Centre interuniversitaire de recherche
en économie quantitative

CIREQ

Cahier 14-2005

CARTEL STABILITY IN A DYNAMIC OLIGOPOLY

Hassan BENCHEKROUN and
Licun XUE

Le **Centre interuniversitaire de recherche en économie quantitative (CIREQ)** regroupe des chercheurs dans les domaines de l'économétrie, la théorie de la décision, la macroéconomie et marchés financiers, la microéconomie appliquée et économie expérimentale et l'économie de l'environnement et des ressources naturelles. Ils proviennent principalement des universités de Montréal, McGill et Concordia. Le CIREQ offre un milieu dynamique de recherche en économie quantitative grâce au grand nombre d'activités qu'il organise (séminaires, ateliers, colloques) et de collaborateurs qu'il reçoit chaque année.

*The **Center for Interuniversity Research in Quantitative Economics (CIREQ)** regroups researchers in the fields of econometrics, decision theory, macroeconomics and financial markets, applied microeconomics and experimental economics, and environmental and natural resources economics. They come mainly from the Université de Montréal, McGill University and Concordia University. CIREQ offers a dynamic environment of research in quantitative economics thanks to the large number of activities that it organizes (seminars, workshops, conferences) and to the visitors it receives every year.*

Cahier 14-2005

CARTEL STABILITY IN A DYNAMIC OLIGOPOLY

Hassan BENCHEKROUN and Licun XUE

CIREQ, Université de Montréal
C.P. 6128, succursale Centre-ville
Montréal (Québec) H3C 3J7
Canada

téléphone : (514) 343-6557
télécopieur : (514) 343-5831
cireq@umontreal.ca
http://www.cireq.umontreal.ca

Dépôt légal, Bibliothèque nationale du Canada, 2005, ISSN 0821-4441
Dépôt légal, Bibliothèque nationale du Québec, 2005, ISBN 2-89382-508-7

Cartel Stability in a Dynamic Oligopoly*

Hassan Benchekroun and Licun Xue†

McGill University and CIREQ, Montreal

This version: April 2005

Abstract

We study the stability of catels in a dynamic oligopoly. We use the differential game model of an oligopoly with sticky prices (Fershtman and Kamien 1987). We show that when firms use closed-loop strategies and the rate of increase of the marginal cost is "small enough", the grand coalition (i.e., when the cartel includes all firms) is stable: it is unprofitable for a firm to exit the cartel. Moreover we show that a cartel of 3 firms is stable for any positive rate of increase of the marginal cost: it is not profitable for an insider firm to exit the coalition, nor it is profitable for an outsider firm to join the coalition. When firms use open-loop strategies we show that no cartel is stable.

JEL Classification code: D43, L13, L12, C72.

Keywords: Stable Cartel, Mergers, Dynamic Oligopoly, Differential Games.

*We thank SSHRC and FQRSC for financial support. We thank Effrosyni Diamantoudi and Ngo Van Long for helpful comments. Usual disclaimers apply.

†Address of correspondence: Department of Economics, McGill University. 855 Sherbrooke Street West, Montreal, Quebec, Canada H3A 2T7. Emails: hassan.benchekroun@mcgill.ca and licun.xue@mcgill.ca. Fax: 514-398-4938.

1 Introduction

It is intuitive to expect that in an oligopolistic market, whenever possible, firms would wish to form a cartel to coordinate their productions and increase their profits. A cartel of a subset of firms that acts as a multiplant firm and competes against the other firms can intuitively be expected to achieve higher overall profits for its members. This intuition can be wrong, however. When firms compete *a la* Cournot it is well known from the merger literature[1] that the formation of a cartel by a subset of firms may reduce its members' profits. Indeed, in a static Cournot oligopoly with linear demand and constant marginal cost, a cartel (or merger) of a subset of firms is profitable only if they represent a significant share of the market prior to the formation of the cartel (see Salant, Switzer and Reynolds, 1983, SSR henceforth). In particular, if there are three or more firms in the industry, a cartel formed by two firms always decreases their total profits. This remains to be true when marginal cost is not constant as long as the cartel does not experience "large" efficiency gain, such is the case when the cost function takes the quadratic form of $cq + \frac{1}{2}q^2$ (where c is constant and q is the output). This result can be intuitively explained by the following: The cartel has an incentive to reduce its members' output relative to their production prior to the cartel. Then the outsiders react by increasing their production, which reduces the profits of the cartel members.

The above important result in static oligopoly theory does not generalize to dynamic oligopolies. In a model of dynamic oligopoly with price dynamics (Fershtman and Kamien, 1987), linear demand and quadratic cost function of the form $cq+\frac{1}{2}q^2$, Dockner and Gaunersdorfer (2001) conduct numerical simulations and obtain a very interesting result: all cartels are profitable[2]. This is in sharp contrast with the conclusions of the static framework[3].

Since all cartels are profitable it is natural to ask which cartel size is more likely to emerge. To address this question we consider a stability criterion. A cartel is stable if no insider firm has an incentive to exit the cartel and no outsider firm would wish to join the

[1] The objective of a cartel is identical to the objective of a merged firm, adopted in the merger literature (Salant, Switzer and Reynolds 1983), when a subset of firms merge. Throughout this paper, the term cartel can be substituted by the term merger.

[2] Benchekroun (2003) shows analytically that such a result generalizes to the case where the number of firms is arbitrarily large: a cartel of 2 firms is profitable even when the number of firms tends to infinity.

