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Global dynamics in a growth model with an exhaustible resource*

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Abstract

We revisit the seminal growth model with exhaustible resources, the so called Dasgupta-Heal-Stiglitz-Solow model (DHSS). For this optimal control problem with two state variables, we explicitly characterize the dynamics of all the variables in the model and from all possible initial values of the stocks.

We determine the condition under which consumption is initially increasing with time and the condition under which initial investment is positive implying that overshooting of man-made capital ocurs. We show that the initial consumption under a utilitarian criterion starts below the maximin rate of consumption if and only the resource is abundant enough and that under a utilitarian criterion, it is not necessarily the present generation that benefits most from a windfall of resources.

Key words: endogenous growth, exhaustible resources, exponential integral

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1 Introduction

We provide a closed form solution to the Dasgupta-Heal-Solow-Stiglitz (from here on DHSS) model. The DHSS model is based on seminal articles by Dasgupta and Heal (1974), Solow (1974) and Stiglitz (1974). It describes an economy with two assets, man-made capital and a nonrenewable resource stock. Together with man-made capital the raw material from the resource is used as an input in the production of a commodity that can be used for consumption and for net investments in man-made capital. In this framework some important questions have been addressed. For instance, in the case where the objective is to maximize the minimum rate of consumption throughout the time horizon, a central question is whether, despite the resource constraint, there exists a sustainable constant positive rate of consumption (Solow, 1974). Another stream of the literature adopts a utilitarian objective and studies the optimal consumption and investment paths that maximize a discounted sum of utility from consumption (Stiglitz, 1974). In the present paper we give the optimal paths for the DHSS economy under a utilitarian objective. In the literature attention has mainly been given to the case where the production function is Cobb-Douglas and instantaneous utility is logarithmic. Even for these specifications no closed form solutions of the optimum have been found so far. There has been some progress regarding the characterization of the optimal solution to the DHSS problem, in particular it has been shown without explicitly finding the solution that consumption can be single peaked[1] (see Pezzey and Withagen, 1998, and Hartwick et al., 2003). However, in the absence of a closed-form solution it is not possible to address other relevant issues such as understanding the relationship between the instant of time where the peak takes place and the initial stocks of capital and the resource and how this phenomenon depends on the model parameters. Moreover, to actually calculate the optimal path more information is needed on the co-state variables associated with the stocks, which amounts to having a complete solution of the model.

In this paper we provide a closed form solution to the DHSS problem using the exponential integral function. The exponential integral belongs to a family of 'special functions' which are extensively used in mathematical physics and probability theory. They are particularly helpful to determine solutions to differential equations encountered in physics (see e.g., Temme, 1996, Ch. 5 and Ch. 7). The use of special functions in economic theory is relatively recent. Boucekkine et al. (2007 and 2008) show that the solution to a two-sector Lucas-Uzawa model of endogenous growth can be expressed in terms of a specific type of 'special functions': the hypergeometric functions. See also Perez-Barahona (2008). In our problem of finding the solution to the canonical growth model with resource constraints, it is another type of special function, the exponential integral[2], that turned out to be instrumental in expressing the

[1] For high rates of pure time preference consumption always decreases over time and for low rates of time preference consumption monotonically increases during an initial interval of time, reaches a maximum, and eventually decreases.

[2] The exponential integral belongs to a class of special functions called the 'confluent hypergeometric functions' which

solution in a closed form. Thus, along with Boucekkine et al. (2007 and 2008), this paper is a proof that special functions can play a key role in analyzing dynamic economic problems and characterizing the transition dynamics of all the variables in a dynamic problem and from all possible initial values of the state variables[3].

We exploit the explicit form of the solution to study the behavior of the optimal consumption and investment paths as functions of the parameters of the model. We compare the solution to the DHSS model, under a utilitarian objective, to the solution of the problem where the objective is to maximize the minimum rate of consumption over the whole time horizon. The solution to the latter problem is called the maximin rate of consumption. Sustainability in the DHSS context requires that the Hartwick rule, i.e. zero genuine savings, holds: the investment rate must equal the extraction rate times the marginal product of the resource. Asheim (1994) shows that this condition should hold at all instants of time. Hence, if at some instant of time Hartwick's rule holds it does not mean that the economy is on a sustainable path. The argument used by Asheim rests on the assumption that in a utilitarian optimum the initial rate of consumption is below the maximin rate of consumption if the pure rate of time preference is small enough, and above the maximin rate of consumption if the pure rate of time preference is large. By continuity there is a rate of time preference for which both initial consumption rates coincide, and for which the utilitarian rate of consumption is increasing for an initial period of time. Asheim (1994) refers to a graph in Dasgupta and Heal (1979) to support this assumption with respect to the rates of time preference. However, Dasgupta and Heal do not provide a formal proof of their claim. Also, no proof is given of continuity. In our case we are able to provide a proof to both claims and show that the ratio of the maximin rate of consumption over the initial rate of consumption is a strictly decreasing continuous function of the rate of discount. Moreover, we show that investments in man-made capital may also be single-peaked. We provide the condition under which overshooting in man-made capital occurs. In particular, this phenomenon arises in relatively natural resource rich economies. Our treatment allows us to study the relationship between the ratio of the maximin rate of consumption over the initial consumption rate under a utilitarian criterion and the stocks of the resource and capital. Using the ratio of the resource stock over the capital stock as an indicator of resource abundance we show that the initial consumption under a utilitarian criterion starts below the maximin rate of consumption if and only the resource is abundant enough and that under a utilitarian criterion, it is not necessarily the present generation that benefits most from a windfall of resources.

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solve the Kummer differential equation, a confluent of the Gauss hypergeometric differential equation. For more details we refer the reader to Temme, 1996, Ch. 5 and Ch. 7.

[3]Dynamic systems in economics, in particular those involving more than one state variable, have been so far treated rigorously but mostly using qualitative techniques such as phase diagrams accompanied with an analytical study of the behaviour in the neighborhood of a steady state, or using numerical techniques.

The outline of the paper is as follows. The model is introduced in section 2. Section 3 contains the characterization of the optimum in a series of lemmata and propositions. The proofs of these are relegated to the mathematical appendix. Section 4 covers some sensitivity analyses and section 5 concludes.

2 The model and preliminary results

Let $K(t)$ and $S(t)$ denote the stock of man-made capital and the nonrenewable resource at instant of time t, respectively. The variables $C(t)$ and $R(t)$ are rates of consumption and resource extraction at instant of time t and are assumed non-negative. Let α be the production elasticity of man made capital ($0 < \alpha < 1$). The rate of pure time preference is ρ. We assume ρ to be strictly positive. The case of zero discounting is extensively treated in Dasgupta and Heal (1979) and is also discussed in Asheim et al. (2007). For any variable $x(t)$ we adopt the convention $\dot{x}(t) = dx(t)/dt$. Consider the following optimal control problem of the DHSS economy, which we refer to as the DHSS optimal growth problem:

$$Max_C \int_0^\infty e^{-\rho t} U\left(C\left(t\right)\right) dt \tag{1}$$

subject to

$$\dot{K}(t) = K(t)^\alpha R(t)^{1-\alpha} - C(t) \tag{2}$$

$$\dot{S}(t) = -R(t) \tag{3}$$

$$K\left(0\right) = K_0 > 0 \text{ and } S\left(0\right) = S_0 > 0. \tag{4}$$

with

$$U\left(C\right) = \begin{cases} \frac{C^{1-\eta}-1}{1-\eta} & \text{for} \quad \eta \neq 1, \eta > 0 \\ \ln C & \text{for} \quad \eta = 1 \end{cases}$$

A solution of the optimal growth problem above is described by a quadruple of paths (C, K, R, S). Let $\lambda\left(t\right)$ and $\mu\left(t\right)$ denote the co-state variables associated with the stock of capital and the natural resource stock, respectively. The current value Hamiltonian is given by

$$H(K, R, C, \lambda, \mu) = U\left(C\left(t\right)\right) + \lambda[K^\alpha R^{1-\alpha} - C] - \mu R$$

The maximum principle yields

$$\lambda(t) = U'\left(C\left(t\right)\right) = C(t)^{-\eta} \tag{5}$$

$$\left(1-\alpha\right) K(t)^\alpha R(t)^{-\alpha} \lambda(t) = \mu(t) \tag{6}$$

$$\dot{\lambda}(t) = \rho\lambda(t) - H_K = \rho\lambda(t) - \alpha K(t)^{\alpha-1} R(t)^{1-\alpha} \lambda(t) \tag{7}$$

$$\dot{\mu}(t) = \rho\mu(t) - H_S = \rho\mu(t) \tag{8}$$

Any solution that satisfies the above system along with the following transversality conditions

$$\lim_{t\to\infty} e^{-\rho t}\lambda\left(t\right)K\left(t\right)=0 \tag{9}$$

$$\lim_{t\to\infty} e^{-\rho t}\mu\left(t\right)S\left(t\right)=0 \tag{10}$$

is a solution to the optimal control problem (1)-(4).

