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The possibility of ordering infinite utility streams*

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Abstract

This paper revisits Diamond's classical impossibility result regarding the ordering of infinite utility streams. We show that if no representability condition is imposed, there do exist strongly Paretian and finitely anonymous orderings of intertemporal utility streams with attractive additional properties. We extend a possibility theorem due to Svensson to a characterization theorem and we provide characterizations of all strongly Paretian and finitely anonymous rankings satisfying the strict transfer principle. In addition, infinite-horizon extensions of leximin and of utilitarianism are characterized by adding an equity-preference axiom and finite translation-scale measurability, respectively, to strong Pareto and finite anonymity. *Journal of Economic Literature* Classification Nos.: D63, D71.

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1 Introduction

Treating generations equally is one of the basic principles in the utilitarian tradition of moral philosophy. As Sidgwick (1907, p. 414) observes, "the time at which a man exists cannot affect the value of his happiness from a universal point of view; and [...] the interests of posteriority must concern a Utilitarian as much as those of his contemporaries". This view, which is formally expressed by the anonymity condition, is also strongly endorsed by Ramsey (1928).

Following Koopmans (1960), Diamond (1965) establishes that anonymity is incompatible with the strong Pareto principle when ordering *infinite* utility streams. Moreover, he shows that if anonymity is weakened to *finite* anonymity—which restricts the application of the standard anonymity requirement to situations where utility streams differ in at most a finite number of components—and a continuity requirement is added, an impossibility results again. Suzumura and Shinotsuka (2003) adapt the well-known strict transfer principle to the infinite-horizon context. They show that this principle is incompatible with strong Pareto and continuity even if the ranking is merely required to be acyclical. Basu and Mitra (2003) show that strong Pareto, finite anonymity and representability by a real-valued function are incompatible.

A natural question to ask is what happens if no continuity or representability assumptions, which are technical rather than ethical in nature, are imposed. Svensson (1980) proves that strong Pareto and finite anonymity are compatible by showing that any ordering extension of an infinite-horizon variant of Suppes' (1966) grading principle satisfies the required axioms. The Suppes grading principle is a quasi-ordering that combines the Pareto quasi-ordering and finite anonymity. Given Arrow's (1951) version of Szpilrajn's (1930) extension theorem, this establishes the compatibility result.

Capitalizing on Svensson's (1980) results, the focus of this paper is on possibilities rather than impossibilities. We show that, if neither representability nor continuity assumptions are imposed, orderings of infinite utility streams with attractive properties that go beyond the grading principle exist. Especially in an infinite-dimensional framework, technical requirements such as representability and continuity can be considered overly demanding and, as a consequence, the observation that the orderings characterized in this paper fail to satisfy them does not appear to be a serious shortcoming. This view is confirmed by the fact that the set of rules we characterize include orderings where violations of representability or continuity only occur when comparing utility streams that differ in infinitely many components. Thus, we think that the results of this paper are very encouraging.

We first strengthen Svensson's result to a characterization theorem by establishing that any ordering satisfying the two axioms must be an extension of the infinite-horizon Suppes grading principle. We then show that the compatibility survives even if the strict transfer principle is added to strong Pareto and finite anonymity. This is accomplished by characterizing all orderings with these properties and establishing the non-emptiness of this class. Finally, we show how some well-known characterization results appearing in finite-population social-choice theory can be extended to the infinite-horizon model. In particular, we employ an equity-preference condition to characterize infinite-horizon versions of the leximin principle and we use a suitable variant of translation-scale measurability to axiomatize analogous extensions of utilitarianism.

2 Definitions

The set of infinite utility streams is $X = \mathbb{R}^{\mathbb{N}}$, where $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{N}$ denotes the set of all natural numbers. A typical element of X is an infinite-dimensional vector $x = (x_1, x_2, \ldots, x_n, \ldots)$ and, for $n \in \mathbb{N}$, we write $x_{-n} = (x_1, \ldots, x_n)$ and $x_{+n} = (x_{n+1}, x_{n+2}, \ldots)$. The standard interpretation of $x \in X$ is that of a countably infinite utility stream where x_n is the utility experienced in period $n \in \mathbb{N}$. Of course, other interpretations are possible—for example, x_n could be the utility of an individual in a countably infinite population.