[3] The literature cited above considers the profitability of mergers. In this paper the outcome (i.e. the equilibrium price, quantities produced by each firm and firms' profits) of the formation of a cartel by a subset is identical to the outcome of merger of this subset of firms. The results we obtain in this paper, regarding the stability of cartels, naturally apply to the stability of the formation of mergers.

cartel. Such a stability notion was first introduced by D'Aspremont, Jacquemin, Gabsewicz, and Weymark (1983) in a price leadership model in which the dominant cartel acts as the leader and firms in the competitive fringe take price as given. Although this stability notion can be adapted to the context of static Cournot competition, it is easy to see that with three or more firms, no cartel is stable in the case of linear demand and constant marginal cost.

We study the stability of cartels in the dynamic oligopoly model with sticky prices. We consider a generalized version the framework of Fershtman and Kamien (1987) and Dockner and Gaunersdorfer (2001). They focus on the case where the coefficient of the quadratic term is one (i.e., the rate of change of the marginal cost is one). We allow for a general quadratic cost function and such a generalization turns out to have important implications in determining the stability of cartels (although all cartels remain profitable). In particular, when the coefficient of the quadratic term is sufficiently low, the grand coalition (i.e., cartel of all firms) is stable; that is, no firm can benefit from exiting the grand coalition. When the rate of change of the marginal cost is one, only coalitions of three firms are stable, regardless of the total number firms. In fact, size-three coalitions remain stable for any general quadratic cost function.

The above results hold when firms use close-loop strategies whereby a firm's strategy specifies a production rate at a given moment as a function of that moment and the level of the price (the state variable) at that moment. For comparison, we also consider open-loop strategies where a firm's strategy corresponds to a production path announced at time zero and, defined over the whole infinite time horizon. It is well known that the steady state of an open-loop equilibrium "coincides with the Cournot equilibrium of the corresponding static game" (Dockner 1992, see also Dockner et al 2000). An open-loop Nash equilibrium is in general not subgame perfect but a closed-loop Nash equilibrium is by construction subgame perfect. As a benchmark, the open-loop case allows us to isolate the impact of feedback strategies on the stability of cartels. We show, among other things, that no cartel is stable when the rate of change in marginal cost is one and when there are three or more firms, the grand coalition is never stable.

The rest of the paper is organized as follows. In section 2 we present the model and the formal definition of the stability of a cartel. In section 3 we provide the open-loop and the closed-loop Nash equilibrium after a cartel forms. In section 4 we examine the stability of cartels.

2 The model

Consider an industry with N identical firms producing an homogeneous product. The production of firm i at time t is denoted $q_i(t)$. The production cost of each firm is given by

$$cq_i(t) + \frac{\gamma}{2}[q_i(t)]^2$$

where $c \geq 0$ and $\gamma > 0$. Note that $\gamma = 1$ in Fershtman and Kamien (1987) and Dockner and Gaunersdorfer (2001).

Let $p(t)$ denote the price of the output. Due to price stickiness, the adjustment process of the price to a change in quantity is given

$$\dot{p}(t) = s\left(a - \sum_i q_i(t) - p(t)\right) \text{ with } p(0) = p_0 \geq 0 \tag{1}$$

where $a > c$ and $0 < s \leq \infty$ is a parameter that captures the speed of the adjustment of the price. (1) implies a linear inverse demand function.

The objective of firm i is to maximize the discounted sum of profits

$$J_i = \int_0^\infty \left(p(t) - c - \frac{\gamma}{2}q_i(t)\right) q_i(t)e^{-rt}dt \tag{2}$$

subject to (1) where $r > 0$ denotes the interest rate.

We consider two strategy spaces. The first set of strategies is the open-loop strategy set where one firm's strategy defines this firm's production path for the whole time horizon. The second set of strategies considered is the set of closed-loop strategies where the production of firm i at time t is allowed to depend on t and the price level at time t, $p(t)$[4].

A closed-loop (open-loop) Nash equilibrium for the pre-cartel game, i.e. the game without formation of a cartel, is defined by a vector of N closed-loop (open-loop) strategies $(\phi_1^*, .., \phi_N^*)$ such that each strategy ϕ_i^* is a best closed-loop (open-loop) response to $\Phi_{-i}^* \equiv \left(\phi_1^*, .., \phi_{i-1}^*, \phi_{i+1}^*, .., \phi_N^*\right)$.

The Cartel

We then consider the possibility of a cartel of M firms, $M \leq N$. Without loss of generality assume that in the case of a cartel of M firms, the insider firms (firms that form the cartel) are the first M firms, i.e. firm $k = 1, .., M$. If $M < N$, the outsider firms are firms j, $j = M + 1, .., N$.

[4] For a formal definition of these strategy sets see for example Fershtman and Kamien (1987), Definition 1 and Definition 3, page 1154.

The objective of a cartel of M firms is to maximize the joint discounted sum of profits of the M firms denoted J^C

$$J^C = \sum_{k=1}^{M} J^k.$$

The cartel takes the production strategies, ϕ_j with $j = M+1, .., N$, of the outsider firms as given and chooses an M-tuple vector of production strategies $(\phi_1, .., \phi_M)$ that solves

$$\max \ J^C$$

subject to (1). The problem of an outsider firm k, with $M+1 \leq k \leq N$, is to maximize J_k subject to (1) by choosing a strategy while taking the strategies of the cartel and $N-M-1$ other outsiders as given.

