

Centre interuniversitaire de recherche
en économie quantitative

CIREQ

Cahier 19-2012

*Proximity-Sensitive Individual
Deprivation Measures*

Walter BOSSERT and
Conchita D'AMBROSIO



Le **Centre interuniversitaire de recherche en économie quantitative (CIREQ)** regroupe des chercheurs dans les domaines de l'économétrie, la théorie de la décision, la macroéconomie et les marchés financiers, la microéconomie appliquée et l'économie expérimentale ainsi que l'économie de l'environnement et des ressources naturelles. Ils proviennent principalement des universités de Montréal, McGill et Concordia. Le CIREQ offre un milieu dynamique de recherche en économie quantitative grâce au grand nombre d'activités qu'il organise (séminaires, ateliers, colloques) et de collaborateurs qu'il reçoit chaque année.

*The **Center for Interuniversity Research in Quantitative Economics (CIREQ)** regroups researchers in the fields of econometrics, decision theory, macroeconomics and financial markets, applied microeconomics and experimental economics, and environmental and natural resources economics. They come mainly from the Université de Montréal, McGill University and Concordia University. CIREQ offers a dynamic environment of research in quantitative economics thanks to the large number of activities that it organizes (seminars, workshops, conferences) and to the visitors it receives every year.*

Cahier 19-2012

Proximity-Sensitive Individual Deprivation Measures

Walter BOSSERT and Conchita D'AMBROSIO

CIREQ, Université de Montréal
C.P. 6128, succursale Centre-ville
Montréal (Québec) H3C 3J7
Canada

téléphone : (514) 343-6557
télécopieur : (514) 343-7221
cireq@umontreal.ca
<http://www.cireqmontreal.com>



Ce cahier a également été publié par le Département de sciences économiques de l'Université de Montréal sous le numéro (2012-16).

This working paper was also published by the Department of Economics of the University of Montreal under number (2012-16).

Dépôt légal - Bibliothèque nationale du Canada, 2012, ISSN 0821-4441

Dépôt légal - Bibliothèque et Archives nationales du Québec, 2012

ISBN-13 : 978-2-89382-641-7

Proximity-sensitive individual deprivation measures*

WALTER BOSSERT

Department of Economics and CIREQ, University of Montreal

walter.bossert@umontreal.ca

CONCHITA D'AMBROSIO

Università di Milano-Bicocca and Econpubblica

conchita.dambrosio@unimib.it

This version: December 21, 2012

Abstract. We propose and characterize a generalization of the classical linear index of individual deprivation based on income shortfalls. Unlike the original measure, our class allows for increases in the income of a higher-income individual to have a stronger impact on a person's deprivation the closer they occur to the income of the individual whose deprivation is being assessed. The subclass of our measures with this property is axiomatized in our second result. *Journal of Economic Literature* Classification No.: D63.

Keywords: Income distribution, relative deprivation, equity.

* Financial support from the Fonds de recherche sur la société et la culture du Québec and the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged.

1 Introduction

Relative deprivation has attracted increasing attention in the past decades when the measurement of individual well-being gained importance not only in the academic context but also in the public discourse and in policy-making circles. The main reason for this is the characteristic at the basis of the concept: the observation that, since individuals do not live in isolation, they determine their well-being also from comparisons with others. Although this consideration appears to be absent from much of standard economic modelling, it has been shown to be one of the main determinants of self-reported satisfaction with income and life; for a survey see, for example, Frey and Stutzer (2002). Measuring relative deprivation is important not only *per se* but also because of its links to major social phenomena such as crime (Stack, 1984), political violence (Gurr, 1968), health status (Wagstaff and van Doorslaer, 2000; Jones and Wildman, 2008), mortality (Salti, 2010) and migration decisions (Stark and Taylor, 1989).

The pioneering contribution in the economics literature on the measurement of relative deprivation is Yitzhaki (1979). Yitzhaki bases his approach on Runciman's (1966, p.10) definition of relative deprivation: "We can roughly say that [a person] is relatively deprived of X when (i) he does not have X, (ii) he sees some other person or persons, which may include himself at some previous or expected time, as having X, (iii) he sees it as feasible that he should have X." Yitzhaki (1979) proposes to consider income as the object of relative deprivation. Individual comparisons, which are at the basis of all indices of relative deprivation proposed since then, are absent in Yitzhaki's (1979) contribution, as he himself acknowledges in Yitzhaki (1980).

The *income shortfall* approach to individual deprivation measurement is due to Hey and Lambert (1980). Motivated by Runciman's remark (1966, p.10) that "[t]he magnitude of a relative deprivation is the extent of the difference between the desired situation and that of the person desiring it," Hey and Lambert (1980) obtain the index proposed by Yitzhaki (1979) by providing an alternative individual-based interpretation of it: each individual compares herself to better-off individuals and the sentiment of deprivation felt with respect to each of them is quantifiable by how short of income she is with respect to the richer individual. The individual index of relative deprivation is the sum of these sentiments divided by the population size. This interpretation has become one of this measure's distinguishing features.

