

Centre interuniversitaire de recherche
en économie quantitative

CIREQ

Cahier 12-2003

*RESOURCE-MONOTONICITY FOR
HOUSE ALLOCATION PROBLEMS*

Lars EHLERS and
Bettina KLAUS

Cahier 12-2003

RESOURCE-MONOTONICITY FOR HOUSE ALLOCATION PROBLEMS

Lars EHLERS¹ and Bettina KLAUS²

- ¹ Centre interuniversitaire de recherche en économie quantitative (CIREQ) and Département de sciences économiques, Université de Montréal
- ² Departament d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona

May 2003

The authors wish to thank Szilvia Pápai for useful comments.

CIREQ, Université de Montréal
C.P. 6128, succursale Centre-ville
Montréal (Québec) H3C 3J7
Canada

téléphone : (514) 343-6557
télécopieur : (514) 343-5831
cireq@umontreal.ca
<http://www.cireq.umontreal.ca>

Université 
de Montréal

 McGill

 Concordia
UNIVERSITY

Abstract

We study a simple model of assigning indivisible objects (e.g., houses, jobs, offices, *etc.*) to agents. Each agent receives at most one object and monetary compensations are not possible. We completely describe all rules satisfying efficiency and *resource-monotonicity*. The characterized rules assign the objects in a sequence of steps such that at each step there is either a dictator or two agents who “trade” objects from their hierarchically specified “endowments.”

Keywords : indivisible objects, resource-monotonicity

1 Introduction

We study the problem of allocating heterogeneous indivisible objects among a group of agents (for instance, houses, jobs, or offices) when monetary compensations are not possible. Agents are assumed to have strict preferences over objects and remaining unassigned. An assignment is an allocation of the objects to the agents such that every agent receives at most one object. A rule associates an assignment to each preference profile. This problem is known as house allocation and it is the subject of recent papers by Abdulkadiroğlu and Sönmez (1998, 1999), Bogomolnaia and Moulin (2001), Ehlers (2002a), Ehlers and Klaus (2002a, 2002b), Ehlers, Klaus, and Pápai (2002), Ergin (2000, 2002), Pápai (2000), and Svensson (1999).¹

We consider situations where resources may change. When the change of the environment is exogenous, it would be unfair if the agents who were not responsible for this change are treated unequally. We apply this idea of solidarity and require that if more resources are available, then as a result either all remaining agents (weakly) gain or they all (weakly) lose. This requirement is called resource-monotonicity (Chun and Thomson, 1988). Some recent studies that consider resource-monotonic allocation for different economic environments are Moulin (1999) and Ehlers (2002b, allocation with single-peaked preferences), Ehlers and Klaus (2000, multiple assignment problems), Hokari (2000, convex TU-games), and Maniquet and Sprumont (2000, fair division).² Essentially all the above mentioned articles study the restrictions resource-monotonicity imposes by either (a) inducing incompatibilities with other desirable properties (efficiency and conditional equal split in Maniquet and Sprumont (2000)) or solution concepts (the nucleolus in Hokari (2000)) or (b) by decisively narrowing down the class of available allocation rules that satisfy resource-monotonicity and other desirable properties (e.g., ordered-path rationing methods in Moulin (1999) and Ehlers (2002b) or serial dictatorships in Ehlers and Klaus (2000)).

We contribute to this line of research by applying resource-monotonicity to house allocation problems. Our main result is the characterization of a class of rules, called mixed dictator-pairwise-exchange rules, by efficiency and resource-monotonicity. Mixed dictator-pairwise-exchange rules are essentially hierarchical since they allow "trading" of the objects by at most two agents at a time. Therefore, our result implies that efficiency and resource-monotonicity are only feasible in the absence of initial individual ownership

¹This list is not exhaustive.

²This list is not exhaustive.

(or, more precisely, in the absence of more than two owners at a time).

In Section 2 we introduce the house allocation problem with variable resources and define our main properties for rules. In Section 3 we first present the class of mixed dictator-pairwise-exchange rules. We state and discuss our characterization of this class of rules by efficiency and resource-monotonicity in the second part of Section 3. We prove our main result in Section 4.

2 House Allocation with Variable Resources

Let N denote a finite set of agents, $|N| \geq 2$. Let K denote a set of potential objects. For technical convenience, we assume $|K| > |N|$.³ Let 0 represent the null object. "Receiving the null object" means "not receiving any object." Each agent $i \in N$ is equipped with a strict preference relation R_i over $K \cup \{0\}$. In other words, R_i is a linear order over $K \cup \{0\}$. Given $x, y \in K \cup \{0\}$, $x P_i y$ means that agent i strictly prefers x to y . Let \mathcal{R} denote the class of all linear orders over $K \cup \{0\}$, and \mathcal{R}^N the set of (preference) profiles $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. Given $K^0 \subseteq K \cup \{0\}$, let $R_{i|K^0}$ denote the restriction of R_i to K^0 and $R_{j|K^0} = (R_{i|K^0})_{i \in N}$. Let $\mathcal{R}_0 \subseteq \mathcal{R}$ denote the class of preference relations where the null object is the worst object. That is, if $R_i \in \mathcal{R}_0$, then all the objects are "goods": for all $x \in K$; $x P_i 0$.

An allocation is a list $a = (a_i)_{i \in N}$ such that for all $i \in N$, $a_i \in K \cup \{0\}$, and none of the objects in K is assigned to more than one agent. Note that 0 , the null object, can be assigned to any number of agents and that not all objects in K have to be assigned. Let \mathcal{A} denote the set of all allocations. Let \mathcal{H} denote the set of all non-empty subsets H of K . A (house allocation) problem consists of a set of objects $H \in \mathcal{H}$ and a preference profile $R \in \mathcal{R}^N$. An (allocation) rule is a function $\sigma : \mathcal{R}^N \times \mathcal{H} \rightarrow \mathcal{A}$ such that for all problems $(R; H) \in \mathcal{R}^N \times \mathcal{H}$:

- (i) $\sigma(R; H) \in \mathcal{A}$ is feasible, i.e., for all $i \in N$, $\sigma_i(R; H) \in H \cup \{0\}$ and
- (ii) $\sigma(R; H)$ is independent of irrelevant objects, i.e., for all $R^0 \in \mathcal{R}^N$ such that $R_{j|H \cup \{0\}} = R^0_{j|H \cup \{0\}}$, $\sigma(R; H) = \sigma(R^0; H)$.