In the sequel we aim at obtaining an explicit solution to the DHSS optimal growth problem. The line of attack can be sketched as follows. Using (6), (7) and (8) yields

$$\dot{\lambda}(t)=\rho\lambda(t)-\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}\left(\mu_0 e^{\rho t}\right)^{\frac{\alpha-1}{\alpha}}\lambda(t)^{\frac{1}{\alpha}} \tag{11}$$

where $\mu_0=\mu(0)$, the initial value of μ, still to be determined. It is easily verified that the solution to this differential equation reads

$$\lambda(t)=e^{\rho t}\left[\lambda_0{}^{\frac{\alpha-1}{\alpha}}+(1-\alpha)\left(\frac{\mu_0}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}t\right]^{\frac{\alpha}{\alpha-1}} \tag{12}$$

where λ_0, the initial value of λ, is still to be determined. From (2), (6) and substituting consumption from (5) we have

$$\dot{K}(t)=K(t)\left(\frac{\mu(t)}{(1-\alpha)\lambda(t)}\right)^{\frac{\alpha-1}{\alpha}}-\lambda(t)^{\frac{-1}{\eta}} \tag{13}$$

and since λ and μ are given functions of time, we solve for K as a function of time, and of the initial λ and μ. Next we can determine $R(t)$ from (6) and solve for the resulting resource stock $S(t)$. Finally, we use the transversality conditions (9) and (10) to solve for the initial values of the co-state variables. Given the strict concavity of the utility and production functions involved, if the optimal growth problem has a solution, it is unique. Hence, if we find a solution satisfying the transversality conditions it is the unique solution to the DHSS optimal growth problem.

It turns out that the solution for K from (13), and hence for R and S, can be expressed in terms of a special function, i.e., the exponential integral defined as

$$E_a\left(z\right)\equiv\int_1^\infty e^{-zu}u^{-a}du$$

with $a\in\mathcal{R}$ and $z>0$ (see e.g., Abramowitz and Stegun, 1972, and Tomme, 1996[4]). A special case of the exponential integral is

$$E_1\left(z\right)\equiv\int_1^\infty e^{-zu}u^{-1}du$$

[4]Both Abramowitz and Stegun (1972) and Tomme (1996) define the exponential integral with a integer and allow for z to be complex with $\mathrm{Re}\left(z\right)>0$. However, the definition extends naturally to allow a to be real or complex. It is this generalized definition of the exponential integral that we use. In our analysis the argument, a, is real.

A simple change of variable allows to have this useful alternate expression for the exponential integrals

$$E_a(z) \equiv z^{a-1} \int_z^{\infty} e^{-v} v^{-a} dv. \tag{14}$$

The function $E_a(z)$ is a strictly decreasing function of z and, finally,

$$E_0(z) = \frac{e^{-z}}{z}.$$

Another property of the exponential integral that will prove useful for our purposes is (see Abramowitz and Stegun,1972, p.229, inequality 5.1.20)

$$\frac{1}{2} e^{-z} \ln\left(1 + \frac{2}{z}\right) < E_1(z) < e^{-z} \ln\left(1 + \frac{1}{z}\right) \tag{15}$$

which implies that

$$\lim_{z \to \infty} E_1(z) = 0 \text{ and } \lim_{z \to 0^+} E_1(z) = \infty.$$

3 Solving the optimal growth problem

In this section, we provide the steps to determine the solution to the set of conditions given by the maximum principle (5)-(8). In view of (12) it will turn out convenient to define

$$\varphi = (1-\alpha)\left(\frac{\mu_0}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}, \; \pi(t) = \lambda_0^{\frac{\alpha-1}{\alpha}} + \varphi t, \; x(t) = \frac{\rho \pi(t)}{\varphi}, \; x_0 = x(0), \; \beta = \frac{1-\frac{\alpha}{\eta}}{1-\alpha} \tag{16}$$

The following observations will be useful for the rest of the analysis: the case $\eta = 1$ implies $\beta = 1$, $\beta \geq 0$ if and only if $\eta \geq \alpha$ and the variable $x(t)$ is an affine function of time with $x(t) = x_0 + \rho t$.

3.1 Consumption

Proposition 1

The optimal consumption path is given by

$$C(t) = e^{\frac{-\rho}{\eta} t} \pi(t)^{\frac{\alpha}{\eta(1-\alpha)}}$$

Proof: This is straightforward from (12) and (5)■

3.2 Man-made capital

From (13), we have that the stock of man-made capital is given by

$$K(t) = K_0 e^{-\int_0^t f(z)dz} + \int_0^t g(z) e^{-\int_z^t f(s)ds} dz \tag{17}$$

where

$$f(t) \equiv -\left(\frac{\mu(t)}{(1-\alpha)\lambda(t)}\right)^{\frac{\alpha-1}{\alpha}} \text{ and } g(t) \equiv -\lambda(t)^{\frac{-1}{\eta}}.$$

Since $\mu(t) = e^{\rho t}\mu_0$ and $\lambda(t) = e^{\rho t}\pi(t)^{\frac{\alpha}{\alpha-1}}$ from (12) and (16), we have

$$f(t) = -\left(\frac{\mu_0}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} \frac{1}{\pi(t)}$$

and

$$\int_z^t f(z)\,dz = -\ln\left(\frac{\pi(t)}{\pi(z)}\right)^{\frac{1}{1-\alpha}} \tag{18}$$

We now determine the second term of the right hand side of equation (17).

Lemma 1

$$\int_0^t g(z)\,e^{-\int_z^t f(s)ds}dz = -\frac{1}{\varphi}e^{\frac{x_0}{\eta}}\pi(t)^{\frac{1}{1-\alpha}}\left(\frac{\varphi}{\rho}\right)^{1-\beta}\left(x_0^{1-\beta}E_\beta\left(\frac{x_0}{\eta}\right) - x(t)^{1-\beta}E_\beta\left(\frac{x(t)}{\eta}\right)\right) \tag{19}$$

where $x(t)$ is defined in (16).

Proof: Appendix A.

We can now derive the path of the capital stock.

Proposition 2

The optimal path of the stock of capital is

$$K(t) = \pi(t)^{\frac{1}{1-\alpha}}\left(K_0\lambda_0^{\frac{1}{\alpha}} - \frac{1}{\varphi}e^{\frac{x_0}{\eta}}\left(\frac{\varphi}{\rho}\right)^{1-\beta}\left(x_0^{1-\beta}E_\beta\left(\frac{x_0}{\eta}\right) - x(t)^{1-\beta}E_\beta\left(\frac{x(t)}{\eta}\right)\right)\right) \tag{20}$$

Proof: Substituting (18) and (19) into (17) gives

$$K(t) = K_0\left(\frac{\pi(t)}{\pi_0}\right)^{\frac{1}{1-\alpha}} - \frac{1}{\varphi}e^{\frac{x_0}{\eta}}\left(\frac{\varphi}{\rho}\right)^{1-\beta}\pi(t)^{\frac{1}{1-\alpha}}\left(x_0^{1-\beta}E_\beta\left(\frac{x_0}{\eta}\right) - x(t)^{1-\beta}E_\beta\left(\frac{x(t)}{\eta}\right)\right).$$

Factoring $\pi(t)^{\frac{1}{(1-\alpha)}}$ and taking into account that $(\frac{1}{\pi_0})^{\frac{1}{1-\alpha}} = \lambda_0^{1/\alpha}$ yields (20)■

3.3 Extraction

It follows from (6) and (16) that

$$R(t) = \left(\frac{1-\alpha}{\mu_0}\right)^{\frac{1}{\alpha}}\pi(t)^{\frac{1}{\alpha-1}}K(t)$$

Direct substitution from Proposition 2 yields

Proposition 3:

The optimal extraction rate is given by

$$R(t) = \left(\frac{1-\alpha}{\mu_0}\right)^{\frac{1}{\alpha}}\left(K_0\lambda_0^{\frac{1}{\alpha}} - \frac{1}{\varphi}e^{\frac{x_0}{\eta}}\left(\frac{\varphi}{\rho}\right)^{1-\beta}\left(x_0^{1-\beta}E_\beta\left(\frac{x_0}{\eta}\right) - x(t)^{1-\beta}E_\beta\left(\frac{x(t)}{\eta}\right)\right)\right) \tag{21}$$

3.4 The resource stock

The optimal path of the stock of the resource is the unique solution to (3) with $S(0) = S_0$.

Proposition 4.

The optimal path of the stock of the resource is given by

$$\begin{aligned} S(t) - S_0 &= -\left(K_0\left(\frac{1-\alpha}{\mu_0}\lambda_0\right)^{\frac{1}{\alpha}} - \frac{1}{\mu_0}e^{\frac{x_0}{\eta}}\left(\frac{\varphi}{\rho}\right)^{1-\beta} x_0^{1-\beta}E_\beta\left(\frac{x_0}{\eta}\right)\right)t \\ &\quad -\frac{1}{\rho\mu_0}e^{\frac{x_0}{\eta}}\left(\frac{\varphi}{\rho}\right)^{1-\beta}\eta^{2-\beta}\left(\Psi\left(\frac{x(t)}{\eta}\right) - \Psi\left(\frac{x_0}{\eta}\right)\right) \end{aligned}$$

with

$$\Psi(x) = x^{2-\beta}\left(E_\beta(x) - E_{\beta-1}(x)\right).$$

Proof: Appendix B.

3.5 Solving for the co-state variables

To fully characterize the optimal paths of consumption, the rate of extraction, and the stocks of capital and the resource we still need to determine μ_0 and λ_0. We use the transversality conditions (9) and (10) to do so.

Lemma 2

Given $x_0 > 0$, the vector (λ_0, μ_0) is given by

$$\lambda_0 = \left(\frac{1}{K_0}\frac{1}{\varphi}e^{\frac{x_0}{\eta}}\left(\frac{\varphi}{\rho}\right)^{1-\beta} x_0^{1-\beta}E_\beta\left(\frac{x_0}{\eta}\right)\right)^{\alpha} \tag{22}$$

$$\mu_0 = \left(\frac{\left(\frac{\eta}{\rho}\right)^{2-\beta}\left(\frac{1}{1-\alpha}\right)^{-\frac{(1-\beta)}{\alpha}}\Psi\left(\frac{x_0}{\eta}\right)}{S_0}\right)^{\frac{\alpha}{(\alpha-1)\beta+1}}. \tag{23}$$

Proof: Appendix C.

