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Strategy-proof Preference Aggregation*

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Abstract. An aggregation rule maps each profile of individual strict preference orderings over a set of alternatives into a social ordering over that set. We call such a rule strategy-proof if misreporting one's preference never produces a social ordering that is strictly between the original ordering and one's own preference. After describing a few examples of manipulable rules, we study in some detail three classes of strategy-proof rules: (i) rules based on a monotonic alteration of the majority relation generated by the preference profile; (ii) rules improving upon a fixed status-quo; and (iii) rules generalizing the Condorcet-Kemeny aggregation method. *Journal of Economic Literature* Classification Number: D71.

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1 Introduction

Social choice theory is primarily concerned with the problem of making collective decisions that reflect the preferences of the members of society.

One part of the theory, which includes voting theory and resource allocation theory, is devoted to *choice rules* mapping profiles of individual preferences into feasible social alternatives. Work in this area has addressed both the problem of designing ethically appealing (in particular, efficient and equitable) choice rules and the problem of inducing agents to reveal their preferences. Regarding the latter problem, an important literature has focused on the possibility of constructing *strategy-proof* rules. The early negative results of Gibbard (1973) and Satterthwaite (1975) regarding the existence of strategy-proof and non-dictatorial choice rules were followed by a number of possibilities for restricted domains of individual preferences; see Sprumont (1995) and Barberà (2011) for surveys.

A second part of social choice theory is interested in *aggregation rules* (also called *social welfare functions*) that map profiles of individual preferences into social orderings of the alternatives. The bulk of the literature in this area has focused on the normative aspect of the preference aggregation problem. If Arrow's (1963) binary independence property is dropped or suitably weakened, several aggregation rules can be recommended on the basis of various efficiency, fairness and coherence properties. For instance, Young (1974) and Nitzan and Rubinstein (1981) axiomatized the Borda aggregation rule, while Young and Levenglick (1978) offered a characterization of the Condorcet-Kemeny rule. More recent contributions study efficient and fair aggregation in models with a specific structure on alternatives and preferences. See, for instance, Dhillon and Mertens (1999) and Sprumont (2012) for the case where alternatives are lotteries, and Fleurbaey and Maniquet (2011) for Arrovian aggregation rules in economic environments.

Contrary to choice rules, aggregation rules have not been much studied from the viewpoint of their robustness to preference misrepresentations. Researchers are clearly aware of the incentive issue and seem to agree that some aggregation rules (such as the Borda rule, for instance) are somehow "more vulnerable to misrepresentations" than others. What prevents a systematic analysis, however, is the lack of a formal notion of robustness of aggregation rules to preference misrepresentations. The classic notion of strategy-proofness, which concerns choice rules, needs to be adapted. The only attempt to formulate a definition applicable to aggregation rules that we are aware of is due to Bossert and Storcken (1992). An aggregation rule is strategy-proof in their sense if misrepresenting one's preference never induces a social ordering which is closer to one's own preference according to the Kemeny distance. The results in Bossert and Storcken (1992) are mainly impossibilities.

In this paper, we propose an alternative definition which is based on the notion of betweenness; see Grandmont (1978). A rule is strategy-proof in our sense if misreporting one's preference never produces a social ordering that is strictly between the original ordering and one's own preference. Interestingly, our definition yields a nontrivial partition of the set of aggregation rules into manipulable and strategy-proof rules. We describe a few examples of manipulable rules and analyze three classes of strategy-proof rules in some detail. These are (i) rules based on a monotonic alteration of the majority relation generated by the preference profile; (ii) rules improving upon a fixed status-quo; and (iii) rules generalizing the Condorcet-Kemeny aggregation method.

2 Setup

Let A be a finite nonempty set containing m alternatives. Let $\mathbb{N}$ be the set of positive integers and let $\mathcal{N}$ denote the set of all finite nonempty subsets of $\mathbb{N}$. Each set $N \in \mathcal{N}$ is interpreted as a potential group of agents—a society.

Agents' preferences over alternatives are assumed to be strict orderings (i.e., complete, reflexive, transitive and antisymmetric binary relations) on A and the set of such preferences is denoted $\mathcal{R}$. We denote by $\widetilde{\mathcal{R}}$ the set of all orderings (i.e., complete, reflexive and transitive relations) on A. Typical elements of $\widetilde{\mathcal{R}}$ are denoted by symbols such as R, R', R_i, R_j. We use the notations aRb and $(a, b) \in R$ interchangeably to indicate that the pair of alternatives (a, b) is in the relation $R \subseteq A \times A$. If $N \in \mathcal{N}$ is a given society, $\mathcal{R}^N$ is the set of possible preference profiles for that society and a typical profile is written R_N.

An *aggregation rule* (a *rule,* for short) transforms each preference profile into a single ordering and, thus, is a function $f\colon \cup_{N\in\mathcal{N}}\mathcal{R}^N \to \widetilde{\mathcal{R}}$. Under this traditional formulation, the social ordering $f(R_N)$ need not be strict. This flexibility is useful (e.g., to guarantee the compatibility of equity properties such as anonymity and neutrality) but one could argue that a strict social ordering is needed to guarantee a unique social choice in every conceivable subset of alternatives. A *strict aggregation rule* is a function $f\colon \cup_{N\in\mathcal{N}}\mathcal{R}^N \to \mathcal{R}$. Formally, strictness of f is defined as follows.

Strictness. $f\left(\cup_{N\in\mathcal{N}}\mathcal{R}^N\right) \subseteq \mathcal{R}$.

The traditional notion of strategy-proofness applies to social choice functions, that is, to mappings from preference profiles into the set of objects over which preferences are defined. An aggregation rule f is not a social choice function: it selects objects—orderings of alternatives—over which individual preferences are not defined. In order to assess the manipulability of such a rule, individual preferences over alternatives must be extended to preferences over *orderings* of alternatives. We use the following well-established notion of *betweenness* (see, for instance, Grandmont, 1978) to define such preferences over orderings. This contrasts with Bossert and Storcken's (1992) definition in terms of the Kemeny distance.

Betweenness. For any $R, R', R'' \in \widetilde{\mathcal{R}}$, R'' is between R and R' (which we write $R'' \in [R, R']$) if and only if $R \cap R' \subseteq R'' \subseteq R \cup R'$.

An ordering R'' is between R and R' if and only if it agrees with R and R' whenever R and R' agree, that is to say,

$$\text{(i) } aR''b \text{ if } (aRb \text{ and } aR'b)$$

and

$$\text{(ii) } \neg(aR''b) \text{ if } (\neg(aRb) \text{ and } \neg(aR'b)).$$

Note that $[R,R'] = [R',R]$. Note also that if R,R',R'' are strict orderings, then the definition can be simplified. In this case, $R'' \in [R,R']$ if and only if $R \cap R' \subseteq R''$.

The *prudent extension* of a strict ordering of alternatives $R \in \mathcal{R}$ is the binary relation $\mathbf{R}$ over orderings of alternatives defined by

$$R''\mathbf{R}R' \;\Leftrightarrow\; R'' \in [R,R'] \tag{1}$$

for all $R'',R' \in \widetilde{\mathcal{R}}$. Thus, R'' is at least as good as R' if *and only if* it is between R' and R. The dependence on the underlying relation is captured by employing the bold-face version of the symbol used for the original relation. For instance, the prudent extension of R^0 is $\mathbf{R}^0$ and so on. It is easy to check that $\mathbf{R}$ is a strict quasiordering (i.e., a reflexive, transitive and antisymmetric relation). Transitivity of $\mathbf{R}$ follows from transitivity of the betweenness relation, namely, the property that, for all $R^0,R,R',R'' \in \widetilde{\mathcal{R}}$,

$$\left(R' \in \left[R^0,R\right] \text{ and } R'' \in \left[R^0,R'\right]\right) \;\Rightarrow\; R'' \in \left[R^0,R\right]. \tag{2}$$

We stress that $\mathbf{R}$ is not a complete relation.

The relation $\mathbf{R}$ is dubbed "prudent" because it contains only the unambiguous comparisons between orderings of alternatives. It is arguably the minimal (i.e., the least complete) relation over orderings that is consistent with R. To understand this, consider first the case where both R' and R'' are strict orderings. For each nonempty set $B \subseteq A$, let $a^*(R',B)$ and $a^*(R'',B)$ denote the unique maximal elements of R' and R'' in B. It is easy to check that $R'' \in [R,R']$ if and only if $a^*(R'',B)Ra^*(R',B)$ for every nonempty $B \subseteq A$. Statement (1) therefore means that an agent with preference R over alternatives finds the ordering R'' at least as good as R' if and only if the choice recommended by R'' in any feasible set is at least good, according to R, as the choice recommended by R'.

Consider now the case where R', R'' need not be strict. Let $A^*(R',B)$ and $A^*(R'',B)$ denote the sets of maximal elements of R' and R'' in B. With a slight abuse of notation, write $A^*(R'',B)RA^*(R',B)$ (and say that the former set is at least as good as the latter) if and only if

(i) bRa for all $b \in A^*(R'',B) \setminus A^*(R',B)$ and all $a \in A^*(R',B) \setminus A^*(R'',B)$

or

(ii) $A^*(R',B) \subseteq A^*(R'',B)$ and bRa for all $b \in A^*(R'',B) \setminus A^*(R',B)$ and all $a \in A^*(R',B)$

or

(iii) $b \in A^*(R',B) \supseteq A^*(R'',B)$ and bRa for all $b \in A^*(R'',B)$ and all $a \in A^*(R',B) \setminus A^*(R'',B)$.

