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Reference groups and individual deprivation*

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Abstract. We provide an axiomatization of Yitzhaki's index of individual deprivation. Our result differs from an earlier characterization due to Ebert and Moyes in the way the reference group of an individual is represented in the model. Ebert and Moyes require the index to be defined for all logically possible reference groups, whereas we employ the standard definition of the reference group as the set of all agents in a society. As a consequence of this modification, some of the axioms used by Ebert and Moyes can no longer be applied and we provide alternative formulations. *Journal of Economic Literature* Classification No.: D63.

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1 Introduction

The measurement of deprivation has been an important topic of investigation in the social sciences at least since Runciman's (1966) contribution. Yitzhaki (1979) proposes an index of individual deprivation that is closely linked to the Gini index of inequality. According to the Yitzhaki index, the deprivation suffered by an individual is the aggregate income shortfall of the individual from the incomes of all those who are richer divided by the population size.

An important modelling choice in designing an indicator of an individual's situation in a society is the definition of the individual's reference group. The term 'reference group' has been used with different interpretations in the past and, in order to avoid ambiguities, it is useful to define and clarify the terms we employ. We view the reference group as the group the members of which a person compares itself to (see Runciman, 1966, Chapter II for a detailed discussion). Yitzhaki (1979) considers a model where income is the variable relevant for the purposes of measuring deprivation and assumes that there is one reference group that applies to everyone—the entire society. This is a plausible choice if the population is homogeneous and individuals are identical but may have different incomes. For the purposes of determining the deprivation of an individual in a given income distribution, however, it is not necessarily the case that the incomes of all members of the reference group influence the value of the index. The comparison group for deprivation measurement (comparison group, for short) is the subgroup of the reference group with respect to which an individual feels deprived in a given distribution, and it is usually composed of the set of agents in the reference group whose income is higher than that of the agent under consideration. Thus, we make a distinction between the reference group and the comparison group of an individual. The reference group includes all agents the individual compares itself to in general (and, thus, not only when considering matters of deprivation), whereas the comparison group is the subset of this set containing those who are richer. Note that the reference group is defined independently of a particular income distribution but, once the reference group is defined, the comparison group relevant for measuring deprivation is determined by the distribution and varies from one distribution to another. We follow Yitzhaki (1979) and define the reference group of an individual as the set of all agents. That the comparison group consists of those members of the reference group who are richer than the individual under consideration is imposed as one of the axioms.

Ebert and Moyes (2000) consider a more general notion of a reference group than

Yitzhaki (1979). They assume that any subset of the population may be a reference group (in their formulation, the individual itself is not a member of the reference group but this is merely a choice of convention). As a consequence, they characterize a generalization of the Yitzhaki index that is defined on a more general domain—it provides an index value not only for every income distribution but for every combination of an income distribution and a reference group. The reference group is allowed to vary independently of the distribution. In their framework, the comparison group relevant for deprivation considerations can be any subset of the set of those with higher incomes.

While this alternative setup provides more generality, it also endows the axioms used in Ebert and Moyes' (2000) characterization with a scope that may be considered too large in some circumstances. In general, especially when combined with the assumption that individuals are identical but may have different incomes, it is difficult to argue that the income of an agent i who is richer than an individual k is relevant for k's deprivation but the income of another agent j who is richer than k (and possibly, in addition to being identical in all other respects, even has the same income as i) is not. To illustrate this issue, note that, for example, the application of Ebert and Moyes' (2000) independence axiom requires that a given individual may be the sole member of the reference group, no matter what the underlying distribution might be. Thus, the domain assumption of Ebert and Moyes (2000) could be considered too permissive in some applications, particularly if we consider the traditional definition of the Yitzhaki index. Similarly, the additive-decomposition axiom employed by Ebert and Moyes (2000) demands that the deprivation of an individual for a given distribution and a given reference group is the sum of the levels of deprivation for the same distribution and two disjoint subgroups of the original reference group. Again, this construction cannot be applied under the standard definition employing a fixed reference group.

The purpose of this paper is to complement the approach of Ebert and Moyes (2000) by providing a characterization of the Yitzhaki index in the standard framework where the reference group is fixed and given by the entire society. Thus, the characterization result of Ebert and Moyes (2000) does not apply because some of their axioms require independent variations in the reference group. We employ those of the axioms of Ebert and Moyes (2000) that can be translated into the Yitzhaki model. It turns out that an analogue of the above-mentioned independence axiom is not needed for our characterization result. In addition, we can dispense with the anonymity axiom they use because it is implied by a natural modification of a normalization condition which, together with the remaining axioms, implies that the index is anonymous.