Our notation for vector inequalities on X is as follows. For all $x, y \in X$, (i) $x \geq y$ if $x_n \geq y_n$ for all $n \in \mathbb{N}$; (ii) $x > y$ if $x \geq y$ and $x \neq y$; (iii) $x \gg y$ if $x_n > y_n$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $x \in X$, $(x_{(1)}, \ldots, x_{(n)})$ is a rank-ordered permutation of x_{-n} such that $x_{(1)} \leq \ldots \leq x_{(n)}$, ties being broken arbitrarily.

$R \subseteq X \times X$ is a weak preference relation on X with strict preference $P(R)$ and indifference relation $I(R)$. A quasi-ordering is a reflexive and transitive relation, and an ordering is a complete quasi-ordering. Analogously, a partial order is an asymmetric and transitive relation, and a linear order is a complete partial order. Let R and R' be relations on X. R' is an extension of R if $R \subseteq R'$ and $P(R) \subseteq P(R')$. If an extension R' of R is an ordering, we call it an ordering extension of R, and if R' is an extension of R that is a linear order, we refer to it as a linear order extension of R. The transitive closure of a relation R is denoted by $\overline{R}$, that is, for all $x, y \in X$, $(x, y) \in \overline{R}$ if there exist $K \in \mathbb{N}$ and $z^0, \ldots, z^K \in X$ such that $x = z^0$, $(z^{k-1}, z^k) \in R$ for all $k \in \{1, \ldots, K\}$ and $z^K = y$.

A finite permutation of $\mathbb{N}$ is a bijection $\rho\colon \mathbb{N} \rightarrow \mathbb{N}$ such that there exists $m \in \mathbb{N}$ with $\rho(n) = n$ for all $n \in \mathbb{N} \setminus \{1, \ldots, m\}$. The corresponding finite permutation matrix $B^\rho = (b^\rho_{ij})_{i,j\in\mathbb{N}}$ is defined by letting, for all $i \in \mathbb{N}$, $b^\rho_{i\rho(i)} = 1$ and $b^\rho_{ij} = 0$ for all $j \in \mathbb{N}\setminus\{\rho(i)\}$. Finite permutation matrices are special cases of finite bistochastic matrices. A finite bistochastic matrix is a matrix $B = (b_{ij})_{i,j\in\mathbb{N}}$ such that there exists $m \in \mathbb{N}$ such that $b_{ij} \geq 0$ for all $i, j \in \{1, \ldots, m\}$, $\sum_{i=1}^m b_{ij} = 1$ for all $j \in \{1, \ldots, m\}$, $\sum_{j=1}^m b_{ij} = 1$ for all $i \in \{1, \ldots, m\}$, $b_{ii} = 1$ for all $i \in \mathbb{N} \setminus \{1, \ldots, m\}$ and $b_{ij} = 0$ for all $i, j \in \mathbb{N} \setminus \{1, \ldots, m\}$ with $i \neq j$. Finite bistochastic matrices will be convenient in the proof of Theorem 2.

The following axioms are used in this paper.

Strong Pareto: For all $x, y \in X$, if $x > y$, then $(x, y) \in P(R)$.

Finite anonymity: For all $x \in X$ and for all finite permutations ρ of $\mathbb{N}$,

$$(B^\rho x, x) \in I(R).$$

Strict transfer principle: For all $x, y \in X$ and for all $n, m \in \mathbb{N}$, if $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$, $y_m > x_m \geq x_n > y_n$ and $x_n + x_m = y_n + y_m$, then $(x, y) \in P(R)$.

Equity preference: For all $x, y \in X$ and for all $n, m \in \mathbb{N}$, if $x_k = y_k$ for all $k \in \mathbb{N}\setminus\{n, m\}$ and $y_m > x_m > x_n > y_n$, then $(x, y) \in R$.

Finite translation-scale measurability: For all $x, y, z \in X$ and for all $m \in \mathbb{N}$, if $x_n = y_n$ for all $n \in \mathbb{N} \setminus \{1, \ldots, m\}$, then

$$(x + z, y + z) \in R \Leftrightarrow (x, y) \in R.$$

Strong Pareto and finite anonymity are the standard axioms used in the literature on ranking infinite utility streams.

The strict transfer principle is the natural analogue of the corresponding condition for finite streams. See also Suzumura and Shinotsuka (2003).

Equity preference is the extension of Hammond's (1976) equity axiom to the infinite-horizon environment. d'Aspremont and Gevers (1977) use a stronger axiom by requiring $(x, y) \in P(R)$ rather than merely $(x, y) \in R$ in the conclusion of the axiom. In the presence of strong Pareto, the two axioms are equivalent. Moreover, strong Pareto and equity preference together imply the following property which, in turn, obviously implies the strict transfer principle.