By feasibility (i), each agent receives an available object or nothing. By independence of irrelevant objects (ii), the rule depends only on preferences

³ If $|K| = 1$, then $|K| > |N|$ is trivially satisfied. We explain how to adjust the results if $|K| = |N|$ at the end of Section 3.

over the set of available objects. The set of available objects includes the null object since it is available in any economy. Given $i \in N$, we call $a_i(R; H)$ the allotment of agent i at $(R; H)$.

First, we require that a rule chooses only (Pareto) efficient allocations.

Efficiency: For all $(R; H) \in R^N \times H$, there is no feasible allocation $a \in A$ such that for all $i \in N$, $a_i R_i a_i(R; H)$, with strict preference holding for some $j \in N$.

When the set of objects varies, then a natural requirement is resource-monotonicity. It describes the effect of a change in the available resource on the welfare of the agents. A rule satisfies resource-monotonicity, if after such a change either all agents (weakly) lose or all (weakly) gain.

It is easy to see that in combination with efficiency, resource-monotonicity means that if for some fixed preference profile and some fixed set of objects, new additional objects are available, then – this being good news – all agents (weakly) gain. Since we study resource-monotonicity together with efficiency we use the latter to formalize resource-monotonicity.

Resource-Monotonicity: For all $R \in R^N$ and all $H; H^0 \in H$, if $H \supseteq H^0$, then for all $i \in N$, $a_i(R; H^0) R_i a_i(R; H)$.

3 Mixed Dictator-Pairwise-Exchange Rules

Our aim is to describe the class of rules that are efficient and resource-monotonic. Each rule belonging to this class allocates the available objects in a sequence of steps as follows: At the first step there is either a dictator who receives for all problems his most preferred object from the set of available objects or there are exactly two agents who divide the set of objects among them and for all problems their allotments result from a pairwise exchange using the division of the objects as endowments. At the second step there is again either a dictator or a pairwise exchange (restricted to the remaining available objects); and so on. Here, we call such a rule a mixed dictator-pairwise-exchange rule (Ehlers, 2002a). In Ehlers, Klaus, and Pápai (2002) we essentially discuss the same class of rules under the name “restricted endowment inheritance rules.” For another interpretation of these rules as so-called efficient priority rules, we refer to Ehlers and Klaus (2002b) and Ergin (2002).

For the formal description we use (endowment) inheritance tables (Pápai, 2000). For each object $x \in K$, a one-to-one function $\mu_x : \{1, \dots, j\} \rightarrow N$ specifies the inheritance of object x . Here agent $\mu_x(1)$ is initially endowed with x . If x is still available after $\mu_x(1)$ received an object, then $\mu_x(2)$

inherits x ; and so on. Let f_1^N denote the set of all one-to-one functions from $f_1; \dots; jNjg$ to N . An inheritance table is a profile $\mu = (\mu_x)_{x \in K}$ specifying the inheritance of each object. We call an inheritance table μ a mixed dictator-pairwise-exchange inheritance table with respect to $S = (S_1; S_2; \dots; S_m)$ if

- (i) $(S_1; S_2; \dots; S_m)$ is a partition of N into singletons and pairs, i.e., for all $t \in f_1; \dots; mg$, $2 \leq |S_t| \leq 2$,
- (ii) row 1 and row jS_tj of the inheritance table contain exactly S_t , i.e., $f_{\mu_x}(1) \in S_t$ and $f_{\mu_x}(jS_tj) \in S_t$, and
- (iii) row $1 + \sum_{i=1}^{t-1} |S_i|$ and row $1 + \sum_{i=1}^t |S_i|$ contain exactly S_t , i.e., for all $t \in f_2; \dots; mg$, $f_{\mu_x}(1 + \sum_{i=1}^{t-1} |S_i|) \in S_t$ and $f_{\mu_x}(1 + \sum_{i=1}^t |S_i|) \in S_t$.

Given $i \in N$ and $H \in H$, let $\text{top}(R_i; H)$ denote agent i 's most preferred object under R_i in $H \cap f_0g$.

Mixed Dictator-Pairwise-Exchange Rules, $\mu^{(\mu; S)}$: Given a mixed dictator-pairwise-exchange inheritance table μ with respect to $S = (S_1; S_2; \dots; S_m)$, for all $(R; H) \in R^N \times H$ the allocation $\mu^{(\mu; S)}(R; H)$ is inductively determined as follows:

Step 1:

- (a) If $S_1 = \{i\}$, then $\mu^{(\mu; S)}(R; H) = \text{top}(R_i; H)$.
- (b) Let $S_1 = \{i; j\}$ ($i \neq j$). If $\text{top}(R_i; H) = \text{top}(R_j; H) = h_1 \in H$ and $\mu_{h_1}(1) = i$, then $\mu^{(\mu; S)}(R; H) = h_1$ and $\mu^{(\mu; S)}(R; H) = \text{top}(R_j; H \setminus \{h_1\})$. Otherwise, $\mu^{(\mu; S)}(R; H) = \text{top}(R_i; H)$ and $\mu^{(\mu; S)}(R; H) = \text{top}(R_j; H)$.

Step t : Let $H_{t-1} = \bigcap_{i=1}^{t-1} f_i^{(\mu; S)}(R; H)g$ denote the set of objects that are assigned up to Step t .