A closed-loop (open-loop) Nash equilibrium of the game when a cartel of M firms forms is thus an N-tuple vector of closed-loop (open-loop) production strategies $\left(\phi_1^M, .., \phi_N^M\right)$ such that

$$J^C\left(\phi_1^M, .., \phi_N^M\right) \geq J^C\left(\phi_1, .., \phi_M, \phi_{M+1}^M .., \phi_N^M\right) \text{ for all } \left(\phi_1, .., \phi_M\right)$$

and

$$J^j\left(\phi_1^M, .., \phi_j^M, .., \phi_N^M\right) \geq J^j\left(\phi_1^M, .., \phi_j, .., \phi_N^M\right) \text{ for all } \phi_j \text{ and all } j = M+1, .., N$$

A cartel of M firms is said to be profitable (for the constituent firms) if

$$J^C\left(\phi_1^M, .., \phi_N^M\right) \geq \sum_{k=1}^{M} J^k\left(\phi_1^*, .., \phi_N^*\right)$$

where $(\phi_1^*, .., \phi_N^*)$ is a closed-loop (open-loop) Nash equilibrium of the game without a cartel (when $M = 1$) and $\left(\phi_1^M, .., \phi_N^M\right)$ is a closed-loop (open-loop) Nash equilibrium when M firms form a cartel. Since we assume that the firms are symmetric we focus on the symmetric equilibrium, i.e. an equilibrium with a cartel of M firms such that

$$\phi_k = \phi_{ins} \text{ for all } k = 1, .., M$$

and

$$\phi_j = \phi_{out} \text{ for all } j = M+1, .., N$$

where ϕ_{ins} and ϕ_{out} are strategies of insiders and outsiders respectively. Given that the marginal cost of a firm is increasing with the quantity produced and that firms are symmetric, the cartel equally splits the overall production and (hence) profits among its members.

Note that by setting $M = 1$ we obtain the case of a symmetric oligopoly with N firms[5].

Stability of a Cartel

When all cartel sizes are profitable, a natural follow up question is: which of the profitable cartels are more likely to emerge? We use a stability criterion to answer this question.

The concept of stability used here is the one proposed by D'Aspremont et al (1983). It will be convenient to introduce the following notations for insiders' and outsiders' profits at the equilibrium of the game with a cartel of M firms in an N-firm oligopoly. Let

$$\Pi_{ins}(M,N) \equiv \frac{J^C\left(\phi_1^M,..,\phi_N^M\right)}{M}$$

and

$$\Pi_{out}(M,N) \equiv J^j\left(\phi_1^M,..,\phi_N^M\right) \text{ for any } j = M+1,..,N.$$

A cartel of M firms is *internally stable* if

$$\Pi_{ins}(M,N) \geq \Pi_{out}(M-1,N); \tag{3}$$

i.e., no insider of cartel has an incentive to unilaterally exit.

A cartel of M firms is *externally stable* if

$$\Pi_{out}(M,N) \geq \Pi_{ins}(M+1,N); \tag{4}$$

i.e., no outsider firm has an incentive to join the cartel of the M firms.

A cartel of M firms is *stable* if it is both internally and externally stable.

Remark 1: If a cartel of M firms is *strictly* internally stable (i.e. $\Pi_{ins}(M,N) > \Pi_{out}(M-1,N)$) then a cartel of $M-1$ is, necessarily, externally unstable.

Remark 2: In the case of the grand coalition, i.e. when $M = N$, only internal stability is relevant.

Remark 3: A cartel of two firms is profitable if and only if a cartel of size 2 is internally stable.

The objective of this paper is to determine, among the profitable cartels, the ones that are stable. To this end we must characterize the equilibrium profits of firms (insiders and outsiders) when a cartel forms.

[5] The symmetric Cournot oligopoly can also be obtained as a special case of this game with 0 insiders and N outsiders.

3 The equilibrium with a cartel

In this section, we determine the equilibrium of the game when M firms form a cartel.

3.1 The case of open-loop strategies

Proposition 1 *There exists a stable steady-state open-loop Nash equilibrium of the post-cartel game. The equilibrium production rates and the price at the steady state are given by*

$$q_{ins}^{OL} = \delta \left(1 + R\right) \tag{5}$$

$$q_{out}^{OL} = \delta \left(1 + MR\right) \tag{6}$$

and

$$p^{OL} = a - \left(N - M\right) q_{out}^{OL} - M q_{ins}^{OL} \tag{7}$$

where $R \equiv \frac{s}{r+s}$ *and* $\delta \equiv \frac{(a-c)}{1+N+R+2MR+MRN-M^2R+MR^2} > 0$.

Proof. The proof is omitted. It is a straightforward generalization of Proposition 1 in Benchekroun (2003) to arbitrary $\gamma \geq 0$. ■

The profits of a merging firm at the steady state are given by:

$$\pi_{ins}^{OL}\left(M, N\right) \equiv \left(p^{OL} - c - \frac{1}{2} q_{ins}^{OL}\right) q_{ins}^{OL}$$

that is

$$\pi_{ins}^{OL}\left(M, N\right) = \frac{1}{2}\delta^2 \left(1 + R + 2MR + 2MR^2\right)\left(1 + R\right) \tag{8}$$

It can be verified that in the limit case where the adjustment speed is infinite, i.e. when price adjusts instantaneously, the steady-state equilibrium price, quantities and profits correspond exactly to the outcome of a one shot static Cournot game. This is in line with Fershtman and Kamien (1987) and is similar to the result in Dockner (1992) where, for an adjustment cost differential game, it is shown that the steady state open-loop equilibrium "coincides with the Cournot equilibrium of the corresponding static game".