Strict equity preference: For all $x, y \in X$ and for all $n, m \in \mathbb{N}$, if $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$ and $y_m > x_m \geq x_n > y_n$, then $(x, y) \in P(R)$.

To see that strict equity preference is implied by strong Pareto and equity preference, suppose that R satisfies the first two axioms, and let $x, y \in X$ and $n, m \in \mathbb{N}$ be such that $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$ and $y_m > x_m \geq x_n > y_n$. Let $z \in X$ be such that $z_k = x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$ and $x_n > z_m > z_n > y_n$. By strong Pareto, $(x, z) \in P(R)$ and by equity preference, $(z, y) \in R$. Thus, transitivity implies $(x, y) \in P(R)$ and strict equity preference is satisfied.

Finite translation-scale measurability imposes restrictions on the informational contents of utility streams. It requires that utilities are unique up to independent translations. This is the natural adaptation of the corresponding axiom known from finite-population social-choice theory to our environment. Note that the axiom only applies to comparisons of utility streams that differ in at most a finite number of components.

Szpilrajn's (1930) fundamental result establishes that every partial order has a linear order extension. Arrow (1951, p. 64) presents a variant of Szpilrajn's theorem stating that every quasi-ordering has an ordering extension; see also Hansson (1968). This implies that the sets of orderings characterized in the theorems of the following sections are non-empty and, therefore, our results indeed show the compatibility of the stated systems of axioms.

3 The infinite-horizon Suppes grading principle

The Suppes (1966) grading principle combines the requirements of strong Pareto and anonymity into a criterion for establishing a partial social ranking. Adapted to the multi-period framework, the Suppes quasi-ordering R_S on X is defined as follows. For all $x, y \in X$, $(x, y) \in R_S$ if there exists a finite permutation ρ of $\mathbb{N}$ such that $x \geq B^\rho y$. Svensson (1980, Theorem 2) shows that any ordering extension of R_S satisfies strong Pareto and finite anonymity. We formulate a stronger result by establishing that these ordering extensions of R_S are the only orderings on X with these properties.

Theorem 1 *An ordering R on X satisfies strong Pareto and finite anonymity if and only if R is an ordering extension of R_S.*

Proof. 'If.' That any ordering extension of R_S satisfies strong Pareto and finite equity is shown in Svensson (1980, Theorem 2).

'Only if.' Now suppose R is an ordering that satisfies the two axioms. We prove that R is an ordering extension of R_S.

Let $x, y \in X$ be such that $(x, y) \in R_S$. Thus, there exists a finite permutation ρ of $\mathbb{N}$ such that $x \geq B^\rho y$. By finite anonymity, $(B^\rho y, y) \in I(R) \subseteq R$. If $x = B^\rho y$, $(x, B^\rho y) \in R$ follows from the reflexivity of R. If $x > B^\rho y$, strong Pareto implies $(x, B^\rho y) \in P(R) \subseteq R$. Because R is transitive, we obtain $(x, y) \in R$ in all cases.

Now let $x, y \in X$ be such that $(x, y) \in P(R_S)$. By definition, there exists a finite permutation ρ of $\mathbb{N}$ such that $x \geq B^\rho y$ and there exists no finite permutation ρ' of $\mathbb{N}$ such that $y \geq B^{\rho'} x$. As shown above, $(x, y) \in R_S$ implies $(x, y) \in R$. If $x = B^\rho y$, letting $\rho' = \rho^{-1}$ immediately yields a contradiction. Thus, $x > B^\rho y$. By strong Pareto, $(x, B^\rho y) \in P(R)$. Finite anonymity implies $(B^\rho y, y) \in I(R)$ and, by transitivity, we obtain $(x, y) \in P(R)$. Thus, R is an extension of R_S.

Because R is an ordering by assumption, this implies that R is an ordering extension of R_S. ■

4 Transfer-sensitive infinite-horizon orderings

Now we examine the consequences of adding the strict transfer principle to our list of axioms. We provide a strengthening of Svensson's (1980) possibility result by showing that the three resulting axioms are compatible. Moreover, we characterize all orderings with these properties.

