Budget-Balance, Fairness and Minimal Manipulability*

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Abstract

A common real-life problem is to fairly allocate a number of indivisible objects and a fixed amount of money among a group of agents. Fairness requires that each agent weakly prefers his consumption bundle to any other agent's bundle. In this context, fairness is incompatible with budget-balance and non-manipulability (Green and Laffont, 1979). Therefore, retaining fairness and budget-balance, non-manipulability has to be abandoned or weakened. We search for the rules which are minimally manipulable among all fair and budget-balanced rules. First, we show for a given utility profile, all fair and budget-balanced rules are either (all) manipulable or (all) non-manipulable. Hence, measures based on counting profiles where a rule is manipulable or considering a possible inclusion of profiles where rules are manipulable do not distinguish fair and budget-balanced rules. Thus, a "finer" measure is needed. Our new concept compares two rules with respect to their degree of manipulability by counting for each profile the number of agents who can manipulate the rule. We show that maximally linked fair allocation rules are the minimally (individually and coalitionally) manipulable fair and budget-balanced allocation rules. Such rules link any agent to the bundle of a pre-selected agent through a sequence of indifferences.

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1 Introduction

Many real-life problems involve the allocation of indivisible objects among agents through price or compensation mechanisms. Examples include the allocation of jobs among workers together with their salaries on labor markets and the assignment of apartments together with their rents on housing markets. The fundamental criterion employed in these problems is fairness (or envy-freeness) meaning that each agent should like his own consumption bundle (consisting of an object and a monetary compensation) at least as well as that of anyone else, see e.g. Alkan, Demange and Gale (1991), Svensson (1983) or Tadenuma and Thomson (1991).

When analyzing this type of allocation problems, the fairness criterion is often coupled with other properties. One such property is non-manipulability which guarantees that no agent can gain by strategic misrepresentation. For this problem, a complete characterization of the class of fair and non-manipulable allocation rules has been provided by Andersson and Svensson (2008), Sun and Yang (2003) and Svensson (2009). Any such rule fixes a maximal compensation for

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each object, and for any profile, a “maximal” fair allocation is chosen without exceeding the fixed compensations for any object. As a result, the allocation rules in this class violate budget-balance. This comes with no surprise since a famous result by Green and Laffont (1979) shows that there exists no allocation mechanism that is non-manipulable, fair and budget-balanced. However, in many fair allocation problems, budget-balance is a necessary requirement. Thus, non-manipulability must be abandoned. Even though this type of problem has been considered previously, by e.g. Aragones (1995), Haake, Raith and Su (2000), Klijn (2000) and Velez (2011), two fundamental issues have still not been investigated. First, although it is known that each fair and budget-balanced allocation rule is manipulable at some preference profile, a characterization of the preference profiles where successful misrepresentations are possible is missing. This paper provides such a characterization for the case when preferences are represented by quasi-linear utility functions. Second, there is a large class of fair and budget-balanced allocation rules but it is not known exactly which rules are “minimally” or “least” manipulable. We address these issues in this paper. We show that measures based on counting profiles where a rule is manipulable and/or considering the inclusion of profiles where a rule is manipulable do not distinguish fair and budget-balanced allocation rules. With respect to those measures, all fair and budget-balanced allocation rules are equally manipulable. Therefore, we introduce a new concept of minimal manipulability by counting for each profile the number of agents who can manipulate the rule. With respect to this measure we show that maximally linked fair allocation rules are minimally manipulable in the class of fair and budget-balanced allocation rules.

To identify the set of preference profiles where profitable manipulations are possible, we use the following key observation from the literature on fair and non-manipulable allocation mechanisms:[1] a necessary and sufficient condition for obtaining fair and non-manipulable outcomes is that for each set of objects M' with compensations/prices different from the reservation compensations/prices, there are at least $|M'|$ distinct agents who are indifferent between the consumption bundle assigned to them and some distinct consumption bundle containing an object belonging to M'. Although this observation cannot be directly applied when budget-balance is required (because there are no reservation compensations/prices), it points to the fact that the structure of indifference relations at chosen allocations is crucial to understand under which circumstances a fair and budget-balanced allocation rule can be manipulated. Due to this observation, indifference relations are described in two different ways. First, an indirect link between agents i and k is described through an indifference chain, i.e., a sequence of agents from i to k such that any agent in this sequence is indifferent between his consumption bundle and the consumption bundle of the next agent in the sequence. The second way is through an indifference component which simply is a maximal set of agents such that any two agents in the set are linked through an indifference chain in this set.

Given the description of the indifference structure, we characterize the preference profiles under which agents and coalitions can gain from manipulation. Two of them stand out. First, we show that a fair and budget-balanced allocation rule is non-manipulable at a specific profile if and only if each agent is included in the unique indifference component. Second, a fair and budget-balanced allocation rule is (coalitionally) non-manipulable at a specific profile if and only if all fair and budget-balanced allocation rules are (coalitionally) non-manipulable at the this profile.

Using these insights, we compare the ease of manipulation or, equivalently, the degree of manipulation in mechanisms which are known to be manipulable. In the early literature (e.g. Moulin, 1980), the primary focus was on restricting the preference domain under which a mechanism is non-manipulable. As described above, we identify such a restricted domain for fair and budget-balanced problems (given quasi-linear preferences). However, this domain is almost empty and for this reason our focus is shifted towards the more recent literature for evaluating the degree of manipulability of manipulable allocation rules. One of these directions (e.g. Aleskerov and Kurbanov, 1999; Kelly, 1988, 1993; Maus, Peters and Storcken, 2007a,b) is the idea of counting the number of preference profiles at which a given mechanism is manipulable. A second direction

[1] See in particular Andersson and Svensson (2008, Lemma 4) and Mishra and Talman (2010, Theorem 2). Similar observations have previously also been made by e.g. Dubey (1982) and Svensson (1991) where the “tightness” of the market is demonstrated to have a significant impact on manipulation possibilities.

(Pathak and Sönmez, 2011) relies on comparing the sets of preference profiles on which any two mechanisms are manipulable. Previous papers have investigated a number of different problems, including voting rules, matching mechanisms and school choice mechanisms. However, we are not aware of any study with attention to fair and budget-balanced rules.

It turns out that all of the above mentioned measures of minimal manipulability are "coarse" in the sense that preference profiles are categorized as manipulable (for all fair and budget-balanced rules) or non-manipulable (for all fair and budget-balanced rules). This is due to the fact that a fair and budget-balanced allocation rule is non-manipulable at a specific profile if and only if all fair and budget-balanced allocation rules are non-manipulable at the very same profile. Given this insight, it is impossible to base a measure of the degree of manipulability solely on properties of the preference domain. All fair and budget-balanced allocation rules will be equally manipulable according to such a measure. For this reason, none of the existing measures are satisfactory when evaluating rules in our context.

In resolving this problem, we introduce a new "finer" measure of minimal manipulability. Because this measure cannot be based solely on the preference domain, a natural approach is to compare two rules via the number of agents who can manipulate the rule at a given preference profile. Then a rule is minimally manipulable (with respect to agents counting) if, for each preference profile, the number of manipulating agents is smaller than or equal to the number of manipulating agents at an arbitrary fair and budget-balanced allocation rule. This guarantees that the minimally manipulable rule is non-manipulable whenever there exists a non-manipulable rule. The main feature of (global) non-manipulability is respected as much as possible in the sense that the ultimate goal of our new notion is to have zero manipulating agents at each preference profile.

To identify the minimally manipulable fair and budget-balanced allocation rule, we consider rules where, given a fixed agent k, the compensations are chosen such that each agent can be linked through an indifference chain to agent k. We show that such allocations always exist. A fair and budget-balanced allocation rule choosing such allocations for each profile is called an agent $k-$linked fair allocation rule. If in addition agent k belongs to an indifference component with maximal cardinality (where different ks may be selected for different profiles), then the rule is said to be a maximally linked fair allocation rule. One our main results demonstrates that maximally linked fair allocation rules are the minimally manipulable (with respect to agents counting) fair and budget-balanced allocation rules. This result turns out to be robust with respect to coalitional manipulations. In the same vein as before, when comparing two mechanisms we count the number of coalitions that can manipulate at a given profile. Again, maximally linked fair allocation rules are least coalitionally manipulable among all fair and budget-balanced allocation rules. Finally, when comparing two rules with respect to inclusion of the agents who can manipulate the rule at a profile (à la Pathak and Sönmez (2011)), we show that linked fair allocation rules are minimally manipulable among all fair and budget-balanced allocation rules. Such rules choose for any profile an arbitrary agent k and then select the agent k-linked fair allocations for this profile. We also depict the general relations between the various concepts of minimal manipulability.

The paper is organized as follows. Section 2 introduces assignment with compensations and fair and budget-balanced allocation rules. Section 3 provides two different ways of describing indifference relations at fair and budget-balanced allocations, namely agent k-lined fair allocations and indifference components. In Section 4, for any fair and budget-balanced allocation rule, we characterize the set of agents and the set of coalitions who can profitably manipulate the rule at this profile. We use these results to compare different rules with respect to their degree of manipulability in Section 5. We show that measures which compare rules via profiles counting or profiles inclusion cannot be used to distinguish among fair and budget-balanced allocation rules. Then we introduce our new criterion of minimal manipulability by counting at each profile the number of agents who can manipulate. Our main result shows that maximally linked fair allocation rules are agents-counting-minimally manipulable among all fair and budget-balanced allocation rules. We show that the same result holds if we compare two rules by counting the number of coalitions who can manipulate the rule at the profile. Finally, we show when comparing rules with respect to inclusion of the set of agents who can manipulate, linked fair allocation rules are agents-inclusion-minimally manipulable among all fair and budget-balanced allocation rules. Subsection 5.2 contains an algorithm for identifying agent $k-$linked allocations (and maximally linked allocations).