In this context a cartel is profitable only if the pre-cartel market share of the member firms is large enough. In particular, when γ tends to 0, the results of SSR emerge.

In the remainder of the paper, as in Fershtman and Kamien (1987) and Dockner and Gaunersdorfer (2001), we shall focus on the limit case where s tends to ∞. This facilitates the tractability of the equilibria we study. We then have

$$\pi_{ins}^{OL}(M,N) = \frac{1}{2}(a-c)^2 \frac{(\gamma+1)(2M\gamma+\gamma+2M+\gamma^2)}{(\gamma^2+\gamma N+\gamma+2M+M\gamma+MN-M^2)^2} \tag{9}$$

and

$$\pi_{out}^{OL}(M,N) = \frac{1}{2}a^2 \frac{(\gamma+M)(2M\gamma+\gamma+2M+\gamma^2)}{(\gamma^2+\gamma N+\gamma+2M+M\gamma+MN-M^2)^2} \tag{10}$$

Since we focus on the case where price adjust instantaneously (i.e.,$s \to \infty$) we can limit our attention to the steady state. In that case we have $\Pi_{out}(M,N) = \frac{\pi_{out}^{OL}(M,N)}{r}$ and $\Pi_{ins}(M,N) = \frac{\pi_{ins}^{OL}(M,N)}{r}$ and therefore we can substitute $\Pi_{out}(M,N)$ and $\Pi_{ins}(M,N)$ in the stability conditions (3) and (4) by $\pi_{out}^{OL}(M,N)$ and $\pi_{ins}^{OL}(M,N)$.

The results of this section will serve as a benchmark that will allow us to isolate and identify the role played by closed-loop strategies.

3.2 The case of closed-loop strategies

We shall focus on the equilibrium production strategies when there is an interior solution. Also, for simplicity we set $c = 0$. This is innocuous and it simplifies the expressions that characterize the close-loop equilibrium we study.

Proposition 2 ***Proposition 3*** *The close-loop equilibrium strategies for the insider and outsider firms are given by*

$$\phi_{ins}(p) = (1-K_c)\,p + E_c \text{ and } \phi_{out}(p) = (1-K_k)\,p + E_k$$

where (K_k, K_c) is a pair that solves the following system (S_K)

$$(S_K)\begin{cases} \frac{M}{\gamma}(1-K_c) - \frac{1}{2}\frac{M}{\gamma}(1-K_c)^2 + K_c\left(-\frac{M}{\gamma}(1-K_c) - \frac{1}{\gamma}(N-M)(1-K_k) - 1\right) = 0 \\ \frac{1}{\gamma}(1-K_k) - \frac{1}{2\gamma}(1-K_k)^2 + K_k\left(-\frac{M}{\gamma}(1-K_c) - \frac{1}{\gamma}(N-M)(1-K_k) - 1\right) = 0 \end{cases}$$

such that

$$MK_c + (N-M)K_k - 1 - N < 0. \tag{11}$$

and (E_k, E_c) is the unique solution to the following linear system (S_E)

$$(S_E)\begin{cases} K_k M \frac{E_c}{\gamma} + ((2(N-M)-1)K_k + MK_c - N - \gamma)\frac{E_k}{\gamma} = aK_k \\ (MK_c - N + (N-M)K_k - \gamma)\frac{E_c}{\gamma} + (N-M)K_c\frac{E_k}{\gamma} = aK_c. \end{cases}$$

Proof. The proof is omitted. It is a straightforward generalization of Proposition 3 in Dockner and Gaunersdorfer (2001) to the case of an arbitrary $\gamma > 0$. ■

Remark 4: Condition (11) ensures the stability of the steady-state equilibrium price.

To determine the steady-state profits of an insider firm, we need to determine the steady-state production of each firm (insider and outsider firms) and the steady-state price as a function of M and N. This requires us to determine solutions to the system (S_K). However, the system (S_K) is nonlinear and an analytical solution of (K_c, K_k) as explicit functions of N and M, when it exists, is in general impossible to obtain[6]. Dockner and Gaunersdorfer (2001) solve the system numerically for specific values of N and M while Benchekroun (2003) shows the existence of a solution to this system.

The steady-state equilibrium price is the solution to

$$a - M\phi_{ins}(p) - (N - M)\phi_{out}(p) - p = 0$$

that is

$$p^{CL} = -\frac{a - ME_c - (N - M)E_k}{MK_c - N + (N - M)K_k - 1}$$

where K_c, K_k, E_c and E_k are respectively given by (S_K) and (S_E).

The profits of an insider firm at the steady state are thus given by

$$\begin{aligned}
\pi_{ins}^{CL}(M, N) &= \left(p^{CL} - c - \frac{1}{2}\phi_{ins}\left(p^{CL}\right)\right)\phi_{ins}\left(p^{CL}\right) \\
\pi_{out}^{CL}(M, N) &= \left(p^{CL} - c - \frac{1}{2}\phi_{out}\left(p^{CL}\right)\right)\phi_{out}\left(p^{CL}\right)
\end{aligned}$$

Dockner and Gaunersdorfer (2001), numerically establish that all cartels are profitable in a 10 firm industry and that a cartel of 2 firms remains profitable when the total number of firms varies between 2 and 10. Benchekroun (2003) shows analytically that this remains true even when the total number of firms in an industry is arbitrarily large.