Again, one can show that $R'' \in [R,R']$ if and only if $A^*(R'',B)RA^*(R',B)$ for every nonempty $B \subseteq A$. Statement (1) means that an agent with preference R finds the ordering

R'' at least as good as R' if and only if he finds the *subset* of possible choices recommended by R'' in any feasible set at least good as the subset recommended by R'.

We are now ready to define our notion of manipulability of an aggregation rule. If $R, R' \in \widetilde{\mathcal{R}}$, let $[R, R'[\, = [R, R'] \setminus \{R'\}$. For $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, $i \in N$ and $R'_i \in \mathcal{R}$, let $(R'_i, R_{N\setminus i})$ denote the preference profile in $\mathcal{R}^N$ obtained from R_N by replacing R_i with R'_i.

Manipulability. There exist $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, $i \in N$ and $R'_i \in \mathcal{R}$ such that $f(R'_i, R_{N\setminus i}) \in [R_i, f(R_N)[$.

As in the traditional definition of strategy-proofness, a rule is strategy-proof if it cannot be manipulated by any of the agents.

Strategy-proofness. f is not manipulable.

This is just the standard notion of strategy-proofness in disguise—provided that preferences are properly defined. To see this explicitly, define $\mathfrak{R} = \{\mathbf{R} \mid R \in \mathcal{R}\}$, the set of preferences over orderings of alternatives which are the prudent extension of some preference over alternatives. Define the function $\mathfrak{f}\colon \cup_{N\in\mathcal{N}} \mathfrak{R}^N \to \widetilde{\mathcal{R}}$ by $\mathfrak{f}(\mathbf{R}_N) = f(R_N)$, where R_N is the profile of preferences whose prudent extension is $\mathbf{R}_N$. This function $\mathfrak{f}$ is a social choice function. According to the standard definition, $\mathfrak{f}$ is strategy-proof if there does not exist $N \in \mathcal{N}$, $\mathbf{R}_N \in \mathfrak{R}^N$, $i \in N$ and $\mathbf{R}'_i \in \mathfrak{R}$ such that $\mathfrak{f}(\mathbf{R}'_i, \mathbf{R}_{N\setminus i})\mathbf{R}_i\mathfrak{f}(\mathbf{R}_N)$ and $\mathfrak{f}(\mathbf{R}'_i, \mathbf{R}_{N\setminus i}) \neq \mathfrak{f}(\mathbf{R}_N)$. Because of (1), f is strategy-proof according to our definition if and only if $\mathfrak{f}$ is strategy-proof in the standard (choice-theoretic) sense.

Because the prudent extension is the minimal relation over orderings consistent with a given preference over alternatives, our definition of strategy-proofness is the weakest meaningful definition applicable to an aggregation rule. Indeed, since misreporting one's preference is most likely to induce a social ordering that is non-comparable to the ordering resulting from honest reporting, the scope for profitable manipulations is minimal.

Yet, it turns out that both the class of strategy-proof rules *and* the class of manipulable rules are very rich and contain many well-known examples. Our definition of strategy-proofness thus provides an interesting test of robustness to strategic behavior. Section 3 describes a few manipulable rules and Sections 4, 5 and 6 study three different classes of strategy-proof rules.

3 Some manipulable rules

We begin with a few examples of aggregation rules that are manipulable.

Example 1. The Borda score of alternative a at profile R_N is

$$\beta(a, R_N) = \sum_{i\in N} |\{c \in A \setminus a \mid aR_ic\}| \,.$$

The *Borda aggregation rule* f^B ranks alternatives according to their Borda score. That is, for all $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$ and $a, b \in A$,

$$af^B(R_N)b \;\Leftrightarrow\; \beta(a, R_N) \geq \beta(b, R_N).$$

To see that this rule is manipulable, suppose $A = \{a, b, c, d\}$ and $N = \{1, 2, 3, 4\}$. Suppose that the individual strict preference orderings over alternatives are $R_1 = abcd$, $R_2 = acdb$, $R_3 = bcad$ and $R_4 = dbac$, where the notation xy means that x is strictly preferred to y. Then $f^B(R_1, R_2, R_3, R_4) = abcd$. But if $R'_4 = bdca$, the Borda rule yields $f^B(R_1, R_2, R_3, R'_4) = bacd \in \left[R_4, f^B(R_1, R_2, R_3, R_4)\right[$.

Since the Borda voting method is known to be highly vulnerable to manipulations, it is not too surprising that the corresponding aggregation rule fails to be strategy-proof –even according to our rather weak definition.

Perhaps more surprisingly, it turns out that many aggregation rules based on the majority relation generated by individual preferences are manipulable as well. For any $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$, define the majority relation $M(R_N)$ on A by

$$aM(R_N)b \;\Leftrightarrow\; |\{i \in N \mid aR_ib\}| \geq |\{i \in N \mid bR_ia\}|.$$

Example 2. The Copeland score of alternative a at profile R_N is

$$\gamma(a, R_N) = |\{c \in A \setminus a \mid aM(R_N)c\}|.$$

The *Copeland aggregation rule* f^C ranks alternatives according to their Copeland score: for all $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$ and $a, b \in A$,

$$af^C(R_N)b \;\Leftrightarrow\; \gamma(a, R_N) \geq \gamma(b, R_N).$$

Suppose $A = \{a, b, c, d\}$ and $N = \{1, \ldots, 25\}$. Let R_N be a profile such that $R_1 = abcd$ and each of the 24 possible strict orderings over alternatives is the preference of one of the remaining agents. Then $M(R_N) = R_1$ and therefore $f^C(R_N) = R_1$. Consider the agent $i \in N$ whose preference is $R_i = badc$ and let $R'_i = bdac$. The majority relation $M(R'_i, R_{N\setminus i})$ coincides with $M(R_N)$ on every unordered pair of alternatives except $\{a, d\}$ because $M(R'_i, R_{N\setminus i}) = (M(R_N) \setminus \{(a, d)\}) \cup \{(d, a)\}$. The new Copeland scores are $\gamma(a, (R'_i, R_{N\setminus i})) = \gamma(b, (R'_i, R_{N\setminus i})) = 2 > 1 = \gamma(c, (R'_i, R_{N\setminus i})) = \gamma(d, (R'_i, R_{N\setminus i}))$. The Copeland aggregation rule therefore recommends the ordering $f^C(R'_i, R_{N\setminus i}) = (ab)(cd)$, where the notation (xy) means that x, y are indifferent. Since $f^C(R'_i, R_{N\setminus i}) = (ab)(cd) \in [badc, abcd[= \left[R_i, f^C(R_N)\right[$, the rule is manipulable.

Example 3. The *long-path rule* f^{LP} is another rule that ranks alternatives on the basis of the majority relation generated by the preference profile. In order to rank an alternative a, the long-path rule takes into account the strength of the alternatives that a beats in majority comparisons. Consider a population N of odd size and a preference profile R_N. Let $\mathbf{M}(R_N)$ be the adjacency matrix of $M(R_N)$ defined by

$$\mathbf{M}_{a,b}(R_N) = \begin{cases} 1 & \text{if } (a, b) \in M(R_N); \\ 0 & \text{otherwise.} \end{cases}$$

The vector of relative strengths $r^*(R_N) = (r^*_a(R_N))_{a \in A}$ is computed as follows. Let $s^0(R_N)$ be the unit vector in $\mathbb{R}^{|A|}$ and, for each $k \in \mathbb{N}$, let $s^k(R_N) = \mathbf{M}^k(R_N) \cdot s^0(R_N)$, where

$\mathbf{M}^k(R_N)$ is the k-fold Cartesian product of $\mathbf{M}(R_N)$, and $r^k(R_N) = \frac{s^k(R_N)}{\left\| s^k(R_N) \right\|}$. By a theorem of Frobenius, the limit of the sequence $(r^k(R_N))_{k\in\mathbb{N}}$ as k tends to infinity exists. We denote it by $r^\infty(R_N)$ and let

$$af^{LP}(R_N)b \Leftrightarrow r_a^\infty(R_N) \geq r_b^\infty(R_N).$$

The method can be generalized to societies of even sizes by breaking ties in the majority relation $M(R_N)$ before computing the vector of relative strengths. In any case, the rule fails to be strategy-proof even for societies of odd sizes.

Consider the two profiles used in Example 2. Obviously, $f^{LP}(R_N) = R_1 = abcd$. Considering an agent $i \in N$ whose preference is $R_i = badc$ and letting $R_i' = bdac$, we obtain $f^{LP}(R_i', R_{N\setminus i}) = abdc \in [badc, abcd[= \left[R_i, f^{LP}(R_N)\right[$, violating strategy-proofness.

4 Alterations of the majority relation

When there are only two alternatives (i.e., $m = 2$), the majority rule $f(R_N) = M(R_N)$ as defined in the previous section is the strategy-proof aggregation rule *par excellence.* It is therefore natural to try and extend this rule to more than two alternatives. The difficulty, of course, is that the majority relation associated with a profile of individual preferences may fail to be transitive. This section analyzes rules which aggregate individual preferences by altering the majority relation they generate so as to make it an ordering.

The relation $M(R_N)$ is complete and reflexive, and it is antisymmetric if $|N|$ is odd. Let $\mathcal{T}$ denote the set of complete, reflexive and antisymmetric relations on A and let $\widetilde{\mathcal{T}}$ denote the set of all complete and reflexive relations on A.[1] The literature offers a large selection of interesting methods for ranking the alternatives in A on the basis of a relation $T \in \mathcal{T}$; see, for instance, Laslier's (1997) extensive survey.[2] Because the majority relation may fail to be antisymmetric when the population size is even, applying such methods to our aggregation problem requires a tie-breaking rule.