- (a) If $S_t = \{i\}$, then $\mu^{(\mu; S)}(R; H) = \text{top}(R_i; H \setminus H_{t-1})$.
- (b) Let $S_t = \{i; j\}$ ($i \neq j$). If $\text{top}(R_i; H \setminus H_{t-1}) = \text{top}(R_j; H \setminus H_{t-1}) = h_t \in H \setminus H_{t-1}$ and $\mu_{h_t}(1 + \sum_{i=1}^{t-1} |S_i|) = i$, then $\mu^{(\mu; S)}(R; H) = h_t$ and $\mu^{(\mu; S)}(R; H) = \text{top}(R_j; H \setminus (H_{t-1} \cup \{h_t\}))$. Otherwise, $\mu^{(\mu; S)}(R; H) = \text{top}(R_i; H \setminus H_{t-1})$ and $\mu^{(\mu; S)}(R; H) = \text{top}(R_j; H \setminus H_{t-1})$.

Mixed dictator-pairwise-exchange rules are a subclass of endowment inheritance rules as discussed in Ehlers, Klaus, and Pápai (2002).⁴ Our main result also applies to the domain R_0^N where all objects are "goods".

⁴Endowment inheritance rules are based on Gale's top trading cycle algorithm. We

Theorem 1. On the domain $\mathbb{R}^N \setminus \{0\}$, mixed dictator-pairwise-exchange rules are the only rules satisfying efficiency and resource-monotonicity.

We present the proof of this characterization in Section 4.

Theorem 1 is a characterization which is based on two relatively mild requirements since many rules satisfy either efficiency or resource-monotonicity. However, there is only a small class of rules satisfying both axioms. Furthermore, note that Theorem 1 is the first characterization in house allocation problems that does not require strategy-proofness: no agent can ever benefit from misrepresenting his preferences. Next we demonstrate that in fact all mixed dictator-pairwise-exchange rules satisfy the stronger non-manipulability property of coalitional strategy-proofness: no group of agents can ever benefit by misrepresenting their preferences.

Given $R \in \mathbb{R}^N$ and $M \subseteq N$, let R_M denote the profile $(R_i)_{i \in M}$. It is the restriction of R to the subset M of agents.

Coalitional Strategy-Proofness: For all $(R; H) \in \mathbb{R}^N \times H$ and all $M \subseteq N$, there exists no $\hat{R}_M \in \mathbb{R}^M$ such that for all $i \in M$, $(\hat{R}_M; R_{-i}; H) R_i \succ_i (R; H)$, with strict preference holding for some $j \in M$:

The following corollary from Theorem 1 is implied by the fact that any mixed dictator-pairwise-exchange rule is an endowment inheritance rule, which is coalitionally strategy-proof (Pápai, 2000).

Corollary 1. If a rule is efficient and resource-monotonic, then it is coalitionally strategy-proof.

In an earlier version of this article we characterized the class of mixed dictator-pairwise-exchange rules by efficiency, resource-monotonicity, and coalitional strategy-proofness. In Theorem 1 we strengthen this result by dropping coalitional strategy-proofness and including the much weaker (and in this context very natural) independence of irrelevant objects in the definition of a rule. The use of coalitional strategy-proofness greatly simplifies the proof, which then is very similar to the proof of the characterization presented in Ehlers, Klaus, and Pápai (2002).

Ehlers, Klaus, and Pápai (2002) characterize the same class of rules by efficiency, strategy-proofness, and the solidarity property population-monotonicity⁵. Similarly as in Ehlers, Klaus, and Pápai (2002) we conclude

omit their somewhat tedious definition and refer the interested reader to Pápai (2000). It is interesting to note that on the domain \mathbb{R}_0^N some inheritance tables that do not satisfy the conditions of a mixed dictator-pairwise-exchange inheritance table may still generate an endowment inheritance rule that equals a mixed dictator-pairwise-exchange rule.

⁵ If some agents leave, then as a result either all remaining agents (weakly) gain or they all (weakly) lose.

that guaranteeing solidarity comes with a price. Whereas without the solidarity property agents can “trade” objects arbitrarily, resource-monotonicity restricts the assignment of individual property rights and therefore “trading” to two agents at a time. This does not come as a surprise, considering that much of the literature on resource-monotonicity deals with impossibilities and incompatibilities (e.g., Hokari (2000), Maniquet and Sprumont (2000), Moulin and Thomson (1988), and Thomson (1994)). In fact, in light of this literature, our results can be regarded as positive results: we were able to identify rules that not only satisfy resource-monotonicity, but also the similarly demanding property of coalitional strategy-proofness. In spite of the hierarchical nature of these rules, they are not unappealing, and they offer flexibility in selecting the hierarchy itself and choosing the splitting of the endowments in the case of “twin-dictators.”

It is easy to see that the efficiency and resource-monotonicity are logically independent. Below we discuss how our results change if $|K| < |N|$.

Adjustments for $|K| \cdot |N|$

Recall that so far we have been assuming $|K| > |N|$ for technical convenience. Now assume $|K| \cdot |N|$ and denote $k \leq |K|$ and $n \leq |N|$. For the larger domain R all our results remain true. However, if $n \leq k$, then on the domain R_0 ; where the null object is always ranked last, we derive a slightly bigger set of rules, which are essentially mixed dictator-pairwise-exchange rules except that, loosely speaking, the last two objects may be arbitrarily inherited.

In particular, these allocation rules still require that the inheritance table reflects that at most two agents trade. However, since every agent who leaves the market receives an object, given the preference domain R_0 , only the first k rows of the inheritance table are relevant if $n > k$. In fact, inheritance of an object by an agent in row $k + 1$ implies that there are no more than two objects left, given that $k + 2$ agents have already received their allotments, and thus independent of the structure of the inheritance table after row k , these rules still do not allow trading by more than two agents. This is illustrated by the following example.

Example 1. Let $N = \{1; 2; 3; 4\}$ and $K = \{x; y; z\}$. Consider the inheri-

tance table $\frac{1}{4}$ specified below.