Define

$$A = \left(\eta\frac{(1-\alpha)^{\frac{1}{\alpha}}}{\rho}\right)^{\frac{\alpha}{1-\alpha}}\frac{S_0}{K_0} \tag{24}$$

and

$$h_\beta(x) = -\left(\frac{x}{\eta}\right)^{\frac{1}{\alpha-1}+\beta-1}\frac{\Psi\left(\frac{x}{\eta}\right)}{E_\beta\left(\frac{x}{\eta}\right)}$$

Lemma 3

For any S_0 and K_0 positive, x_0 is the unique solution to $h_\beta(x_0) = A$.

Proof: Appendix D.

The determination of the optimal solution to the DHSS optimal growth problem is now complete. For any given positive values of S_0 and R_0 we solve for x_0 using Lemma 3, then derive the initial values of the co-state variables λ_0 and μ_0 from Lemma 2. We obtain the time paths of all the model's variables from Propositions 1-4.

4 Sensitivity analysis

We exploit the analytical tractability of the solution to the DHSS model to establish some key features of the optimal paths. We focus on the consumption and the investment paths.

4.1 The optimal consumption path

We first study the conditions under which consumption is increasing for some initial period of time. We highlight the role of the parameters of the model, like the pure rate of time preference and the initial stocks of capital and the natural resource, on the possibility that consumption may rise for an initial period of time. For the ease of exposition only, we focus on the case of a logarithmic utility function: $\eta = 1$, i.e. $U(C) = \ln C$. From the specification of consumption in Proposition 1 it follows that the time at which maximum consumption is reached is

$$t^* = -\frac{1}{\rho\varphi}\left(\varphi\frac{\alpha}{\alpha-1} + \rho\pi_0\right) = \frac{1}{\rho}\left(\frac{\alpha}{1-\alpha} - x_0\right) \tag{25}$$

Clearly $t^* > 0$ iff $x_0 < \frac{\alpha}{1-\alpha}$ which holds iff $A > \tilde{A} \equiv h_1\left(\frac{\alpha}{1-\alpha}\right) > 0$. Indeed, we show in Appendix E that h_1 is a strictly decreasing function of x and therefore, given A and x_0 such that $h_1(x_0) = A$ we have

$$x_0 < \frac{\alpha}{1-\alpha} \text{ iff } A > \tilde{A}.$$

Therefore, consumption is initially (i.e., for $t < t^*$) increasing over time, if and only if $A > \tilde{A}$. For any $\alpha \in (0,1)$ we have $A \equiv \frac{S_0}{K_0}\left(\frac{(1-\alpha)^{\frac{1}{\alpha}}}{\rho}\right)^{\frac{\alpha}{1-\alpha}} > \tilde{A}$ when S_0/K_0 is large enough or ρ is small enough. More precisely, from lemma 3 and (25) and using the implicit function theorem we have

$$\frac{dx_0}{d\rho} = \frac{\frac{dA}{d\rho}}{h_1'(x_0)} > 0 \text{ since } h_1'(x_0) < 0 \text{ and } \frac{dA}{d\rho} < 0 \tag{26}$$

and thus

$$\frac{dt^*}{d\rho} = -\frac{1}{\rho^2}\left(\frac{\alpha}{1-\alpha} - x_0\right) - \frac{1}{\rho}\frac{dx_0}{d\rho} < 0.$$

Also observe that the time where the peak takes place goes to infinity as the rate of pure time preference goes to zero. Indeed, $\rho \to 0$ implies $A \to \infty$, which implies that in the optimum $h_1(x_0) \to \infty$, so that, according to Appendix D, $x_0 \to 0$. Therefore, for each given instant of time, the difference between the optimal rate of consumption with zero discounting can be made arbitrarily close to the utilitarian optimum with discounting by choosing the rate of time preference small enough. Note, however, that this convergence does not imply convergence over the entire trajectory. Consequently, by choosing the rate of time preference small enough, we can postpone the moment in time at which consumption decreases below the maximin rate of consumption.

We can also determine

$$\frac{dt^*}{dA} = \frac{1}{\rho}\left(-\frac{dx_0}{dA}\right) = -\frac{1}{\rho h_1'(x_0)} > 0$$

and therefore an increase of S_0/K_0 implies a larger t^*. Note that the existence of a phase where consumption is increasing with time depends on the ratio of S_0 and K_0 and not the absolute values of S_0 or K_0.

The peak of consumption can be expressed, after manipulations, as

$$C^* = \rho^{\frac{1}{1-\alpha}} \alpha^{-\frac{\alpha}{\alpha-1}} e^{\frac{\alpha}{\alpha-1}} \frac{S_0}{\left(e^{-x_0} - x_0 E_1(x_0)\right)}$$

or, in terms of the initial stock of capital, as

$$C^* = \rho \left(\frac{\alpha}{1-\alpha}\right)^{-\frac{\alpha}{\alpha-1}} e^{\frac{\alpha}{\alpha-1}} K_0 \frac{1}{E_1(x_0)} \left(\frac{1}{x_0}\right)^{\frac{1}{1-\alpha}}.$$

Consider now the rate of consumption that solves the following growth problem

$$Max_C \{Min\ U(C)\} \tag{27}$$

subject to (2)-(4). It can be shown (Solow, 1974) that, provided that $\frac{1}{2} < \alpha < 1$, the solution to this problem, referred to as the maximin rate of consumption, is

$$\tilde{C} = \alpha (2\alpha - 1)^{\frac{1-\alpha}{\alpha}} S_0^{\frac{1-\alpha}{\alpha}} K_0^{\frac{2\alpha-1}{\alpha}}.$$

The main criticism of the utilitarian criterion is that it discounts future consumption. It is intuitive to think that the solution under such a criterion, which favors present consumption relative to future consumption, would result in larger initial consumption than any consumption rate that would be sustained at all time. We show below that this is not true in its generality. There exists a rate of pure time preference $\hat{\rho} > 0$ such that if $\rho < \hat{\rho}$ the initial rate of consumption is below the maximin rate of consumption. The initial rate of consumption in the utilitarian framework is

$$C_0 \equiv C(0) = \frac{1}{\lambda_0} = \frac{\rho K_0}{x_0 e^{x_0} E_1(x_0)} \tag{28}$$

The analysis of the ratio $\tilde{C}/C_0$ as a function of x_0 allows to determine the behaviour of $\tilde{C}/C_0$ as a function of ρ as well as S_0/K_0 in a compact way. The ratio $\tilde{C}/C_0$ can be written as

$$\tilde{C}/C_0 = \frac{\alpha\left(2\alpha-1\right)^{\frac{1-\alpha}{\alpha}}}{\left(1-\alpha\right)^{\frac{1}{\alpha}}}\xi\left(x_0\right)$$

where

$$\xi\left(x_0\right) = \left(\frac{1}{x_0 e^{x_0} E_1(x_0)} - 1\right)^{\frac{1-\alpha}{\alpha}} e^{x_0} E_1\left(x_0\right).$$

Lemma 4:

We have $\frac{d\xi}{dx_0} < 0$ for all $x_0 > 0$ with $\lim_{x_0 \to \infty} \xi\left(x_0\right) = 0$ and $\lim_{x_0 \to 0^+} \xi\left(x_0\right) = \infty$

Proof: See Appendix F.

We can now link the ratio $\tilde{C}/C_0$ to ρ and to S_0/K_0 through $h_1\left(x_0\right) = A$ with A given in (24).

Proposition 5a:

The ratio $\tilde{C}/C_0$ is a strictly decreasing function of ρ. Moreover there exists $\hat{\rho} > 0$ such that $\tilde{C}/C_0 > 1$ iff $\rho < \hat{\rho}$.

Proof: From (24) we have

$$\frac{dx_0}{d\rho} = \frac{\frac{dA}{d\rho}}{h_1'(x_0)} > 0 \tag{29}$$

since $\frac{dA}{d\rho} < 0$ and in Appendix E we show that $h_1'\left(x\right) < 0$ for all $x > 0$.

Using Lemma 4 along with $\frac{dx_0}{d\rho} > 0$ and the fact that $\lim_{\rho \to \infty}\left(x_0\right) = \infty$ and $\lim_{\rho \to 0^+}\left(x_0\right) = 0$ completes the proof ■

Asheim (1994) obtains this result using the assumption that the consumption path in a utilitarian optimum has the following properties: (i) consumption is a continuous function of the rate of pure time preference ρ and (ii) there exists $\hat{\rho} > 0$ such that if $\rho < \hat{\rho}$ the initial rate of consumption is below the maximin rate of consumption. These, plausible, properties have been assumed for instance by Asheim (1994) to show that if at some instant of time Hartwick's rule holds it does not mean that the economy is on a sustainable path. Our analysis provides a proof of both properties and shows monotonicity of $\tilde{C}/C_0$ with respect to ρ. In the existing litterature these properties were shown to hold in the case where the production function is Cobb-Douglas with constant returns to scale where the production elasticity of capital is assumed equal to the elasticity of marginal utility (see e.g., Pezzey and Withagen (1998) and Hartwick et al. (2003)). In our analysis we do not rely on this assumption. Moreover, we can give closed form solutions of all other relevant variables, which will be further exploited.

Indeed, our treatment also allows to determine the relationship between the ratio $\tilde{C}/C_0$ and the ratio S_0/K_0.

Proposition 5b:

The ratio $\tilde{C}/C_0$ is a strictly decreasing function of S_0/K_0. Moreover there exists a ratio $\widehat{S_0/K_0} > 0$ such that $\tilde{C}/C_0 > 1$ iff $S_0/K_0 > \widehat{S_0/K_0}$.

