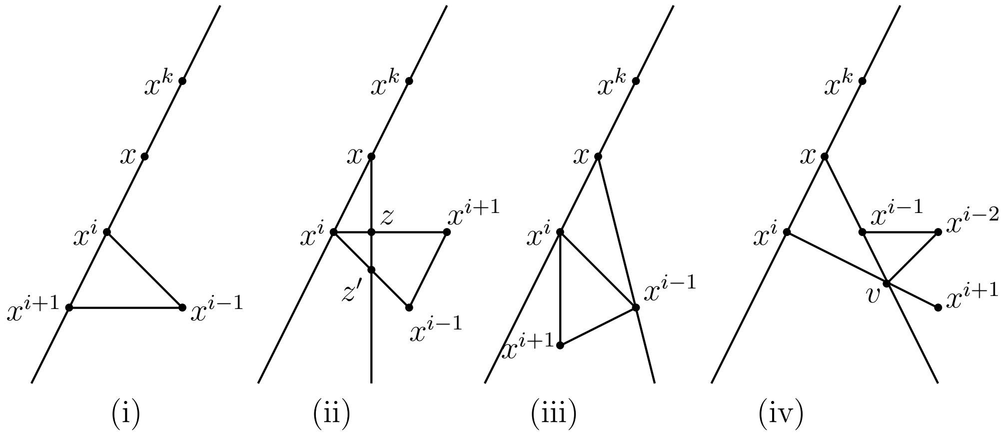

Figure 3: The four cases in the proof of Theorem 4.5

and intersecting the segments $[x, x^{i+1}]$ and $[x, x^{i-1}]$ in points z and z', respectively. Then $zR_\varphi z'$ by Theorem 3.1, $z'R_\varphi x^i$ by Theorem 3.1 since $x^{i-1}R_\varphi x^i$, and $x^i R_\varphi z$ by Theorem 3.1 since $x^i R_\varphi x^{i+1}$. Hence, we have a cycle of length three, which is a contradiction to Lemma 4.4.

Case (iii): x^{i+1} is not a point on the line through x^i and x^k, and the half-line $[x^i, x^{i-1}, \rightarrow)$ is in the convex hull of the half-lines $[x^i, x, \rightarrow)$ and $[x^i, x^{i+1}, \rightarrow)$; and x^{i+1} is separated from x^k by the line through x and x^{i-1}. (Observe that $x^{i-1} \notin [x, x^k, \rightarrow)$ since otherwise $x^{i-1}R_\varphi x^k$ by Theorem 3.1, a contradiction.) The proof for this case is identical to the proof of case (i).

Case (iv): x^{i+1} is not a point on the line through x^i and x^k, and the half-line $[x^i, x^{i-1}, \rightarrow)$ is in the convex hull of the half-lines $[x^i, x, \rightarrow)$ and $[x^i, x^{i+1}, \rightarrow)$; and x^{i+1} and x^k are on the same side of the line through x and x^{i-1}. To deal with this case, let v be the point of intersection of $[x^i, x^{i+1}]$ and $[x, x^{i-1}, \rightarrow)$. By Theorem 3.1, $vR_\varphi x^{i+1}$. Consider the point x^{i-2} (recall that $m \geq 4$). Since for any point $y \in [x^{i-1}, v] \cup [v, x^{i-2}]$ the line segment $[x, y]$ intersects the line segment $[x^{i-2}, x^{i-1}]$ in some point $\hat{y}$, Theorem 3.1 implies $\hat{y}R_\varphi y$ for any such point y. Hence $\varphi(\mathrm{conv}(\{x^{i-2}, x^{i-1}, v\})) \in [x^{i-2}, x^{i-1}]$, and therefore IIA implies $\varphi(\mathrm{conv}(\{x^{i-2}, x^{i-1}, v\})) = x^{i-2}$. We now have $\ldots R_\varphi x^{i-2} R_\varphi v R_\varphi x^{i+1} R_\varphi \ldots$, hence a cycle of length $m-1$. This contradiction takes care of case (iv) and completes the proof of the proposition. □

Unfortunately, Theorem 4.5 does not extend to higher dimensions. This can be shown by a modification of the extension of an example of Gale