2 Assignment with Compensations

2.1 Agents, Allocations, and Preferences

Let $N = \{1, \ldots, n\}$ and $M = \{1, \ldots, m\}$ denote the set of agents and objects, respectively. The number of agents and objects are assumed to coincide, i.e. $|N| = |M|$.[2] Each agent $i \in N$ consumes exactly one object $j \in M$ together with some amount of money. A consumption bundle is a pair $(j, x_j) \in M \times \mathbb{R}$ where x_j is the monetary compensation received when consuming object j. An allocation (a, x) is a list of $|N|$ consumption bundles where $a : N \to M$ is a mapping assigning object a_i to agent $i \in N$, and where $x \in \mathbb{R}^M$ (or $x : M \to \mathbb{R}$) assigns the amount x_j of money for the object $j \in M$. An allocation (a, x) is *feasible* if $a_i \neq a_j$ whenever $i \neq j$ for $i, j \in N$, and $\sum_{j \in M} x_j \leq 0$.[3] If $\sum_{j \in M} x_j = 0$, then the allocation (a, x) satisfies *budget-balance*. Let $\mathcal{A}$ denote the set of feasible and budget-balanced allocations.

Each agent $i \in N$ has preferences over consumption bundles (j, x_j) which are represented by continuous utility functions $u_i : M \times \mathbb{R}^M \to \mathbb{R}$. We will write $u_{ij}(x)$ instead of $u_i(j, x)$ to denote the utility of agent $i \in N$ when consuming object $j \in M$ and receiving compensation x_j in the distribution vector x. The utility function is assumed to be quasi-linear and strictly increasing (or monotonic) in money, i.e.

$$u_{ij}(x) = v_{ij} + x_j \text{ for some } v_{ij} \in \mathbb{R}.$$

A list of utility functions $u = (u_i)_{i \in N}$ is a preference profile. We also adopt the notational convention of writing $u = (u_C, u_{-C})$ for $C \subseteq N$. The set of preference profiles with utility functions having the above properties is denoted by $\mathcal{U}$.

Let $u \in \mathcal{U}$ and (a, x) be a feasible allocation. Then (a, x) is *efficient* if there exists no feasible allocation (b, y) such that $u_{ib_i}(y) \geq u_{ia_i}(x)$ for all $i \in N$ with strict inequality holding for some $i \in N$. Obviously, if (a, x) is efficient, then (a, x) is budget-balanced.

Throughout the paper we focus on feasible allocations satisfying budget-balance.[4] For convenience, in the following "allocation" stands for "feasible allocation satisfying budget-balance".

2.2 Fair Allocation Rules

The fundamental concept of fairness corresponds to envy-freeness which was first introduced by Foley (1967). It says that each agent weakly prefers his consumption bundle to any other agent's bundle.

Definition 1. For a given profile $u \in \mathcal{U}$, an allocation (a, x) is *fair* if $u_{ia_i}(x) \geq u_{ia_j}(x)$ for all $i, j \in N$. Let $F(u)$ denote the set of fair allocations for a given profile $u \in \mathcal{U}$.

Under fairness, for feasible allocations efficiency is equivalent to budget-balance.[5]

The following is a well-known property of fair allocations (see e.g. Svensson, 2009): if two allocations are fair at a given profile, then one may interchange both the assignment of objects and the monetary distribution without losing fairness. Obviously, this result holds for fair allocations satisfying budget-balance.

Lemma 1. Suppose that allocations (a, x) and (b, y) are fair at profile $u \in \mathcal{U}$. Then allocations (a, y) and (b, x) are also fair at profile $u \in \mathcal{U}$.

[2]If $|N| > |M|$, then we simply add $|N| - |M|$ null objects with zero value for all agents.

[3]All our results remain true if the budget constraint is replaced by $\sum_{j \in M} x_j \leq x_0$ for an arbitrary $x_0 \in \mathbb{R}$.

[4]When budget-balance is relaxed to $\sum_{j \in M} x_j \leq 0$, then general non-manipulability results are possible, see e.g. Andersson and Svensson (2008) or Sun and Yang (2004).

[5]This is due to the fact that any fair allocation must assign the objects efficiently.

An *allocation rule* is a non-empty correspondence φ choosing for each profile $u \in \mathcal{U}$ a non-empty set of allocations, $\varphi(u) \subseteq \mathcal{A}$, such that $u_{ib_i}(y) = u_{ia_i}(x)$ for all $i \in N$ and all $(a,x),(b,y) \in \varphi(u)$. Hence, the various allocations in the set $\varphi(u)$ are utility equivalent. Such a correspondence is called *essentially single-valued.* It is important to note that alternatively we may consider *single-valued* allocation rules choosing for each profile $u \in \mathcal{U}$ a unique allocation. All our results remain unchanged for single-valued allocation rules.

An allocation rule φ is called *fair* if for any profile $u \in \mathcal{U}$, $\varphi(u) \subseteq F(u)$. The following is a useful property of fair allocation rules.

Lemma 2. Let φ be a fair allocation rule and $u \in \mathcal{U}$. If $(a,x),(b,y) \in \varphi(u)$, then $x = y$.

Proof. Since $(a,x),(b,y) \in \varphi(u)$, we have $u_{ia_i}(x) = u_{ib_i}(y)$ for all $i \in N$. By fairness, $u_{ia_i}(x) \geq u_{ib_i}(x)$. Thus, $u_{ib_i}(y) \geq u_{ib_i}(x)$ and $y_{b_i} \geq x_{b_i}$. Similarly, we obtain $x_{b_i} \geq y_{b_i}$. Hence, $x = y$, the desired conclusion. □

An important implication of Lemma 2 is that for fair allocation rules, a unique distribution of money is chosen for any given preference profile. Hence, often for the study of fair allocation rules it is sufficient to consider its induced distributions of money.

3 Indifferences at Fair and Budget-Balanced Allocations

It is well established that the possibility for agents to manipulate a fair allocation rule depends on the structure of the indifference relations at the allocation(s) chosen by the rule.[6] For example, in the single-item Vickrey auction, the number of indifference relations is maximized at the final allocation (i.e., all agents with the second highest bid are indifferent between being assigned the item or not). Because there exists no fair and budget-balanced allocation rule that is non-manipulable (Laffont and Green, 1979), a systematic description of the indifference structure at fair and budget-balanced allocations is the key to understand under which preference profiles a fair and budget-balanced rule can or cannot be manipulated. Below we provide such a description. To achieve this task, we will introduce the concepts of agent $k-$linked (fair) allocations and indifference components.

3.1 Agent $k-$linked Allocations

The following will facilitate the description of indifference relations at fair and budget-balanced allocations.

Definition 2. Let $(a,x) \in \mathcal{A}$.

(i) For any $i,j \in N$, we write $i \rightarrow_{(a,x)} j$ if $u_{ia_i}(x) = u_{a_j}(x)$.

(ii) An *indifference chain* at allocation (a,x) consists of a tuple of distinct agents $g = (i_0, i_1, ..., i_k)$ such that $i_0 \rightarrow_{(a,x)} i_1 \rightarrow_{(a,x)} \cdots \rightarrow_{(a,x)} i_k$.

Note that $i \rightarrow_{(a,x)} j$ means that agent i is indifferent between his consumption bundle and agent j's consumption bundle, and agent i is directly linked via indifference to agent j at allocation (a,x). An indifference chain at an allocation is simply a sequence of agents such that any agent in the sequence is indifferent between his bundle and the bundle of the agent following him in the sequence. Indifference chains indirectly link agents via indifference in a sequence of directly linked agents.

The following concept of agent k-linked allocations will play an important role.

Definition 3. Let $(a,x) \in \mathcal{A}$.

[6] See for example, Andersson and Svensson (2008), Andersson, Svensson and Yang (2010) or Mishra and Talman (2010) for theoretical results, and Sankaran (1994) or Mishra and Parkes (2010) for efficient procedures to calculate allocations with the maximum number of indifference relations.

(i) Agent $i \in N$ is *linked* to agent $k \in N$ at allocation (a, x) if there exists an indifference chain $(i_0, ..., i_t)$ at allocation (a, x) with $i = i_0$ and $i_t = k$.

(ii) Allocation (a, x) is *agent $k-$linked* if each agent $i \in N$ is linked to agent $k \in N$.

Thus, at an agent $k-$linked allocation, each agent is linked to agent $k \in N$ through some indifference chain. Our first theorem establishes the existence of a k-linked (fair and budget-balanced) allocation for all $k \in N$ and all $u \in \mathcal{U}$. The proof shows that any utility maximizing allocation for agent $k \in N$ in $F(u)$ is agent k-linked. In Section 5 we provide an alternative and constructive existence proof via an algorithm for computing agent $k-$linked allocations.

Theorem 1. For each profile $u \in \mathcal{U}$ and each $k \in N$, there exists an agent $k-$linked allocation in $F(u)$. Moreover, any allocation that maximizes the utility of agent k in $F(u)$ is an agent $k-$linked allocation.

Proof. Note that an agent $k \in N$ utility maximizing allocation exists in $F(u)$ for each profile $u \in \mathcal{U}$ since $F(u)$ is compact. Thus, it remains to show that any allocation in $F(u)$ which maximizes the utility of agent $k \in N$ is agent $k-$linked.