Proof: From (24) we have

$$\frac{dx_0}{d\left(S_0/K_0\right)}=\frac{1}{h_1'(x_0)}\left(\frac{(1-\alpha)^{\frac{1}{\alpha}}}{\rho}\right)^{\frac{\alpha}{1-\alpha}}<0 \tag{30}$$

The proof that $h_1'(x)<0$ for all $x>0$ is given in Appendix E.

Using Lemma 4 along with $\frac{dx_0}{d(S_0/K_0)}<0$ and the fact that $\lim_{S_0/K_0\to\infty}(x_0)=0$ and $\lim_{S_0/K_0\to 0^+}(x_0)=\infty$ completes the proof ■

The ratio S_0/K_0 can be considered as an indicator of resource abundance. Proposition 5b states that the initial consumption under a utilitarian criterion starts below the maximin rate of consumption if and only the resource is abundant enough. This is not a priori intuitive. The utilitarian criterion is generally considered as biased towards present generations and therefore under such a criterion it may be intuitive to expect that abundance of resources will be heavily exploited by present generations at the detriment of future generations. We have shown that a more abundant resource increases both the maximin rate of consumption and the initial consumption rate under a utilitarian criterion. However, the latter increase is smaller than the former. Thus, under a utilitarian criterion, it is not necessarily the present generation that benefits most from a windfall of resources.

4.2 The optimal investment path

The optimal investment path can be obtained by direct substitution of $C(t)$, $K(t)$ and $R(t)$

$$\begin{aligned}\dot{K}(t) &= \pi(t)^{\frac{\alpha}{1-\alpha}}\left(\frac{1-\alpha}{\mu_0}\right)^{\frac{1-\alpha}{\alpha}}K_0\lambda_0^{1/\alpha} \\ &\quad -\frac{1}{\varphi}e^{\frac{x_0}{\eta}}\left(\frac{\varphi}{\rho}\right)^{1-\beta}x_0^{1-\beta}E_\beta\left(\frac{x_0}{\eta}\right)-x(t)^{1-\beta}E_\beta\left(\frac{x(t)}{\eta}\right)-\left(e^{-\rho t}\pi(t)^{-\frac{\alpha}{\alpha-1}}\right)^{\frac{1}{\eta}}\end{aligned} \tag{31}$$

The investment path can also initially increase over time and decline after reaching a peak and eventually become negative. This was shown earlier by Asheim (1994). In particular he proves that if the rate of time preference is such that $C_0=\tilde{C}$ this will be the behavior of the investment path. Here we provide a numerical example that illustrates the investment and consumption patterns as a function of resource abundance. More precisely we set $\eta=1$, $K_0=1$, $\alpha=0.6$ and $\rho=0.03$ and we plot the optimal investment and consumption paths under a utilitarian objective and the maximin consumption rate for different values of the stock of the resource.

We first set $S_0=0.5$, we have that $C_0<\tilde{C}$ and that investment is initially increasing over time and declines after reaching a peak and eventually becomes negative, see Figure 1. When we set $S_0=0.1$, we have $C_0>\tilde{C}$ and investment is always decreasing over time and eventually becomes negative, see Figure 2. There exists a threshold stock of the resource $\tilde{S}_0$ for which $C_0=\tilde{C}$. For our numerical example the approximate value of $\tilde{S}_0$ is 0.1825. In Figure 3 we plot the case where $S_0=0.1825$.

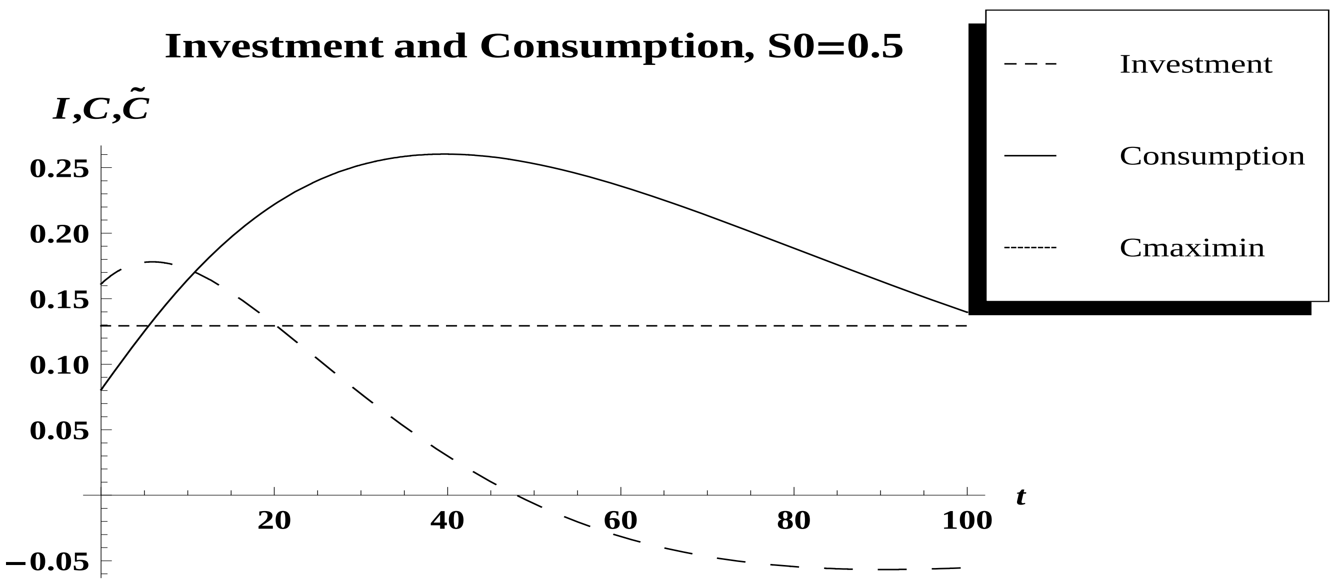
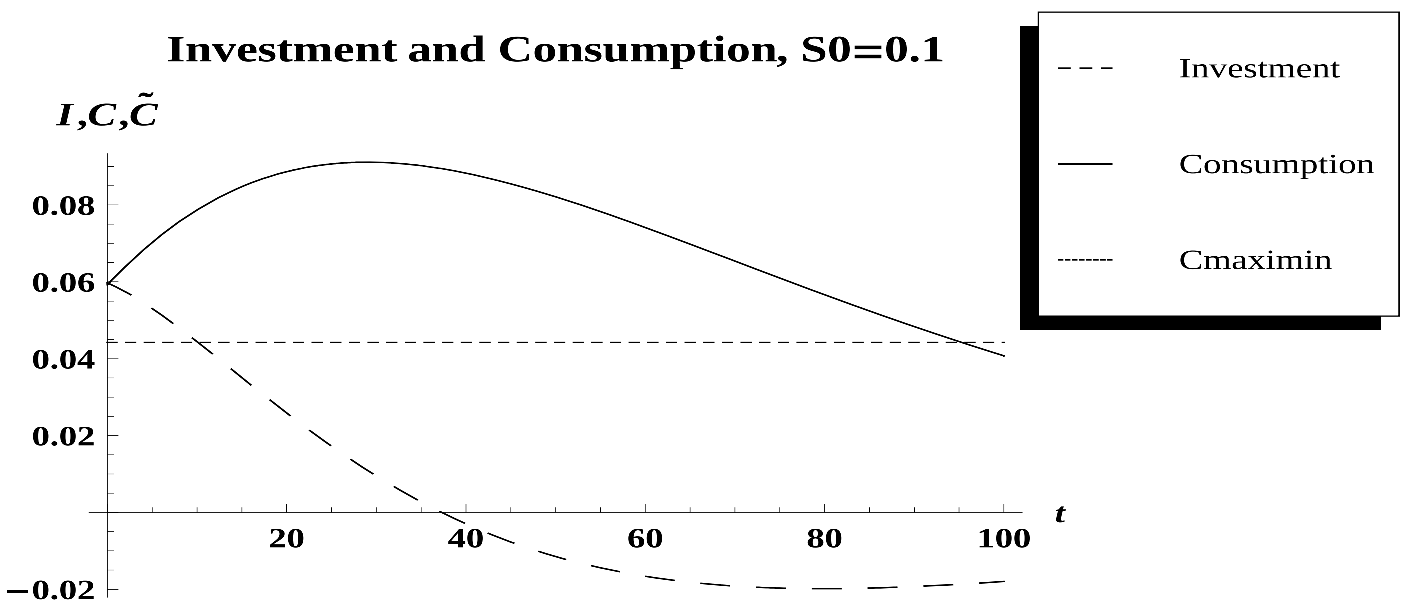

Investment and Consumption, S0=0.5
$I,C,\tilde{C}$
0.25
0.20
0.15
0.10
0.05
−0.05
20
40
60
80
100
t
Investment
Consumption
Cmaximin

Figure 1:



Investment and Consumption, S0=0.1
$I, C, \tilde{C}$
0.08
0.06
0.04
0.02
−0.02
20
40
60
80
100
t
Investment
Consumption
Cmaximin

Figure 2:

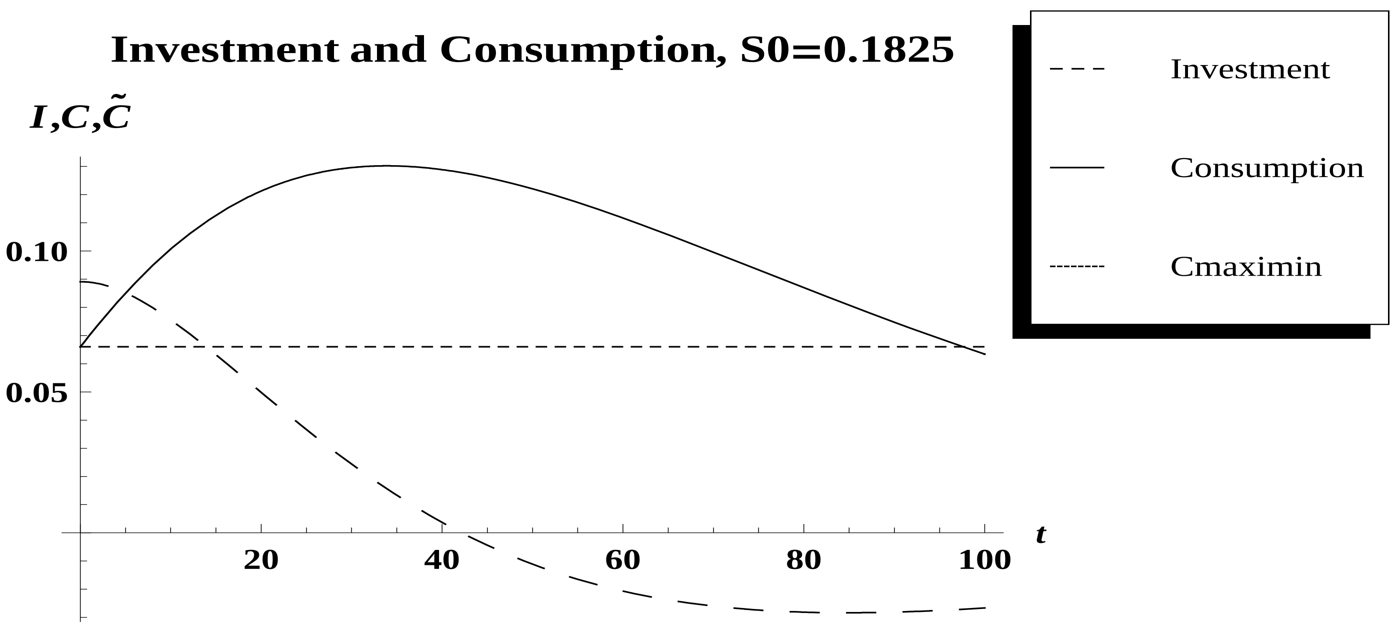

Investment and Consumption, S0=0.1825
I,C,$\tilde{C}$
0.10
0.05
20
40
60
80
100
t
Investment
Consumption
Cmaximin

Figure 3:

Note that investment becomes negative before the moment beyond which the optimal consumption path under the utilitarian objective falls below $\tilde{C}$ forever. We obtained this qualitative result for all numerical simulations we have conducted.

It can be shown that, when $\alpha = \frac{1}{2}$, investment at time zero is equal to

$$\dot{K}(0) = -\frac{1}{2}\frac{2e^{x_0}x_0E_1(x_0) - x_0}{e^{x_0}x_0E_1(x_0) - 1}S_0.$$

From (45) (see Appendix F) we know the denominator is negative. Therefore, we have positive investment at time zero iff $e^{x_0}E_1(x_0) > \frac{1}{2}$. Moreover, also from (45) (see Appendix F), we have

$$\frac{d\left(e^{x_0}E_1(x_0)\right)}{dx_0} = -\frac{1}{x_0} + e^{x_0}E_1(x_0) < 0.$$

Therefore, there exists $\bar{x}_0(\approx 1.289)$ such that overshooting in man made capital occurs (i.e., $\dot{K}(0) > 0$) iff $x_0 < \bar{x}_0$, i.e. iff ρ or K_0 small enough or S_0 large enough. Thus, it is relatively resource rich countries that are more likely to overshoot in man made capital.

5 Conclusion

We have given a closed form solution to the seminal model of endogenous growth with exhaustible resources, based on Dasgupta and Heal (1974), Solow (1974) and Stiglitz (1974). For this two-state variables optimal control problem, we give a closed form representation of the dynamics of all the variables in the model and from all possible initial values of the state variables. We establish several features that the solution may exhibit. In particular, we determine the condition under which the consumption is initially increasing with time and the condition under which initial investment is positive. We have shown that the initial consumption under a utilitarian criterion starts below the maximin rate of consumption if and only the resource is abundant enough and that under a utilitarian criterion, it is not necessarily the present generation that benefits most from a windfall of resources. We also provide an example where investment is initially positive increasing with time, reaches a maximum, declines and eventually becomes negative.

Appendix A: Proof of Lemma 1

From (18) we have

$$e^{-\int_z^t f(s)ds} = \left(\frac{\pi(t)}{\pi(z)}\right)^{\frac{1}{1-\alpha}}$$

Hence, since $g(t) = -\lambda(t)^{\frac{-1}{\eta}}$ and $\lambda(t) = e^{\rho t}\pi(t)^{\frac{\alpha}{\alpha-1}}$ by definition, it holds that

$$\begin{aligned}\int_0^t g(z)\, e^{-\int_z^t f(s)ds} dz &= \int_0^t -\frac{1}{\left(e^{\rho z}\pi(z)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{1}{\eta}}}\left(\frac{\pi(t)}{\pi(z)}\right)^{\frac{1}{1-\alpha}} dz \\ &= -\pi(t)^{\frac{1}{1-\alpha}}\int_0^t e^{-\frac{1}{\eta}\rho z}\pi(z)^{-\frac{1}{1-\alpha}-\frac{\alpha}{\alpha-1}\frac{1}{\eta}}dz\end{aligned}$$

From (15) we have $x(t) \equiv \frac{\rho\pi(t)}{\varphi}$, $z = \frac{\pi(z)-\pi_0}{\varphi}$ and $d\pi(z) = \varphi dz$. Hence

$$\begin{aligned}\int_0^t g(z)\, e^{-\int_z^t f(s)ds} dz &= -\pi(t)^{\frac{1}{1-\alpha}}\int_{\pi_0}^{\pi(t)} e^{-\frac{1}{\eta}\rho\frac{\pi(z)-\pi_0)}{\varphi}}\pi(z)^{-\beta}\frac{d\pi(z)}{\varphi} \\ &= -\frac{1}{\varphi}e^{\frac{1}{\eta}\rho\frac{\pi_0}{\varphi}}\pi(t)^{\frac{1}{1-\alpha}}\int_{\pi_0}^{\pi(t)} e^{-\frac{1}{\eta}\frac{\rho}{\varphi}u}u^{-\beta}du \\ &= -\frac{1}{\varphi}e^{\frac{x_0}{\eta}}\pi(t)^{\frac{1}{1-\alpha}}\int_{\pi_0}^{\pi(t)} e^{-\frac{1}{\eta}\frac{\rho}{\varphi}u}u^{-\beta}du\end{aligned}$$

Consider the following change of variable

$$w = \frac{u}{\pi_0} = \frac{\rho}{\varphi x_0}u$$

Then $dw = \frac{\rho}{\varphi x_0}du$. Hence

$$\begin{aligned}\int_{\pi_0}^{\pi(t)} e^{-\frac{1}{\eta}\frac{\rho}{\varphi}u}u^{-\beta}du &= \int_1^{\frac{\pi(t)}{\pi_0}} e^{-\frac{x_0}{\eta}w}w^{-\beta}\left(\frac{\varphi x_0}{\rho}\right)^{-\beta}\frac{\varphi x_0}{\rho}dw \\ &= \left(\frac{\varphi x_0}{\rho}\right)^{-\beta+1}\int_1^{\frac{x(t)}{x_0}} e^{-\frac{x_0}{\eta}w}w^{-\beta}dw \\ &= \left(\frac{\varphi x_0}{\rho}\right)^{-\beta+1}\left(\int_1^{\infty} e^{-\frac{x_0}{\eta}w}w^{-\beta}dw - \int_{\frac{x(t)}{x_0}}^{\infty} e^{-\frac{x_0}{\eta}w}w^{-\beta}dw\right)\end{aligned}$$

We also have

$$\int_1^{\infty} e^{-\frac{x_0}{\eta}w}w^{-\beta}dw = E_\beta\left(\frac{x_0}{\eta}\right) \tag{32}$$

Let $\omega = \frac{x_0 w}{x(t)}$. Then $d\omega = \frac{x_0}{x(t)}dw$ and

$$\begin{aligned}\int_{\frac{x(t)}{x_0}}^{\infty} e^{-\frac{x_0}{\eta}w}w^{-\beta}dw &= \int_1^{\infty} e^{-\frac{x(t)}{\eta}\omega}\omega^{-\beta}\left(\frac{x(t)}{x_0}\right)^{-\beta}\frac{x(t)}{x_0}d\omega \\ &= \left(\frac{x(t)}{x_0}\right)^{-\beta+1}\int_1^{\infty} e^{-\frac{x(t)}{\eta}\omega}\omega^{-\beta}d\omega \\ &= \left(\frac{x(t)}{x_0}\right)^{-\beta+1}E_\beta\left(\frac{x(t)}{\eta}\right)\end{aligned}$$

Hence

$$\int_1^{\frac{x(t)}{x_0}} e^{-\frac{x_0}{\eta}w} w^{-\beta} dw = E_\beta\left(\frac{x_0}{\eta}\right) - \left(\frac{x(t)}{x_0}\right)^{-\beta+1} E_\beta\left(\frac{x(t)}{\eta}\right)$$

So

$$\int_0^t g(z)\, e^{-\int_z^t f(s)ds} dz = -\frac{1}{\varphi} e^{\frac{x_0}{\eta}} \pi(t)^{\frac{1}{1-\alpha}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} \left(x_0^{1-\beta} E_\beta\left(\frac{x_0}{\eta}\right) - x(t)^{1-\beta} E_\beta\left(\frac{x(t)}{\eta}\right)\right) \blacksquare$$