It can be shown that the discounted sum of profits of an insider firm and an outsider firm can be respectively written as

$$G_{ins} = \frac{1}{2}\frac{(2E_k - E_c - 2a\gamma + NE_c)E_c}{\gamma r} \tag{12}$$

and

$$G_{out} = \frac{1}{2}\frac{(E_k - 2E_c - 2a\gamma + 2NE_c)E_k}{\gamma r}. \tag{13}$$

We now turn to the main question of the paper: Are these cartels stable?

[6]For example, substitution of K_c from the second equation of the system (S_K) into the first equation yields a polynomial of degree 4 in K_k for which the roots can be determined explicitly but are too complex to offer any insight, see Dockner and Gaunersdorfer (2001).

4 Stability

4.1 The case of open-loop strategies

When firms use open-loop strategies, profitability of a cartel depends on γ. With small γ, for a cartel to be profitable, it has to represent a significant pre-cartel market share. For example, when $\gamma = 1$, a minimum market share of 69% is necessary for a cartel to be profitable. However, simulations indicate that no cartel is stable, which is consistent with the analytical results in this subsection.

Proposition 4 *The grand coalition is never stable when $N \geq 3$.*

Proof. The internal stability condition of the grand coalition can be written as

$$\pi_{ins}(N, N) - \pi_{out}(N-1, N) > 0$$

which reduces to

$$\frac{z^2 \delta(N)}{(\gamma^2 + 2N\gamma + 3N - 3)^2 (\gamma + 2N)} < 0$$

where

$$\begin{aligned}\delta(N) \equiv & \left(\frac{1}{2}N - 1\right)\gamma^3 + \left(\frac{5}{2} - \frac{11}{2}N + 2N^2\right)\gamma^2 + \\ & \left(2N^3 - 6N^2 + 3N + 1\right)\gamma + 11N + 2N^3 - \frac{9}{2} - \frac{17}{2}N^2\end{aligned}$$

We shall show that $\delta > 0$ for all $N \geq 3$. The expression for δ is a polynomial of γ. It is easy to show that the coefficient of each monomial in γ is strictly positive for $N \geq 3$. Therefore $\delta > 0$ for all $\gamma \geq 0$. Consequently, the grand coalition is never internally stable. ■

Note that the above proposition implies that in the SSR setup, where $\gamma = 0$ (i.e., the marginal cost is constant), the grand coalition is never stable.

The stability of arbitrary cartel sizes for arbitrary values of the parameter γ proves to be difficult to establish analytically. We study the stability of arbitrary cartel sizes for two specific values of the parameter $\gamma : \gamma = 1$ (the value used in Fershtman and Kamien (1987)) and $\gamma = 0$ (which gives the SSR model). In both cases we show that no cartel is stable:

Proposition 5 *When $\gamma = 1$, no cartel is stable.*

Proof. When $\gamma = 1$, the internal stability condition implies

$$(a-c)^2 \frac{\Delta(M, N)}{(-2 - N - 3M - MN + M^2)^2 (2 - 5M - MN + M^2)^2} < 0 \tag{14}$$

where

$$\begin{aligned}\Delta(M,N) &= \left(2M^4 - M^3 - 2M^2 - M\right)N^2 \\ &\quad + \left(4M - 4M^5 + 18M^4 - 6M^2 - 20M^3\right)N \\ &\quad -8 + 18M^2 + 54M^4 - 17M^5 - 77M^3 + 20M + 2M^6.\end{aligned}$$

This quadratic function of N is strictly convex (U-shaped) because

$$\left(2M^4 - M^3 - 2M^2 - M\right) = M\left((2M-1)\left(M^2-1\right) - 2\right)$$

which is strictly positive for all $M \geq 2$. Moreover we show that Δ is strictly increasing in N therefore for $M < N$. Indeed, we examine the sign of the slope of the quadratic function Δ for $M < N$

$$\begin{aligned}\frac{\partial \Delta}{\partial N} &= 2\left(2M^4 - M^3 - 2M^2 - M\right)N \\ &\quad + \left(4M - 4M^5 + 18M^4 - 6M^2 - 20M^3\right).\end{aligned}$$

By convexity of Δ with respect to N we have that $\frac{\partial \Delta}{\partial N}$ is strictly increasing in N and therefore for $M < N$ we have

$$\frac{\partial \Delta}{\partial N}(M,N) > \frac{\partial \Delta}{\partial N}(M,M) = 4M(2M+1)\left(2M^2 - 4M + 1\right) > 0.$$

Therefore, when $M < N$ we have

$$\Delta(M,N) > \Delta(M,M) = 32M^4 - 84M^3 + 22M^2 - 8 + 20M > 0 \text{ for all } M \geq 3.$$

Consequently, the condition (14) is thus never satisfied and any cartel $M \geq 3$ is internally unstable.

For $M = 2$, the internal stability condition requires $14N^2 - 16N - 64 \leq 0$. This is never satisfied for any $N \geq 3$. So a cartel of 2 firms is also internally unstable (i.e. a cartel of two firms is unprofitable). ■

We now consider the case $\gamma = 0$. Note that by studying stability for $\gamma = 0$ we determine the stability of a cartel of M firms in the SSR setup where the marginal cost is constant. Numerical simulations indicate that no cartel is stable when firms are using open-loop strategies.