(1960) for consumer theory in Peters and Wakker (1991); see also Bossert (1994). Peters and Wakker (1991) consider, for $k = 3$, the subset $\Sigma \subseteq \mathcal{C}$, consisting of all elements of $C \in \mathcal{C}$ that satisfy (i) $x \geq \mathbf{0}$ for all $x \in C$; (ii) $y \in C$ for all $x, y \in \mathbb{R}^3$ with $x \in C$ and $\mathbf{0} \leq y \leq x$; and (iii) $x > \mathbf{0}$ for all $x \in \mathrm{PO}(C)$. Here, $\mathrm{PO}(C) := \{x \in C \mid \forall y \in C, y \geq x \Rightarrow y = x\}$ is the Pareto optimal subset of C.[3] Then a choice function $\varphi\colon \Sigma \to \mathbb{R}^3$ is constructed that satisfies IIA and is continuous (and Pareto optimal), but admits a cycle of length four. It can be verified that this construction is invariant under a translation over a vector with equal coordinates. More precisely, let $\mathbf{1} := (1, 1, 1) \in \mathbb{R}^3$. Then, for any sets $C, C' \in \Sigma$ such that there is a number $\alpha \in \mathbb{R}$ with $\mathrm{PO}(C) = \{x + \alpha\mathbf{1} \mid x \in \mathrm{PO}(C')\}$, it holds that $\varphi(C) = \varphi(C') + \alpha\mathbf{1}$. Therefore, φ can be extended to all of $\mathcal{C}$ while preserving IIA and CON as follows. For any set $C' \in \mathcal{C}$, take a number $\alpha \in \mathbb{R}$ such that $C := \{x + \alpha\mathbf{1} \mid x \in C'\}$ satisfies (i) and (iii) above, and extend this set C to a set $\hat{C} := \{x \in \mathbb{R}^3 \mid \exists y \in C \text{ such that } \mathbf{0} \leq x \leq y\}$. Then $\hat{C} \in \Sigma$. Now define $\varphi(C') := \varphi(\hat{C}) - \alpha\mathbf{1}$.

5 Representability

As mentioned earlier, the one-dimensional case is special because rationalizability-representability is guaranteed; see the discussion in Section 3. Furthermore, the previous section has established that, in higher dimensions, IIA is, in general, not sufficient for rationalizability even in the presence of CON. Therefore, in order to obtain rationalizability-representability, we strengthen IIA to the acyclicity of R_φ – that is, the strong axiom of revealed preference. In this case, $\overline{R_\varphi}$ (or any of its extensions; see Szpilrajn, 1930, and Richter, 1966) can be used as a rationalization.

The continuity property CIC is not sufficient to guarantee the rationalizability-representability of φ if added to the acyclicity of R_φ. This is established in the following example.

Example 5.1 Let $k = 2$ and define the relation R on $\mathbb{R}^2$ by

$$xRy :\Leftrightarrow [|x_1| < |y_1|] \text{ or } [|x_1| = |y_1| \text{ and } |x_2| \leq |y_2|]$$

for all $x, y \in \mathbb{R}^2$. This is a single-peaked choice function with peak $p = \mathbf{0}$. Now let, for all $C \in \mathcal{C}$, $\varphi(C)$ be the unique best element in C according

[3] Of course, in our context Pareto optimality has no special appeal.

to R. This choice function is well-defined because C is non-empty, compact and convex. Thus, φ is rationalizable and R_φ is equal to R and therefore acyclic (which, of course, also implies IIA). CIC is satisfied but CON is not. Because of the lexicographic nature of this example, φ is not rationalizable-representable. $\square$

To obtain a representation theorem, we assume below that R_φ is acyclic and φ satisfies CON. We first establish two preliminary results. For $x \in \mathbb{R}^k$ and $\varepsilon > 0$, we use $\mathcal{B}(x, \varepsilon)$ to denote the open ε-ball around x, that is, the set of points in $\mathbb{R}^k$ that have Euclidean distance smaller than ε to x.

Lemma 5.2 *Let $k \geq 2$, let $x, z \in \mathbb{R}^k$ be such that $x \neq z$ and $zR_\varphi x$ and let $y \in [z, x, \rightarrow) \setminus [z, x]$. If a choice function $\varphi: \mathcal{C} \rightarrow \mathbb{R}^k$ satisfies IIA and CON, then there is an $\varepsilon > 0$ such that $xR_\varphi v$ for all $v \in \mathcal{B}(y, \varepsilon)$.*