The most fundamental change required in moving from the framework considered by Ebert and Moyes (2000) to that of Yitzhaki (1979) is a reformulation of the additive-decomposition axiom. We define an axiom that is analogous in spirit and, at the same time, can be applied in the model where the comparison group for the measurement of individual deprivation is given by the entire set of individuals who are richer in a distribution.

2 Basic definitions

We use $\mathbb{N}$ to denote the set of all positive integers and $\mathbb{R}$ ($\mathbb{R}_{++}$) is the set of all (all positive) real numbers. For $n \in \mathbb{N}$, $\mathbb{R}^n_+$ is the set of n-dimensional vectors with non-negative components and $\mathbf{1}_n$ is the vector consisting of n ones. There is a fixed set $N = \{1, \ldots, n\}$ of $n \geq 2$ individuals and their incomes are recorded in an income distribution $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$. The restriction to non-negative incomes is not crucial; allowing for negative incomes, as Ebert and Moyes (2000) do, would not change our results. For $y, z \in \mathbb{R}^n_+$ and a subset M of N, the vector $x = (y|_M, z|_{N \setminus M})$ is defined as follows. For all $i \in N$,

$$x_i = \begin{cases} y_i & \text{if } i \in M, \\ z_i & \text{if } i \in N \setminus M. \end{cases}$$

An individual measure of deprivation for individual $k \in N$ is a function $D_k\colon \mathbb{R}^n_+ \to \mathbb{R}$. Letting $B_k(y) = \{j \in N \mid y_j > y_k\}$ denote the set of individuals with a higher income than k, Yitzhaki's (1979) index of individual deprivation D^Y_k is defined as follows. For all $y \in \mathbb{R}^n_+$,

$$D^Y_k(y) = \frac{1}{n} \sum_{j \in B_k(y)} (y_j - y_k).$$

The interpretation of Yitzhaki's index is straightforward. It calculates individual k's deprivation as the aggregate income shortfall from the incomes of all those who are richer than k divided by the population size. Thus, the set $B_k(y)$ forms the group of agents with respect to whom the individual feels deprived—the comparison group.

3 Axioms

We employ variants of some of the axioms used by Ebert and Moyes (2000), suitably formulated for our framework where the reference group is the entire society. First, we note that their anonymity axiom is not required because we use a slightly modified version of normalization that, together with the remaining axioms, implies anonymity. Moreover, their independence axiom is not needed in the traditional Yitzhaki framework either. This is convenient because independence cannot be formulated in our setting: the axiom requires that an individual can be chosen as the only individual in the reference group, independently of the income distribution. The only remaining axiom that cannot be adapted in a straightforward manner to the Yitzhaki framework is additive decomposition. However, it is possible to define a suitable version in our setting and we will return to it in detail after defining the other axioms. We do not provide a detailed discussion of them because they are motivated by the same considerations as the versions of Ebert and Moyes (2000).

The first axiom is a focus axiom, requiring that the income levels of those who are at or below k's income level are irrelevant. This property parallels Sen's (1976) focus axiom for poverty measures and it formalizes the idea that the comparison group consists of all members of the reference group who are richer than k.

Focus. For all $y, z \in \mathbb{R}^n_+$ such that $B_k(y) = B_k(z)$ and $y_j = z_j$ for all $j \in B_k(y) \cup \{k\}$,

$$D_k(y) = D_k(z).$$

Translation invariance requires that the index is absolute, that is, invariant with respect to equal absolute changes in all incomes.

Translation invariance. For all $y \in \mathbb{R}^n_+$ and for all $\delta \in \mathbb{R}$ such that $(y + \delta \mathbf{1}_n) \in \mathbb{R}^n_+$,

$$D_k(y + \delta \mathbf{1}_n) = D_k(y).$$

Linear homogeneity demands that an equal proportional change in all incomes changes individual deprivation in the same proportion.

Linear homogeneity. For all $y \in \mathbb{R}^n_+$ and for all $\lambda \in \mathbb{R}_{++}$,

$$D_k(\lambda y) = \lambda D_k(y).$$

Normalization requires that a specific income distribution has a degree of individual deprivation of $1/n$. This axiom could be replaced by alternative normalizations. What is crucial is that a positive level of deprivation is achieved for some distribution; otherwise we cannot rule out the degenerate measure where individual deprivation is equal to zero for all distributions. Because we do not specify the identity of the individual who has an income of one in the axiom statement, we do not need an anonymity requirement in our result.