$\frac{1}{4}_x$	$\frac{1}{4}_y$	$\frac{1}{4}_z$
1	1	1
2	3	4
3	4	2
4	2	3

Note that when preferences are restricted to R_0 , $\frac{1}{4}$ satisfies efficiency, resource-monotonicity, and coalitional strategy-proofness. Furthermore, note that even though the second and third rows contain three different agents each, an agent in the second row inherits an object only if agent 1 has already received an object (which is different from the null object, since the null object is ranked last). Thus, exactly two agents inherit objects in the second row, and trading by three agents is excluded by this rule. }

There are two difficulties regarding the specification of mixed dictator-pairwise-exchange inheritance tables when $n \geq k$. The first one has to do with the uniqueness of the inheritance tables. When agents may rank the null object first (for preferences in R), each mixed dictator-pairwise-exchange inheritance table $\frac{1}{4}$ uniquely defines a mixed dictator-pairwise-exchange rule, in the sense that for two different mixed dictator-pairwise-exchange inheritance tables there always exists at least one preference profile at which the resulting allocations differ. This follows since, in situations where agents want to consume the null object and thus can leave the market without any assignment, each entry in a mixed dictator-pairwise-exchange inheritance table $\frac{1}{4}$ matters, given that the potential inheritance indicated by each entry is realized in certain cases. By contrast, if agents always rank the null object last (the case we are examining here), a mixed dictator-pairwise-exchange inheritance table may not uniquely define a rule.⁶ Note that in Example 1 the entire last two rows are redundant: since 2, 3, and 4 (in fact, only two of them) inherit one object each from 1, given the second row of the table, further inheritances will not occur at any preference profile.

The second difficulty is that when inheritances indicated by entries in the last (k -th) row of the table are not redundant, they may depend on the allocation of the objects to agents who have already received their assignments. For example, if agents 1 and 2 share exclusively the first two rows of an inheritance table, given the same setup as in Example 1, then we may

⁶ This possibility of non-uniqueness is reflected in the definition of the more general class of endowment inheritance rules. For a further discussion of uniqueness see Pápai (2000).

specify the inheritance of object z (which is not redundant in this case) as follows: let 3 inherit z if 1 receives x and 2 receives y , and let 4 inherit z if 1 receives y and 2 receives x . This “history-dependent” specification of inheritance, unlike in any other case, does not violate resource-monotonicity in the current case, since on the preference domain R_0 agents 3 or 4 will never inherit from 1 or 2 if there is less than the full set of three objects to allocate. Note that this type of conditional inheritance cannot be described by an inheritance table, and that in fact these rules do not form a subclass of the endowment inheritance rules.⁷

Both of the above difficulties are avoided if $n < k$, that is, if the full set of objects contains more objects than there are agents. The reason for this is that in this case resource-monotonicity has implications for the last two rows of the inheritance table as well, even on the smaller preference domain R_0 , which precludes agents who have endowments from leaving the market without an object. Note, furthermore, that the rules satisfying the required properties when $n \geq k$ are not significantly different from the mixed dictator-pairwise-exchange rules: the differences only concern the allocation of the last two objects, and these rules only offer more flexibility in choosing the last two recipients, while trade is still restricted to at most two agents. Therefore, in order to avoid the addition of tedious details, we state and prove our theorem on the preference domain R_0 for the less general case of $n < k$. Finally note that we do not compromise much with this assumption, since it is a very reasonable assumption in the context of variable resources: it simply says that we have more potentially available objects than the fixed number of agents, but, since resources may vary, it may be the case that the actually available set of objects contains fewer objects than the number of the agents.

4 Proof of Theorem 1

It is easy to verify that mixed dictator-pairwise-exchange rules satisfy efficiency and resource-monotonicity since no more than two agents “trade” at any step. In proving the converse, let ρ be a rule satisfying efficiency and resource-monotonicity. We give all proofs for the domain R_0 , since the proof for the larger domain R is completely analogous.

Recall that $|K_j| > |N_j|$ and since all agents always rank the null object

⁷The hierarchical exchange rules defined in Pápai (2000) include “history-dependent” inheritances and thus further explanations may be found there.

last, for all $H \in \mathcal{H}$ and all $R; R^0 \in R_0^N$ such that $R_j H = R^0_j H$ we have

$$\varphi(R; H) = \varphi(R^0; H): \quad (1)$$

Equation (1) reflects independence of irrelevant objects on R_0 , denoted IIO in the sequel, as specified in the definition of a rule.

We prove that we can calculate allocations assigned by φ in a sequence of steps that correspond to the algorithm for a unique mixed dictator-pairwise-exchange rule.

1. At most two agents trade in Step 1

Let $R \in R_0^N$. For all $h \in K$, we define $f^h(1) = i$ if and only if $\varphi_i(R; fhg) = h$. By efficiency, $f^h(1)$ is well-defined. Note that (1) and the fact that the null object is ranked last imply that the definition of $f^h(1)$ is independent from the choice of R . We call agent $f^h(1)$ the dictator over object h and define $\mu_h(1) = f^h(1)$.

The first lemma proves that the first row of the inheritance table contains at most two agents.

Lemma 1. $\{f^h(1) \mid h \in K\} \leq 2$.

Proof. Suppose that $\{f^h(1) \mid h \in K\} \geq 3$. Let $1; 2; 3 \in N$, $a; b; c \in K$, and

$$\begin{aligned} \varphi_1(R; fag) &= a; \\ \varphi_2(R; fbg) &= b; \\ \varphi_3(R; fcg) &= c; \end{aligned}$$

Let $R^0 \in R_0^N$ be such that

$$\begin{aligned} b P_1^0 c P_1^0 a, \\ a P_2^0 b P_2^0 c, \\ a P_3^0 c P_3^0 b. \end{aligned}$$

By $R_j fag = R^0_j fag$, $\varphi_1(R^0; fag) = a$. Similarly, $\varphi_2(R^0; fbg) = b$ and $\varphi_3(R^0; fcg) = c$. Hence, by efficiency and resource-monotonicity,

$$\begin{aligned} \varphi_1(R^0; fa; bg) &= b; \\ \varphi_2(R^0; fa; bg) &= a; \\ \varphi_1(R^0; fa; cg) &= c; \\ \varphi_3(R^0; fa; cg) &= a. \end{aligned}$$

Now resource-monotonicity yields the desired contradiction φ
 $\varphi_2(R^0; fa; b; cg) = a$ and $\varphi_3(R^0; fa; b; cg) = a$.