Proof. By CON, there is a $\delta > 0$ such that $\varphi([w, x]) \in [w, x)$ for all $w \in \mathcal{B}(z, \delta)$. By Theorem 3.1, $xR_\varphi v$ for all $w \in \mathcal{B}(z, \delta)$ and $v \in [w, x, \rightarrow) \setminus [w, x]$. We can choose $\varepsilon > 0$ sufficiently small so that, for each $v \in \mathcal{B}(y, \varepsilon)$, there is a $w \in \mathcal{B}(z, \delta)$ such that $v \in [w, x, \rightarrow) \setminus [w, x]$. Then $xR_\varphi v$ for all $v \in \mathcal{B}(y, \varepsilon)$. $\square$

Lemma 5.3 *Let $k \geq 2$ and let $x, w \in \mathbb{R}^k$ be such that $x \neq w$ and $xR_\varphi w$. If a choice function $\varphi: \mathcal{C} \rightarrow \mathbb{R}^k$ satisfies IIA and CON, then there is an $a \in \mathbb{Q}^k \setminus \{x, w\}$ such that $x\overline{R_\varphi}aR_\varphi w$.*

Proof. By CON, there is a $\delta > 0$ such that $||\varphi([v, w]) - x|| < ||x - w||/3$ for all $v \in \mathcal{B}(x, \delta)$.

If there is no $z \in \mathbb{R}^k \setminus \{x\}$ with $zR_\varphi x$, then x is a peak of φ and we choose $y \in \mathcal{B}(x, \delta)$ arbitrarily. Otherwise, let $z \in \mathbb{R}^k \setminus \{x\}$ with $zR_\varphi x$ and choose $y \in \mathcal{B}(x, \delta) \cap [z, x, \rightarrow) \setminus [z, x]$. By Lemma 5.2, we can choose $\varepsilon > 0$ such that $\mathcal{B}(y, \varepsilon) \subseteq \mathcal{B}(x, \delta)$ and $xR_\varphi v$ for all $v \in \mathcal{B}(y, \varepsilon)$. Since the set $\{[v, w] \in \mathbb{R}^k \mid v \in \mathcal{B}(y, \varepsilon)\}$ has full dimension, we can take a $\bar{v} \in \mathcal{B}(y, \varepsilon)$ such that $[\bar{v}, w]$ contains a point $a \in \mathbb{Q}^k \setminus \{w\}$ with $||a - w|| < ||x - w||/3$. Since $||\varphi([\bar{v}, w]) - x|| < ||x - w||/3$ and $||a - w|| < ||x - w||/3$, Theorem 3.1 implies $\varphi([\bar{v}, w])R_\varphi a$ and $aR_\varphi w$. Since $\bar{v} \in \mathcal{B}(x, \delta)$, $xR_\varphi\varphi([\bar{v}, w])$. So $x\overline{R_\varphi}aR_\varphi w$. $\square$

We now obtain our representation result.

Theorem 5.4 *Let $k \geq 2$. If a choice function $\varphi: \mathcal{C} \rightarrow \mathbb{R}^k$ satisfies CON and is such that R_φ is acyclic, then φ is rationalizable-representable.*

Proof. The acyclicity of R_φ implies that φ is rationalizable and $\overline{R_\varphi}$ is a rationalization of φ. We complete the proof by establishing the existence of a representation u of $\overline{R_\varphi}$.

Lemma 5.3 straightforwardly implies

$$x \neq y \text{ and } x\bar{R}_\varphi y \;\Rightarrow\; \exists a \in \mathbb{Q}^k \setminus \{x, y\} \text{ such that } x\overline{R_\varphi}a\overline{R_\varphi}y \tag{2}$$

for all $x, y \in \mathbb{R}^k$. Let $\mathbb{Q}^k = \{a_1, a_2, \ldots\}$ be an enumeration of the (countable) set $\mathbb{Q}^k$. Define $u\colon \mathbb{R}^k \to \mathbb{R}$ by

$$u(x) = \sum_{k \in \mathbb{N} \;|\; x\overline{R_\varphi}a_k} 2^{-k}$$

for all $x \in \mathbb{R}^k$. By (2) and the antisymmetry and transitivity of $\overline{R_\varphi}$, u represents $\overline{R_\varphi}$.[4] $\square$

The reason why CIC is sufficient for a representation result in the one-dimensional case but not for higher dimensions is quite intuitive. In the one-dimensional case, variations along a straight line are sufficient to span a full-dimensional neighborhood of a point but, of course, this is not the case in higher dimensions.

A property that is often associated with generalizations of single-peaked preferences to higher dimensions is strong quasi-concavity. Our final theorem establishes that if a representation of $\overline{R_\varphi}$ exists and CON is satisfied (IIA of course follows), then this function is strongly quasi-concave.