Let $u \in \mathcal{U}$, $k \in N$ and (a, x) be an agent k utility maximizing allocation in $F(u)$. By contradiction, suppose that (a, x) is not $k-$linked, i.e., that there is an agent $l \in N$ which is not linked to agent k. Let

$$G = \{i \in N : \ i \text{ is linked to } k \text{ at } (a, x)\} \cup \{k\}.$$

Because $k \in G$ and $l \in N - G$, both G and $N - G$ are non-empty. Moreover, by construction, $u_{ia_i}(x) > u_{ia_j}(x)$ if $i \in N - G$ and $j \in G$. From the Perturbation Lemma in Alkan, Demange and Gale (1991) it then follows that there exists another allocation $(b, y) \in F(u)$ such that $y_{a_i} > x_{a_i}$ for all $i \in G$.[7] Then by fairness and monotonicity in money, we have

$$u_{ib_i}(y) \geq u_{ia_i}(y) = v_{ia_i} + y_{a_i} > v_{ia_i} + x_{a_i} = u_{ia_i}(x) \text{ for all } i \in G.$$

Because $k \in G$, it follows that $u_{kb_k}(y) > u_{ka_k}(x)$, which contradicts the fact that (a, x) maximizes k's utility in $F(u)$. Hence, any allocation maximizing the utility of agent k in $F(u)$ is an agent $k-$linked allocation. □

Given $k \in N$ and $u \in \mathcal{U}$, let $\psi^k(u) \subseteq F(u)$ denote the set of all fair and budget-balanced allocations which are agent $k-$linked at profile u. By Theorem 1, $\psi^k(u)$ is non-empty. Our next result shows that ψ^k is an allocation rule, i.e., that for any profile all agents are indifferent between all agent k-linked fair allocations. The allocation rule ψ^k will be called the *agent $k-$linked fair allocation rule*, henceforth.

Proposition 1. ψ^k is an allocation rule.

Proof. Let $u \in \mathcal{U}$. To prove the result we need to establish that if $(a, x), (b, y) \in \psi^k(u)$, then $u_{ia_i}(x) = u_{ib_i}(y)$ for all $i \in N$. By Theorem 1, we may suppose without loss of generality that (b, y) is an allocation that maximizes the utility of agent k in $F(u)$.

We first demonstrate the analogue of Lemma 2 for agent k-linked fair allocations: if $(a, x), (b, y) \in \psi^k(u)$, then $x = y$. To see this, note that (a, y) is also fair by Lemma 1. First, we show that (a, y) is agent $k-$linked if (b, y) is agent $k-$linked. Fairness implies

$$u_{ia_i}(y) = u_{ib_i}(y) \text{ for all } i \in N. \tag{1}$$

Since (b, y) maximizes the utility of agent k in $F(u)$, (1) implies that (a, y) also maximizes the utility of agent k in $F(u)$. Thus, by Theorem 1, (a, y) is agent $k-$linked. Hence, without loss of generality we may assume $a = b$.

[7] Because preferences are quasi-linear, this can be simply done by infinitesimally increasing equally the compensations of $\{a_i : i \in G\}$ and infinitesimally decreasing equally the compensations of $\{a_i : i \in N - G\}$ (while preserving budget-balance).

Suppose that the fair allocations (a,x) and (a,y) are agent $k-$linked but $x \neq y$. Then by budget-balance and $x \neq y$, there must be two non-empty groups of agents:

$$\begin{aligned} A &= \{i \in N : x_{a_i} > y_{a_i}\}, \\ B &= \{i \in N : x_{a_i} \leq y_{a_i}\}. \end{aligned}$$

Note that for all $i \in A$ and all $j \in B$, $u_{ia_i}(x) > u_{ia_i}(y) \geq u_{ia_j}(y) \geq u_{ia_j}(x)$. Hence, no agent in A can be linked to any agent in B at allocation (a,x). Because (a,x) is agent $k-$linked, we must have $k \in A$. Let $j \in B$ and $i \in A$. By fairness and monotonicity in money,

$$u_{ja_j}(y) \geq u_{ja_j}(x) \geq u_{ja_i}(x) > u_{ja_i}(y).$$

Thus, at allocation (a,y) no agent in B can be linked to any agent in A. Hence, by $k \in A$, allocation (a,y) cannot be agent $k-$linked which contradicts our assumption.

Let $(a,x),(b,y) \in \psi^k(u)$ and $i \in N$. By the above, we have $x = y$. Obviously, if $a_i = b_i$, then $u_{ia_i}(x) = u_{ib_i}(y)$. If $a_i \neq b_i$, then by fairness both $u_{ia_i}(x) \geq u_{ib_i}(x)$ and $u_{ia_i}(y) \leq u_{ib_i}(y)$. Hence, by $x = y$, we have $u_{ia_i}(x) = u_{ib_i}(y)$, the desired conclusion. □

Note that Proposition 1 and Theorem 1 imply that the set of agent $k-$linked fair allocations and the set of allocations maximizing the utility of agent k in $F(u)$ coincide. Hence, all agents are indifferent between all fair allocations which maximize agent k's utility in $F(u)$.

3.2 Indifference Components

In the above, indifference relations were described by means of sequences of linked agents. We next introduce a more demanding notion of indifference structures, namely indifference components. In each such component any two agents are linked through an indifference chain in this component and there is no superset of this component where any two agents are linked.

Definition 4. Let $(a,x) \in \mathcal{A}$. An *indifference component* at allocation (a,x) is a non-empty set $G \subseteq N$ such that for all $i,k \in G$ there exists an indifference chain at (a,x) in G, say $g = (i_0,...,i_k)$ with $\{i_0,\ldots,i_k\} \subseteq G$, such that $i = i_0$ and $i_k = k$, and there exists no $G' \supsetneq G$ satisfying the previous property at allocation (a,x).

The next result states an important characteristic of indifference components, namely that if there are two allocations that are fair and budget-balanced at some profile $u \in \mathcal{U}$ and if there is an indifference component at one of these allocations, then the very same indifference component must be present at the other allocation. In other words, indifference components at fair and budget-balanced allocations only depend on the preference profile $u \in \mathcal{U}$ because they are invariant with respect to the selected fair and budget-balanced allocation.

Lemma 3. Suppose that allocations (a,x) and (b,y) are fair and budget-balanced at profile $u \in \mathcal{U}$. If G is an indifference component at allocation (a,x), then G is an indifference component at allocation (b,y).

Proof. By Lemma 1, we know that (a,y) is fair. First we show that the indifference component G is present at (a,y).

Because G is an indifference component at (a,x), G consists of indifference chains $g = (i_0,i_1,\ldots,i_k)$ such that $i_k \to_{(a,x)} i_0$. Thus, we have $i_0 \to_{(a,x)} i_1 \to_{(a,x)} \cdots \to_{(a,x)} i_k \to_{(a,x)} i_0$. We show $i_0 \to_{(a,y)} i_1 \to_{(a,y)} \cdots \to_{(a,y)} i_k \to_{(a,y)} i_0$.

For any $i \in N$, let $\Delta_{a_i} = y_{a_i} - x_{a_i}$. To obtain a contradiction, suppose that we do not have $i_0 \to_{(a,y)} i_1 \to_{(a,y)} \cdots \to_{(a,y)} i_k \to_{(a,y)} i_0$, say $u_{i_0a_{i_0}}(x) = u_{i_0a_{i_1}}(x)$ but $u_{i_0a_{i_0}}(y) > u_{i_0a_{i_1}}(y)$. Thus, $\Delta_{a_{i_0}} > \Delta_{a_{i_1}}$. Now, fairness is respected among the agents in G at allocation (a,y) only if

$$\Delta_{a_{i_j}} \geq \Delta_{a_{i_{j+1}}} \text{ for all } j \in \{0,...,k-1\}, \tag{2}$$

$$\Delta_{a_{i_k}} \geq \Delta_{a_{i_0}}. \tag{3}$$

From (2) and $\Delta_{a_{i_0}} > \Delta_{a_{i_1}}$, we obtain $\Delta_{a_{i_0}} > \Delta_{a_{i_k}}$. Hence, (3) is not satisfied. Thus, allocation (a, y) cannot be fair, which contradicts our assumption. Hence, $i_0 \to_{(a,y)} i_1 \to_{(a,y)} \cdots \to_{(a,y)} i_k \to_{(a,y)} i_0$. Note that there exists no $G' \supsetneq G$ such that G' is an indifference component at (a, y) because otherwise, using the previous arguments, any two agents in G' are connected through some indifference chain at (a, x) in G' which contradicts the definition of G being an indifference component at (a, x). Thus, the indifference component G is present at (a, y).

Next, we show that G must be also an indifference component at (b, y). Fairness implies

$$u_{ia_i}(y) = u_{ib_i}(y) \text{ for all } i \in N. \tag{4}$$

Let $j, k \in G$ and suppose that $j \to_{(a,y)} k$. Let $a_k \neq b_k$ and $l_1 \in N$ be such that $a_{l_1} = b_k$. Obviously, (4) implies $k \to_{(a,y)} l_1$. More generally, let $l_1, \ldots, l_t$ be such that $a_{l_r} = b_{l_{r-1}}$ with $r = 2, \ldots, t$ and $a_k = b_{l_t}$. Note that such a "cycle" exists because $|N| = |M|$. Now obviously we have $k \to_{(a,y)} l_1$, $l_r \to_{(a,y)} l_{r+1}$ for all $r = 1, \ldots, t-1$, and $l_t \to_{(a,y)} k$. Since $k \in G$ and G is an indifference component at (a, y), we must have $\{l_1, \ldots, l_t\} \subseteq G$.