Let $a, b \in K$ be such that $a \notin b$. Let $\hat{R} \in R_0^N$ be such that $a \hat{P}_{f^a(1)} b$ and for all $i \in N \setminus f^a(1)$, $b \hat{P}_i a$. We define $f_b^a(2) = i$ if and only if $\varphi_i(\hat{R}; fa; bg) = b$.

The following argument shows that $f_b^a(2) \in N \setminus f^a(1)$. By IIO, $\varphi_{f^a(1)}(\hat{R}; fag) = a$. Thus, by $a \hat{P}_{f^a(1)} b$ and resource-monotonicity, $\varphi_{f^a(1)}(\hat{R}; fa; bg) = a$. Then efficiency implies that $f_b^a(2) \in N \setminus f^a(1)$. Similarly as in Lemma 1 the following holds.

Lemma 2. $\varphi_{f_b^a(2)} \succ b \succ K \setminus fagj$.

Assume that agent i is the dictator over objects a and b , i.e., $i = f^a(1) = f^b(1)$. Then, a second dictator of a third object does not depend on which of the first two objects agent i picks.

Lemma 3. Let $a, b \in K$ be such that $f^a(1) = f^b(1)$. Then for all $c \in K \setminus fa; bg$ we have $f_c^a(2) = f_c^b(2)$.

Proof. Without loss of generality, let $f^a(1) = f^b(1) = 1$ and $f_c^a(2) = 2$. Let $R \in R_0^N$ be such that $b P_1 a P_1 c$ and for all $i \in N \setminus 1$, $\text{top}(R_i; K) = c$. By $f^a(1) = 1$, IIO, and resource-monotonicity, $\varphi_1(R; fa; cg) = a$. By $f_c^a(2) = 2$ and $R_j|_{fa; cg} = \hat{R}_j|_{fa; cg}$, $\varphi_2(R; fa; cg) = c$. By resource-monotonicity, $\varphi_2(R; fa; b; cg) = c$ (otherwise 2 would lose). Since $f^b(1) = 1$, $\varphi_1(R; fb; cg) = b$. Thus, by resource-monotonicity, $\varphi_2(R; fb; cg) = c$ (otherwise, some agent $j \in N \setminus 1, 2$ gains when object a is removed). Thus, by IIO, $f_c^b(2) = 2 = f_c^a(2)$. φ

By Lemma 1 we have two cases. In Case 1 we can derive the allotment for one agent by a "dictator step" in the first step of the definition of a mixed dictator-pairwise-exchange rule. Case 2 deals with a pairwise exchange step.

Case 1: Step 1 is a dictator step, i.e., without loss of generality, for all $h \in K$, $f^h(1) = 1$. Thus, for all $h \in K$, $\varphi_h(1) = 1$ and $S_1 = f1g$.

Note that resource-monotonicity implies that agent 1 receives for all economies his most preferred object. First, we show that the definition of $f_c^a(2)$ is independent from $R_{N \setminus 1g}$. Let $f_c^a(2) = 2$.

Lemma 4. For all $R \in R_0^N$, if $a P_1 c$, then $\varphi_2(R; fa; cg) = c$.

Proof. Let $b \in K \setminus fa; cg$. Let $\hat{R} \in R_0^N$ be such that $a \hat{P}_1 b \hat{P}_1 c$ and for all $i \in N \setminus 1$, $\hat{R}_i|_{fa; cg} = R_i|_{fa; cg}$ and $c \hat{P}_i b$. By Lemma 3 and $f^a(1) = f^b(1) = 1$, we have $f_c^a(2) = f_c^b(2) = 2$. By IIO,

$$\varphi(\hat{R}; fa; cg) = \varphi(R; fa; cg): \quad (2)$$

By IIO and the construction of \hat{R} , $'_1(\hat{R}; fb; cg) = b$ and $'_2(\hat{R}; fb; cg) = c$. By resource-monotonicity, $'_1(\hat{R}; fa; b; cg) = a$ (since $f^a(1) = 1$) and therefore, $'_2(\hat{R}; fa; b; cg) = c$. Since $f^a(1) = 1$, $'_1(\hat{R}; fa; cg) = a$. Thus, by resource-monotonicity, $'_2(\hat{R}; fa; cg) = c$ (otherwise, some agent $j \notin \{1, 2\}$ gains when object b is removed). Hence, by (2), $'_2(R; fa; cg) = 2$. \square

The next lemma is the important step for Case 1 in proving that the next row of the inheritance table contains at most two agents.

Lemma 5. $jff_c^a(2) \in \{a, c\} \subseteq K$ and $a \notin cgj$. \square

Proof. Suppose that $jff_c^a(2) \in \{a, c\} \subseteq K$ and $a \notin cgj$. \square Hence, $|Nj| \geq 4$. By Lemmas 2 and 3, for some $a \in K$ we must have $jff_x^a(2) \in \{x, 2\} \subseteq K \cap faggj = 2$ and for some $h \in K \cap fag$, $f_a^h(2) \geq f_x^a(2) \in \{x, 2\} \subseteq K \cap fagg$. Let

$$\overline{ff_a^h(2); f_b^a(2); f_c^a(2)g} = 3: \quad (3)$$

Without loss of generality, let $h \notin b$. By Lemma 3, $f_a^b(2) = f_a^h(2)$. Since $|Nj| \geq 4$ and $|Kj| > |Nj|$, there exists $d \in K \cap fag$. By Lemma 3, we have $f_a^d(2) = f_a^b(2)$, $f_b^d(2) = f_b^a(2)$, and $f_c^d(2) = f_c^a(2)$. Thus, by (3), $\overline{ff_h^d(2) \in \{h, 2\} \subseteq K \cap fdgg} \geq 3$, which contradicts Lemma 3. \square

Case 2: Step 1 is a pairwise exchange step, i.e., without loss of generality, $ff^h(1) \in \{h, 2\} \subseteq Kg = \{1, 2\}g$. Thus, for all $h \in K$, $\%_h(1) \subseteq \{1, 2\}g$ and $S_1 = \{1, 2\}g$.⁸

First note the following feature of mixed dictator-pairwise-exchange rules. Suppose that $f^a(1) = 1$ and for all $h \in K \cap fag$, $f^h(1) = 2$, i.e., 1 owns object a only. Then 1 either receives a or he inherits another object from 2 or he exchanges a for another object. In particular, 2 never “physically” inherits object a .