Fix a strict ordering $\succeq$ on A and define the binary relation $M_\succeq(R_N)$ on A by letting $aM_\succeq(R_N)b$ if and only if

$$|\{i \in N \mid aR_ib\}| > |\{i \in N \mid bR_ia\}|$$

or

$$|\{i \in N \mid aR_ib\}| = |\{i \in N \mid bR_ia\}| \text{ and } a \succeq b.$$

Observe that $M_\succeq(R_N) \in \mathcal{T}$. Moreover, $M_\succeq(R_N) = M(R_N)$ if $|N|$ is odd. We can now provide a formal definition of a *majority-based aggregation rule.*[3]

[1] A *tournament* on A is a complete, transitive and asymmetric relation. Formally, elements of $\mathcal{T}$ are not tournaments. But each $T \in \mathcal{T}$ can be identified with the tournament $T \setminus \{(a,a) \mid a \in A\}$. It is convenient to focus on $\mathcal{T}$ rather than on the set of tournaments because $\mathcal{T}$ includes the strict orderings, that is, $\mathcal{R} \subseteq \mathcal{T}$.

[2] Deriving a ranking from a relation $T \in \widetilde{\mathcal{T}}$ is notoriously harder and the methods found in the literature are generally less satisfactory.

[3] Strictly speaking, it would be more precise, but also cumbersome, to use the term *majority-based aggregation rule with ties broken according to* $\succeq$. We use the simpler formulation for ease of exposition.

Definition 1 *An aggregation rule f is majority-based if for all $N, M \in \mathcal{N}$, for all $R_N \in \mathcal{R}^N$ and for all $R'_M \in \mathcal{R}^M$,*

$$M_{\succeq}(R_N) = M_{\succeq}(R'_M) \;\Rightarrow\; f(R_N) = f(R'_M).$$

An *alteration* is a function $\varphi\colon \mathcal{T} \to \widetilde{\mathcal{R}}$ such that $\varphi(R) = R$ if $R \in \mathcal{R}$. For any $T \in \mathcal{T}$, we refer to $\varphi(T)$ as the alteration of T (under φ). Extend the definition of betweenness to all relations in $\widetilde{\mathcal{T}}$ as follows. If $T, T', T'' \in \widetilde{\mathcal{T}}$, then $T'' \in [T, T']$ if and only if $T \cap T' \subseteq T'' \subseteq T \cup T'$. An alteration φ is *agreement-monotonic* (*monotonic,* for short) if for all $R \in \mathcal{R}$ and $T_1, T_2 \in \mathcal{T}$,

$$T_2 \in [R, T_1[\;\Rightarrow\; \varphi(T_1) \notin [R, \varphi(T_2)[\,. \tag{3}$$

In other words, if a relation T_2 agrees more with a given strict ordering than another relation T_1, then the alteration of T_1 should not agree more with that ordering than the alteration of T_2. Examples will be given after Proposition 1 below. We can now define the notion of a *monotonic majority-alteration rule.*

Definition 2 *An aggregation rule f is a monotonic majority-alteration rule if there is a monotonic alteration φ such that $f(R_N) = \varphi(M_{\succeq}(R_N))$ for all $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$.*

Among the majority-based rules, monotonic majority-alteration rules can be characterized by combining strategy-proofness and unanimity. We first introduce a notion of efficiency suitable for our framework. Given a preference profile $R_N \in \mathcal{R}^N$ generating the corresponding profile of prudent extensions $\mathbf{R}_N \in \mathfrak{R}^N$, an ordering $R \in \widetilde{\mathcal{R}}$ is *efficient* if there is no $R' \in \widetilde{\mathcal{R}}$ such that $R' \neq R$ and $R' \mathbf{R}_i R$ for all $i \in N$. Since the prudent extensions are incomplete relations, efficiency is a relatively weak requirement. Given the definition of the prudent extensions, an ordering R is efficient if $\cap_{i \in N} [R_i, R[= \varnothing$. Thus, we define the efficiency of f as follows.

Efficiency. For all $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$,

$$\cap_{i \in N} [R_i, f(R_N)[= \varnothing.$$

A consequence of efficiency is *unanimity,* a very weak property.

Unanimity. For all $N \in \mathcal{N}$, $R \in \mathcal{R}$ and $R_N \in \mathcal{R}^N$,

$$R_i = R \text{ for all } i \in N \;\Rightarrow\; f(R_N) = R.$$

A warning is in order. A common axiom in the literature requires that the social ordering should deem alternative a at least as good as b whenever all agents find a at least as good as b. We refer to this axiom as binary unanimity.

Binary unanimity. For all $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$,

$$\cap_{i \in N} R_i \subseteq f(R_N).$$

Binary unanimity is usually called the (weak) Pareto principle but that term is inadequate here because agents' preferences over alternatives have been extended to preferences over orderings of alternatives. The proper formulation of the Pareto principle now requires that a dominated *ordering* of alternatives should never be recommended. This is precisely efficiency (in the sense of our definition). Binary unanimity and efficiency are independent conditions: neither of them implies the other. However, binary unanimity does imply unanimity.

We now obtain the following characterization of the monotonic majority-alteration rules.

Proposition 1 *An aggregation rule is majority-based, unanimous and strategy-proof if and only if it is a monotonic majority-alteration rule.*

Proof. By convention, whenever a pair $(a, b) \in A \times A$ is considered in this proof, it is assumed that $a \neq b$.

"If." Let φ be a monotonic alteration and let f be the aggregation rule generated by φ (given the fixed tie-breaking strict ordering $\succeq$), i.e., $f(R_N) = \varphi(M_{\succeq}(R_N))$ for all $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$. It is clear that f is majority-based and unanimous. To check strategy-proofness, fix $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$. For $(a, b) \in A \times A$, the definition of $M_{\succeq}$ implies

$$(a, b) \in M_{\succeq}((R'_i, R_{N\setminus i})) \cap R_i \;\Rightarrow\; (a, b) \in M_{\succeq}(R_N). \tag{4}$$

Since the relations $M_{\succeq}((R'_i, R_{N\setminus i}))$, R_i and $M_{\succeq}(R_N)$ are all antisymmetric, (4) means that $M_{\succeq}(R_N) \in \left[R_i, M_{\succeq}((R'_i, R_{N\setminus i}))\right]$. If $M_{\succeq}(R_N) = M_{\succeq}((R'_i, R_{N\setminus i}))$, then

$$f(R'_i, R_{N\setminus i}) = \varphi(M_{\succeq}((R'_i, R_{N\setminus i}))) = \varphi(M_{\succeq}(R_N)) = f(R_N)$$

and agent i does not benefit from misreporting her preference. If $M_{\succeq}(R_N) \neq M_{\succeq}((R'_i, R_{N\setminus i}))$, then $M_{\succeq}(R_N) \in \left[R_i, M_{\succeq}((R'_i, R_{N\setminus i}))\right[$ and by monotonicity of φ,

$$f(R'_i, R_{N\setminus i}) = \varphi(M_{\succeq}((R'_i, R_{N\setminus i}))) \notin [R_i, \varphi(M_{\succeq}(R_N))[= [R_i, f(R_N)[\,,$$

so that agent i again does not benefit from misreporting her preference.

"Only if." Let f be a majority-based, unanimous and strategy-proof aggregation rule. Since f is majority-based, there exists a function $\varphi \colon \mathcal{T} \to \widetilde{\mathcal{R}}$ such that $f(R_N) = \varphi(M_{\succeq}(R_N))$ for all $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$. Since f is unanimous, $\varphi(R) = R$ for all $R \in \mathcal{R}$, that is, φ is an alteration. It remains to be proven that φ is monotonic. Suppose, to the contrary, that there exist $R^0 \in \mathcal{R}$ and $T_1, T_2 \in \mathcal{T}$ such that

$$T_2 \in \left[R^0, T_1\right[\quad \text{and} \quad \varphi(T_1) \in \left[R^0, \varphi(T_2)\right[. \tag{5}$$

For $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$ and $(a, b) \in A \times A$, let

$$n((a, b), R_N) = |\,\{i \in N \mid (a, b) \in R_i\}\,| - |\,\{i \in N \mid (b, a) \in R_i\}\,|.$$

Let $\mathcal{N}_0 = \{N \in \mathcal{N} \mid |N| \text{ is odd}\}$. Notice that $n((a, b), R_N) \geq 1$ if $(a, b) \in M(R_N)$ and $N \in \mathcal{N}_0$. We continue by establishing the following lemma.

Lemma 1 *Let $T \in \mathcal{T}$ and let $Q \subseteq T$. Then there exist $N \in \mathcal{N}_0$ and $R_N \in \mathcal{R}^N$ such that*

(i) $M(R_N) = T$;

(ii) $n((a,b), R_N) \geq 3$ for all $(a,b) \in Q$;

(iii) $n((a,b), R_N) = 1$ for all $(a,b) \in T \setminus Q$.

Part (i) of this lemma is McGarvey's (1953) theorem. Parts (ii) and (iii) together state that one can construct the profile generating T to guarantee a "large" majority on every pair in a given subset of T and a "thin" majority on every pair in its complement.

Proof of Lemma 1.

Step 1. We show that for each $T \in \mathcal{T}$ there exist $N \in \mathcal{N}_0$ and $R_N \in \mathcal{R}^N$ such that $n((a,b), R_N) = 1$ for all $(a,b) \in T$.