Now by (4), we have $u_{jb_j}(y) = u_{ja_j}(y) = u_{ja_k}(y) = u_{jb_{l_t}}(y)$ which implies $j \to_{(b,y)} l_t$. Note that by construction, we also have $l_1 \to_{(b,y)} k$ and $l_r \to_{(b,y)} l_{r-1}$ for all $r = 2, \ldots, t$. This means that j and k are connected through the indifference chain $j \to_{(b,y)} l_t \to_{(b,y)} l_{t-1} \to_{(b,y)} \cdots \to_{(b,y)} l_1 \to_{(b,y)} k$ in G under (b, y). Because this is true for any $j, k \in G$ such that $j \to_{(a,y)} k$, it also follows that any two agents belonging to G must be connected through an indifference chain in G at (b, y). Furthermore, there can be no $G' \supsetneq G$ satisfying this property under (b, y) because by the same argument G' would also satisfy this property under (a, x), which would contradict the definition of an indifference component. □

The existence of indifference components is closely related to the presence of isolated groups (or coalitions): a group of agents $C \subseteq N$ is isolated if no agent outside this group can be linked to any agent in C. In Section 5 we provide an algorithm for identifying an isolated group containing agent $k \in N$.

Definition 5. A group of agents $C \subseteq N$ is *isolated* at allocation (a, x) if $i \not\to_{(a,x)} j$ for all $i \in N - C$ and all $j \in C$.

The following relates isolated groups and indifference components.

Lemma 4. Let φ be a fair and budget-balanced allocation rule, $u \in \mathcal{U}$ and $(a, x) \in \varphi(u)$. If $N - G$ is the (possibly empty) isolated group with maximal cardinality at allocation (a, x), then G is an indifference component at allocation (a, x).

Proof. We first show that all $i, j \in G$ can be linked via an indifference chain in G. Suppose not, i.e. there exist $i, j \in G$ such that i cannot be linked to j via some indifference chain G. Let

$$H = \{k \in G : k \text{ can be linked to } j \text{ via some indifference chain in } G\}.$$

Since $i \in G - H$, we have $G - H \neq \emptyset$. Because no agent in $G - H$ can be linked to any agent in H, by construction, it follows that the set $(N - G) \cup H$ is isolated and $|(N - G) \cup H| > |N - G|$, which contradicts the assumption that $N - G$ is the isolated group with maximal cardinality at allocation $(a, x) \in \varphi(u)$.

Now, the proof follows directly because the group $N - G$ is isolated at allocation (a, x), i.e., $i \not\to_{(a,x)} j$ for all $i \in G$ and all $j \in N - G$. Consequently, there is no $G' \supsetneq G$ such that G' is an indifference component by Definition 4. □

Below we introduce some notation which will be helpful for later purposes. By Lemma 3, the set of indifference components is identical for all fair and budget-balanced allocations in $F(u)$. Let

$$\mathcal{G}(u) = \{G \subseteq N : G \text{ is an indifference component at all } (a, x) \in F(u)\},$$

denote the set of all indifference components of fair and budget-balanced allocations at profile $u \in \mathcal{U}$. Note that for any $k \in N$ there exists $G \in \mathcal{G}(u)$ with $k \in G$.

4 Manipulability and Non-Manipulability

The description of the indifference structure at fair allocations from the previous section will allow us to determine the (non-)manipulation possibilities of fair allocation rules.

Definition 6. An allocation rule φ is *manipulable* at a profile $u \in \mathcal{U}$ by an agent $i \in N$ if there exists a profile $(\hat{u}_i, u_{-i}) \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(\hat{u}_i, u_{-i})$ such that $u_{ib_i}(y) > u_{ia_i}(x)$. If the allocation rule φ is not manipulable by any agent at profile $u \in \mathcal{U}$, then φ is *non-manipulable* at profile $u \in \mathcal{U}$.

Remark 1. Since allocation rules may choose sets of allocations, one may alternatively employ a more conservative notion of manipulability: φ is strongly manipulable at a profile $u \in \mathcal{U}$ by an agent $i \in N$ if there exists a profile $(\hat{u}_i, u_{-i}) \in \mathcal{U}$ such that $u_{ib_i}(y) > u_{ia_i}(x)$ for all $(a, x) \in \varphi(u)$ and all $(b, y) \in \varphi(\hat{u}_i, u_{-i})$. From Svensson (2009, Proposition 3 and its proof) it follows that for any fair allocation rule φ and any profile $u \in \mathcal{U}$, φ is strongly manipulable at profile $u \in \mathcal{U}$ by $i \in N$ if and only if φ is manipulable at profile $u \in \mathcal{U}$ by $i \in N$. Hence, we may use the conservative notion of manipulability instead of ours.

It is well-known (Green and Laffont, 1979) that any fair and budget-balanced rule φ is manipulable for some profile $u \in \mathcal{U}$. Even though we are primarily interested in manipulation by individuals, it will be interesting to describe some of our later results in terms of manipulation by coalitions. We adopt the following version of coalitional manipulability and coalitional non-manipulability.[8] As usual, a coalition is a non-empty subset of N.

Definition 7. An allocation rule φ is *(coalitionally) manipulable* at a profile $u \in \mathcal{U}$ by a coalition $C \subseteq N$ if there is a profile $(\hat{u}_C, u_{-C}) \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(\hat{u}_C, u_{-C})$ such that $u_{ib_i}(y) > u_{ia_i}(x)$ for all $i \in C$. If the allocation rule φ is not manipulable by any coalition at profile u, then φ is *coalitionally non-manipulable* at profile $u \in \mathcal{U}$.

The following result describes the relation between isolated groups and the possibility to manipulate φ at a specific profile. We show that any coalition contained in an isolated group can manipulate the fair and budget-balanced allocation rule.

Lemma 5. Let φ be a fair and budget-balanced allocation rule, $u \in \mathcal{U}$ and $(a, x) \in \varphi(u)$. If the non-empty group $G \subseteq N$ is isolated at allocation (a, x), then each coalition $C \subseteq G$ can manipulate φ at profile $u \in \mathcal{U}$.

Proof. Let $(a, x) \in \varphi(u)$, and suppose that $G \subseteq N$ is a non-empty isolated coalition, i.e., that both $i \not\to_{(a,x)} j$ and $u_{ia_i}(x) > u_{ia_j}(x)$ for all $i \in N - G$ and all $j \in G$. Now simultaneously all compensations for objects a_i $(i \in G)$ can be increased by the same amount and all compensations for objects a_j $(j \in N - G)$ can be decreased by the same amount without losing budget-balance and fairness. Hence, there is a number $\tau > 0$ and $(a, y) \in F(u)$ such that $u_{ia_i}(y) > u_{ia_i}(x) + \tau$ for all $i \in G$ (and $y_{a_i} > x_{a_i} + \tau$ for all $i \in G$). Fix $0 < \varepsilon < \tau$ and define for any $i \in G$ the function $\hat{u}_i$ as follows: for all $j \in M$ and all $x' \in \mathbb{R}^M$, let

$$\hat{u}_{ij}(x') = (-y_j + \varepsilon_{ij}) + x'_j, \tag{5}$$

where $\varepsilon_{ij} = 0$ if $j \neq a_i$ and $\varepsilon_{ia_i} = \varepsilon > 0$. Note that $\hat{v}_{ij} = -y_j + \varepsilon_{ij}$. Let $C \subseteq G$ and $\hat{u}_C = (\hat{u}_i)_{i \in C}$. By construction of $\hat{u}_C$, we have $(a, y) \in F(\hat{u}_C, u_{-C})$.[9]

Let $(b, z) \in \varphi(\hat{u}_C, u_{-C})$. We first show $b_i = a_i$ for all $i \in C$. Let $\delta_j = z_j - y_j$ for all $j \in M$. Without loss of generality, order M such that $\delta_j \geq \delta_{j+1}$ for all $j = 1, \ldots, |M| - 1$.

If $z = y$, then by fairness, $\hat{u}_{ib_i}(y) = \hat{u}_{ia_i}(y)$ for all $i \in C$. Since for all $i \in C$, $\hat{u}_{ia_i}(y) = \varepsilon$ and $\hat{u}_{ij}(y) = 0$ for $j \neq a_i$, we obtain $b_i = a_i$ for all $i \in C$.

[8] Again, in the same vein as Remark 1, we may use a more conservative notion of coalitional manipulability where all deviating agents are strictly better off after the deviation for any of the chosen allocations. This would not change any of our results.

[9] Note that for all $i \in C$, $\hat{u}_{ia_i}(y) = \varepsilon$ and $\hat{u}_{ij}(y) = 0$ for $j \neq a_i$.

If $z \neq y$, then by budget-balance of both (b, z) and (a, y), $\delta_1 > 0$ and $\delta_n < 0$. Let $(j_l)_l$ be a subsequence of $(1, \dots, n)$ such that $j_l < j_{l+1}$, $\delta_{j_l} > \delta_{j_{l+1}}$ and $\delta_j = \delta_{j_l}$ if $j_l \leq j < j_{l+1}$. Let $S_l = \{i \in N : j_l \leq a_i < j_{l+1}\}$. Then for $i \in S_l$:

$$u_{ia_i}(z) = u_{ia_i}(y) + \delta_{a_i} \geq u_{ib_i}(y) + \delta_{a_i} > u_{ib_i}(y) + \delta_{b_i} = u_{ib_i}(z) \text{ if } b_i \geq j_{l+1} \text{ and } i \in N - C,$$
$$\hat{u}_{ia_i}(z) = z_{a_i} - y_{a_i} + \varepsilon = \delta_{a_i} + \varepsilon > \delta_{b_i} = \hat{u}_{ib_i}(z) \text{ if } b_i \geq j_{l+1} \text{ and } i \in C.$$

Thus, by fairness, for all l, $i \in S_l$ implies $j_l \leq b_i < j_{l+1}$. Moreover, for $i \in C$, $\hat{u}_{ia_i}(z) = \delta_{a_i} + \varepsilon > \delta_{b_i} = \hat{u}_{ib_i}(z)$ if $b_i \neq a_i$ and $b_i \geq j_l$. Hence, by fairness, $b_i = a_i$ for all $i \in C$.