To define this class of orderings, consider first the following relation R_T. For all $x, y \in X$, $(x, y) \in R_T$ if there exist $n, m \in \mathbb{N}$ such that $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$, $y_m > x_m \geq x_n > y_n$ and $x_n + x_m = y_n + y_m$. This relation captures the requirements imposed by the strict transfer principle. Clearly, $P(R_T) = R_T$. Note that if $(x, y) \in R_T$, then there exists a finite bistochastic matrix B such that $x = By$. This matrix is obtained by letting $b_{nm} = b_{mn} = (x_n - y_n)/(y_m - y_n)$, $b_{nn} = b_{mm} = (x_m - y_n)/(y_m - y_n)$, $b_{ni} = b_{in} = b_{m_i} = b_{im} = 0$ for all $i \in \mathbb{N} \setminus \{n, m\}$, $b_{ii} = 1$ for all $i \in \mathbb{N} \setminus \{n, m\}$ and $b_{ij} = 0$ for all $i, j \in \mathbb{N} \setminus \{n, m\}$ with $i \neq j$.

Because, in addition, we want our ordering to satisfy strong Pareto and finite anonymity, the relation R_S must be respected as well. Finally, because we only consider transitive relations, the transitive closure of the union of these two relations appears in the definition of the relevant class of orderings. Clearly, the transitive closure $\overline{R_S \cup R_T}$ of $R_S \cup R_T$ is a quasi-ordering: reflexivity follows from the reflexivity of R_S and transitivity is satisfied by definition. We obtain the following characterization of the class of all ordering extensions of $\overline{R_S \cup R_T}$.

Theorem 2 *An ordering R on X satisfies strong Pareto, finite anonymity and the strict transfer principle if and only if R is an ordering extension of $\overline{R_S \cup R_T}$.*

Proof. 'If.' We first prove that $\overline{R_S \cup R_T}$ is an extension of both R_S and R_T. It is immediate that $R_S \subseteq \overline{R_S \cup R_T}$ and $R_T \subseteq \overline{R_S \cup R_T}$, so we only need to establish the set inclusions

$$P(R_S) \subseteq P(\overline{R_S \cup R_T}) \tag{1}$$

and

$$P(R_T) \subseteq P(\overline{R_S \cup R_T}). \tag{2}$$

To prove (1), suppose that $(x, y) \in P(R_S)$. This implies $(x, y) \in \overline{R_S \cup R_T}$. By way of contradiction, suppose that $(y, x) \in \overline{R_S \cup R_T}$. Thus, there exist a finite permutation ρ of $\mathbb{N}$, $K \in \mathbb{N}$ and $z^0, \ldots, z^K \in X$ such that $x > B^\rho y$, $y = z^0$, $(z^{k-1}, z^k) \in R_S \cup R_T$ for all $k \in \{1, \ldots, K\}$ and $z^K = x$. Let $k \in \{1, \ldots, K\}$. If $(z^{k-1}, z^k) \in R_S$, it follows that there exists a finite permutation ρ^k of $\mathbb{N}$ such that $z^{k-1} \geq B^{\rho^k} z^k$. If $(z^{k-1}, z^k) \in R_T$, it follows that there exists a finite bistochastic matrix B such that $z^{k-1} = B z^k$. Suppose first that, whenever $(z^{k-1}, z^k) \in R_S$, we have $z^{k-1} = B^{\rho^k} z^k$ for some finite permutation ρ^k. Because the set of finite bistochastic matrices is closed under matrix multiplication, it follows that $y = B^0 x$ for some finite bistochastic matrix B^0. Let $m \in \mathbb{N}$ be such that $b^0_{ii} = b^\rho_{ii} = 1$ for all $i \in \mathbb{N} \setminus \{1, \ldots, m\}$. Because $y = B^0 x$, it follows that $\sum_{i=1}^m y_i = \sum_{i=1}^m x_i$. But $x > B^\rho y$ implies $\sum_{i=1}^m x_i > \sum_{i=1}^m y_i$, a contradiction. If some of the inequalities are strict, an analogous contradiction emerges. Therefore, $(y, x) \in \overline{R_S \cup R_T}$ is impossible and (1) follows. The proof of (2) is analogous.

Next, we prove that any ordering extension of $\overline{R_S \cup R_T}$ satisfies the required axioms. Suppose R is such an ordering extension.

We begin with strong Pareto. Suppose that $x > y$ for some $x, y \in X$. This implies $(x, y) \in P(R_S)$ and, by (1), $(x, y) \in P(\overline{R_S \cup R_T})$. Because R is an ordering extension of $\overline{R_S \cup R_T}$, it follows that $(x, y) \in P(R)$ and strong Pareto is satisfied.