Proposition 6 *Assume $\gamma = 0$ and $N \geq 3$. No cartel is stable.*

Proof. The internal stability condition $\pi_{ins}(M,N) > \pi_{out}(M-1,N)$ implies that

$$\frac{a^2}{4M+4MN-4M^2+M^3+MN^2-2M^2N} > \frac{a^2}{6N-6M-2MN+M^2+N^2+9}$$

i.e.,

$$(M-1)N^2 + \left(6M-2M^2-6\right)N + M^3 - 5M^2 - 9 + 10M < 0$$

This quadratic function of N is also strictly increasing in N. Indeed,

$$\begin{aligned} 2(M-1)N + \left(6M-2M^2-6\right) &> 2(-1+M)M + \left(6M-2M^2-6\right) \\ &= 4M-6 > 0 \text{ for } M \geq 2. \end{aligned}$$

Since the quadratic function is strictly increasing in N for $N \geq M \geq 2$ we have

$$\begin{aligned} &(M-1)N^2 + \left(6M-2M^2-6\right)N + M^3 - 5M^2 - 9 + 10M \\ > \; &(M-1)M^2 + \left(6M-2M^2-6\right)M + M^3 - 5M^2 - 9 + 10M \\ = \; &4M-9 > 0 \text{ for all } M \geq 3 \end{aligned}$$

Therefore we conclude that for all $N \geq M \geq 3$ a cartel of M firms is not internally stable.

For $M=2$ the internal stability conditions gives

$$\begin{aligned} &(-1+M)N^2 + \left(6M-2M^2-6\right)N + M^3 - 5M^2 - 9 + 10M \\ = \; &N^2 - 2N - 1 < 0 \end{aligned}$$

which is never true for all $N \geq 3$. Therefore, a cartel of 2 firms is never internally stable unless it leads to a monopoly. ■

4.2 The case of closed-loop strategies

In contrast to the open-loop case, simulations indicate that all cartels are profitable regardless of γ, when we consider close-loop equilibrium; moreover, stable cartels do exist in this case. One surprising result we obtain is that the grand coalition (i.e., the cartel of all firms) is stable when γ (the rate of change in marginal cost) is low. In addition, cartels of size 3 is stable regardless of γ.

4.2.1 Stability of the grand coalition

We show that the stability of the grand coalition depends on the value of the rate of change in the marginal cost, γ.

Proposition 7 *For any $N \geq 2$ there exists $\bar{\gamma}_N > 0$ such that for all $\gamma \in (0, \bar{\gamma}_N)$ the grand coalition is stable.*

Proof. To prove the claim above we first note that for any $N > 0$ and for any $0 \leq \gamma < \infty$ a multiplant monopoly earns strictly positive profits per plant. We then show that when γ tends to zero, the profits of a firm that would exit the grand coalition, i.e. an outsider to the coalition of $N-1$ firms, tend to zero . Therefore when γ is close enough to zero no firm would wish to exit the grand coalition. Thus the proof consists of showing that $\lim_{\gamma \to 0} G_{out} = 0$ where G_{out} is given by 13. This is done in Appendix 1 ■

We note that this result contrasts with the result of Proposition 1 derived for the open-loop case. The difference is solely due to the nature of the strategies used. Kamien and Fershtman show that the outcome of a closed-loop game is closer to a competitive outcome than the outcome of an open-loop game[7].

The smaller the value of the positive cost parameter γ the closer is the outcome of the closed-loop game to the outcome where firms are price takers. In the limit case where firm's marginal cost is constant and the firms are price takers, the price is equal to the constant marginal cost and firms's profits are zero. In the closed-loop equilibrium when γ is small enough the profits of all competing firms become also very small (see Table 1)[8]. If a firm exits the grand-coalition, it would face an exacerbated competition due to the feedback effect and would get lower profits than if it had remained in the coalition.

We note however that as γ becomes larger, the grand coalition is no longer stable. This is shown in the table below for the case that $N = 10$ and $a = 100$.

Table 1

γ	$\pi_{ins}\,(M = N = 10)$	$\pi_{out}\,(M = 9, N = 10)$
0.001	250.0	1.927
0.19	247.65	240.68
0.20	247.52	248.47
1	238.10	415.97
10	166.67	197.45

From these simulations we get that the critical value of γ beyond which the grand coalition in a 10 firms industry stops being stable is approximately equal to 0.20.

[7] Simulations show that this remains true even in the case where M firms merge.

[8] We actually obtain a rather interesting result. The profits of firms are inversely U shaped functions of the cost parameter γ. When γ is small enough, firms are able to achieve larger profits when γ increases. This surprising outcome was also obtained in static frameworks: see Seade (1985).

4.2.2 Another stable cartel size

It is interesting to note that the grand coalition may not be the unique stable cartel. Numerical simulations[9] indicate that *for all* $\gamma > 0$ a cartel of size 3 is stable for all $N \geq 4$.

In the first set of simulations we set $\gamma = 0.1, N = 10$ and $a = 100$. We obtain that 3 and 10 (the grand coalition are the two stable cartels:

Table 2

M	1	2	**3**	4	5	6	7	8	9	**10**
π_{ins}	6.02	6.12	**6.38**	6.85	7.63	8.92	11.20	15.92	30.18	**248.88**
π_{out}	6.02	6.32	**6.97**	8.13	10.17	13.97	22.01	44.07	152.60	

In the second set of simulations we set $\gamma = 1, N = 10$ and $a = 100$. We obtain that 3 is the only stable cartel:

Table 3

M	1	2	**3**	4	5	6	7	8	9	10
π_{ins}	49.02	49.61	**51.29**	54.26	58.98	66.27	77.87	97.73	136.90	238.10
π_{out}	49.02	50.97	**55.17**	62.45	74.48	94.85	132.00	209.60	416.00	

In the third set of simulations we set $\gamma = 5, N = 10$ and $a = 100$. We obtain that 3 is the only stable cartel size:

Table 4

M	1	2	**3**	4	5	6	7	8	9	10
π_{ins}	121.4	122.0	**123.9**	127.2	132.0	138.7	147.7	160.0	176.7	200.0
π_{ins}	121.4	123.7	**128.6**	136.5	148.2	164.8	188.6	223.1	275.0	

Other simulations confirm that 3 is a stable cartel size for all $\gamma \geq 0$.