Appendix B: Proof of Proposition 4

The resource extraction path is given by (21):

$$R(t) = \left(\frac{1-\alpha}{\mu_0}\right)^{\frac{1}{\alpha}} \left(K_0 \lambda_0^{1/\alpha} - \frac{1}{\varphi} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} \left(x_0^{1-\beta} E_\beta\left(\frac{x_0}{\eta}\right) - x(t)^{1-\beta} E_\beta\left(\frac{x(t)}{\eta}\right)\right)\right) \tag{33}$$

$$\begin{aligned} S(t) - S(0) &= -\int_0^t R(s)ds \qquad (34) \\ &= -\left(\frac{1-\alpha}{\mu_0}\right)^{\frac{1}{\alpha}} \int_0^t \left(K_0\lambda_0^{1/\alpha} - \frac{1}{\varphi} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} x_0^{1-\beta} E_\beta\left(\frac{x_0}{\eta}\right)\right) dz \\ &\quad + \left(\frac{1-\alpha}{\mu_0}\right)^{\frac{1}{\alpha}} \frac{1}{\varphi} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} \int_0^t x(z)^{1-\beta} E_\beta\left(\frac{1}{\eta} x(z)\right) dz \\ &= -\left(\frac{1-\alpha}{\mu_0}\right)^{\frac{1}{\alpha}} \left(K_0\lambda_0^{1/\alpha} - \frac{1}{\varphi} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} x_0^{1-\beta} E_\beta\left(\frac{x_0}{\eta}\right)\right) t \\ &\quad - \left(\frac{1-\alpha}{\mu_0}\right)^{\frac{1}{\alpha}} \frac{1}{\varphi} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} \int_0^t x(z)^{1-\beta} E_\beta\left(\frac{1}{\eta} x(z)\right) dz \end{aligned}$$

To complete the determination of the path of the stock of resource we need to determine $\int_0^t x(z)^{1-\beta} E_\beta\left(\frac{1}{\eta} x(z)\right) dz$. We have

$$x(t) = \frac{\rho\left(\lambda_0^{\frac{\alpha-1}{\alpha}} + \varphi t\right)}{\varphi} = \frac{\rho \lambda_0^{\frac{\alpha-1}{\alpha}}}{\varphi} + \rho t$$

Consider the following change of variable: $\tau(z) = \frac{1}{\eta}\left(\frac{\rho\lambda_0^{\frac{\alpha-1}{\alpha}}}{\varphi} + \rho z\right) = \frac{1}{\eta} x(z)$. Then $\left(\eta\tau - \frac{\rho\lambda_0^{\frac{\alpha-1}{\alpha}}}{\varphi}\right)\frac{1}{\rho} = z$

and $\eta\frac{1}{\rho}d\tau = dz$ and therefore

$$\begin{aligned}
\int_0^t x(z)^{1-\beta} E_\beta\left(\frac{1}{\eta}x(z)\right)dz &= \int_{\frac{x_0}{\eta}}^{\frac{1}{\eta}x(t)} \eta^{1-\beta}\tau^{1-\beta}E_\beta(\tau)\,\eta\frac{1}{\rho}d\tau \\
&= \frac{1}{\rho}\eta^{2-\beta}\int_{\frac{x_0}{\eta}}^{\frac{1}{\eta}x(t)} \tau^{1-\beta}E_\beta(\tau)\,d\tau \\
&= \frac{1}{\rho}\eta^{2-\beta}\left(\int_0^{\frac{1}{\eta}x(t)} \tau^{1-\beta}E_\beta(\tau)\,d\tau - \int_0^{\frac{x_0}{\eta}} \tau^{1-\beta}E_\beta(\tau)\,d\tau\right) \\
&= \frac{1}{\rho}\eta^{2-\beta}\left(\Psi\left(\frac{x(t)}{\eta}\right) - \Psi\left(\frac{x_0}{\eta}\right)\right) \qquad (35)
\end{aligned}$$

where

$$\int_0^B \tau^{1-a}E_a(\tau)\,d\tau = \Psi(B) + \Gamma(2-a) \qquad (36)$$

where $\Gamma(.)$ is the Gamma function and

$$\Psi(B) \equiv B^{2-a}\left(E_a(B) - E_{a-1}(B)\right)$$

To show (36), we first use (14) and then integrate by parts. We have

$$\int_0^B \tau^{1-a}E_a(\tau)\,d\tau = \int_0^B F(\tau)\,d\tau = BF(B) - \int_0^B \tau F'(\tau)\,d\tau$$

where

$$F(\tau) = \int_\tau^\infty e^{-t}t^{-a}\,dt.$$

This gives

$$\int_0^B \tau^{1-a}E_a(\tau)\,d\tau = B^{2-a}E_a(B) + \int_0^B \tau^{1-a}e^{-\tau}\,d\tau$$

or

$$\int_0^B \tau^{1-a}E_a(\tau)\,d\tau = B^{2-a}E_a(B) + \int_0^\infty \tau^{1-a}e^{-\tau}\,d\tau - \int_B^\infty \tau^{1-a}e^{-\tau}\,d\tau.$$

Using (14), the last part of the right-hand side can be substituted by $B^{2-a}E_{a-1}(B)$ and we have

$$\int_0^B \tau^{1-a}E_a(\tau)\,d\tau = B^{2-a}E_a(B) + \Gamma(2-a) - B^{2-a}E_{a-1}(B)$$

where Γ is the Gamma function. Thus, using (35), (34) and noting that $\left(\frac{1-\alpha}{\mu_0}\right)^{\frac{1}{\alpha}}\frac{1}{\varphi}$ simplifies into $\frac{1}{\mu_0}$ completes the proof■

Appendix C: Proof of Lemma 2

$$
\begin{aligned}
0 &= \lim_{t\to\infty} e^{-\rho t}\lambda(t) K(t) \\
&= \lim_{t\to\infty} \left(\lambda_0^{1-\frac{1}{\alpha}} + \varphi t\right)^{\frac{\alpha}{\alpha-1}} \pi(t)^{\frac{1}{1-\alpha}} \left(K_0\lambda_0^{1/\alpha} - \frac{1}{\varphi} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} \left(x_0^{1-\beta} E_\beta\left(\frac{x_0}{\eta}\right) - x(t)^{1-\beta} E_\beta\left(\frac{x(t)}{\eta}\right)\right)\right) \\
&= \lim_{t\to\infty} \pi(t) \left(K_0\lambda_0^{1/\alpha} - \frac{1}{\varphi} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} \left(x_0^{1-\beta} E_\beta\left(\frac{x_0}{\eta}\right) - x(t)^{1-\beta} E_\beta\left(\frac{x(t)}{\eta}\right)\right)\right) \qquad (37)
\end{aligned}
$$

The series expansion of the exponential integral when z tends to infinity (see Abramowitz and Stegun (1972), 5.1.51) reads

$$
E_\beta(z) = e^{-z}\left(\frac{1}{z} - \beta\left(\frac{1}{z}\right)^2 + O\left(\left(\frac{1}{z}\right)^3\right)\right) \qquad (38)
$$

Therefore

$$
\lim_{t\to\infty} x(t)^{2-\beta} E_\beta\left(\frac{x(t)}{\eta}\right) = \lim_{t\to\infty} x(t)^{2-\beta} e^{-x(t)} \left(\frac{1}{x(t)} + O\left(\left(\frac{1}{x(t)}\right)^2\right)\right)
$$

For any $\beta \in \mathcal{R}$ we have $\lim_{t\to\infty} x(t)^{2-\beta} e^{-x(t)} = \lim_{x\to\infty} x^{2-\beta} e^{-x} = 0$ which implies

$$
\begin{aligned}
\lim_{t\to\infty} \pi(t) \frac{1}{\varphi} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} x(t)^{1-\beta} E_\beta\left(\frac{x(t)}{\eta}\right) &= \lim_{t\to\infty} \frac{\varphi x(t)}{\rho} \frac{1}{\varphi} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} x(t)^{1-\beta} E_\beta\left(\frac{x(t)}{\eta}\right) \\
&= \lim_{t\to\infty} \frac{1}{\rho} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} x(t)^{2-\beta} E_\beta\left(\frac{x(t)}{\eta}\right) = 0
\end{aligned}
$$

The transversality condition gives

$$
\lim_{t\to\infty} \pi(t) \left(K_0\lambda_0^{1/\alpha} - \frac{1}{\varphi} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} x_0^{1-\beta} E_\beta\left(\frac{x_0}{\eta}\right)\right) = 0
$$

This is satisfied if

$$
K_0\lambda_0^{1/\alpha} - \frac{1}{\varphi} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} x_0^{1-\beta} E_\beta\left(\frac{x_0}{\eta}\right) = 0 \qquad (39)
$$

This is true for all $\eta > 0$.

Next we prove (23). We start again from Proposition 4 and take the transversality condition for K into account. Recalling that

$$
\Psi(x) = x^{2-\beta}\left(E_\beta(x) - E_{\beta-1}(x)\right)
$$

we find

$$
-S(0) = \lim_{t\to\infty} -\frac{1}{\rho\mu_0} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} \eta^{2-\beta} \left(\Psi\left(\frac{x(t)}{\eta}\right) - \Psi\left(\frac{x_0}{\eta}\right)\right)
$$

Using the expansion (38) above gives

$$\lim_{t\to\infty} \Psi(x) = \lim_{x\to\infty} \left(x^{2-\beta}\left(E_\beta(x) - E_{\beta-1}(x)\right)\right) = 0$$

So, the transversality condition becomes

$$S_0 = -\frac{1}{\rho\mu_0} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} (\eta)^{2-\beta}\, \Psi\left(\frac{x_0}{\eta}\right)$$

which, after substitution of φ and noting that $\frac{\alpha}{((\alpha-1)\beta+1)} = \eta$, gives

$$\mu_0 = \left(-\frac{\left(\frac{\eta}{\rho}\right)^{2-\beta}\left(\frac{1}{1-\alpha}\right)^{-\frac{(1-\beta)}{\alpha}} \Psi\left(\frac{x_0}{\eta}\right)}{S_0}\right)^{\eta} \blacksquare \tag{40}$$

Appendix D: Proof of Lemma 3

The proof is divided in two steps: (i) proof that x_0 must be solution to $h_\beta(x) = A$ and (ii) proof that there exists a solution to $h_\beta(x) = A$.