To establish finite anonymity, let $x \in X$ and let ρ be any finite permutation of $\mathbb{N}$. This implies $(B^\rho x, x) \in I(R_S)$ and, because $R_S \subseteq \overline{R_S \cup R_T} \subseteq R$, we obtain $(B^\rho x, x) \in I(R)$.

Finally, we prove that the strict transfer principle is satisfied. Suppose $x, y \in X$ and $n, m \in \mathbb{N}$ are such that $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$, $y_m > x_m \geq x_n > y_n$ and $x_n + x_m = y_n + y_m$. This implies $(x, y) \in P(R_T)$ and, by (2) and the assumption that R is an ordering extension of $\overline{R_S \cup R_T}$, we obtain $(x, y) \in P(R)$.

'Only if.' Suppose R satisfies the three axioms of the theorem statement. To prove that R is an ordering extension of $\overline{R_S \cup R_T}$, suppose first that $(x,y) \in \overline{R_S \cup R_T}$. By definition, there exist $K \in \mathbb{N}$ and $z^0, \ldots, z^K \in X$ such that $x = z^0$, $(z^{k-1}, z^k) \in R_S \cup R_T$ for all $k \in \{1, \ldots, K\}$ and $z^K = y$. By Theorem 1, $(z^{k-1}, z^k) \in R$ follows whenever $(z^{k-1}, z^k) \in R_S$ and, by the strict transfer principle, $(z^{k-1}, z^k) \in R$ follows whenever $(z^{k-1}, z^k) \in R_T$. Because R is transitive, it follows that $(x,y) \in R$.

Now suppose that $(x,y) \in P(\overline{R_S \cup R_T})$. By definition, there exist $K \in \mathbb{N}$ and $z^0, \ldots, z^K \in X$ such that $x = z^0$, $(z^{k-1}, z^k) \in R_S \cup R_T$ for all $k \in \{1, \ldots, K\}$ and $z^K = y$. Moreover, at least one of these preferences must be strict because otherwise we would have $(y,x) \in \overline{R_S \cup R_T}$, contradicting $(x,y) \in P(\overline{R_S \cup R_T})$. If the strict preference is such that $(z^{k-1}, z^k) \in P(R_S)$, $(z^{k-1}, z^k) \in P(R)$ follows from Theorem 1. If the strict preference is such that $(z^{k-1}, z^k) \in P(R_T)$, $(z^{k-1}, z^k) \in P(R)$ follows immediately from the strict transfer principle. Therefore, in either case, the transitivity of R implies $(x,y) \in P(R)$. This completes the proof that R is an ordering extension of $\overline{R_S \cup R_T}$. ■

5 Infinite-horizon leximin

If the strict transfer principle is replaced by equity preference (which, in the presence of strong Pareto, is a strengthening), the only remaining orderings are infinite-horizon versions of the leximin criterion. Let $n \in \mathbb{N}$. We denote the usual leximin ordering on $\mathbb{R}^n$ by R^n_ℓ, that is, for all $x, y \in X$,

$$(x_{-n}, y_{-n}) \in R^n_\ell \quad \Leftrightarrow \quad \begin{array}{l} x_{-n} \text{ is a permutation of } y_{-n} \text{ or there exists } m \in \{1, \ldots, n\} \text{ such that} \\ x_{(k)} = y_{(k)} \text{ for all } k \in \{1, \ldots, n\} \setminus \{m, \ldots, n\} \text{ and } x_{(m)} > y_{(m)}. \end{array}$$

Again, let $n \in \mathbb{N}$ and define a relation $R^n_L \subseteq X \times X$ by letting, for all $x, y \in X$, $(x,y) \in R^n_L$ if $(x_{-n}, y_{-n}) \in R^n_\ell$ and $x_{+n} \geq y_{+n}$. It is straightforward to verify that R^n_L is a quasi-ordering for all $n \in \mathbb{N}$. Finally, let $R_L = \cup_{n \in \mathbb{N}} R^n_L$. This relation is a quasi-ordering but it is not complete—some infinite utility streams are not ranked by R_L. Our next result characterizes all ordering extensions of R_L.

Theorem 3 *An ordering R on X satisfies strong Pareto, finite anonymity and equity preference if and only if R is an ordering extension of R_L.*

Proof. 'If.' First, we prove that, for all $n, m \in \mathbb{N}$ such that $m > n$,

$$R^n_L \subseteq R^m_L \tag{3}$$

and

$$P(R_L^n) \subseteq P(R_L^m). \tag{4}$$

Let $n, m \in \mathbb{N}$ be such that $m > n$.