Theorem 5.5 *Let $k \geq 2$. If a choice function $\varphi\colon \mathcal{C} \to \mathbb{R}^k$ satisfies CON and a function $u\colon \mathbb{R}^k \to \mathbb{R}$ represents $\overline{R_\varphi}$, then u is strongly quasi-concave.*

Proof. Let $x \in \mathbb{R}^k$ and let $T := \{y \in \mathbb{R}^k \mid u(y) \geq u(x)\}$. Let $z, z' \in T$ with $z \neq z'$, $0 < \alpha < 1$ and $z'' := \alpha z + (1 - \alpha)z'$. By Theorem 3.1, $z''R_\varphi z$ or $z''R_\varphi z'$. Hence $u(z'') \geq u(x)$, so that $z'' \in T$. This shows that T is convex. If $[z, z']$ contains an interior point of T, then by convexity of T all points of (z, z') are interior. Now suppose that $[z, z']$ contains no interior point of T. Let $y := \varphi([z, z'])$ where, without loss of generality, $y \neq z$. By CON, there is an $\varepsilon > 0$ small enough such that $||\varphi([v, z]) - y|| \leq ||z - y||/4$ for all $v \in \mathcal{B}(y, \varepsilon)$. Let $y' := y/4 + 3z/4$. Take $\delta > 0$ such that for all $w \in \mathcal{B}(y', \delta)$ there is a point $v \in \mathcal{B}(y, \varepsilon)$ with $w \in [v, z]$. Then Theorem 3.1 implies $wR_\varphi z$ and therefore $u(w) \geq u(z) \geq u(x)$ for all $w \in \mathcal{B}(y', \delta)$. Hence, $\mathcal{B}(y', \delta) \subseteq T$, so that y' is an interior point of T, a contradiction. $\square$

[4] This argument, using (2), is a variation on Lemma II in Debreu (1954) for partial orders. See also Jaffray (1975).

6 Conclusion

Single-peaked preferences play an important role in the economics and political-science literatures. In this paper, we have examined single-peakedness from a choice-theoretic perspective, thus providing new insights on the foundations of this notion. In particular, we have characterized a class of single-peaked choice functions by contraction consistency (IIA) and a weak (collinear) continuity condition. For this class we have obtained detailed results on rationalizability and representability. Our results appear to provide strong support of the use of single-peakedness because the two axioms characterizing single-peaked choice are widely accepted in the relevant literature.

A Appendix: Proof of Theorem 4.2

Let $n \in \mathbb{N}$ with $n \geq 3$ and consider a convex n-polygon $\mathcal{P}$ in $\mathbb{R}^2$ with consecutive vertices $x^1, \ldots, x^n$. This means that $\mathcal{P}$ is the convex hull of $\{x^1, \ldots, x^n\}$, every x^i is an extreme point, and the boundary is the union of the line segments $[x^i, x^{i+1}]$ for $i = 1, \ldots, n$ with $x^{n+1} := x^1$. A point $\hat{x} \in \mathcal{P}$ is called a *Brocard point*[5] if $\measuredangle \hat{x}x^1x^2 = \measuredangle \hat{x}x^2x^3 = \ldots = \measuredangle \hat{x}x^{n-1}x^n = \measuredangle \hat{x}x^nx^1$. Denote this common angle size by $\alpha_{\hat{x}}$.

Denote by $O(C)$ the area of a set C in $\mathbb{R}^2$. For points $x, y, z \in \mathbb{R}^2$ denote by $\triangle(xyz)$ the triangle with vertices x, y, and z. Recall that $A(n) = \frac{n-2}{2n}\pi$ is equal to $(1/2)\measuredangle x^1x^2x^3$ if $\mathcal{P}$ is a regular polygon.

Theorem A.1 *Let $\hat{x}$ be a Brocard point in $\mathcal{P}$ with common angle size $\alpha_{\hat{x}}$. Then*

(i) $\cot \alpha_{\hat{x}} = \left(\sum_{i=1}^{n} ||x^{i+1} - x^i||^2 \right) / 4\, O(\mathcal{P})$;

(ii) $\alpha_{\hat{x}} \leq A(n)$.