(i) We have

$$S_0 = -\frac{1}{\rho\mu_0} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} \eta^{2-\beta}\Psi\left(\frac{x_0}{\eta}\right)$$

and

$$K_0\lambda_0^{1/\alpha} = \frac{1}{\varphi} e^{\frac{x_0}{\eta}} \left(\frac{\varphi}{\rho}\right)^{1-\beta} x_0^{1-\beta} E_\beta\left(\frac{x_0}{\eta}\right) \tag{41}$$

Therefore

$$\frac{S_0}{K_0} = -\lambda_0^{\frac{1}{\alpha}} \frac{\varphi}{\mu_0} \frac{\eta^{2-\beta}\Psi\left(\frac{x_0}{\eta}\right)}{\rho x_0^{1-\beta} E_\beta\left(\frac{x_0}{\eta}\right)} \tag{42}$$

with

$$\varphi = (1-\alpha)\left(\frac{\mu_0}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} \text{ and thus } \frac{\varphi}{\mu_0} = \left(\frac{\mu_0}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}-1} = \left(\frac{\mu_0}{1-\alpha}\right)^{-\frac{1}{\alpha}}$$

Substituting $\frac{\varphi}{\mu_0}$ into (42) gives

$$\frac{S_0}{K_0} = -\left(\frac{\lambda_0}{\mu_0}\right)^{\frac{1}{\alpha}} \left(\frac{1}{1-\alpha}\right)^{-\frac{1}{\alpha}} \frac{\eta^{2-\beta}\Psi\left(\frac{x_0}{\eta}\right)}{\rho x_0^{1-\beta} E_\beta\left(\frac{x_0}{\eta}\right)}.$$

Using the following relationship

$$\left(\frac{x_0}{\rho}\right)^{\frac{\alpha}{\alpha-1}} (1-\alpha)^{\frac{1}{\alpha-1}} = \frac{\lambda_0}{\mu_0}$$

yields

$$\left(\frac{(1-\alpha)^{\frac{1}{\alpha}}}{\rho}\right)^{\frac{\alpha}{1-\alpha}} \eta^{\beta-2}\frac{S_0}{K_0} = -x_0^{\frac{1}{\alpha-1}+\beta-1} \frac{\Psi\left(\frac{x_0}{\eta}\right)}{E_\beta\left(\frac{x_0}{\eta}\right)}$$

or

$$\left(\eta\frac{(1-\alpha)^{\frac{1}{\alpha}}}{\rho}\right)^{\frac{\alpha}{1-\alpha}}\frac{S_0}{K_0}=-\left(\frac{x_0}{\eta}\right)^{\frac{1}{\alpha-1}+\beta-1}\frac{\Psi\left(\frac{x_0}{\eta}\right)}{E_\beta\left(\frac{x_0}{\eta}\right)}$$

Recall that

$$A=\left(\eta\frac{(1-\alpha)^{\frac{1}{\alpha}}}{\rho}\right)^{\frac{\alpha}{1-\alpha}}\frac{S_0}{K_0}$$

and

$$h_\beta\left(x\right)=-\left(\frac{1}{\eta}x\right)^{\frac{1}{\alpha-1}+\beta-1}\frac{\Psi\left(\frac{1}{\eta}x\right)}{E_\beta\left(\frac{1}{\eta}x\right)}$$

then x_0 solves

$$h_\beta\left(x_0\right)=A$$

This completes (i).

(ii) We now argue that

$$\lim_{x_0\to0^+}h_\beta\left(x_0\right)=\infty \text{ and } \lim_{x_0\to\infty}h_\beta\left(x_0\right)=0$$

which given the continuity of h_β over $(0,\infty)$ proves, the existence of a sulution.

We start by rewriting $h_\beta\left(x\right)$ using the recurrence relationship (see Abramowitz and Stegun (1972), 5.1.14)

$$E_\beta(z)=\frac{1}{\beta-1}\left(e^{-z}-zE_{\beta-1}(z)\right)$$

which gives

$$\begin{aligned}\frac{\Psi\left(x\right)}{E_\beta\left(x\right)} &= x^{2-\beta}\left(1-\frac{E_{\beta-1}(x)}{E_\beta(x)}\right)\\ &= x^{2-\beta}\left(1-\frac{e^{-x}-\left(\beta-1\right)E_\beta(x)}{xE_\beta(x)}\right)\end{aligned}$$

and thus

$$\begin{aligned}h_\beta\left(x\right) &= -\left(x\right)^{\frac{\alpha}{\alpha-1}}\left(1-\frac{E_{\beta-1}(x)}{E_\beta(x)}\right)\\ &= -\left(x\right)^{\frac{\alpha}{\alpha-1}}\left(1-\frac{e^{-x}-\left(\beta-1\right)E_\beta(x)}{xE_\beta(x)}\right)\end{aligned}$$

or

$$h_\beta\left(x\right)=\left(x\right)^{\frac{\alpha}{\alpha-1}}\left(\frac{e^{-x}}{xE_\beta(x)}-\frac{\left(\beta-1\right)}{x}-1\right)$$

From Abramowitz and Stegun (1972), 5.1.51, we have that $E_\beta\left(z\right)$ is asymptotically equal (and we use the symbol $\sim$) to

$$E_\beta\left(z\right)\sim\frac{e^{-z}}{z}\left(1-\frac{\beta}{z}+\frac{\beta\left(\beta+1\right)}{z^2}-\frac{\beta\left(\beta+1\right)\left(\beta+2\right)}{z^3}...\right) \tag{43}$$

and therefore

$$z^{\frac{1}{1-\alpha}} e^{z} E_{\beta}(z) \sim z^{\frac{\alpha}{1-\alpha}} \left(1 - \frac{\beta}{z} + \frac{\beta(\beta+1)}{z^{2}} - \frac{\beta(\beta+1)(\beta+2)}{z^{3}} ...\right) \tag{44}$$

This combined with $\alpha \in (0,1)$ gives

$$\lim_{z\to\infty} z^{\frac{1}{1-\alpha}} e^{z} E_{\beta}(z) = \infty \text{ and thus } \lim_{x\to\infty} h_{\beta}(x) = 0.$$

We now turn to the case where $x \to 0^{+}$. Let n_{β} denote the smallest integer strictly smaller than β. It is straightforward to show that $E_{\beta}(x)$ is a strictly decreasing function of β. Hence, since $n_{\beta} \geq \beta - 1$, we have

$$h_{\beta}(x) = x^{\frac{\alpha}{\alpha-1}} \left(\frac{E_{\beta-1}(x)}{E_{\beta}(x)} - 1\right) \geq x^{\frac{\alpha}{\alpha-1}} \left(\frac{E_{n_{\beta}}(x)}{E_{\beta}(x)} - 1\right)$$

We have for all $\beta \in (0,1]$

$$\lim_{x\to 0^{+}} \frac{E_{n_{\beta}}(x)}{E_{\beta}(x)} = \frac{E_{0}(x)}{E_{\beta-n_{\beta}}(x)} = \lim_{x\to 0^{+}} \frac{1}{x e^{x} E_{\beta-n_{\beta}}(x)}$$

Using L'Hopital's rule we have, for $\beta > 1$

$$\lim_{x\to 0^{+}} \frac{E_{n_{\beta}}(x)}{E_{\beta}(x)} = \lim_{x\to 0^{+}} \frac{E_{n_{\beta}-1}(x)}{E_{\beta-1}(x)} = ... = \lim_{x\to 0^{+}} \frac{E_{0}(x)}{E_{\beta-n_{\beta}}(x)} = \lim_{x\to 0^{+}} \frac{1}{x e^{x} E_{\beta-n_{\beta}}(x)}$$

and similarly for $\beta \leq 0$

$$\lim_{x\to 0^{+}} \frac{E_{n_{\beta}}(x)}{E_{\beta}(x)} = \lim_{x\to 0^{+}} \frac{E_{n_{\beta}+1}(x)}{E_{\beta+1}(x)} = ... = \lim_{x\to 0^{+}} \frac{E_{0}(x)}{E_{\beta-n_{\beta}}(x)} = \lim_{x\to 0^{+}} \frac{1}{x e^{x} E_{\beta-n_{\beta}}(x)}.$$

We now show that

$$\lim_{x\to 0^{+}} \frac{1}{x e^{x} E_{\beta-n_{\beta}}(x)} = \infty$$

Note that $\beta - n_{\beta} \in (0,1]$. We distinguish between $\beta - n_{\beta} \in (0,1)$ and $\beta - n_{\beta} = 1$.