To prove (3), suppose that $(x, y) \in R_L^n$. By definition, $(x_{-n}, y_{-n}) \in R_\ell^n$ and $x_{+n} \geq y_{+n}$. Hence $(x_{-m}, y_{-m}) \in R_\ell^m$ and $x_{+m} \geq y_{+m}$, that is, $(x, y) \in R_L^m$.

To establish (4), suppose that $(x, y) \in P(R_L^n)$. By definition, at least one of the following two statements is true:

$$(x_{-n}, y_{-n}) \in P(R_\ell^n) \quad \text{and} \quad x_{+n} \geq y_{+n}; \tag{5}$$

$$(x_{-n}, y_{-n}) \in R_\ell^n \quad \text{and} \quad x_{+n} > y_{+n}. \tag{6}$$

By (3), it follows that $(x, y) \in R_L^m$. To prove that $(x, y) \in P(R_L^m)$, suppose, by way of contradiction, that $(y, x) \in R_L^m$. Then, by definition,

$$(x_{-n}, y_{-n}) \in I(R_\ell^n) \text{ and } x_{+n} = y_{+n},$$

contradicting (5) and (6).

Next, we prove that R_L is a quasi-ordering. Reflexivity is immediate because, for all $x \in X$, $(x, x) \in R_L^n$ for all $n \in \mathbb{N}$ and hence $(x, x) \in R_L$. To prove that R_L is transitive, suppose that $(x, y), (y, z) \in R_L$. By definition, there exist $n, m \in \mathbb{N}$ such that $(x, y) \in R_L^n$ and $(y, z) \in R_L^m$. Let $k = \max\{n, m\}$. By (3), $(x, y), (y, z) \in R_L^k$ and by the transitivity of R_L^k, $(x, z) \in R_L^k$ which, in turn, implies $(x, z) \in R_L$.

We now show that, for all $x, y \in X$,

$$(x, y) \in P(R_L) \Leftrightarrow \exists n \in \mathbb{N} \text{ such that } (x, y) \in P(R_L^n). \tag{7}$$

Suppose first that $(x, y) \in P(R_L)$. By definition, there exists $n \in \mathbb{N}$ such that $(x, y) \in R_L^n$. Moreover, $(y, x) \notin R_L^n$ because otherwise we obtain $(y, x) \in R_L$ by definition and thus a contradiction to our hypothesis that $(x, y) \in P(R_L)$. Hence $(x, y) \in P(R_L^n)$.

Conversely, suppose that there exists $n \in \mathbb{N}$ such that $(x, y) \in P(R_L^n)$. Suppose there exists $m \in \mathbb{N}$ such that $(y, x) \in R_L^m$. Because $(x, y) \in P(R_L^n)$, (4) implies $n > m$. But then (3) implies $(y, x) \in R_L^n$, a contradiction. We conclude that $(x, y) \in R_L^n$ and $(y, x) \notin R_L^m$ for all $m \in \mathbb{N}$. By definition, this implies $(x, y) \in P(R_L)$.

Now let R be an ordering extension of R_L. We complete the proof of the 'if' part by showing that R satisfies the required axioms.

To establish that strong Pareto is satisfied, suppose that $x, y \in X$ are such that $x > y$. Let $n = \min\{m \in \mathbb{N} \mid x_m > y_m\}$. By definition, $(x, y) \in P(R_L^n)$. By (7), $(x, y) \in P(R_L)$ and, because R is an ordering extension of R_L, we obtain $(x, y) \in P(R)$.

Next, we show that finite anonymity is satisfied. Let $x \in X$ and let ρ be a finite permutation of $\mathbb{N}$. By definition, there exists $m \in \mathbb{N}$ such that $\rho(n) = n$ for all $n \in \mathbb{N} \setminus \{1, \ldots, m\}$. By definition of R_L^m, $(B^\rho x, x) \in I(R_L^m)$. By definition of R_L, this implies $(B^\rho x, x) \in I(R_L)$. Because R is an ordering extension of R_L, we obtain $(B^\rho x, x) \in I(R)$.

Finally, we show that equity preference is satisfied. Consider $x, y \in X$ and $n, m \in \mathbb{N}$ such that $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{n, m\}$ and $y_m > x_m > x_n > y_n$. Let $j = \max\{n, m\}$. By definition of R_L^j, we obtain $(x, y) \in R_L^j$. By (7), $(x, y) \in R_L$ and, because R is an ordering extension of R_L, $(x, y) \in R$.