Normalization. For all $y \in \mathbb{R}^n_+$ such that there exists $j \in N \setminus \{k\}$ with $y_j = 1$ and $y_i = 0$ for all $i \in N \setminus \{j\}$,

$$D_k(y) = 1/n.$$

Additive decomposition is a separability property. The version of Ebert and Moyes (2000) postulates that, for any income distribution, deprivation for that distribution and any reference group is equal to the sum of the levels of deprivation that result if the reference group is divided into two subgroups, keeping the income distribution unchanged (the case where one of the subgroups is empty is covered by the axiom). Clearly, if the reference group is fixed and given by the entire society, the axiom does not apply except in degenerate cases. However, a natural analogue is obtained by considering distributions where the individuals in each of two subgroups of the comparison group have the same income as k (and, therefore, do not contribute to k's deprivation) and then apply the additivity requirement using these distributions.

Additive decomposition. For all $y \in \mathbb{R}^n_+$ and for all $B^1, B^2 \subseteq B_k(y)$ such that $B^1 \cap B^2 = \emptyset$ and $B^1 \cup B^2 = B_k(y)$,

$$D_k(y) = D_k(y_k \mathbf{1}_n|_{B^1}, y|_{N \setminus B^1}) + D_k(y_k \mathbf{1}_n|_{B^2}, y|_{N \setminus B^2}).$$

4 A characterization of the Yitzhaki index

The axioms in the previous section (which are independent, as can be seen easily by suitably adapting the relevant examples used in the independence proof of Ebert and Moyes, 2000) characterize the Yitzhaki index D^Y_k.

Theorem 1. *An individual deprivation index D_k satisfies focus, translation invariance, linear homogeneity, normalization and additive decomposition if and only if $D_k = D^Y_k$.*

Proof. That D^Y_k satisfies the axioms of the theorem statement is straightforward to verify. Conversely, suppose D_k is an individual deprivation index satisfying the axioms.

Consider first distributions of the form $(y_j \mathbf{1}_n|_{\{j\}}, y_k \mathbf{1}_n|_{N \setminus \{j\}})$ where $y_j > y_k$. Thus, there exists an individual $j \in N$ such that $B(y_j \mathbf{1}_n|_{\{j\}}, y_k \mathbf{1}_n|_{N \setminus \{j\}}) = \{j\}$ and everyone other than j has the same income as agent k. Translation invariance with $\delta = -y_k$ implies

$$D_k(y_j \mathbf{1}_n|_{\{j\}}, y_k \mathbf{1}_n|_{N\setminus\{j\}}) = D_k((y_j - y_k)\mathbf{1}_n|_{\{j\}}, 0\mathbf{1}_n|_{N\setminus\{j\}}).$$

Let $f_k^j(y_j - y_k) = D_k((y_j - y_k)\mathbf{1}_n|_{\{j\}}, 0\mathbf{1}_n|_{N\setminus\{j\}})$. Linear homogeneity with $\lambda = 1/(y_j - y_k)$ implies $f_k^j(y_j - y_k) = f_k^j(1)(y_j - y_k)$. Substituting back, we obtain

$$D_k(y_j \mathbf{1}_n|_{\{j\}}, y_k \mathbf{1}_n|_{N\setminus\{j\}}) = f_k^j(1)(y_j - y_k).$$

Using normalization, it follows that $f_k^j(1) = 1/n$ and, thus,

$$D_k(y_j \mathbf{1}_n|_{\{j\}}, y_k \mathbf{1}_n|_{N\setminus\{j\}}) = \frac{1}{n}(y_j - y_k). \tag{1}$$

Now let $y \in \mathbb{R}_+^n$ be arbitrary. By the focus axiom, we can without loss of generality assume that $y_i = y_k$ for all $i \in N \setminus B_k(y)$. Focus and repeated application of additive decomposition together imply

$$D_k(y) = \sum_{j \in B_k(y)} D_k(y_j \mathbf{1}_n|_{\{j\}}, y_k \mathbf{1}_n|_{N\setminus\{j\}})$$

and, by (1), we obtain

$$D_k(y) = \frac{1}{n} \sum_{j \in B_k(y)} (y_j - y_k) = D_k^Y(y). \quad \blacksquare$$

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