It remains to prove that $u_{ib_i}(z) > u_{ia_i}(x)$ for all $i \in C$, i.e., φ is manipulable at u by coalition C. From the above, we have $a_i = b_i$ for all $i \in C$. Since φ is fair, we have $(b, z) \in F(\hat{u}_C, u_{-C})$. Now we have for all $i \in C$ with $b_i \neq 1$,

$$\hat{u}_{ib_i}(z) = \hat{u}_{ia_i}(z) = z_{ia_i} - y_{ia_i} + \varepsilon \geq z_{i1} - y_{i1} = \hat{u}_{i1}(z). \tag{6}$$

Because $\delta_j = z_j - y_j$, it follows from the above condition that $\delta_{b_i} \geq \delta_1 - \varepsilon$ for $i \in C$ with $b_i \neq 1$. Note that this inequality holds trivially if $b_i = 1$ because $\varepsilon > 0$. Now this fact, the definition of δ_j and our choice of $0 < \varepsilon < \tau$, $\delta_1 \geq 0$ and $a_i = b_i$ for all $i \in C$, yield for all $i \in C$

$$\begin{aligned}
u_{ia_i}(x) &< u_{ia_i}(y) - \tau \\
&= u_{ib_i}(y) - \tau \\
&= v_{ib_i} + z_{b_i} - (z_{b_i} - y_{b_i}) - \tau \\
&= u_{ib_i}(z) - \delta_{b_i} - \tau \\
&\leq u_{ib_i}(z) - \delta_1 - (\tau - \varepsilon), \\
&< u_{ib_i}(z),
\end{aligned}$$

where the first inequality follows from $u_{ia_i}(y) > u_{ia_i}(x) + \tau$, the first equality from $a_i = b_i$ for $i \in C$, the second inequality from $-\delta_{b_i} \leq -(\delta_1 - \varepsilon)$, and the last inequality from $\delta_1 \geq 0$ and $\tau > \varepsilon$. Hence, $u_{ia_i}(x) < u_{ib_i}(z)$ for all $i \in C$, which is the desired conclusion. □

Our next result shows that the agent $k-$linked fair allocation rule cannot be manipulated by any coalition containing agent k. The intuition behind this is as follows. If agent k can successfully manipulate the allocation rule, then by fairness agent k must be assigned a consumption bundle where the monetary compensation increases. Since each agent is linked to agent k, then each agent must be assigned a consumption bundle where the monetary compensation increases, because if this is not the case then fairness is violated at the new allocation. But then the budget must be exceeded. Hence, agent k cannot manipulate. The same intuition holds for any fair allocation rule choosing only agent k-linked fair allocations for some profile.

Lemma 6. Let φ be a fair and budget-balanced allocation rule, $k \in N$ and $u \in \mathcal{U}$. If $\varphi(u) \subseteq \psi^k(u)$, then no coalition $C \subseteq N$ containing agent k can manipulate φ at profile $u \in \mathcal{U}$.

Proof. Let $C \subseteq N$ be such that $k \in C$. Suppose that φ is manipulable at profile $u \in \mathcal{U}$ by coalition C. Then there is a profile $(\hat{u}_C, u_{-C}) \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(\hat{u}_C, u_{-C})$ such that $u_{ib_i}(y) > u_{ia_i}(x)$ for all $i \in C$. Note that $\varphi(u) \subseteq \psi^k(u)$ and $(a, x) \in \psi^k(u)$.

By fairness, $u_{ia_i}(x) \geq u_{ib_i}(x)$ for all $i \in C$. Hence, for all $i \in C$, $u_{ib_i}(y) > u_{ib_i}(x)$ and $y_{b_i} > x_{b_i}$. Because (b, y) satisfies budget-balance, we must have $C \subsetneq N$. Since $k \in C$ and (a, x) is an agent $k-$linked fair allocation, there exists $i \in N - C$ and $j \in C$ such that $i \to_{(a,x)} j$. Now by $y_{a_j} > x_{a_j}$ $(j \in C)$ and $u_{ia_i}(x) = u_{ia_j}(x)$, fairness and monotonicity in money imply

$$u_{ib_i}(y) \geq u_{ia_j}(y) > u_{ia_j}(x) = u_{ia_i}(x) \geq u_{ib_i}(x).$$

Hence, $y_{b_i} > x_{b_i}$. Let $C^1 = C \cup \{i \in N : i \to_{(a,x)} j \text{ for some } j \in C\}$. Thus, we have $y_{b_i} > x_{b_i}$ for all $i \in C^1$.

Using the same arguments it follows that for each $i \in N$ such that $i \to_{(a,x)} j$ for some $j \in C^1$, we have $y_{b_i} > x_{b_i}$. For any l, let $C^{l+1} = C^l \cup \{i \in N : i \to_{(a,x)} j \text{ for some } j \in C^l\}$.

Because (a, x) is agent $k-$linked, for some t we obtain $C^t = N$ and $y_{b_i} > x_{b_i}$ for all $i \in C^t$, which is contradiction to budget-balance of (b, y). Hence, C cannot manipulate φ at profile $u \in \mathcal{U}$. □

The following theorem identifies all preference profiles $u \in \mathcal{U}$ at which any fair and budget-balanced allocation rule is (coalitionally) non-manipulable.

Theorem 2. A fair and budget-balanced allocation rule φ is (coalitionally) non-manipulable at profile $u \in \mathcal{U}$ if and only if N is the unique indifference component at profile $u \in \mathcal{U}$ (i.e., $\mathcal{G}(u) = \{N\}$).

Proof. The "only if" part follows directly from Lemma 5 since there always is an isolated group unless N is the unique indifference component by Lemma 4. To prove the "if" part, note that if N is the unique indifference component, any $(a, x) \in F(u)$ is agent i-linked for any $i \in N$ by Lemma 3. Since $\varphi(u) \subseteq F(u)$, Lemma 6 implies that no coalition containing $i \in N$ can manipulate ψ at profile $u \in \mathcal{U}$. Hence, φ is both non-manipulable at profile $u \in \mathcal{U}$ and coalitionally non-manipulable at profile $u \in \mathcal{U}$. □

The above results allow us to demonstrate that a fair and budget-balanced allocation rule is non-manipulable at a profile if and only if all fair and budget-balanced allocation rules are non-manipulable at this profile. Furthermore, the same equivalence holds when considering coalitional non-manipulability instead of individual non-manipulability.

Theorem 3. Let φ and ψ be two arbitrary fair and budget-balanced allocation rules. Then φ is (coalitionally) non-manipulable at profile $u \in \mathcal{U}$ if and only if ψ is (coalitionally) non-manipulable at profile $u \in \mathcal{U}$.

Proof. Follows directly from Lemma 3 and Theorem 2. □

Note that for any $i \in N$, there is a unique indifference component $G \in \mathcal{G}(u)$ such that $i \in G$ (where $G = \{i\}$ is possible), i.e., any agent is included in exactly one indifference component at any profile $u \in \mathcal{U}$. Given this observation, we determine for any profile the precise number of agents and coalitions who *can* manipulate the agent k-linked fair allocation rule. Specifically, we demonstrate that ψ^k can be manipulated by less than 50% of all coalitions at any profile.

Theorem 4. Let $k \in N$ and $u \in \mathcal{U}$.

(i) If $k \in S \in \mathcal{G}(u)$, then ψ^k is manipulable by exactly $|N| - |S|$ agents and exactly $2^{|N|-|S|} - 1$ coalitions at profile $u \in \mathcal{U}$.

(ii) ψ^k is manipulable by less than 50% of all coalitions at any profile $u \in \mathcal{U}$.

Proof. To prove (i), note that since S is an indifference component, for all $i \in S$ and all $(a, x) \in \psi^k(u)$, allocation (a, x) is agent i-linked. From Lemma 6 it then follows that no coalition containing agent $i \in S$ can manipulate ψ^k at profile $u \in \mathcal{U}$. Thus, at most $2^{|N|-|S|} - 1$ coalitions can manipulate ψ^k at profile $u \in \mathcal{U}$. Lemma 5 guarantees that this bound is tight, i.e., that exactly $2^{|N|-|S|} - 1$ coalitions can manipulate ψ^k at profile $u \in \mathcal{U}$. Because there are exactly $|N| - |S|$ non-empty singleton coalitions in the class of coalitions that can gain by manipulation, it follows that exactly $|N| - |S|$ agents can manipulate ψ^k at profile $u \in \mathcal{U}$.

To prove (ii), note that $|S| \geq 1$. Because $2^{|N|-|S|} \leq 2^{|N|-1}$ for any $|S| \geq 1$, it follows from Part (i) of this theorem that ψ^k can be manipulated at profile $u \in \mathcal{U}$ by at most $2^{|N|-1} - 1$ coalitions. Since there are $2^{|N|} - 1$ non-empty coalitions of N and $2^{|N|} - 1 = 2(2^{|N|-1} - 1) + 1$, less than 50% of all coalitions can manipulate ψ^k at profile $u \in \mathcal{U}$. □

Therefore, if the agent $k-$linked fair allocation rule is adopted, then in order to calculate the exact number of manipulating agents and coalitions at a given profile, one only needs to know the number of agents that are included in the indifference component containing agent k. Because indifference components are invariant with respect to the chosen fair allocation (Lemma 3) it is sufficient to find an arbitrary agent k-linked fair allocation at the given preference profile to find the exact number of manipulating agents and coalitions. This task can be achieved, for example by using the algorithm in Klijn (2000). Because this algorithm is polynomially bounded, this is not even computationally hard. We will provide an algorithm for identifying agent $k-$linked fair allocations in Subsection 5.2.