'Only if.' Suppose R is an ordering on X satisfying the three axioms of the theorem statement. Fix $n \in \mathbb{N}$ and $z \in X$ and define the relation $Q(n, z) \subseteq \mathbb{R}^n \times \mathbb{R}^n$ as follows. For all $x, y \in X$,

$$(x_{-n}, y_{-n}) \in Q(n, z) \Leftrightarrow ((x_{-n}, z_{+n}), (y_{-n}, z_{+n})) \in R.$$

$Q(n, z)$ is an ordering because R is. Furthermore, it is clear that

$$(x_{-n}, y_{-n}) \in P(Q(n, z)) \Leftrightarrow ((x_{-n}, z_{+n}), (y_{-n}, z_{+n})) \in P(R) \tag{8}$$

for all $x, y \in X$. The three axioms imply that $Q(n, z)$ must satisfy the n-person versions of the axioms and, using Hammond's (1976, Theorem 7.2) characterization of n-person leximin (see also d'Aspremont and Gevers, 1977, Theorem 5), it follows that

$$Q(n, z) = R_\ell^n. \tag{9}$$

Because n and z were chosen arbitrarily, (9) is true for all $n \in \mathbb{N}$ and for any $z \in X$.

By way of contradiction, suppose R is not an ordering extension of R_L. There are two possible cases.

Case 1. There exist $x, y \in X$ such that $(x, y) \in R_L$ and $(y, x) \in P(R)$. By definition of R_L, there exists $n \in \mathbb{N}$ such that $(x, y) \in R_L^n$, that is,

$$(x_{-n}, y_{-n}) \in R_\ell^n \text{ and } x_{+n} \geq y_{+n}.$$

Hence, by (9),

$$(x_{-n}, y_{-n}) \in Q(n, z) \text{ and } x_{+n} \geq y_{+n}$$

for all $z \in X$. Choosing $z = y$ and using the definition of $Q(n, z)$, it follows that $((x_{-n}, y_{+n}), (y_{-n}, y_{+n})) \in R$. Because $x_{+n} \geq y_{+n}$, reflexivity (if $x_{+n} = y_{+n}$) or the conjunction of strong Pareto and transitivity (if $x_{+n} > y_{+n}$) implies $((x_{-n}, x_{+n}), (y_{-n}, y_{+n})) = (x, y) \in R$, a contradiction.

Case 2. There exist $x, y \in X$ such that $(x, y) \in P(R_L)$ and $(y, x) \in R$. By (7), there exists $n \in \mathbb{N}$ such that $(x, y) \in P(R_L^n)$. Thus, (5) or (6) is true. If (5) holds, (9) implies

$$(x_{-n}, y_{-n}) \in P(Q(n, z)) \text{ and } x_{+n} \geq y_{+n}$$

for all $z \in X$. Setting $z = y$ and using (8), we obtain $((x_{-n}, y_{+n}), (y_{-n}, y_{+n})) \in P(R)$ and, using reflexivity or strong Pareto and transitivity as in case 1, we obtain $(x, y) \in P(R)$, a contradiction. If (6) holds, we proceed as in case 1. ■

6 Infinite-horizon utilitarianism

The technique employed in the previous section to characterize infinite-horizon versions of leximin can also be applied to a characterization of utilitarian orderings. This is an interesting observation because it demonstrates that the necessary violations of continuity and representability are restricted to comparisons of genuinely different infinite utility streams—streams differing in at most finitely many components can be ranked using well-behaved criteria. To define infinite-horizon utilitarian orderings, we begin by letting, for all $n \in \mathbb{N}$ and for all $x, y \in X$,

$$(x_{-n}, y_{-n}) \in R_u^n \Leftrightarrow \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} y_i$$

and

$$(x, y) \in R_U^n \Leftrightarrow (x_{-n}, y_{-n}) \in R_u^n \text{ and } x_{+n} \geq y_{+n}.$$

Clearly, R_U^n is a quasi-ordering for all $n \in \mathbb{N}$. Now define $R_U = \cup_{n \in \mathbb{N}} R_U^n$. As is the case for R_L, R_U is a quasi-ordering but it is not complete. However, as is straightforward to verify, if x and y differ in at most a finite number of components, they are comparable according to R_U.

Our final result characterizes all ordering extensions of R_U.