The following lemma shows that if agent 1 owns at least two objects, then he inherits his objects to agent 2.

Lemma 6. If $f^a(1) = f^b(1) = 1$, then $f_b^a(2) = 2$.

Proof. By Case 2, there is some $c \in K \cap fag; bg$ such that $f^c(1) = 2$. Let $R \in R_0^N$ be such that aP_1cP_1b and for all $i \in N \cap \{1, 2\}$, bP_1cP_1a . By efficiency, resource-monotonicity, $f^b(1) = 1$, and $f^c(1) = 2$, we have $'_1(R; fb; cg) = c$ and $'_2(R; fb; cg) = b$. By resource-monotonicity, we have $'_1(R; fa; b; cg) = a$ (since $f^a(1) = 1$) and $'_2(R; fa; b; cg) = b$. Since $f^a(1) = 1$, $'_1(R; fa; bg) = a$. Thus, by resource-monotonicity, $'_2(R; fa; bg) = b$ (otherwise, some agent

⁸ If $\%_h(1) = 1$, then $\%_h(2) = 2$ and if $\%_h(1) = 2$, then $\%_h(2) = 1$.

$j \notin \{1, 2\}$ would gain when object c is removed). Thus, by IIO, $f_b^a(2) = 2$, the desired conclusion. \square

Lemma 6 implies that whenever there are only two objects, then agents 1 and 2 receive them and the other agents receive the null object. By efficiency and resource-monotonicity in Case 2 the first step is a pairwise exchange.

Next we define $f_c^{ab}(3)$. Let $R \in R_0^N$ be such that $a P_1 b P_1 c$, $b P_2 a P_2 c$, and for all $i \in N \setminus \{1, 2\}$, $c P_i a P_i b$. Notice that at R , since the first step is a pairwise exchange, agent 1 receives object a and agent 2 object b whenever a and b are present. Thus, $'_1(R; fa; b; cg) = a$ and $'_2(R; fa; b; cg) = b$. We define $f_c^{ab}(3) = j$ if and only if $'_j(R; fa; b; cg) = c$. By efficiency, $f_c^{ab}(3)$ is well-defined.

Next, we show that the definition of $f_c^{ab}(3)$ is independent from object b that agent 2 picks. Let $d \in K \setminus \{a, b, c\}$.

Lemma 7. $f_c^{ab}(3) = f_c^{ad}(3)$.

Proof. Let $R \in R_0^N$ be such that $a P_1 b P_1 d P_1 c$, $d P_2 b P_2 a P_2 c$, and for all $i \in N \setminus \{1, 2\}$, $c P_i a P_i b P_i d$. By definition and IIO, $'_1(R; fa; c; dg) = a$, $'_2(R; fa; c; dg) = d$, and $'_{f_c^{ad}(3)}(R; fa; c; dg) = c$. By resource-monotonicity, $'_1(R; fa; b; c; dg) = a$, $'_2(R; fa; b; c; dg) = d$, and $'_{f_c^{ad}(3)}(R; fa; b; c; dg) = c$. Since the first step is a pairwise exchange, $'_1(R; fa; b; cg) = a$ and $'_2(R; fa; b; cg) = b$. Thus, by resource-monotonicity, $'_{f_c^{ad}(3)}(R; fa; b; cg) = c$ (otherwise, some agent $j \notin \{1, 2\}$ gains when object d is removed). Hence, by definition and IIO, $f_c^{ab}(3) = f_c^{ad}(3)$, the desired conclusion. \square

Similar to Lemma 7 we can prove the following lemma.

Lemma 8. $f_c^{ab}(3) = f_c^{db}(3)$.

Lemmas 7 and 8 together with IIO imply that $f_c^{ab}(3) = f_c^{ba}(3)$,⁹ i.e., the definition of the third agent is independent from whether agent 1 picks object a and agent 2 object b or vice versa. Furthermore, Lemmas 7 and 8 imply

$$\sum_{a \in K} f_a^{hh^0}(3) \cdot j \cdot h; h^0 \in K \setminus \{a\} \text{ and } h \notin h^0 = 1: \quad (4)$$

Next we show that if agent 1 owns object a , then the definition of $f_c^{ab}(3)$ is independent from whether agent 2 prefers object a to b or object b to a .

Lemma 9. Let $f^a(1) = 1$ and $R \in R_0^N$ be such that $a P_1 b P_1 c$, $a P_2 b P_2 c$, and for all $i \in N \setminus \{1, 2\}$, $c P_i a P_i b$. Then $'_{f_c^{ab}(3)}(R; fa; b; cg) = c$.

⁹ We have $f_c^{ab}(3) = f_c^{ad}(3) = f_c^{bd}(3) = f_c^{ba}(3)$.

Proof. Let $d \in K \setminus \{a, b, c, g\}$. Let $R^0 \in \mathcal{R}$ be such that $a \in P_1^0, b \in P_1^0, d \in P_1^0, c, d \in P_2^0, a \in P_2^0, b \in P_2^0, c$, and for all $i \in \{1, 2\}, g, c \in P_i^0, a \in P_i^0, d \in P_i^0, b$.