The corollary below follows from the above results.

Corollary 1. (i) ψ^k cannot be manipulated by agent k at any profile $u \in \mathcal{U}$.

(ii) For any two distinct agents $i, j \in N$, there exists no fair and budget-balanced allocation rule φ such that neither i nor j can manipulate φ at any profile $u \in \mathcal{U}$.

Note that Lemma 6 implies that the agent k-linked fair allocation rule cannot be manipulated by any coalition containing k at any profile. In particular, the agent k-linked fair allocation rule is not manipulable by agent k at any profile $u \in \mathcal{U}$, which is the first part of Corollary 1. The second part of Corollary 1 is easy to verify and left to the reader.

Corollary 1 has the same flavor as the corresponding results in two-sided matching (with men and women): (i) for any agent there exists a stable matching rule which is not manipulable by this agent at any profile; and (ii) there is no stable matching rule which cannot be manipulated by at least one man and at least one woman (Ma, 1995).

5 Minimal Manipulability

Any fair and budget-balanced allocation rule is manipulable at some profile $u \in \mathcal{U}$. Thus, retaining fairness and budget-balance, non-manipulability has to be abandoned or weakened. A natural question is whether there is a "minimally (or least) manipulable" allocation rule among all fair and budget-balanced rules. Several recent contributions[10] use a notion of the degree of manipulability in order to compare the ease of manipulation in allocation mechanisms that are known to be manipulable. The common feature is that these results (except for Theorem 4 in Pathak and Sönmez (2011)) use measures for the degree of manipulability which are based on the preference domain.

To describe the various notions of minimal manipulability in detail, given an allocation rule φ, let $\mathcal{U}^\varphi \subseteq \mathcal{U}$ denote the subset of preference profiles at which φ is manipulable (by some agent). In addition, let $P^\varphi(u)$ denote the set of agents who can manipulate the allocation rule φ at profile $u \in \mathcal{U}$.

In Definitions 8-12, we make weak comparisons of two rules and "more" stands for "weakly more" (like "preferred" stands for "weakly preferred").

Definition 8 (Profiles counting)**.** Let φ and ψ be two allocation rules.

(a) φ is profiles-counting-more manipulable than ψ if $|\mathcal{U}^\varphi| \geq |\mathcal{U}^\psi|$; and

(b) φ and ψ are profiles-counting-equally manipulable if $|\mathcal{U}^\varphi| = |\mathcal{U}^\psi|$.

Note that any two rules can be compared regarding their manipulability with respect to profiles counting. The following partial comparison has been proposed by Pathak and Sönmez (2011).

Definition 9 (Profiles inclusion)**.** Let φ and ψ be two allocation rules.

(a) φ is profiles-inclusion-more manipulable than ψ if $\mathcal{U}^\varphi \supseteq \mathcal{U}^\psi$; and

(b) φ and ψ are profiles-inclusion-equally manipulable if $|\mathcal{U}^\varphi| = |\mathcal{U}^\psi|$.

[10] See e.g. Aleskerov and Kurbanov (1999), Kelly (1988, 1993), Maus, Peters and Storcken (2007a,b) or Pathak and Sönmez (2011).

Note that if φ is profiles-inclusion-more manipulable than ψ, then φ is profiles-counting-more manipulable than ψ. However, neither of these measures can be used to distinguish fair and budget-balanced allocation rules with respect to their degree of manipulability.

Proposition 2. Let φ and ψ be two fair and budget-balanced allocation rules. Then (i) φ and ψ are profiles-counting-equally manipulable, and (ii) φ and ψ are profiles-inclusion-equally manipulable.

Proof. By Theorem 3, both $\mathcal{U}^\varphi = \mathcal{U}^\psi$ and $|\mathcal{U}^\varphi| = |\mathcal{U}^\psi|$, which yields the desired conclusion. □

Since all fair and budget-balanced rules are equally manipulable if the measure is based only on the cardinality and/or set inclusions of subsets in the preference domain, a "finer" notion is needed. Note that an equivalent way of stating (global) non-manipulability (or "strategy-proofness") is the following. Allocation rule φ is (globally) non-manipulable if

$$|P^\varphi(u)| = 0 \text{ for all } u \in \mathcal{U}. \tag{7}$$

Given the fact that (7) never can be satisfied for fair and budget-balanced rules and the above insights, it is natural to search for rules where $|P^\varphi(u)|$ is minimized for each profile $u \in \mathcal{U}$. This guarantees that the rule is non-manipulable whenever a non-manipulable rule exists for a specific profile, and that the core idea of (global) non-manipulability is respected as much as possible.

Definition 10 (Agents counting)**.** Let φ and ψ be two allocation rules. Then φ is agents-counting-more manipulable than ψ if $|P^\varphi(u)| \geq |P^\psi(u)|$ for all $u \in \mathcal{U}$.

The corresponding notion with respect to inclusion was introduced by Pathak and Sönmez (2011).

Definition 11 (Agents inclusion)**.** Let φ and ψ be two allocation rules. Then φ is agents-inclusion-more manipulable than ψ if $P^\varphi(u) \supseteq P^\psi(u)$ for all $u \in \mathcal{U}$.

While it is clear that these measures are partial comparisons of allocation rules, the following depicts the relations among the various measures of the degree of manipulability. For any two allocation rules φ and ψ, we have:[11]

$$\begin{aligned} & \varphi \text{ is agents-inclusion-more manipulable than } \psi \\ \Rightarrow\ & \varphi \text{ is agents-counting-more manipulable than } \psi \\ \Rightarrow\ & \varphi \text{ is profiles-inclusion-more manipulable than } \psi \\ \Rightarrow\ & \varphi \text{ is profiles-counting-more manipulable than } \psi. \end{aligned}$$

Note that these relations between the different concepts are general and do not depend on our specific model.

Remark 2. Note that Definitions 8, 9, 10, and 11 (weakly) compare two rules with respect to their manipulability. Naturally, any of these concepts would strongly compare two rules A and B, if A is comparable to B but B is not comparable to A. In other words, under a strong comparison in Definition 8 (a) requires a strict inequality, in Definition 9 (a) a strict inclusion, in Definition 10 a strict inequality for some profile, and in Definition 11 a strict inclusion for some profile. Actually, as the careful reader may check, Pathak and Sönmez (2011)'s second concept makes (only) a strong comparison in the vein of Definition 11 but requires in addition $\mathcal{U}^\varphi \supsetneq \mathcal{U}^\psi$. Of course, again by Theorem 3, in this sense no two fair and budget-balanced rules would be strongly comparable.

[11] In showing $\mathcal{U}^\varphi \supseteq \mathcal{U}^\psi$ for the second implication, note that for any $u \in \mathcal{U}^\varphi$ we have $0 = |P^\varphi(u)| \geq |P^\psi(u)| \geq 0$. Thus, both $|P^\psi(u)| = 0$ and $u \in \mathcal{U}^\psi$.

5.1 (Maximally) Linked Fair Allocation Rules

In determining the least manipulable fair and budget-balanced allocation rules, for agent $k-$linked fair allocation rules, not only the preference profile, but also the selection of $k \in N$ may influence the manipulability possibilities. In the search for the agents-counting-minimally manipulable fair and budget-balanced allocation rule, it is important to select the right $k \in N$ for any given profile $u \in \mathcal{U}$. For this reason, the selection of agent k will be endogenously determined by the profile $u \in \mathcal{U}$. The general idea is first to select an indifference component with maximal cardinality, and then some agent k belonging to this indifference component and finally the set of agent $k-$linked fair allocations.

Recall that $\mathcal{G}(u)$ denotes the set of all indifference components at fair allocations for profile $u \in \mathcal{U}$. Let

$$\bar{\mathcal{G}}(u) = \{G \in \mathcal{G}(u) : |G| \geq |G'| \text{ for all } G' \in \mathcal{G}(u)\},$$

denote the set of indifference components with maximal cardinality. Let

$$\bar{G}(u) = \cup_{G \in \bar{\mathcal{G}}(u)} G,$$

denote the union of all indifference components with maximal cardinality.

A *selection* is a function $\kappa : \mathcal{U} \rightarrow N$. The linked fair allocation rule ϕ^{κ} based on $\kappa : \mathcal{U} \rightarrow N$ is defined as follows: for all $u \in U$, $\phi^{\kappa}(u) = \psi^{\kappa(u)}(u)$. In other words, a linked fair allocation rule selects for each u an agent $\kappa(u)$ and chooses all $\kappa(u)$-linked fair allocations. Note that by Proposition 1, ϕ^{κ} is a well-defined allocation rule because $\psi^{k}(u)$ is essentially single-valued for any $k \in N$ and any $u \in \mathcal{U}$. Furthermore, we will say that an allocation rule φ is a *linked* fair allocation rule if there exists a selection κ such that for all $u \in \mathcal{U}$ we have $\varphi(u) \subseteq \phi^{\kappa}(u)$.