There are comparable results in the literature on the stability of cartels in a static framework where it is shown that only cartels of *small* sizes are stable.

Donsimoni et al (1986) show that in D'Aspremont et al.'s price leadership model with competitive fringe, results similar as ours emerge with linear demand function and quadratic cost function without the linear term. Either there is a unique stable cartel $M < N$ when

[9] Numerical simulations were carried out with MuPad Pro 3.0.

the firms are not too cost efficient relative to demand or there are two stable cartels N and $M < N$ otherwise. In particular, when cost function of each firm is $C(q) = \frac{1}{2}q^2$ and there are at least four firms, the only stable cartel size is three. Note, however, the demand function used by Donsimoni et al. (1986) is $D(p) = N(a - bp)$, which is a function of N. Nonetheless, the similarity between their results and ours is intriguing.

Shaffer (1995) considers the case where the cartel acts as a Stackelberg leader and the outsider firms constitute a Cournot fringe. She investigates how the size of the stable cartel is related to the number of firms in a setting with linear demand and constant marginal cost. Konishi and Lin (1999) proves the existence of a stable cartel with general demand and cost functions. In an example they show that with demand function $D(p) = a - Q$ and cost function $C(q) = \frac{1}{2}q^2$, size of stable cartel increase with N. This is in contrast to that in D'Aspremont et al.'s price leadership model with the same demand function and cost function, the size of stable cartel is 3 for $N > 5$.

Diamantoudi (2005) shows that if firms are endowed with foresight, larger cartels are stable. The stability proposed in Diamantoudi (2005) captures the foresight of any firm that contemplates leaving (joining) a cartel. In particular, each firm anticipates that after it leaves the grand coalition, other firms may also leave afterwards and consequently, its profits may decrease ultimately to a level below those of an insider of the grand coalition.

5 Concluding Remarks

We have shown that for an oligopoly where firms use open-loop strategies, the grand coalition is unstable. We have also shown for specific values of γ that no cartel is stable. Simulations indicate that no cartel is stable in general. Note that a large cartel may be profitable, but each firm earns even larger profits by exiting the cartel. Like the static Cournot equilibrium, open-loop equilibrium in the dynamic model we analyze cannot be used to explain profitable *stable* cartels.

We have shown that when firms use closed-loop strategies, the grand coalition is stable for small enough γ. The closed-loop effect renders not only all cartels profitable but also the grand coalition stable. For larger values of γ the grand coalition is no longer stable; however the coalition size of 3 emerges as the only coalition size that is stable.

It could also be interesting to investigate the profitability and stability of cartels when firms play a non-linear equilibrium. It is known that the game studied in this paper, i.e. a Cournot competition with sticky prices, admits a continuum of equilibria with non-linear strategies (see Tsutsui and Mino (1990)).

Appendix 1

Proof of Proposition 6: We show in this appendix that $\lim_{\gamma\to 0} G_{out} = 0$ where G_{out} is given by 13. From Proposition 2, the equations that determine the parameters are obtained by setting $M = N - 1$, which yields:
for K_c and K_k,

$$\frac{1}{2}\frac{N - 2NK_c - 2\gamma K_c + 2K_cK_k - K_c^2 + NK_c^2 - 1}{\gamma} = 0 \tag{15}$$

$$\frac{1}{2}\frac{2NK_cK_k - 2\gamma K_k - 2K_cK_k - 2NK_k + K_k^2 + 1}{\gamma} = 0 \tag{16}$$

for E_c and E_k,

$$K_k (N-1) \frac{E_c}{\gamma} + (K_k + (N-1) K_c - N - \gamma) \frac{E_k}{\gamma} = aK_k \tag{17}$$

$$((N-1) K_c - N + K_k - \gamma) \frac{E_c}{\gamma} + K_c \frac{E_k}{\gamma} = aK_c \tag{18}$$

and for G_{ins} and G_{out},

$$G_{ins} = \frac{1}{2}\frac{(2E_k - E_c - 2a\gamma + NE_c) E_c}{\gamma r} \tag{19}$$

$$G_{out} = \frac{1}{2}\frac{(E_k - 2E_c - 2a\gamma + 2NE_c) E_k}{\gamma r} \tag{20}$$

To show that $\lim_{\gamma\to 0} G_{ins} = 0$ and $\lim_{\gamma\to 0} G_{out} = 0$, it suffices to show that $\lim_{\gamma\to 0} \frac{E_c}{\gamma} < \infty$ and $\lim_{\gamma\to 0} \frac{E_k}{\gamma} < \infty$ as these would imply that $\lim_{\gamma\to 0} E_c = 0$ and $\lim_{\gamma\to 0} E_k = 0$. To this end, we first solve for $\frac{E_c}{\gamma}$ and $\frac{E_k}{\gamma}$ for equations (17) and (18) and then compute $\lim_{\gamma\to 0} \frac{E_c}{\gamma}$ and $\lim_{\gamma\to 0} \frac{E_c}{\gamma}$:

$$\begin{aligned}
\lim_{\gamma\to 0} \frac{E_c}{\gamma} &= a\frac{K_c ((N-1) K_c - N)}{D} \\
\lim_{\gamma\to 0} \frac{E_k}{\gamma} &= \lim a\frac{(K_k - N) K_k}{D}
\end{aligned}$$

where $D \equiv 2NK_c - 2N^2K_c + (N-1)\,K_cK_k + (N-K_k)^2 + (N-1)^2\,K_c^2$. It can be shown that the system for K_k and K_c admits only finite solutions[10]. Then it suffices to show that $D \neq 0$.