By Lemma 7, $f_c^{ad}(3) = f_c^{ab}(3)$. Thus, by definition of R^0 and IIO, we have $\sigma_{f_c^{ad}(3)}(R^0; fa; c; dg) = \sigma_{f_c^{ab}(3)}(R^0; fa; c; dg) = c$. By resource-monotonicity, $\sigma_{f_c^{ab}(3)}(R^0; fa; b; c; dg) = c$. Since the first step is a pairwise exchange, $\sigma_1(R^0; fa; b; cg) = a$ and $\sigma_2(R^0; fa; b; cg) = b$. Thus, by resource-monotonicity, $\sigma_{f_c^{ab}(3)}(R^0; fa; b; cg) = c$ (otherwise, some agent $j \in \{1, 2\}; f_c^{ab}(3)g$ gains when object d is removed). By construction, $R^0_{jfa;b;cg} = R_{jfa;b;cg}$, and by IIO, $\sigma(R^0; fa; b; cg) = \sigma(R; fa; b; cg)$, the desired conclusion. \square

Similarly to the proof of Lemma 9 we can show that whenever agents 1 and 2 receive objects a and b , always the same agent receives object c if $H = \{fa; b; cg\}$, even if agent 1 or agent 2 rank object c as their second best object among the objects in $H = \{fa; b; cg\}$ (and not as their worst as assumed in Lemma 9). This fact and Lemma 9 imply that the definition of $f_c^{ab}(3)$ is independent from the preferences of agents 1 and 2 (given that they strictly prefer both objects to c).

The following lemma is similar to Lemma 2.

Lemma 10. $\sigma_{f_h^{ab}(3)}(j; h) \in K \setminus \{a, b, g\}$. \square

The next lemma is similar to Lemma 5. We prove that the third row of the inheritance table contains at most two agents.

Lemma 11. $\sigma_{f_h^{hh^0}(3)}(j; h; h^0; h) \in K$ and $\sigma_{f_h^{hh^0}(3)}(j; h; h^0; hg) = 3g$. \square

Proof. Suppose that $\sigma_{f_h^{hh^0}(3)}(j; h; h^0; h) \in K$ and $\sigma_{f_h^{hh^0}(3)}(j; h; h^0; hg) = 3g$. Hence, $|jN_j| \geq 5$ and $|jK_j| > 5$. By (4) there exist $a, b, c \in K$ such that $\sigma_{f_a^{bc}(3); f_b^{ac}(3); f_c^{ab}(3)}(g) = 3$ and $\sigma_{jfa; b; cg} = 3$. Let $d, e \in K \setminus \{a, b, c, g\}$. By (4), we have $f_a^{de}(3) = f_a^{bc}(3)$, $f_b^{de}(3) = f_b^{ac}(3)$, and $f_c^{de}(3) = f_c^{ab}(3)$. But then $\sigma_{f_h^{de}(3)}(j; h) \in K \setminus \{d, e, g\}$, which contradicts Lemma 10. \square

We already proved that whenever agents 1 and 2 receive objects a and b , always the same agent receives object c if $H = \{fa; b; cg\}$ and for all agents $i \in \{1, 2\}, g, c \in P_i, a \in P_i, b$. Finally, we show that the agent who receives the object c does not depend on the preferences of agents in $\{1, 2\}$.

Lemma 12. Let $R \in \mathcal{R}_0^N$ be such that $\sigma_1(R; fa; b; cg) = a$ and $\sigma_2(R; fa; b; cg) = b$. Then $\sigma_{f_c^{ab}(3)}(R; fa; b; cg) = c$.

Proof. Since $|jK_j| > |jN_j|$ we assume that $|jK_j| \geq 5$ (otherwise $|jN_j| = 3$ and the conclusion of Lemma 12 is trivial). Let $d, e \in K \setminus \{a, b, c, g\}$. By IIO, we

may suppose that c is ranked worst in $fc; d; eg$ for agents 1 and 2, 1 prefers a to both d and e , and 2 prefers b to both d and e . By (4) we have

$$f_c^{de}(3) = f_c^{ab}(3): \quad (5)$$

Let $R^0 \in R_0^N$ be such that $R_{f1;2g}^0 = R_{f1;2g}$ and for all $i \in N \setminus \{1; 2; g\}$, $R_{ijfa;b;cg}^0 = R_{ijfa;b;cg}$ and $c \succ_i^0 d \succ_i^0 e$. By IIO,

$$v(R^0; fa; b; cg) = v(R; fa; b; cg): \quad (6)$$

By definition and (5), $v_{f_c^{ab}(3)}(R^0; fc; d; eg) = c$. By resource-monotonicity, $v_1(R^0; fa; b; c; d; eg) = a$ and $v_2(R^0; fa; b; c; d; eg) = b$. Thus, by resource-monotonicity, $v_{f_c^{ab}(3)}(R^0; fa; b; c; d; eg) = c$. Again, by resource-monotonicity, $v_{f_c^{ab}(3)}(R^0; fa; b; cg) = c$ (otherwise, some agent $j \notin \{1; 2; f_c^{ab}(3); g\}$ would gain when objects d and e are removed). Hence, by (6), $v_{f_c^{ab}(3)}(R; fa; b; cg) = c$. \square

Cases 1 and 2 imply that at Step 2 at most two agents trade (Lemmas 5 and 11) independently of the preference profile and the allotments of the dictator(s) at Step 1 (Lemmas 4, 9, and 12). Similarly as for Step 1, we can now prove that Step 2 is either a dictator step or a pairwise exchange step and that again at Step 3, if it exists, at most two agents trade independently of the preference profile and the allotments of the dictator(s) at Steps 1 and 2, etc.

2. General Induction Step: In the first part of the proof, we have shown that v allocates the objects through a dictatorship or a pairwise exchange in both Step 1 and Step 2. As already indicated above, in proving the general induction step we use the arguments of the first part of the proof (they give a lot of insight how the axioms work) and the following definitions.

Suppose that we have defined the mixed dictator-pairwise exchange rule up to Step t . Let $S_1; S_2; \dots; S_t$ be the members of the ordered partition up to Step t , where each member is a singleton or a pair. Let $s = \sum_{i=1}^t |S_i|$ (so, the first s rows of the mixed dictator-pairwise-exchange inheritance table are defined). Without loss of generality, let $[_{i=1}^t S_i = f1; 2; \dots; sg$.