Let $T \in \mathcal{T}$ and construct R_N recursively as follows. Choose $(a^1, b^1) \in T$. Let $R^1 \in \mathcal{R}$ be a strict preference ordering such that $(a^1, b^1) \in R^1$. Let $N^1 = \{1\}$ and $R_{N^1} = R_1 = R^1$. By construction, $n((a^1, b^1), R_{N^1}) = 1$. Next, fix $k > 1$ and suppose we have constructed $N^{k-1} \in \mathcal{N}_0$ and $R_{N^{k-1}} \in \mathcal{R}^{N^{k-1}}$ such that $n((a,b), R_{N^{k-1}}) = 1$ for $k-1$ distinct pairs $(a,b) \in T$. Choose $(a^k, b^k) \in T \setminus M(R_{N^{k-1}})$. Let $R, \widetilde{R} \in \mathcal{R}$ be such that $(a^k, b^k) \in R \cap \widetilde{R}$ and, for all $(x,y) \neq (a^k, b^k)$, $(x,y) \in R$ if and only if $(y,x) \in \widetilde{R}$. Such preferences are easily constructed: denoting by $c^1, \ldots, c^{m-2}$ the alternatives other than a^k, b^k, it suffices to let $R = a^k b^k c^1 \ldots c^{m-2}$ and $\widetilde{R} = c^{m-2} \ldots c^1 a^k b^k$ as in McGarvey (1953). Let $i, j \in \mathbb{N} \setminus N^{k-1}$, $R_i = R$, $R_j = \widetilde{R}$, $N^k = N^{k-1} \cup \{i, j\}$ and $R_{N^k} = (R_{N^{k-1}}, R_{\{i,j\}})$. By construction $N^k \in \mathcal{N}_0$ and $n((a,b), R_{N^k}) = 1$ for k distinct pairs $(a,b) \in T$.

Step 2. Let $N \in \mathcal{N}_0$ and $R_N \in \mathcal{R}^N$ be such that $n((a,b), R_N) = 1$ for all $(a,b) \in T$. We modify R_N to make the majority large on every pair in Q while keeping it thin on $T \setminus Q$.

The argument is similar to the one in Step 1; we only sketch it. Choose $(a^1, b^1) \in Q$. Let $R^1, \widetilde{R}^1 \in \mathcal{R}$ be such that $(a^1, b^1) \in R^1 \cap \widetilde{R}^1$ and, for all $(x,y) \neq (a^1, b^1)$, $(x,y) \in R^1$ if and only if $(y,x) \in \widetilde{R}^1$. Add to N two agents with preferences $R^1, \widetilde{R}^1$ and call the resulting profile R_{N^1}. Then $n((a^1, b^1), R_{N^1}) = 3$ and $n((x,y), R_{N^1}) = n((x,y), R_N) = 1$ for all $(x,y) \neq (a^1, b^1)$. Repeating this construction eventually yields a profile R_{N^*} such that $n((a,b), R_{N^*}) = 3$ for all $(a,b) \in Q$ and $n((a,b), R_{N^*}) = 1$ for all $(a,b) \in T \setminus Q$. This completes the proof of the lemma.

Returning to the proof of Proposition 1, we apply Lemma 1 to $T = T_2$ and $Q = T_2 \cap T_1$, where T_1, T_2 are the relations in (5): there exist $N \in \mathcal{N}_0$ and $R_N \in \mathcal{R}^N$ such that $M(R_N) = T_2$, $n((a,b), R_N) \geq 3$ for all $(a,b) \in T_2 \cap T_1$, and $n((a,b), R_N) = 1$ for all $(a,b) \in T_2 \setminus T_1$. Moreover, we can assume that R_N is a *diversified* profile: for every $R \in \mathcal{R}$ there exists some agent $i \in N$ such that $R_i = R$ (if this is not the case, simply add to R_N a *balanced profile,* i.e., one where for each $R \in \mathcal{R}$ there exists one and only one agent i such that $R_i = R$).

Pick $i \in N$ such that $R_i = R^0$ and let $R'_i = (R^0)^{-1} = \{(y,x) \mid (x,y) \in R^0\}$ be the opposite of the strict ordering R^0. We claim that

$$M(R'_i, R_{N \setminus i}) = T_1. \tag{6}$$

To see this, note that for all $(a,b) \in A \times A$,

$$\begin{aligned}(a,b) \in T_1 \cap T_2 &\Rightarrow n((a,b), R_N) \geq 3 \\ &\Rightarrow n((a,b), (R_i', R_{N\setminus i})) \geq 1 \\ &\Rightarrow (a,b) \in M(R_i', R_{N\setminus i})\end{aligned}$$

and

$$\begin{aligned}(a,b) \in T_1 \setminus T_2 &\Rightarrow (b,a) \in T_2 \setminus T_1 \\ &\Rightarrow n((b,a), R_N) = 1 \\ &\Rightarrow n((b,a), (R_i', R_{N\setminus i})) = -1 \\ &\Rightarrow n((a,b), (R_i', R_{N\setminus i})) = 1 \\ &\Rightarrow (a,b) \in M(R_i', R_{N\setminus i}),\end{aligned}$$

where the third implication holds for the following reason. Since $(b,a) \in T_2 \setminus T_1$, the assumption $T_2 \in [R^0, T_1[$ implies $(b,a) \in R^0 = R_i$, hence $(a,b) \in (R^0)^{-1} = R_i'$.

We have just shown that $(a,b) \in T_1$ implies $(a,b) \in M(R_i', R_{N\setminus i})$, i.e., $T_1 \subseteq M(R_i', R_{N\setminus i})$. Since $T_1 \in \mathcal{T}$ and $M(R_i', R_{N\setminus i}) \in \mathcal{T}$ (because $N \in \mathcal{N}_0$), (6) follows.

Now agent i gains by misreporting R_i' at profile R_N. To see this, note that

$$\begin{aligned}f(R_i', R_{N\setminus i}) &= \varphi(M_{\succeq}((R_i', R_{N\setminus i}))) \\ &= \varphi(M(R_i', R_{N\setminus i})) \\ &= \varphi(T_1) \in \left[R^0, \varphi(T_2)\right[\end{aligned}$$

and

$$\begin{aligned}\left[R^0, \varphi(T_2)\right[&= [R_i, \varphi(M(R_N))[\\ &= [R_i, \varphi(M_{\succeq}(R_N))[\\ &= [R_i, f(R_N)[\end{aligned}$$

and, therefore, $f(R_i', R_{N\setminus i}) \in [R_i, f(R_N)[$ which contradicts strategy-proofness. ■

To illustrate the usefulness of Proposition 1, we now give two examples of monotonic alterations.

Example 4: Lexicographic alterations. Let $\succsim$ be a strict ordering on the set of unordered pairs of alternatives $\mathcal{P}(A) = \{\{a,b\} \mid a,b \in A \text{ and } a \neq b\}$. Denote by $\left\{a^k, b^k\right\}$ the unordered pair ranked kth according to the strict ordering $\succsim$. Thus

$$\left\{a^1, b^1\right\} \succsim \left\{a^2, b^2\right\} \succsim \ldots \succsim \left\{a^K, b^K\right\},$$

where $K = \frac{m(m-1)}{2}$. Given $T \in \mathcal{T}$, the $\succsim$-*lexicographic alteration* of T is the strict ordering R computed by altering T recursively as follows. Let $T(0) = T$. Given $T(h)$, let $k(h)$ be the largest integer k such that the subrelation

$$\{(a,b) \in T(h) \mid \text{ there exists } k' \in \{1,\ldots,k\} \text{ such that } \{a^{k'}, b^{k'}\} = \{a,b\}\}$$

is acyclic. If $k(h) < K$, then we let

$$T(h+1) = \left[T(h) \cup \left\{(a^{k(h)+1}, b^{k(h)+1})\right\}\right] \setminus \left\{(b^{k(h)+1}, a^{k(h)+1})\right\}$$

if $(b^{k(h)+1}, a^{k(h)+1}) \in T(h)$, and

$$T(h+1) = \left[T(h) \cup \left\{(b^{k(h)+1}, a^{k(h)+1})\right\}\right] \setminus \left\{(a^{k(h)+1}, b^{k(h)+1})\right\}$$

if $(a^{k(h)+1}, b^{k(h)+1}) \in T(h)$. If $k(h) = K$, then $T(h)$ is a strict ordering and we let $R = T(h)$. This algorithm terminates in at most K steps. If T itself is a strict ordering, then $R = T(0) = T$.

As an illustration, suppose $A = \{a, b, c, d\}$, $T = \{(a,b), (a,c), (b,c), (c,d), (d,a), (d,b)\}$ and $\{b,c\} \succsim \{c,d\} \succsim \{a,d\} \succsim \{a,b\} \succsim \{a,c\} \succsim \{b,d\}$. Then $k(0) = 3$ and $T(1)$ is obtained by reversing in T the preference over the fourth unordered pair according to $\succsim$, namely $\{a,b\}$. This yields

$$T(1) = \{(a,c), (b,a), (b,c), (c,d), (d,a), (d,b)\}.$$

Now, $k(1) = 4$ and we reverse the preference over $\{a,c\}$ in $T(1)$ to get

$$T(2) = \{(b,a), (b,c), (c,a), (c,d), (d,a), (d,b)\}.$$

Next, $k(2) = 5$, and we reverse the preference over the last unordered pair according to $\succsim$, namely $\{b,d\}$, to get

$$T(3) = \{(b,a), (b,c), (b,d), (c,a), (c,d), (d,a)\}.$$

This is a strict ordering (indeed, $T(3) = bcda$), hence $R = T(3)$.

To check that every lexicographic alteration φ is agreement-monotonic, fix such an alteration with associated strict ordering $\succsim$ on $\mathcal{P}(A)$ and let $R \in \mathcal{R}$ and $T_1, T_2 \in \mathcal{T}$ be such that $T_2 \in [R, T_1[$. Let $\left\{a^{k_1}, b^{k_1}\right\}$ be the first unordered pair according to $\succsim$ on which T_1, T_2 disagree. Since $T_2 \in [R, T_1[$, we have, say,

$$(a^{k_1}, b^{k_1}) \in R \cap T_2 \text{ and } (b^{k_1}, a^{k_1}) \in T_1.$$

Since T_1, T_2 agree on all unordered pairs $\left\{a^k, b^k\right\}$ such that $k < k_1$, the recursive construction of φ ensures that $\varphi(T_1)$ and $\varphi(T_2)$ also agree on these pairs. Since T_1, T_2 disagree on $\left\{a^{k_1}, b^{k_1}\right\}$, at most one of them can be altered by φ on that unordered pair.