Suppose that $\beta - n_{\beta} \in (0,1)$. The asymptotic behavior of $E_{\beta-n_{\beta}}(z)$ when $z \to 0$ is given by

$$E_{\beta-n_{\beta}}(z) = z^{\beta-n_{\beta}-1}\Gamma(1-(\beta-n_{\beta})) - \sum_{n=0}^{\infty} \frac{(-1)^{n} z^{n}}{n!\,(1-(\beta-n_{\beta})+n)}$$

which is found by using the following relationship

$$E_{a}(z) = z^{a-1}\Gamma(1-a) - z^{a-1}\gamma(1-a,z)$$

where Γ is the Gamma function and γ is called the incomplete Gamma function, and by the asymptotic behavior γ in the neighborhoud of zero (see Temme, 1996, p. 279). Therefore,

$$e^{x} x E_{\beta-n_{\beta}}(x) = e^{x} x^{\beta-n_{\beta}}\Gamma(1-(\beta-n_{\beta})) - e^{x} x \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{n}}{n!\,(1-(\beta-n_{\beta})+n)}$$

and since $\beta - n_{\beta} \in (0,1)$

$$\lim_{x\to 0^{+}} e^{x} x E_{\beta}(x) = 0$$

or

$$\lim_{x\to 0^+}\left(\frac{1}{e^x x E_{\beta}(x)}-1\right)=\infty$$

and thus

$$\lim_{x\to 0^+} h_\beta(x)=\infty.$$

For the case where $\beta - n_\beta = 1$ (i.e., β integer) we have

$$\lim_{x\to 0^+}\frac{1}{xe^x E_1(x)}$$

using

$$\frac{1}{2}e^{-x_0}\ln\left(1+\frac{2}{x_0}\right) < E_1(x_0) < e^{-x_0}\ln\left(1+\frac{1}{x_0}\right)$$

gives

$$\frac{1}{2}x\ln\left(1+\frac{2}{x_0}\right) < xe^x E_1(x_0) < x\ln\left(1+\frac{1}{x_0}\right)$$

thus

$$\frac{1}{x\ln\left(1+\frac{1}{x_0}\right)} < \frac{1}{xe^x E_1(x_0)} < \frac{2}{x\ln\left(1+\frac{2}{x_0}\right)}$$

since

$$\lim_{x\to 0^+}\frac{1}{x\ln\left(1+\frac{1}{x}\right)}=\infty$$

we have

$$\lim_{x\to 0^+}\frac{1}{xe^x E_1(x)}=\infty$$

and thus

$$\lim_{x\to 0^+} h_\beta(x)=\infty.$$

To sum-up we have,

$$\lim_{x_0\to 0^+} h_\beta(x_0)=\infty \text{ and } \lim_{x_0\to\infty} h_\beta(x_0)=0$$

where $h_\beta(.)$ is a continuous function of x over $(0,\infty)$.Therefore there exists at least one solution $x_0>0$ to $h_\beta(x_0)=A$, for any $A>0$. Taking into account that the solution of the optimal control problem under consideration is unique, we thereby also establish uniqueness of x_0■

Appendix E: Proof that $h_1'(x)<0$ for all $x>0$.

We compute the derivative of the function h_1 and find

$$h_1'(x)=\frac{\left(e^{-x}\right)^2(1-\alpha)-\left(e^{-x}x(1-\alpha)+e^{-x}\right)E_1(x)+\left(E_1(x)\right)^2 x\alpha}{\left(E_1(x)\right)^2 x_0^{\frac{1}{1-\alpha}+1}(1-\alpha)}$$

The sign of the denominator is positive. Therefore the sign of $h'(x)$ is the same as the sign of the numerator, denoted by $N(x)$. Using the property of the exponential integral that

$$\frac{1}{2}e^{-x}\ln\left(1+\frac{2}{x}\right) < E_1(x) < e^{-x}\ln\left(1+\frac{1}{x}\right)$$

it holds for all $x \in (0, \infty)$ that

$$N(x) < e^{-2x} Z(x)$$

where

$$Z(x) \equiv \left((1-\alpha) - (x(1-\alpha)+1) \frac{1}{2} \ln\left(1+\frac{2}{x}\right) + \left(\ln\left(1+\frac{1}{x}\right)\right)^2 x\alpha \right).$$

For the derivative of $Z(x)$ we get

$$\begin{aligned} Z'(x) &= -\frac{1}{2}(1-\alpha)\ln(1+\frac{2}{x}) + \frac{x(1-\alpha)+1}{x(x+2)} \\ &\quad +\alpha\left(\ln(1+\frac{1}{x})\right)^2 - 2\alpha\frac{1}{1+x}\ln(1+\frac{1}{x}) \end{aligned}$$

The second derivative is

$$Z''(x) = -2\frac{2x - 3x\alpha + x^2 - 2x^2\alpha + \left(4x\alpha + 4x^2\alpha + x^3\alpha\right)\ln\frac{1}{x}(x+1) + 1}{(x+1)^2 (x+2)^2 x^2}$$

Using

$$\frac{\frac{1}{x}}{1+\frac{1}{x}} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$$

we have

$$\begin{aligned} &\left(2x - 3x\alpha + x^2 - 2x^2\alpha + \left(4x\alpha + 4x^2\alpha + x^3\alpha\right)\ln(\frac{1}{x}(x+1)) + 1\right) \\ > &\left(2x - 3x\alpha + x^2 - 2x^2\alpha + \left(4x\alpha + 4x^2\alpha + x^3\alpha\right)\frac{\frac{1}{x}}{1+\frac{1}{x}} + 1\right) \end{aligned}$$

and thus

$$Z''(x) < (-2)\frac{\left(2x - 3x\alpha + x^2 - 2x^2\alpha + \left(4x\alpha + 4x^2\alpha + x^3\alpha\right)\frac{\frac{1}{x}}{1+\frac{1}{x}} + 1\right)}{(x+1)^2 (x+2)^2 x^2}$$

which after simplifications becomes

$$Z''(x) < (-2)\frac{(3+\alpha)x + (3-\alpha)x^2 + x^3(1-\alpha) + 1}{(x+1)^3 (x+2)^2 x^2} < 0$$

So $Z''(x) < 0$ for all $\alpha \in [0,1]$ and all $x \in (0, \infty)$ and therefore

$$Z'(x) \in (Z'(\infty), Z'(0))$$

with

$$\lim_{x\to\infty} Z'(x) = 0 \text{ and } \lim_{x\to 0^+} Z'(x) = \infty$$

that is $Z'(x) > 0$ for all $x \in (0, \infty)$ and therefore

$$Z(x) \in (Z(0), Z(\infty))$$

with

$$\lim_{x \to \infty} Z(x) = 0 \text{ and } \lim_{x \to 0^+} Z(x) = -\infty$$

Since $Z(x) < 0$ all $x \in (0, \infty)$ we have $N(x) < 0$ and $h_1'(x) < 0$ all $x \in (0, \infty)$■

Appendix F: Proof of Lemma 4

We show that (i) $\frac{d(x_0 e^{x_0} E_1(x_0))}{dx_0} > 0$ with $0 < x_0 e^{x_0} E_1(x_0) < 1$ for all $x_0 > 0$ and (ii) $\frac{d(e^{x_0} E_1(x_0))}{dx_0} < 0$ with $0 < e^{x_0} E_1(x_0)$ for all $x_0 > 0$.

(i) We have $\frac{d(x_0 e^{x_0} E_1(x_0))}{dx_0} = -1 + (1 + x_0) e^{x_0} E_1(x_0)$. Using the fact that (see Abramowitz and Stegun (1972) p. 229 Inequality 5.1.19)

$$\frac{1}{z+1} < e^z E_1(z) < \frac{1}{z} \text{ for } z > 0 \tag{45}$$

we have

$$\frac{1}{x_0 + 1} < e^{x_0} E_1(x_0)$$

gives $\frac{d(x_0 e^{x_0} E_1(x_0))}{dx_0} > 0$ and therefore $\frac{1}{x_0 e^{x_0} E_1(x_0)} - 1$ is a strictly decreasing function of x_0 over the domain $(0, \infty)$ with

$$\lim_{x \to 0^+} \frac{1}{x e^x E_1(x)} = \infty$$

using

$$\frac{1}{2} e^{-x_0} \ln\left(1 + \frac{2}{x_0}\right) < E_1(x_0) < e^{-x_0} \ln\left(1 + \frac{1}{x_0}\right)$$

gives

$$\frac{1}{2} x_0 \ln\left(1 + \frac{2}{x_0}\right) < x_0 e^{x_0} E_1(x_0) < x_0 \ln\left(1 + \frac{1}{x_0}\right)$$

thus

$$\frac{1}{x_0 \ln\left(1 + \frac{1}{x_0}\right)} < \frac{1}{x_0 e^{x_0} E_1(x_0)} < \frac{2}{x_0 \ln\left(1 + \frac{2}{x_0}\right)}$$

since

$$\lim_{x \to 0^+} \frac{1}{x \ln\left(1 + \frac{1}{x}\right)} = \infty$$

we have

$$\lim_{x \to 0^+} \frac{1}{x e^x E_1(x)} = \infty$$

and

$$\lim_{x_0 \to \infty} \left(\frac{2}{x_0 \ln\left(1 + \frac{2}{x_0}\right)}\right) = \lim_{z \to 0^+} \left(\frac{z}{\ln(1+z)}\right) = 1$$

therefore $\frac{1}{xe^{x}E_{1}(x)}-1$ is a strictly decreasing function from $(0,\infty)$ into $(0,\infty)$.

(ii) $\frac{d(e^{x_0}E_1(x_0))}{dx_0}=-\frac{1}{x_0}+e^{x_0}E_1(x_0)$ from

$$e^{x_0}E_1\left(x_0\right)<\frac{1}{x_0}$$

we have $\frac{d(e^{x_0}E_1(x_0))}{dx_0}<0$ for all $x_0>0$ with

$$\frac{1}{2}\ln\left(1+\frac{2}{x_0}\right)<e^{x_0}E_1\left(x_0\right)<\ln\left(1+\frac{1}{x_0}\right)$$

and therefore

$$\lim_{x\to 0^{+}}e^{x}E_1\left(x\right)=\infty$$

and

$$\lim_{x\to\infty}e^{x}E_1\left(x\right)=0 \blacksquare$$

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