Theorem 4 *An ordering R on X satisfies strong Pareto, finite anonymity and finite translation-scale measurability if and only if R is an ordering extension of R_U.*

Proof. 'If.' All steps in the 'if' part of the proof of Theorem 3 go through if R_ℓ^n, R_L^n and R_L are replaced with R_u^n, R_U^n and R_U, respectively, except, of course, for the proof of equity preference. It remains to establish that any ordering extension of R_U satisfies finite translation-scale measurability. Let $x, y, z \in X$ and $m \in \mathbb{N}$ be such that $x_n = y_n$

for all $n \in \mathbb{N} \setminus \{1, \ldots, m\}$. By definition, x and y are ranked by R_U because they differ in at most a finite number of components, and the same is true for $x+z$ and $y+z$. Thus, because R is an ordering extension of R_U, the ranking of x and y ($x+z$ and $y+z$, respectively) according to R is the same as the ranking of x and y ($x+z$ and $y+z$, respectively) according to R_U, and we obtain

$$\begin{aligned}
(x+z, y+z) \in R \quad &\Leftrightarrow \quad (x+z, y+z) \in R_U \\
&\Leftrightarrow \quad \exists n \in \mathbb{N} \text{ such that } (x+z, y+z) \in R_U^n \\
&\Leftrightarrow \quad \exists n \in \mathbb{N} \text{ such that } \sum_{i=1}^{n}(x_i + z_i) \geq \sum_{i=1}^{n}(y_i + z_i) \text{ and} \\
&\qquad\qquad\qquad\qquad\qquad\quad x_{+n} + z_{+n} \geq y_{+n} + z_{+n} \\
&\Leftrightarrow \quad \exists n \in \mathbb{N} \text{ such that } \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} y_i \text{ and } x_{+n} \geq y_{+n} \\
&\Leftrightarrow \quad \exists n \in \mathbb{N} \text{ such that } (x, y) \in R_U^n \\
&\Leftrightarrow \quad (x, y) \in R_U \\
&\Leftrightarrow \quad (x, y) \in R.
\end{aligned}$$

'Only if.' Again, all steps of the 'only-if' part of Theorem 3 go through, except that the set of n-person axioms that are implied for the relation $Q(n, z)$ is composed of n-person strong Pareto, n-person anonymity and n-person translation-scale measurability. Now we can invoke Theorem 3 of d'Aspremont and Gevers (1977), which remains true if cardinal unit comparability is weakened to translation-scale measurability—see, for instance, Theorem 12 in Blackorby, Bossert and Donaldson [2002], to conclude that $Q(n, z) = R_u^n$. The remainder of the proof is identical to that of Theorem 3. ■

The restriction of finite translation-scale measurability to utility streams that differ in at most a finite number of components is important for the conclusion of Theorem 4. Without that restriction, the non-constructive technique employed in the proof does not allow us to conclude that any arbitrary ordering extension of R_U satisfies the resulting stronger axiom. It is interesting to compare this feature of the axiom to a related observation regarding finite anonymity: as shown by Diamond (1965), limiting the scope of the finite-anonymity axiom is crucial as well because, without the restriction to finite permutations, an incompatibility with strong Pareto emerges.

7 Concluding remarks

The results of this paper establish the existence of orderings of infinite utility streams satisfying attractive properties. In addition, we provide characterizations of various classes of such orderings. Given the nature of the proofs, we do not provide explicit constructions of these orderings. However, this feature is by no means unique to our approach. Extending quasi-orderings to orderings often requires non-constructive techniques; see, for example, Richter's (1966) use of Szpilrajn's (1930) extension theorem in the context of rational choice.

A plausible conclusion to be drawn from this paper is that impossibility results such as those of Diamond (1965), Basu and Mitra (2003) and Suzumura and Shinotsuka (2003) are, to a large extent, caused by continuity or representability assumptions. Without these rather restrictive requirements, evaluation rules satisfying attractive axioms can be characterized. Most notably, even orderings such as the infinite-horizon variants of utilitarianism characterized in the previous section become available and, therefore, violations of representability or continuity are limited to genuinely infinite utility streams. In our view, this confirms that the state of affairs in this area is not as disappointing and negative as has been suggested by the impossibility results of many earlier contributions.

The technique employed to characterize infinite-horizon versions of leximin and utilitarianism appears to be very powerful and applicable to the extension of other finite-population social-choice rules. However, as the discussion at the end of the previous section demonstrates, care needs to be taken in formulating suitable extensions of finite-population axioms and, thus, the methodology employed here cannot be applied in a mechanical fashion. We hope that our approach will stimulate further research in the area of intergenerational social choice by identifying alternative sets of attractive axioms and characterizing the social orderings that satisfy them.

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