If neither T_1 nor T_2 is altered, then

$$(a^{k_1}, b^{k_1}) \in R \cap \varphi(T_2) \text{ and } (b^{k_1}, a^{k_1}) \in \varphi(T_1),$$

implying that $\varphi(T_1) \notin [R, \varphi(T_2)[$, as desired.

If exactly one of T_1, T_2 is altered, then $\varphi(T_1)$ and $\varphi(T_2)$ agree on all unordered pairs $\left\{a^k, b^k\right\}$ such that $k \leq k_1$. In that case, let $\left\{a^{k_2}, b^{k_2}\right\}$ be the second unordered pair according to $\succsim$ on which T_1, T_2 disagree and repeat the argument above until reaching an unordered pair where neither T_1 nor T_2 is altered.

For the next example, we introduce some further definitions that will also be of use in Section 6. For any $T_1, T_2 \in \mathcal{T}$, define

$$\begin{aligned}\mathbf{A}(T_1,T_2) &= (T_1 \cap T_2) \cup (A \times A) \setminus (T_1 \cup T_2);\\ \mathbf{D}(T_1,T_2) &= (T_1 \setminus T_2) \cup (T_2 \setminus T_1).\end{aligned}$$

These are, respectively, the set of pairs of alternatives on which T_1, T_2 agree and the set of pairs on which they disagree. The (Kemeny) distance between T_1 and T_2 is

$$d(T_1,T_2) = |\mathbf{D}(T_1,T_2)|.$$

For any $T \in \mathcal{T}$, define

$$\mathcal{R}^*(T) = \{R \in \mathcal{R} \mid d(T,R) \leq d(T,R') \text{ for all } R' \in \mathcal{R}\},$$

the set of strict orderings whose distance to T is minimal.

Example 5: Slater alterations. Given a tie-breaking strict ordering $\gtrapprox$ on $\mathcal{R}$, the $\gtrapprox$-*Slater alteration* is the alteration φ which assigns to each $T \in \mathcal{T}$ the strict ordering R ranked first in $\mathcal{R}^*(T)$ according to $\gtrapprox$. See Slater (1961).

We claim that every Slater alteration φ is agreement-monotonic. The proof uses the following observation. For all $T, T', T'' \in \mathcal{T}$,

$$d(T,T'') = d(T,T') + |\mathbf{A}(T,T') \cap \mathbf{D}(T',T'')| - |\mathbf{D}(T,T') \cap \mathbf{D}(T',T'')|. \tag{7}$$

To see why this is true, note that

$$\mathbf{D}(T',T'') = [\mathbf{A}(T,T') \cap \mathbf{D}(T',T'')] \cup [\mathbf{D}(T,T') \cap \mathbf{A}(T',T'')].$$

Since the two sets inside the brackets are disjoint,

$$d(T,T'') = |\mathbf{A}(T,T') \cap \mathbf{D}(T',T'')| + |\mathbf{D}(T,T') \cap \mathbf{A}(T',T'')|. \tag{8}$$

On the other hand, since $\mathbf{A}(T',T'')$ and $\mathbf{D}(T',T'')$ partition $A \times A$, we have

$$d(T,T') = |\mathbf{D}(T,T')| = |\mathbf{D}(T,T') \cap \mathbf{A}(T',T'')| + |\mathbf{D}(T,T') \cap \mathbf{D}(T',T'')|,$$

hence

$$|\mathbf{D}(T,T') \cap \mathbf{A}(T',T'')| = d(T,T') - |\mathbf{D}(T,T') \cap \mathbf{D}(T',T'')|.$$

Substituting this expression into (8) yields (7).

Fix a Slater alteration φ with tie-breaking strict ordering $\gtrapprox$ on $\mathcal{R}$ and let $R \in \mathcal{R}$ and $T_1, T_2 \in \mathcal{T}$ be such that $T_2 \in [R, T_1[$. Note that $T_1 \neq T_2$. By observation (7) we have, for (possibly identical) $i, j \in \{1, 2\}$,

$$d(T_i, \varphi(T_j)) = d(T_i, R) + |\mathbf{A}(T_i, R) \cap \mathbf{D}(R, \varphi(T_j))| - |\mathbf{D}(T_i, R) \cap \mathbf{D}(R, \varphi(T_j))|.$$

Since $d(T_1, \varphi(T_1)) \leq d(T_1, \varphi(T_2))$, this implies

$$|\mathbf{D}(T_1, R) \cap \mathbf{D}(R, \varphi(T_2))| - |\mathbf{D}(T_1, R) \cap \mathbf{D}(R, \varphi(T_1))|$$

$$\leq |\mathbf{A}(T_1,R)\cap \mathbf{D}(R,\varphi(T_2))| - |\mathbf{A}(T_1,R)\cap \mathbf{D}(R,\varphi(T_1))|. \tag{9}$$

Likewise, since $d(T_2,\varphi(T_2)) \leq d(T_2,\varphi(T_1))$, we get

$$|\mathbf{A}(T_2,R)\cap \mathbf{D}(R,\varphi(T_2))| - |\mathbf{A}(T_2,R)\cap \mathbf{D}(R,\varphi(T_1))|$$

$$\leq |\mathbf{D}(T_2,R)\cap \mathbf{D}(R,\varphi(T_2))| - |\mathbf{D}(T_2,R)\cap \mathbf{D}(R,\varphi(T_1))|. \tag{10}$$

Suppose, by way of contradiction, that $\varphi(T_1) \in [R,\varphi(T_2)[$. Then

$$\mathbf{A}(R,\varphi(T_1)) \supset \mathbf{A}(R,\varphi(T_2)) \text{ and } \mathbf{D}(R,\varphi(T_1)) \subset \mathbf{D}(R,\varphi(T_2))$$

and since $T_2 \in [R,T_1[$ we also have

$$\mathbf{A}(T_2,R) \supset \mathbf{A}(T_1,R) \text{ and } \mathbf{D}(T_2,R) \subset \mathbf{D}(T_1,R). \tag{11}$$

Because of these strict inclusions, (9) can be rewritten as

$$|\mathbf{D}(T_1,R)\cap [\mathbf{D}(R,\varphi(T_2)) \setminus \mathbf{D}(R,\varphi(T_1))]\,|$$

$$\leq |\mathbf{A}(T_1,R)\cap [\mathbf{D}(R,\varphi(T_2)) \setminus \mathbf{D}(R,\varphi(T_1))]\,|, \tag{12}$$

and (10) becomes

$$|\mathbf{A}(T_2,R)\cap [\mathbf{D}(R,\varphi(T_2)) \setminus \mathbf{D}(R,\varphi(T_1))]\,|$$

$$\leq |\mathbf{D}(T_2,R)\cap [\mathbf{D}(R,\varphi(T_2)) \setminus \mathbf{D}(R,\varphi(T_1))]\,|. \tag{13}$$

If $d(T_1,\varphi(T_1)) < d(T_1,\varphi(T_2))$ or $d(T_2,\varphi(T_2)) < d(T_2,\varphi(T_1))$, then (9) or (10) is a strict inequality. Then (12) or (13) is strict, which is impossible because (11) implies that the right side of (9) is less than or equal to the left side of (10) and the right side of (10) is less than or equal to the left side of (9).

If $d(T_1,\varphi(T_1)) = d(T_1,\varphi(T_2))$ and $d(T_2,\varphi(T_2)) = d(T_2,\varphi(T_1))$, then we must have $\varphi(T_1) \gtrapprox \varphi(T_2)$ and $\varphi(T_2) \gtrapprox \varphi(T_1)$, which is impossible as well since $T_1 \neq T_2$.

We conclude this section with three remarks.

Remark 1. Every monotonic alteration φ must be *minimal* in the following sense: for all $T \in \mathcal{T}$, $[T,\varphi(T)[\cap \mathcal{R} = \varnothing$. Indeed, suppose there exist $T \in \mathcal{T}$ and $R \in \mathcal{R}$ such that $R \in [T,\varphi(T)[$. Observe that $T \neq R$ (for otherwise $R \in [R,\varphi(R)[= [R,R[= \varnothing$). Define $T_1 = R$ and $T_2 = T$, and let $R_1 = (\varphi(T))^{-1}$, the opposite of the strict ordering $\varphi(T)$. Then $R_1 = (\varphi(T_2))^{-1}$, and since $\varphi(T_1) \in [(\varphi(T_2))^{-1},\varphi(T_2)]$, we have $\varphi(T_1) \in [R_1,\varphi(T_2)]$. Moreover, $\varphi(T_1) \neq \varphi(T_2)$ (otherwise we would have $\varphi(R) = \varphi(T)$, hence $R = \varphi(T)$, contradicting the assumption that $R \in [T,\varphi(T)[$). Therefore,

$$\varphi(T_1) \in [R_1,\varphi(T_2)[\,. \tag{14}$$

Moreover, it is easily verified that $R \in [T,\varphi(T)[$ implies $T \in [(\varphi(T))^{-1},R]$, hence $T_2 \in [R_1,T_1]$. Since $T_2 = T \neq R = T_1$, we obtain

$$T_2 \in [R_1,T_1[\,. \tag{15}$$

Statements (14) and (15) together imply that φ is not monotonic.