A *maximal selection* is a function $\kappa : \mathcal{U} \rightarrow N$ such that for all $u \in \mathcal{U}$ we have $\kappa(u) \in \bar{G}(u)$. The maximally linked fair allocation rule ϕ^{κ} is the linked fair allocation rule based on κ. Furthermore, we will say that an allocation rule φ is a *maximally linked* fair allocation rule if there exists a maximal selection κ such that for all $u \in \mathcal{U}$ we have $\varphi(u) \subseteq \phi^{\kappa}(u)$. Note that the function κ is a systematic selection from $\bar{\mathcal{G}}(u)$. The meaning of "systematic selection" is that there is a well defined rule for selecting k. This rule can be arbitrary and all our results hold independently of this rule. For example, the rule could be based on a randomized selection from $\bar{G}(u)$ or simply the k with the lowest or highest index in $\bar{G}(u)$.

The next result establishes that maximally linked fair allocation rules are agents-counting-minimally manipulable among all fair and budget-balanced allocation rules.

Theorem 5. Let φ be an arbitrary fair and budget-balanced allocation rule and let ϕ^{κ} be a maximally linked fair allocation rule. Then φ is agents-counting-more manipulable than ϕ^{κ}.

Proof. Suppose that $(a, x) \in \varphi(u)$ and $(b, y) \in \phi^{\kappa}(u)$, and let $N - G$ be a (possibly empty) isolated group with maximal cardinality at allocation $(a, x) \in \varphi(u)$. Then G is an indifference component at allocations (a, x) and (b, y) by Lemma 3 and 4.

Note first that all agents in the isolated coalition $N - G$ can manipulate φ by Lemma 5. Consequently, at least $|N - G|$ agents can manipulate φ. Hence, to conclude the proof we need to show that at most $|N - G|$ agents can manipulate ϕ^{κ}.

Suppose that $\kappa(u)$ belongs to the indifference component $\hat{G} \subseteq \bar{G}(u)$, and note that $|\hat{G}| \geq |G|$ by construction of ϕ^{κ}. Since $\phi^{\kappa}(u) \subseteq \psi^{k}(u)$ for all $k \in \hat{G}$, it now follows from Lemma 6 that no agent $k \in \hat{G}$ can manipulate ϕ^{κ} at profile $u \in \mathcal{U}$. Thus, at most $|N - \hat{G}|$ agents can manipulate ϕ^{κ}. The conclusion then follows directly from the observation that $|\hat{G}| \geq |G|$ implies $|N - \hat{G}| \leq |N - G|$. □

Note that by Theorem 5 (and its proof), (i) any fair and budget balanced allocation rules can be compared to a maximally linked fair allocation rule via agents-counting-manipulability, and (ii) any fair and budget-balanced allocation rule, which is not maximally linked fair, is strongly agents-counting-more manipulable (with a strict inequality for some profile in Definition 10) than any maximally linked fair allocation rule.

In checking the robustness of Theorem 5 we consider the degree of coalitional manipulability. Using the same arguments as above, by Theorem 3 it is in general impossible to define a fair and budget-balanced rule to be less coalitionally manipulable than some other fair and budget-balanced rule if the measure is based only on the cardinality and/or set inclusions of subsets in the preference domain. Hence, in the spirit of Definition 10, let $Q^{\varphi}(u)$ denote the coalitions $C \subseteq N$ that can manipulate the allocation rule φ at profile $u \in \mathcal{U}$. We adopt the following notion.

Definition 12. Let φ and ψ be two allocation rules. Then φ is more coalitionally manipulable than ψ if $|Q^{\varphi}(u)| \geq |Q^{\psi}(u)|$ for all $u \in \mathcal{U}$.

The following result states that maximally linked fair allocation rules are minimally coalitionally manipulable among all fair and budget-balanced allocation rules. This result be seen as an extension of Theorem 5 from minimal individual manipulability to minimal coalitional manipulability, i.e., that Theorem 5 is robust with respect to coalitional manipulations.

Theorem 6. Let φ be a fair and budget-balanced allocation rule and ϕ^{κ} be a maximally linked fair allocation rule. Then φ is more coalitionally manipulable than ϕ^{κ}.

Proof. Suppose that $(a, x) \in \varphi(u)$ and $(b, y) \in \phi^{\kappa}(u)$, and let $N - G$ be the (possibly empty) isolated group with maximal cardinality at allocation $(a, x) \in \varphi(u)$. Then G is an indifference component at allocations (a, x) and (b, y) by Lemmas 3 and 4.

Note first that all coalitions in the isolated group $N - G$ can manipulate φ by Lemma 5. Consequently, at least $2^{|N-G|} - 1$ coalitions can manipulate φ. Hence, to conclude the proof we need to show that at most $2^{|N-G|} - 1$ coalitions can manipulate ϕ^{κ}. Suppose now that $\kappa(u)$ belongs to the indifference component $\hat{G} \subseteq \bar{G}(u)$, and note that $|\hat{G}| \geq |G|$ by construction of ϕ^{κ}. It now follows from Lemma 6 and the construction of ϕ^{κ} that at most $2^{|N-\hat{G}|} - 1$ coalitions can manipulate ϕ^{κ}. The conclusion then follows directly from the observation that $|\hat{G}| \geq |G|$ implies $|N| - |\hat{G}| \leq |N| - |G|$. □

Finally we will establish that linked fair allocation rules are agents-inclusion-minimally manipulable among all fair and budget-balanced allocation rules.[12] We show that any fair and budget balanced allocation rule is agents-inclusion-more manipulable than some linked fair allocation rule.

Theorem 7. Let φ be an arbitrary fair and budget-balanced allocation rule. Then there exists a selection $\kappa : \mathcal{U} \to N$ such that φ is agents-inclusion-more manipulable than ϕ^{κ}.

Proof. We construct $\kappa : \mathcal{U} \to N$ as follows: for all $u \in \mathcal{U}$, if for some $k \in N$, $\varphi(u) \subseteq \psi^{k}(u)$, then we set $\kappa(u) = k$, and otherwise $\kappa(u)$ can be arbitrary.

Let $u \in \mathcal{U}$. If for all $k \in N$, $\varphi(u) \nsubseteq \psi^{k}(u)$, then any agent $i \in N$ belongs to an isolated group. Now by Lemma 5, $P^{\varphi}(u) = N$. Since $\phi^{\kappa}(u) \subseteq \psi^{\kappa(u)}(u)$, now by Lemma 6, $P^{\phi^{\kappa}}(u) \subseteq N - \{\kappa(u)\}$. Hence, $P^{\varphi}(u) \supseteq P^{\phi^{\kappa}}(u)$.

If for some $k \in N$, $\varphi(u) \subseteq \psi^{k}(u)$, then by construction of κ, we also have $\phi^{\kappa}(u) \subseteq \psi^{k}(u)$. But now we have $P^{\varphi}(u) \supseteq P^{\phi^{\kappa}}(u)$.

Hence, for all $u \in \mathcal{U}$, $P^{\varphi}(u) \supseteq P^{\phi^{\kappa}}(u)$, and φ is agents-inclusion-more manipulable than ϕ^{κ}, the desired conclusion. □

Similar as above for agents-counting-minimal manipulability, Theorem 7 is robust with respect to coalitional manipulability (by considering inclusions of the set of coalitions which can manipulate the rule at a profile).

By the proof of Theorem 7, any fair and budget-balanced allocation rule, which is not linked fair, is strongly agents-inclusion-more manipulable (with a strict inclusion for some profile in Definition 11) than some linked fair allocation rule. An important consequence of Theorem 7 is that for any $k \in N$, agent k-linked fair allocation rules are agents-inclusion-minimally manipulable among all fair and budget balanced allocation rules.

[12]The careful reader may note that Theorem 7 is the only new result which is not included in Andersson, Ehlers and Svensson (2010).

5.2 Identifying Agent $k-$linked Allocations

Given the results concerning minimal manipulability, it is important to find an algorithm for identifying agent k-linked fair allocations. Once such allocations are identified, it is also possible to identify $\mathcal{G}(u)$, and as a consequence, maximally linked fair allocations. We provide an algorithm that achieves this task. In similarity with Aragones (1995), our algorithm cannot start at an arbitrary feasible allocation. Instead, we suppose that an arbitrary fair and budget-balanced allocation is known for the given profile. This assumption is not restrictive since arbitrary such allocations can be identified in polynomial time as demonstrated by Klijn (2000).[13]

Given that a fair and budget-balanced allocation (a, x) is known for a given profile $u \in \mathcal{U}$, Lemma 3 can be used to find the set $\mathcal{G}(u)$. More explicitly, if allocation (a, x) is known, all indifference components that are present at this allocation will also be present at each allocation that is fair and budget-balanced for the same profile by Lemma 3. It is therefore an easy task to identify the components containing the most agents. The following example demonstrates the principle, and it will be used throughout to illustrate the main ideas and concepts.

Example 1. Let $N = \{1, 2, 3, 4, 5\}$ and $M = \{1, 2, 3, 4, 5\}$. Let the values of the objects for the agents in the profile u be given by the matrix:

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \\ v_{21} & v_{22} & v_{23} & v_{24} & v_{25} \\ v_{31} & v_{32} & v_{33} & v_{34} & v_{35} \\ v_{41} & v_{42} & v_{43} & v_{44} & v_{45} \\ v_{51} & v_{52} & v_{53} & v_{54} & v_{55} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{8}$$

For these valuations the allocation (a, x) where $a_i = i$ and $x_{a_i} = 0$ for all $i \in N$ is fair and budget-balanced. There are two indifference chains present at this allocation, namely $2 \rightarrow_{(a,x)} 1$ and $4 \rightarrow_{(a,x)} 3$. Any indifference component consists of a single agent, and we have $\mathcal{G}(u) = \{\{i\} \mid i \in N\}$.[14] Consequently, $\bar{G}(u) = N$. □

Before we provide the main algorithm, we first state a simple algorithm that always identifies an isolated group H (Definition 5) at allocation (a, x) containing agent k.