Proof. For any triangle $\triangle(xyz)$ we have the familiar formula $O(\triangle(xyz)) = (1/2)||y - x|| \cdot ||z - x|| \sin \measuredangle yxz$. Hence,

$$\begin{aligned} O(\mathcal{P}) &= O(\triangle(\hat{x}x^1x^2)) + O(\triangle(\hat{x}x^2x^3)) + \ldots + O(\triangle(\hat{x}x^nx^1)) \\ &= (1/2)||x^1 - \hat{x}|| \cdot ||x^2 - x^1|| \sin \measuredangle \hat{x}x^1x^2 \end{aligned}$$

[5] See Honsberger (1995) or Weisstein (2005).

$$
\begin{aligned}
& +(1/2)||x^2-\hat{x}|| \cdot ||x^3-x^2|| \sin \measuredangle \hat{x} x^2 x^3 \\
& + \ldots + (1/2)||x^n-\hat{x}|| \cdot ||x^1-x^n|| \sin \measuredangle \hat{x} x^1 x^2 \\
= \; & (1/2) \sin \alpha_{\hat{x}} \left[\, ||x^1-\hat{x}|| \cdot ||x^2-x^1|| + ||x^2-\hat{x}|| \cdot ||x^3-x^2|| \right. \\
& \left. + \ldots + ||x^n-\hat{x}|| \cdot ||x^1-x^n|| \,\right]
\end{aligned}
$$

hence

$$
\sin \alpha_{\hat{x}} = \frac{2\, O(\mathcal{P})}{\sum_{i=1}^{n} ||x^i-\hat{x}|| \cdot ||x^{i+1}-x^i||} \,. \tag{3}
$$

For any triangle $\triangle(xyz)$ we moreover have the familiar formula

$$
\cos \measuredangle yxz = \frac{||y-x||^2 + ||z-x||^2 - ||y-z||^2}{2||y-x|| \cdot ||z-x||} \,.
$$

Hence

$$
\cos \alpha_{\hat{x}} = \frac{||x^{i+1}-x^i||^2 + ||\hat{x}-x^i||^2 - ||x^{i+1}-\hat{x}||^2}{2||x^{i+1}-x^i|| \cdot ||\hat{x}-x^i||}
$$

for all $i \in \{1, \ldots, n\}$. By adding these expressions for all $i = 1, \ldots, n$, we obtain

$$
\cos \alpha_{\hat{x}} = \frac{\sum_{i=1}^{n} ||x^{i+1}-x^i||^2}{2 \sum_{i=1}^{n} ||x^i-\hat{x}|| \cdot ||x^{i+1}-x^i||}. \tag{4}
$$

Combining (3) and (4), we obtain

$$
\cot \alpha_{\hat{x}} = \frac{\cos \alpha_{\hat{x}}}{\sin \alpha_{\hat{x}}} = \left(\sum_{i=1}^{n} ||x^{i+1}-x^i||^2 \right) / 4\, O(\mathcal{P}) \,,
$$

which proves (i).

For (ii), note that $\alpha_{\hat{x}}$ is maximal if its cotangens value is minimal. Among polygons of fixed circumference, a regular n-polygon minimizes the sum of the squares of the edges (since this sum-function is convex) and maximizes the area.[6] Hence, by (i), $\alpha_{\hat{x}}$ is maximal for a regular n-polygon. This implies (ii). □

We now prove Theorem 4.2, which is a slight extension of Theorem A.1.

[6] This is a classical result known as the *Isoperimetric Theorem*, cf. Weisstein (2006).

Proof of Theorem 4.2. Consider the half-line $\ell = [x^2, x^1, \rightarrow)$ through x^1 starting from x^2 and the half-line $\ell' = [x^{n-1}, x^n, \rightarrow)$ through x^n starting from x^{n-1}.

If ℓ and ℓ' do not intersect, then we can find a point $z \notin \mathcal{P}$ on ℓ' sufficiently far from x^n such that $\measuredangle \hat{x}zx^1 < \alpha$. Since $\measuredangle \hat{x}x^nx^1 \geq \alpha$, by a continuity consideration there must be a point $y \in [x^n, z]$ such that $\measuredangle \hat{x}yx^1 = \alpha$. Then $\hat{x}$ is a Brocard point in the convex n-polygon with vertices $x^1, \ldots, x^{n-1}, y$ with $\alpha_{\hat{x}} = \alpha$. By Theorem A.1(ii), $\alpha \leq A(n)$.

If ℓ and ℓ' intersect, say in some point z, then it is easy to see that $\measuredangle \hat{x}zx^1 < \alpha$. The argument continues as in the first case. □

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