Remark 2. Monotonic majority-alteration rules are efficient. To see why, fix a monotonic alteration φ and suppose, by way of contradiction, that there exist $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$ and $R \in \mathcal{R}$ such that $R \in \cap_{i \in N} [R_i, \varphi(M_{\succeq}(R_N))[$. Then

$$(\cup_{i \in N} R_i) \cap \varphi(M_{\succeq}(R_N)) \subseteq R \subseteq (\cap_{i \in N} R_i) \cup \varphi(M_{\succeq}(R_N))$$

which implies

$$M_{\succeq}(R_N) \cap \varphi(M_{\succeq}(R_N)) \subseteq R \subseteq M_{\succeq}(R_N) \cup \varphi(M_{\succeq}(R_N)).$$

Because $R \neq \varphi(M_{\succeq}(R_N))$, this yields

$$R \in [M_{\succeq}(R_N), \varphi(M_{\succeq}(R_N))[,$$

contradicting the minimality of φ proved in Remark 1.

Remark 3. A weakness of the lexicographic alterations of Example 4 is that the aggregation rules they generate may violate binary unanimity. The example given earlier where

$$T = \{(a, b), (a, c), (b, c), (c, d), (d, a), (d, b)\}$$

and

$$\{b, c\} \succsim \{c, d\} \succsim \{a, d\} \succsim \{a, b\} \succsim \{a, c\} \succsim \{b, d\}$$

illustrates this point. Indeed, $T = M(R_N)$ for $N = \{1, 2, 3, 4, 5\}$ and the profile defined by $R_1 = abcd$, $R_2 = R_3 = dabc$ and $R_4 = R_5 = cdab$. The strict ordering selected at R_N by the aggregation rule based on the $\succsim$-lexicographic alteration is $bcda$. This violates binary unanimity since $(a, b) \in \cap_{i \in N} R_i$.

By contrast, the aggregation rules based on Slater alterations in Example 5 satisfy binary unanimity. This is the case because a Slater alteration of a relation $T \in \mathcal{T}$ cannot rank b above $a \neq b$ if a *covers* b in the sense of Miller (1980), i.e., $(a, b) \in T$ and $(a, c) \in T$ whenever $(b, c) \in T$.

5 Status-quo rules

We now turn to rules that use more information than the majority relation generated by the profile of individual preferences. In this section, we study a class of rules that aggregate individual preferences by Pareto-improving upon a given strict ordering which serves as a status-quo solution. Given an arbitrary $R^0 \in \mathcal{R}$, Guilbaud and Rosenstiehl (1963) prove that $(\mathcal{R}, \mathbf{R}^0)$ is a lattice (recall that the strict quasiordering $\mathbf{R}^0$ is the prudent extension of R^0). In particular, every collection $\{R^1, \ldots, R^T\} \subseteq \mathcal{R}$ has a unique minimal common upper bound, i.e., a strict ordering R such that

(i) $R \mathbf{R}^0 R^t$ for $t = 1, \ldots, T$

and

$$\text{(ii)} \ \left[R'\mathbf{R}^0 R^t \text{ for } t = 1, \ldots, T\right] \ \Rightarrow \ R'\mathbf{R}^0 R.$$

We denote this unique strict ordering by $\vee_{t=1}^{T} R^t$. When we need to emphasize the dependence on the reference ordering R^0, we write $\vee_{t=1}^{T}(R^0)R^t$ instead of $\vee_{t=1}^{T} R^t$.

The graph in Figure 1 illustrates this lattice structure for the case where $A = \{a, b, c, d\}$ and $R^0 = bdac$. Two strict orderings are directly connected by an edge in this graph if and only if they differ on a single pair of alternatives. It can be seen that $R'' \in [R, R']$ if and only if R'' lies on some shortest path linking R and R'. In particular, $R'\mathbf{R}^0 R$ if and only if R' lies on a shortest path from R^0 to R. The unique maximal element of $\mathbf{R}^0$ in $\mathcal{R}$ is $R^0 = bdac$ and its unique minimal element is the opposite ordering *cadb*. If $R^1 = bcad$ and $R^2 = acdb$, then $[R^0, R^1]$ consists of the strict orderings identified by a bold dot and $[R^0, R^2]$ consists of those identified by a circle. In the figure, $R^1 \vee R^2 = bacd$, the unique minimal element of $\mathbf{R}^0$ in $[R^0, R^1] \cap [R^0, R^2]$.

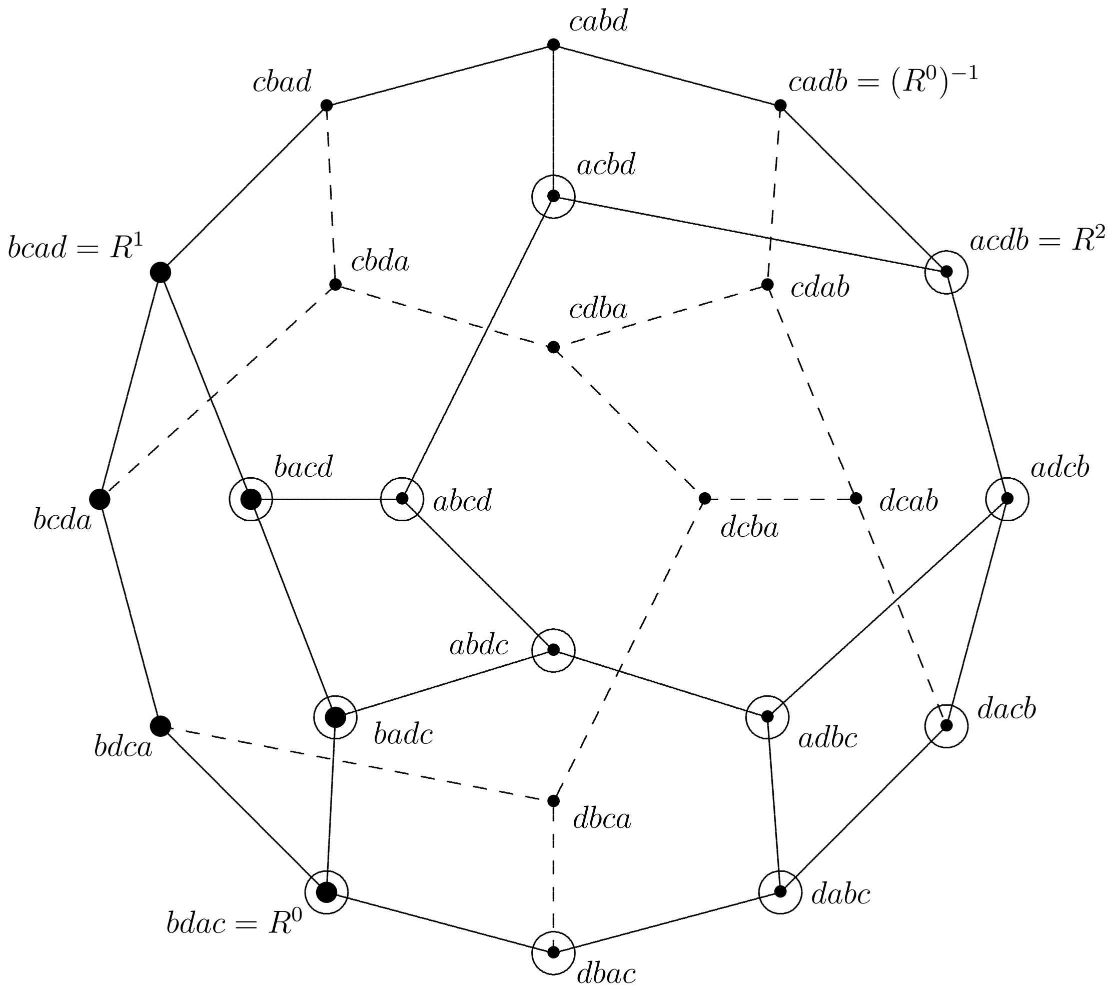

Figure 1: The lattice $(\mathcal{R}, \mathbf{R}^0)$ for $A = \{a, b, c, d\}$ and $R^0 = bdac$

We can now introduce the *status-quo* aggregation rules.

Definition 3 *An aggregation rule f is a status-quo rule if there exists $R^0 \in \mathcal{R}$ such that, for all $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$, $f(R_N) = \vee_{i\in N}(R^0)R_i$.*

Observe that, by definition, a status-quo rule is a strict aggregation rule.

Our next result establishes that the status-quo rules are strategy-proof.

Proposition 2 *Every status-quo rule is strategy-proof.*

Proof. Fix $R^0 \in \mathcal{R}$ and let f be the status-quo rule based on R^0, i.e., $f(R_N) = \vee_{i\in N}(R^0)R_i$ for all $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$. To simplify notation, we drop the reference to R^0 in the rest of the proof.

We begin with an elementary observation. For all $R, R^1, R^2 \in \mathcal{R}$,

$$R \in \left[R^1, R^2\right] \;\Rightarrow\; \left[R, R^1\right] \cap \left[R, R^2\right] = \{R\}\,. \tag{16}$$

To see this, suppose there exists $R' \in [R, R^1] \cap [R, R^2]$ such that $R' \neq R$. Then there exist distinct alternatives a, b such that aRb and $bR'a$. Since $R' \in [R, R^1] \cap [R, R^2]$, we must have bR^1a and bR^2a, hence $R \notin [R^1, R^2]$. This contradiction establishes (16).