We define the $(s + 1)$ st row as follows. For all $H^s \in \mu^k$ such that $|H^s| = s$ and all $c \in K \setminus H^s$, order H^s in an arbitrary manner, say $H^s =$

$f_{h_1; h_2; \dots; h_s} g$, and let $R \in \mathbb{R}_0^N$ be such that

$$\begin{aligned} h_1 & P_1 h_2 P_1 h_3 P_1 \dots P_1 h_s P_1 c, \\ h_2 & P_2 h_1 P_2 h_3 P_2 \dots P_2 h_s P_2 c, \\ h_3 & P_3 h_1 P_3 h_2 P_3 \dots P_3 h_s P_3 c, \\ h_4 & P_4 h_1 P_4 h_2 P_4 \dots P_4 h_s P_4 c, \\ & \vdots \\ h_s & P_s h_1 P_s h_2 P_s \dots P_s h_{s-1} P_s c, \end{aligned}$$

and for all $i \in \{1, \dots, s\}$,

$$c P_i h_1 P_i h_2 P_i \dots P_i h_s.$$

Note that for all $i \in \{1, \dots, s\}$, $f_{i_s}(R; H^s [fcg]) = h_s$. We define¹⁰

$$f_c^{(h_1; \dots; h_s)}(s+1) = j, \quad f_j(R; H^s [fcg]) = c.$$

Let $d \in \text{Kn}(H^s [fcg])$. Similarly to Lemma 7 we can show the following.

Lemma 13. $f_c^{(d; h_2; \dots; h_s)}(s+1) = f_c^{(h_1; h_2; \dots; h_s)}(s+1)$.

Using Lemma 13 it follows that $f_c^{(h_1; h_2; \dots; h_s)}(s+1)$ does not depend on the order of H^s . For example, $f_c^{(h_2; h_1; h_3; \dots; h_s)}(s+1) = f_c^{(h_1; h_2; \dots; h_s)}(s+1)$. Then $f_c^{H^s}(s+1) = f_c^{(h_1; h_2; \dots; h_s)}(s+1)$ is well-defined. Similarly to (4) we have then

$$\bar{f} f_c^{H^s}(s+1) j H^s \mu \text{Knfcg} \text{ and } |j H^s| = |sg| = 1:$$

Then, similarly to Lemmas 9 and 11 in Case 2 of the first part of the proof, we can show that the $(s+1)$ st row of the inheritance table contains at most two agents (Lemma 14), independently from the order of H^s and the set of s objects H^s we pick.

Lemma 14. $\bar{f} f_h^{H^s}(s+1) j H^s [fhg] \mu K$ and $|j H^s [fhg]| = |s+1g| = 2$:

Then similarly to Lemma 12 it follows that $f_c^{H^s}(s+1)$ receives c at any profile where agents $f_1; \dots; s_g$ receive H^s in the first t steps.

Lemma 15. Let $H^s \mu K$ be such that $|j H^s| = s$, $c \in \text{Kn} H^s$, and $R \in \mathbb{R}_0^N$ be such that $f_{i-1}(R; H^s [fcg]); \dots; f_s(R; H^s [fcg])g = H^s$. Then $f_c^{H^s(s+1)}(R; H^s [fcg]) = c$.

Finally, from resource-monotonicity it follows that Step $s+1$ is a dictator step or a pairwise exchange step.

¹⁰Note that before we wrote $f_c^{ab}(3)$ instead of $f_c^{(a;b)}(3)$.

References

- Abdulkadiroğlu, A., and T. Sönmez (1998): Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems, *Econometrica* 66, 689–701.
- Abdulkadiroğlu, A., and T. Sönmez (1999): House Allocation with Existing Tenants, *Journal of Economic Theory* 88, 233–260.
- Bogomolnaia, A., and H. Moulin (2001): A New Solution to the Random Assignment Problem, *Journal of Economic Theory* 100, 295–328.
- Chun, Y., and W. Thomson (1988): Monotonicity Properties of Bargaining Solutions when Applied to Economics, *Mathematical Social Sciences* 15, 11–27.
- Ehlers, L. (2002a): Coalitional Strategy-Proof House Allocation, *Journal of Economic Theory* 105, 298–317.
- Ehlers, L. (2002b): Resource-Monotonic Allocation when Preferences are Single-Peaked, *Economic Theory* 20, 113–131.
- Ehlers, L., and B. Klaus (2000): Coalitional Strategy-Proofness, Resource-Monotonicity, and Separability for Multiple Assignment Problems, *Social Choice and Welfare*, forthcoming.
- Ehlers, L., and B. Klaus (2002a): Consistent House Allocation, Working Paper.
- Ehlers, L., and B. Klaus (2002b): Efficient Priority Rules, Working Paper.
- Ehlers, L., B. Klaus, and S. Pápai (2002): Strategy-Proofness and Population-Monotonicity for House Allocation Problems, *Journal of Mathematical Economics* 83, 329–339.
- Ergin, H. I. (2000): Consistency in House Allocation Problems, *Journal of Mathematical Economics* 34, 77–97.
- Ergin, H. I. (2002): Efficient Resource Allocation on the Basis of Priorities, *Econometrica* 70, 2489–2497.
- Hokari, T. (2000): The Nucleolus is not Aggregate-Monotonic on the Domain of Convex Games, *International Journal of Game Theory* 29, 133–137.

- Maniquet, F. and Y. Sprumont (2000): On Resource Monotonicity in the Fair Division Problem, *Economics Letters* 68, 299–302.
- Moulin, H. (1999): Rationing a commodity along ordered paths, *Journal of Economic Theory* 84, 41–72.
- Moulin, H., and W. Thomson (1988): Can Everyone Benefit from Growth? Two Difficulties, *Journal of Mathematical Economics* 17, 339–345.
- Pápai, S. (2000): Strategyproof Assignment by Hierarchical Exchange, *Econometrica* 68, 1403–1433
- Svensson, L.-G. (1999): Strategy-Proof Allocation of Indivisible Goods, *Social Choice and Welfare* 16, 557–567.
- Thomson, W. (1994): Resource-Monotonic Solutions to the Problem of Fair Division when Preferences are Single-Peaked, *Social Choice and Welfare* 11, 205–223.