Algorithm 1 (Isolated Groups). Let allocation (a, x) be an arbitrary fair and budget-balanced allocation at profile $u \in \mathcal{U}$ and let $k \in N$. Introduce an iteration counter t and set $t = 0$. Let $K^0 = \{k\}$. For each iteration $t = 1, 2, \ldots$:

Step t. Define $K^t = K^{t-1} \cup \{i \in N - K^{t-1} \mid i \rightarrow_{(a,x)} j \text{ for some } j \in K^{t-1}\}$. If $K^t = K^{t-1}$, then stop. Otherwise continue to Step $t + 1$.

Obviously, if $K^t = N$ for some t, then (a, x) is agent k-linked and Algorithm 1 verifies whether a given allocation is agent k-linked.

Lemma 7. Let $u \in \mathcal{U}$ and $K^0 = \{k\}$. Algorithm 1 identifies an isolated group containing agent k in at most $|N|$ iterations.

Proof. Assume that the algorithm terminates at Step t. If $K^t \neq N$, then $u_{ia_i}(x) > u_{ia_j}(x)$ (or $i \not\rightarrow_{(a,x)} j$) for all $i \in N - K^t$ and all $j \in K^t$ by construction of the algorithm. Thus, K^t is isolated by Definition 5. Note that $k \in K^t$ since $\{k\} = K^0 \subseteq K^t$.

Finally, let T be the last step of the algorithm, and note that because $|K^t| - |K^{t-1}| \geq 1$ as long as $1 \leq t < T$, it is clear that the algorithm terminates in at most $|N|$ number of iterations. □

We next illustrate Algorithm 1 using Example 1.

[13] See Haake, Raith and Su (2000) for a similar procedure.

[14] For example, if $v_{12} = 1$, then there is one indifference component consisting of agents 1 and 2.

Example 2 (Example 1 continued)**.** Start with $K^0 = \{1\}$. Then Algorithm 1 terminates in two steps, i.e.:

Step 1. From (8), it is clear that $i \rightarrow_{(a,x)} 1$ only for $i = 2$. Hence, $K^1 = \{1\} \cup \{2\} = \{1, 2\}$.

Step 2. From (8), it is clear that $i \not\rightarrow_{(a,x)} j$ for all $i \in N - K^1$ and all $j \in K^1$. Hence, $K^2 = K^1$ and Algorithm 1 terminates. □

Both the distribution and the assignment are fixed in Algorithm 1. Note that for any agent $k-$linked fair allocation (b, y) and any fair allocation (a, x), allocation (a, y) is also agent $k-$linked and fair.[15] Thus, without loss of generality, in the algorithm below the assignment of objects remains unchanged. We next provide an algorithm for identifying an agent $k-$linked fair allocation given that the distribution is allowed to change.

Algorithm 2 (**Agent** ***k*****-linked Fair Allocation**)**.** Let allocation (a, x) be fair and budget-balanced. Introduce an iteration counter t and let x^t denote the distribution in iteration t. Set $t = 0$ and initialize the distribution at $x^0 = x$. Let $K^0 = \{k\}$. For each iteration $t = 1, 2, \ldots$:

Step t. For the given allocation (a, x^{t-1}) run Algorithm 1 and let N^t be the set identified in the last step of Algorithm 1. If $N - N^t = \emptyset$, then stop. Otherwise, let $\lambda^t_{ij} = u_{ia_i}(x^{t-1}) - u_{ia_j}(x^{t-1})$ for each $i \in N - N^t$ and each $j \in N^t$. Define $\lambda^t = \min_{i\in N-N^t, j\in N^t} \lambda^t_{ij}$. Let the distribution x^t be given by:

$$\begin{aligned} x^t_{a_i} &= x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t \text{ for each } i \in N - N^t, \\ x^t_{a_j} &= x^{t-1}_{a_j} + \frac{|N - N^t|}{|N|} \cdot \lambda^t \text{ for each } j \in N^t, \end{aligned}$$

and continue to Step $t + 1$. □

Theorem 8. For each $u \in \mathcal{U}$, Algorithm 2 identifies an agent k-linked fair allocation in at most $|N|$ number of iterations.

Proof. Note first that the adjustment of the distribution at Step t from x^{t-1} to x^t respects budget balance because by construction of x^t,

$$\sum_{i\in N} x^t_{a_i} = \sum_{i\in N} x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t \cdot |N - N^t| + \frac{|N - N^t|}{|N|} \cdot \lambda^t \cdot |N^t| = \sum_{i\in N} x^{t-1}_{a_i},$$

and allocation (a, x^0) is budget-balanced.

The adjustment of the distribution at Step t from x^{t-1} to x^t also respects fairness because the assignment a is held constant and the construction of x^t guarantees that if $u_{ia_i}(x^{t-1}) \geq u_{ia_j}(x^{t-1})$ then $u_{ia_i}(x^t) \geq u_{ia_j}(x^t)$. If $i, j \in N^t$ or $i, j \in N - N^t$ this follows directly since the adjustments of $x^{t-1}_{a_i}$ and $x^{t-1}_{a_j}$ are identical. In the case when $i \in N^t$ and $j \in N - N^t$, the result follows since $x^{t-1}_{a_i}$ is increased and $x^{t-1}_{a_j}$ is decreased. In the last case when $i \in N - N^t$ and $j \in N^t$, the conclusion follows by definition of λ^t and $\lambda^t_{ij} = u_{ia_i}(x^{t-1}) - u_{ia_j}(x^{t-1})$, i.e.:

$$\begin{aligned} u_{ia_i}(x^t) &= v_{ia_i} + x^t_{a_i} = v_{ia_i} + x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t \geq v_{ia_i} + x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t_{ij} = \\ &= u_{ia_i}(x^{t-1}) - \lambda^t_{ij} + \frac{|N - N^t|}{|N|} \cdot \lambda^t_{ij} = u_{ia_j}(x^{t-1}) + \frac{|N - N^t|}{|N|} \cdot \lambda^t_{ij} \\ &\geq v_{ia_j} + x^{t-1}_{a_j} + \frac{|N - N^t|}{|N|} \cdot \lambda^t = v_{ia_j} + x^t_{a_j} = u_{ia_j}(x^t). \end{aligned}$$

[15]Note that by Lemma 1, (a, y) is fair. But now $u_{ka_k}(y) = u_{kb_k}(y)$ and both (a, y) and (b, y) maximize the utility of agent k in $F(u)$. Now from Theorem 1 it follows that (a, y) is agent k-linked.

Thus, at Step t in the algorithm (a, x^t) satisfies budget-balance and fairness. It remains to prove that the algorithm terminates in at most $|N|$ iterations at an agent k-linked fair allocation.

By construction of N^t, each agent $i \in N^t$ must belong to an indifference chain $g = (i, ..., k)$. Note that at Step t, for $i \in N - N^t$ and $j \in N^t$ such that $\lambda_{ij}^t = \lambda^t$, all the above inequalities become equalities and we obtain $u_{ia_i}(x^t) = u_{ia_j}(x^t)$, $i \rightarrow_{(a,x^t)} j$ and $i \in N^{t+1}$. Note that $N^t \subseteq N^{t+1}$ because for any $i, j \in N^t$ such that $i \rightarrow_{(a,x^{t-1})} j$ we also have $i \rightarrow_{(a,x^t)} j$. Thus, $|N^{t+1}| - |N^t| \geq 1$ as long as $N - N^t \neq \emptyset$. Now it is clear that the algorithm will terminate in at most $|N|$ number of iterations and that the resulting fair allocation is agent $k-$linked. □

Example 3 (Example 1 continued)**.** Recall that $K^0 = \{1\}$, $a_i = i$ and $x_{a_i}^0 = 0$ for all $i \in N$.

Step 1. From Example 2 we know that $N^1 = \{1, 2\}$ (and $N - N^1 = \{3, 4, 5\}$). From matrix (8), it is also easy to see that $\lambda_{3j}^1 = 1$, $\lambda_{4j}^1 = 2$ and $\lambda_{5j}^1 = 3$ for all $j \in N^1$. Thus, $\lambda^1 = 1$, so $x^1 = (x_1^1, x_2^1, x_3^1, x_4^1, x_5^1) = (\frac{3}{5}, \frac{3}{5}, -\frac{2}{5}, -\frac{2}{5}, -\frac{2}{5})$.

Step 2. Given the distribution x^1 identified in Step 1 the following holds:

$$[v_{ij} + x_j^1]_{i,j \in N} = \begin{bmatrix} \frac{8}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\ \frac{8}{5} & \frac{8}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} & \frac{8}{5} & \frac{8}{5} & -\frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} & \frac{13}{5} \end{bmatrix}$$

Thus, when we run Algorithm 1, agent 3 is first included in N^2 (because agent 3 is indifferent between objects 1, 2 and 3) and then agent 4 is included in N^2 (because agent 4 is indifferent between objects 3 and 4). Hence, $N^2 = \{1, 2, 3, 4\}$. Now, $\lambda_{51}^2 = \lambda_{52}^2 = 2$ and $\lambda_{53}^2 = \lambda_{53}^2 = 3$. Thus, $\lambda^2 = 2$, and as a consequence, $x^2 = (x_1^2, x_2^2, x_3^2, x_4^2, x_5^2) = (1, 1, 0, 0, -2)$.

Step 3. By construction of x^2, agent 5 is indifferent between objects 1, 2 and 5 at allocation (a, x^2). Thus, $N^3 = N$ and Algorithm 2 terminates at Step 3. □

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