Now suppose, contrary to what Proposition 2 claims, that there exist $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, $i \in N$ and $R'_i \in \mathcal{R}$ such that

$$(\vee_{j\in N\setminus i}R_j) \vee R'_i \in [R_i, \vee_{j\in N}R_j[\,. \tag{17}$$

By definition, $(\vee_{j\in N\setminus i}R_j) \vee R'_i \in \left[R^0, \vee_{j\in N\setminus i}R_j\right]$. On the other hand, (17) and $\vee_{j\in N}R_j \in [R^0, R_i]$ together imply $(\vee_{j\in N\setminus i}R_j) \vee R'_i \in [R^0, R_i]$ thanks to the transitivity property (2). Thus $(\vee_{j\in N\setminus i}R_j) \vee R'_i \in [R^0, R_i] \cap \left[R^0, \vee_{j\in N\setminus i}R_j\right]$. Therefore, by definition of $\vee_{j\in N}R_j$, we have $\vee_{j\in N}R_j \succsim (\vee_{j\in N\setminus i}R_j) \vee R'_i$, i.e.,

$$(\vee_{j\in N\setminus i}R_j) \vee R'_i \in \left[R^0, \vee_{j\in N}R_j\right]. \tag{18}$$

From (17) and (18) we get $[R_i, \vee_{j\in N}R_j[\cap [R^0, \vee_{j\in N}R_j] \neq \varnothing$. But since $\vee_{j\in N}R_j \in [R^0, R_i]$, (16) implies that $[R_i, \vee_{j\in N}R_j[\cap [R^0, \vee_{j\in N}R_j] = \varnothing$, a contradiction. ■

If the status-quo ordering R^0 in Definition 3 is allowed to depend on the population N, the resulting aggregation rules are also strategy-proof; this is evident because the proof of Proposition 2 utilizes a fixed N.

Status-quo rules can be characterized by the properties of efficiency and population monotonicity. We call a rule population-monotonic if the departure of a subset of agents induces a new social ordering that all the remaining agents find at least as good as the original ordering (according to the prudent extension of their preferences over alternatives). Since these prudent extensions are incomplete relations, population monotonicity is a strong requirement.

Population monotonicity. For all $N, N' \in \mathcal{N}$ such that $N' \subseteq N$ and for all $R_N \in \mathcal{R}^N$,

$$f(R_{N'}) \in \cap_{i\in N'} \left[R_i, f(R_N)\right].$$

Proposition 3 *An aggregation rule f is strict, efficient and population-monotonic if and only if f is a status-quo rule.*

Proof. We begin by noting the following property of the betweenness relation. For all $R^0, R, R', R'' \in \mathcal{R}$,

$$\left(R' \in \left[R, R^0\right] \text{ and } R'' \in \left[R', R^0\right]\right) \Rightarrow R' \in [R, R''] . \tag{19}$$

To prove (19), let $R' \in [R, R^0]$ and $R'' \in [R', R^0]$, and suppose $R' \notin [R, R'']$. Then there exist $a, b \in A$ such that $(a, b) \in R \cap R''$ and $(b, a) \in R'$. Since $R' \in [R, R^0]$, we must have $(b, a) \in R^0$, contradicting the assumption that $R'' \in [R', R^0]$.

Step 1. We prove that every status-quo rule is efficient and population-monotonic.

Fix $R^0 \in \mathcal{R}$ and consider the status-quo rule $f(R_N) = \vee_{i\in N}(R^0)R_i$. In what follows, we write $\vee_{i\in N}R_i$ instead of $\vee_{i\in N}(R^0)R_i$.

To establish efficiency, let $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$, and suppose, by way of contradiction, that there exists some $R \in \mathcal{R}$ such that

$$R \in \cap_{i\in N} \left[R_i, \vee_{j\in N}R_j\right[. \tag{20}$$

By definition,

$$\vee_{j\in N}R_j \in \cap_{j\in N} \left[R^0, R_j\right] . \tag{21}$$

By the transitivity property (2), (20) and (21) together imply $R \in \cap_{i\in N} [R^0, R_i]$. Hence by definition of $\vee_{j\in N}R_j$,

$$R \in \left[R^0, \vee_{j\in N}R_j\right] . \tag{22}$$

But (20), (21) and (22) are incompatible. Indeed, (20) implies that there exist distinct $a, b \in A$ such that $(a, b) \in R$ and $(b, a) \in \vee_{j\in N}R_j$. Then (22) implies $(a, b) \in R^0$, hence by (21) $(b, a) \in \cap_{j\in N}R_j$. Now (20) implies $(b, a) \in R$, a contradiction.

To prove population monotonicity, let $N, N' \in \mathcal{N}$ be such that $N' \subseteq N$, let $R_N \in \mathcal{R}^N$ and let $j \in N'$. Since $\vee_{i\in N}R_i \in \cap_{i\in N} [R^0, R_i] \subseteq \cap_{i\in N'} [R^0, R_i]$, the definition of $\vee_{i\in N'}R_i$ implies

$$\vee_{i\in N}R_i \in \left[R^0, \vee_{i\in N'}R_i\right] .$$

By definition,

$$\vee_{i\in N'}R_i \in \left[R^0, R_j\right] ,$$

hence by property (19)

$$\vee_{i\in N'}R_i \in [\vee_{i\in N}R_i, R_j] ,$$

as desired.

Step 2. Let f be a strict, efficient and population-monotonic aggregation rule. We show that f is a status-quo rule.

Recall that a profile R_N is diversified if for every $R \in \mathcal{R}$ there exists some agent $i \in N$ such that $R_i = R$.

Step 2.1. For any two diversified profiles R_N and $R'_{N'}$, we have $f(R_N) = f(R'_{N'})$.

To prove this claim, let R_N and $R'_{N'}$ be two diversified profiles. Let $N^0 \in \mathcal{N}$ be a society of size $|N^0| \geq |\mathcal{R}|$ such that $N^0 \cap N = N^0 \cap N' = \varnothing$. Let $R^0_{N^0}$ be a diversified profile for N^0. Consider the profile $(R_N, R^0_{N^0})$ for society $N \cup N^0$ and the profile $(R'_{N'}, R^0_{N^0})$ for society $N' \cup N^0$. Because R_N and $R'_{N'}$ are diversified profiles, population monotonicity implies

$$f(R_N) = f(R_N, R^0_{N^0}) \text{ and } f(R'_{N'}) = f(R'_{N'}, R^0_{N^0}).$$

Because $R^0_{N^0}$ is diversified, population monotonicity also implies

$$f(R^0_{N^0}) = f(R_N, R^0_{N^0}) \text{ and } f(R^0_{N^0}) = f(R'_{N'}, R^0_{N^0}).$$

These four equalities together imply that $f(R_N) = f(R'_{N'})$.

Step 2.2. Step 2.1 implies that there exists some $R^* \in \mathcal{R}$ such that $f(R_N) = R^*$ for every diversified profile R_N. We show that $f(R_N) = \vee_{i\in N}(R^*)R_i$ for all $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$.

Let $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$. Let $N^0 \in \mathcal{N}$ be such that $|N^0| \geq |\mathcal{R}|$ and $N^0 \cap N = \varnothing$. Let $R^0_{N^0}$ be a diversified profile for N^0. By Step 2.1, $f(R_N, R^0_{N^0}) = R^*$. By population monotonicity, $f(R_N) \in \cap_{i\in N}\left[R_i, f(R_N, R^0_{N^0})\right] = \cap_{i\in N}\left[R_i, R^*\right]$. By definition of $\vee_{i\in N}(R^*)R_i$,

$$\vee_{i\in N}(R^*)R_i \in \cap_{i\in N}\left[R_i, R^*\right] \tag{23}$$

and

$$f(R_N) \in \left[\vee_{i\in N}(R^*)R_i, R^*\right]. \tag{24}$$

Applying property (19) to statements (23) and (24) yields

$$\vee_{i\in N}(R^*)R_i \in \cap_{i\in N}\left[R_i, f(R_N)\right].$$

If $f(R_N) \neq \vee_{i\in N}(R^*)R_i$, then $\vee_{i\in N}(R^*)R_i \in \cap_{i\in N}\left[R_i, f(R_N)\right[$, contradicting the assumption that f is efficient. Thus $f = \vee_{i\in N}(R^*)R_i$. ■

Proposition 3 is in the spirit of a number of earlier results showing that, in pure public decision models, efficiency and population monotonicity generally lead to choice procedures based on a status-quo; see, e.g., Thomson (1993) and Gordon (2007).

A drawback of the status-quo rules is that they violate binary unanimity. For instance, if $A = \{a, b, c\}$, $R^0 = abc$, $R_1 = bca$ and $R_2 = cab$, then $\vee_{i\in\{1,2\}}(R^0)R_i = R^0$. We have $(c, a) \in R_1 \cap R_2$, yet $(c, a) \notin R^0$.

Moreover, status-quo rules fail to be even weakly neutral. At a balanced profile R_N where for each $R \in \mathcal{R}$ there exists one and only one agent $i \in N$ such that $R_i = R$, social indifference is the only appealing ordering and should be recommended. Yet the status-quo rule based on the strict ordering R^0 chooses $f(R_N) = \vee_{i\in N}(R^0)R_i = R^0$.

Generalizing the status-quo rules to allow the reference preference to be a weak ordering appears to be difficult. If $R^0 \in \widetilde{\mathcal{R}}$, it is unclear whether every collection $\left\{R^1, \ldots, R^T\right\} \subseteq \widetilde{\mathcal{R}}$ has a unique minimal common upper bound with respect to the quasi-ordering $\mathbf{R}^0$. A simple case, however, emerges when the reference ordering is the universal indifference relation $R^0 = A \times A$. It is easy to show that every collection $\left\{R^1, \ldots, R^T\right\} \subseteq \widetilde{\mathcal{R}}$ has a unique minimal common upper bound of the collection $\left\{R^1, \ldots, R^T\right\} \subseteq \widetilde{\mathcal{R}}$ with respect to $\mathbf{R}^0$. This unique ordering is $\overline{\cup_{t=1}^T R^t}$, the transitive closure of $\cup_{t=1}^T R^t$. The corresponding aggregation rule is strategy-proof. We give a direct proof below.