Assume in negation that $D = 0$. Equations (15) and (16) imply that

$$N - 2NK_c - 2\gamma K_c + 2K_cK_k - K_c^2 + NK_c^2 - 1 = 0$$

and

$$2NK_cK_k - 2\gamma K_k - 2K_cK_k - 2NK_k + K_k^2 + 1 = 0.$$

When γ tends to zero we have

$$N - 2NK_c + 2K_cK_k - K_c^2 + NK_c^2 - 1 = 0 \tag{21}$$

and

$$2\,(N-1)\,K_cK_k - 2NK_k + K_k^2 + 1 = 0. \tag{22}$$

From (21), we can obtain

$$2K_cK_k = 2NK_c - N + K_c^2 - NK_c^2 + 1 \tag{23}$$

and from (22),

$$(N-K_k)^2 = N^2 - 1 - 2\,(N-1)\,K_cK_k \tag{24}$$

Substituting (21) and (22) into $D = 0$ gives

$$\frac{1}{2}\,(N-1)\left(3N - 6NK_c - 3K_c^2 + 3NK_c^2 + 1\right) = 0.$$

If $N > 1$, we have

$$3N - 6NK_c - 3K_c^2 + 3NK_c^2 + 1 = 0,$$

which implies that

$$K_c = \frac{1}{3N-3}\left(3N - \sqrt{3}\sqrt{2N+1}\right). \tag{25}$$

Substituting the above into equation (21) gives

$$K_k = \frac{1}{3N+1}\left(2N + \frac{2}{3}\sqrt{6N+3}\right). \tag{26}$$

[10] As noted in footnote 4 any solution (K_k, K_c) to the system (S) is such that K_k (or K_c) corresponds to a root of a polynomial of degree 4 in K_k (or K_c). Infinity is not a root to a polynomial of non negative degree.

Substituting (25) into equation (22) gives

$$K_k^2 - 2NK_k + K_k\left(2N - \frac{2}{3}\sqrt{6N+3}\right) + 1 = 0.$$

The above equation has two roots

$$\begin{aligned} K_k &= \frac{1}{3}\sqrt{3}\sqrt{2N+1} - \frac{1}{3}\sqrt{6}\sqrt{N-1} \text{ and} \\ K_k &= \frac{1}{3}\sqrt{3}\sqrt{2N+1} + \frac{1}{3}\sqrt{6}\sqrt{N-1}. \end{aligned}$$

It is easy to verify that neither of these roots coincides with (26). This leads to the conclusion that $D \neq 0$. Therefore $\lim_{\gamma\to 0}\frac{E_c}{\gamma} < \infty$ and $\lim_{\gamma\to 0}\frac{E_k}{\gamma} < \infty$ ■

References

[1] Benchekroun, H. (2003). "On the Closed-loop Effect and the Profitability of Horizontal Mergers", *Canadian Journal of Economics* 36, 546-565.

[2] D'Aspremont, C., Jacquemin A., Gabsewicz J. and Weymark J. (1983). "On the Stability of Collusive Price Leadership," *Canadian Journal of Economics* 16, 17-25.

[3] Diamantoudi, E. (2005), "Stable Cartel Revisited", *Economic Theory* 26, 907 - 921.

[4] Donsimoni, M.-P., Economides N. S., and Polemarchakis H. M. (1986). "Stable Cartels," *International Economic Review* 27, 317-327.

[5] Dockner E. (1992),"A Dynamic Theory of Conjectural Variations", *Journal of Industrial Economics*, 40: 377-395.

[6] Dockner E. and A. Gaunersdorfer (2001), "On the Profitability of Horizontal Mergers in Industries with Dynamic Competition", *Japan and the World Economy*, 13: 195-216.

[7] Dockner E., S. Jorgensen, N.V. Long and G. Sorger (2000), *Differential Games In Economics and Management Science*. Cambridge University Press.

[8] Ferhstman, C. and M. Kamien (1987), "Dynamic duopolistic competition with sticky prices", *Econometrica* 55, 1151-64.

[9] Konishi, H. and Ping Lin (1999), "Stable Cartels with a Cournot Fringe in a Symmetric Oligopoly", *Keio Economic Studies*, 36: 1-10.

[10] Seade, J. (1985) "Profitable Cost Increases and the Shifting of Taxation: Equilibrium Responses of Markets in Oligopoly", working paper no. 260, Department of Economics, University of Warwick.

[11] Salant, S.W., S. Switzer and R.J. Reynolds (1983),"Losses from horizontal merger: The Effects of an Exogenous Change in Industry Structure on Cournot-Nash Equilibrium", *Quarterly Journal of Economics*, 98: 185-199.

[12] Shaffer, S. (1995). "Stable Cartels with a Cournot Fringe," *Southern Economic Journal* 61, 744-754.

[13] Tsutsui S. and K. Mino (1990),"Nonlinear Strategies in Dynamic Duopolistic Competition with Sticky Prices", *Journal of Economic Theory*, 52: 136-161.