Proposition 4 *The aggregation rule* $f(R_N) = \overline{\cup_{i \in N} R_i}$ *is strategy-proof.*

Proof. Suppose, contrary to the claim, that there exist $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, an agent in N (without loss of generality, let this be agent 1) and $R_1' \in \mathcal{R}$ such that $\overline{R_1' \cup \left(\cup_{i \in N \setminus 1} R_i\right)} \in \left[R_1, \overline{\cup_{i \in N} R_i}\right[$. Then

$$R_1 = R_1 \cap \left(\overline{\cup_{i \in N} R_i}\right) \subseteq \overline{R_1' \cup \left(\cup_{i \in N \setminus 1} R_i\right)} \subseteq R_1 \cup \left(\overline{\cup_{i \in N} R_i}\right) = \overline{\cup_{i \in N} R_i}.$$

Since $\overline{R_1' \cup \left(\cup_{i \in N \setminus 1} R_i\right)} \neq \overline{\cup_{i \in N} R_i}$, we conclude that

$$R_1 \subseteq \overline{R_1' \cup \left(\cup_{i \in N \setminus 1} R_i\right)} \subset \overline{\cup_{i \in N} R_i}. \tag{25}$$

Choose a pair $(a, b) \in \left(\overline{\cup_{i \in N} R_i}\right) \setminus \left(\overline{R_1' \cup \left(\cup_{i \in N \setminus 1} R_i\right)}\right)$. Then there are alternatives $x_0 = a, x_1, \ldots, x_T = b$ such that

$$(x_t, x_{t+1}) \in \cup_{i \in N} R_i \text{ for } t = 0, \ldots, T-1. \tag{26}$$

Since $(a, b) \notin \overline{R_1' \cup \left(\cup_{i \in N \setminus 1} R_i\right)}$, there is $t^* \in \{1, \ldots, T\}$ such that $(x_{t^*-1}, x_{t^*}) \in R_1$. Then the first inclusion in (25) implies that there are alternatives $y_0 = x_{t^*-1}, y_1, \ldots, y_K = x_{t^*}$ such that

$$(y_t, y_{t+1}) \in R_1' \cup \left(\cup_{i \in N \setminus 1} R_i\right) \text{ for } t = 0, \ldots, K-1. \tag{27}$$

Let

$$\begin{aligned} z_0 = x_0 = a, \ldots, z_{t^*-1} = x_{t^*-1}, \\ z_{t^*} = y_0, \ldots, z_{t^*+K} = y_K = x_{t^*+1}, \\ z_{t^*+K+1} = x_{t^*+2}, \ldots, z_{T+K-1} = x_T = b. \end{aligned}$$

From (26) and (27),

$$(z_t, z_{t+1}) \in R_1' \cup \left(\cup_{i \in N \setminus 1} R_i\right) \text{ for } t = 0, \ldots, T+K-2,$$

that is, $(a, b) \in \overline{R_1' \cup \left(\cup_{i \in N \setminus 1} R_i\right)}$, a contradiction. ■

It is straightforward to verify that the rule $f(R_N) = \overline{\cup_{i \in N} R_i}$ is efficient (in the sense of our definition). It is also anonymous and neutral (according to the obvious definitions) but fails to satisfy binary unanimity.

6 The Condorcet-Kemeny rule

A well-known procedure, suggested by Condorcet, formally defined by Kemeny (1959), and axiomatized by Young and Levenglick (1978), consists in choosing a strict ordering that minimizes the sum of the Kemeny distances to the individual preference orderings. We consider a variant of this procedure where ties are broken according to an exogenous strict ordering over the set of strict orderings. In the following definition of the *Condorcet-Kemeny* rules, the Kemeny distance d is defined as in Section 4, as are the agreement and disagreement sets $\mathbf{A}$ and $\mathbf{D}$.

Definition 4 *Let* $\succsim$ *be a strict ordering on* $\mathcal{R}$. *For* $N \in \mathcal{N}$ *and* $R_N \in \mathcal{R}^N$, *let*

$$\mathcal{R}^*(R_N) = \{R \in \mathcal{R} \mid \sum_{i\in N} d(R, R_i) \leq \sum_{i\in N} d(R', R_i) \text{ for all } R' \in \mathcal{R}\}.$$

The $\succsim$*-Condorcet-Kemeny rule is the aggregation rule* f *which assigns to each profile* $R_N \in \cup_{N\in\mathcal{N}}\mathcal{R}^N$ *the strict ordering ranked first in* $\mathcal{R}^*(R_N)$ *according to* $\succsim$.

As proven in the following proposition, the Condorcet-Kemeny rules are strategy-proof.

Proposition 5 *For every strict ordering* $\succsim$ *on* $\mathcal{R}$, *the* $\succsim$*-Condorcet-Kemeny rule is strategy-proof.*

Proof. Let $\succsim$ be a strict ordering on $\mathcal{R}$ and let f be the $\succsim$-Condorcet-Kemeny rule. Let $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, $i \in N$ and $R'_i \in \mathcal{R}$. Since $R_{N\setminus i}$ is fixed throughout the proof, we slightly abuse notation and write $f(R_i)$ rather than $f(R_i, R_{N\setminus i})$ and $f(R'_i)$ instead of $f(R'_i, R_{N\setminus i})$. We must show that $f(R'_i) \notin [R_i, f(R_i)[$.

By definition of $f(R_i)$,

$$\sum_{j\in N\setminus i} d(f(R_i), R_j) + d(f(R_i), R_i) \leq \sum_{j\in N\setminus i} d(f(R'_i), R_j) + d(f(R'_i), R_i). \tag{28}$$

Applying observation (7) with $T = f(R_i)$, $T' = R_i$ and $T'' = R'_i$, we obtain

$$\begin{aligned} d(f(R_i), R'_i) &= d(f(R_i), R_i) + |\mathbf{A}(f(R_i), R_i) \cap \mathbf{D}(R_i, R'_i)| \\ &\quad -|\mathbf{D}(f(R_i), R_i) \cap \mathbf{D}(R_i, R'_i)|. \end{aligned}$$

Applying (7) with $T = f(R'_i)$, $T' = R_i$ and $T'' = R'_i$, it also follows that

$$\begin{aligned} d(f(R'_i), R'_i) &= d(f(R'_i), R_i) + |\mathbf{A}(f(R'_i), R_i) \cap \mathbf{D}(R_i, R'_i)| \\ &\quad -|\mathbf{D}(f(R'_i), R_i) \cap \mathbf{D}(R_i, R'_i)|. \end{aligned}$$

Suppose, by way of contradiction, that $f(R'_i) \in [R_i, f(R_i)[$. Then $\mathbf{A}(f(R'_i), R_i) \supseteq \mathbf{A}(f(R_i), R_i)$ and $\mathbf{D}(f(R'_i), R_i) \subseteq \mathbf{D}(f(R_i), R_i)$. It therefore follows from the two statements just derived that

$$d(f(R_i), R'_i) - d(f(R_i), R_i) \leq d(f(R'_i), R'_i) - d(f(R'_i), R_i). \tag{29}$$

Adding inequalities (28) and (29) yields

$$\sum_{j\in N\setminus i} d(f(R_i), R_j) + d(f(R_i), R'_i) \leq \sum_{j\in N\setminus i} d(f(R'_i), R_j) + d(f(R'_i), R'_i). \tag{30}$$

By definition of $f(R'_i)$, (30) must be an equality, that is,

$$\sum_{j\in N\setminus i} d(f(R_i), R_j) + d(f(R_i), R'_i) = \sum_{j\in N\setminus i} d(f(R'_i), R_j) + d(f(R'_i), R'_i), \tag{31}$$

and $f(R'_i) \precsim f(R_i)$. Subtracting (29) from (31) yields the weak inequality opposite to (28). Therefore (28) must in fact be an equality, hence by definition of $f(R_i)$, we must have $f(R_i) \precsim f(R'_i)$, contradicting $f(R'_i) \precsim f(R_i)$ since $f(R'_i) \neq f(R_i)$. ■

We conclude with two remarks.

Remark 4. Every rule f minimizing the sum of the distances to the individual preference orderings is efficient. To show this, suppose $R \in \cap_{i \in N} [R_i, f(R_N)[$. Then $d(R, R_i) < d(f(R_N), R_i)$ for all $i \in N$, hence $\sum_{i \in N} d(R, R_i) < \sum_{i \in N} d(f(R_N), R_i)$, a contradiction.

Remark 5. Condorcet-Kemeny rules can be generalized. For each $N \in \mathcal{N}$ and $i \in N$, choose an arbitrary function $\delta_{N,i}\colon A \times A \to \mathbb{R}_{++}$ and define, for all $R, R' \in \mathcal{R}$,

$$d_{N,i}(R, R') = \sum_{(a,b) \in \mathbf{D}(R,R')} \delta_{N,i}(a, b).$$

For each $R_N \in \mathcal{R}^N$, let

$$\mathcal{R}^\delta(R_N) = \{R \in \mathcal{R} \mid \sum_{i \in N} d_{N,i}(R, R_i) \leq \sum_{i \in N} d_{N,i}(R', R_i) \text{ for all } R' \in \mathcal{R}\}.$$

Given a strict ordering $\precsim$ on $\mathcal{R}$, choose for each profile R_N the strict ordering that is ranked first in $\mathcal{R}^\delta(R_N)$ according to $\precsim$. The proof of Proposition 5 can be adapted to show that this rule is strategy-proof.

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