A variety of aggregate indices of relative deprivation based on income shortfalls have been proposed following these early contributions. Chakravarty and Chakraborty (1984)

generalize the aggregate index of relative deprivation proposed by Yitzhaki (1979). Berrebi and Silber (1985) and Imedio-Olmedo, Parrado-Gallardo and Bárcena-Martín (2012) show that many of the commonly used inequality indices can be written as aggregate indices of relative deprivation. The interpretation of a generalization of the Gini index (namely, the single-parameter Gini; see Donaldson and Weymark, 1980, and Weymark, 1981) as indices of aggregate relative deprivation is due to Duclos (2000) and Verme (2011). Chakravarty, Chattopadhyay and Majumder (1995), Chakravarty (1997) and Chakravarty and Moyes (2003) propose deprivation quasi-orderings. Kakwani (1984) introduces a useful graphical device, the relative deprivation curve, to represent income shortfalls as a proportion of mean income, and proves that the area under this curve is the Gini coefficient. While the majority of the contributions on relative deprivation propose various aggregate indices, mainly adapting results from inequality measurement, there also are a relatively small number of papers that explicitly deal with individual deprivation. For instance, Ebert and Moyes (2000) and Bossert and D'Ambrosio (2006) characterize the individual deprivation index that is linear in income shortfalls (Yitzhaki, 1979; Hey and Lambert, 1980).

Some authors who deal with individual deprivation focus on the task of capturing the *intensity* of deprivation felt by an individual in the comparison to those who are better off by enriching measures that are based on income shortfalls. Among other features, their contributions can be viewed as addressing the *feasibility* aspect of deprivation underlined by Runciman (1966). According to Runciman (1966, p.10), “[t]he qualification of ‘feasibility’ is obviously imprecise, but it is necessary in order to exclude fantasy wishes. A man may say with perfect truth that he wants to be as rich as the Aga Khan [...] but to include these under the heading of relative deprivation would rob the term of its value.” A similar position on feasibility can be found in Gurr (1968, p.1104) who states that “[r]elative deprivation is defined as actors’ perceptions of discrepancy between their value expectations (the goods and conditions of the life to which they believe they are justifiably entitled) and their value capabilities (the amounts of those goods and conditions that they think they are able to get and keep).” The question of how to deal with the feasibility aspect is a subtle issue. One possible response is to simply reduce Yitzhaki’s (1979) proposed comparison group of *all* richer individuals by eliminating individuals who are ‘much richer’ (such as the Aga Khan in the above Runciman quote) altogether. However, such a rather drastic move would seem to have problems of its own. First of all, it is by no means obvious how the term ‘much richer’ can be defined properly and it seems doubtful that an unambiguous and widely acceptable definition of that notion can be formulated.

Moreover, even if we may not want to accord the same standing to individuals who are ‘considerably’ richer, it seems unwarranted to exclude them entirely from consideration. It seems desirable to us that there be at least *some* increase in individual deprivation if, *ceteris paribus*, the income of a richer person increases; we consider this an essential monotonicity requirement, as do the other authors dealing with this issue. A more adequate response that we (along with all other relevant contributions that we are aware of) endorse is to find a way of assigning more significance to a richer individual depending on how close her income is to that of the person under consideration. However, as will become clear in the following section, we depart significantly from the earlier literature by retaining a structure that is based on income shortfalls.

Contributions that are close to our own as far as the feasibility issue is concerned include those of Paul (1991), Chakravarty and Chattopadhyay (1994), Podder (1996) and Esposito (2010). Among these, Esposito (2010) is the only one that provides a characterization of the individual deprivation index that is being proposed. All of these authors abandon the income shortfall approach in the sense that they either operate within a *utility* shortfall framework as that mentioned in Hey and Lambert (1980) or focus on income *ratios* rather than income differences. One of the objectives of the present paper is to show that these modifications are not necessary in order to address the feasibility problem: to ensure that higher incomes have a higher impact on individual deprivation the closer they are to the income of the individual in question, the income shortfall approach can be retained. We provide a characterization of a class of individual indices with this property in addition to axiomatizing a more general class.

Section 2 contains a review of the individual indices of relative deprivation proposed in the literature as well as our generalization of the linear income shortfall index. Section 3 presents a characterization of this generalized class, and the subclass that we specifically endorse in this paper is axiomatized in Section 4. Section 5 concludes.

2 Individual deprivation indices

We consider a fixed population of $n + 1$ individuals and analyze the deprivation of a fixed individual (for convenience, individual $n + 1$) where n can be any positive integer greater than one. Our results remain true if the case in which n equals one is included. However, this case is somewhat degenerate and axioms such as anonymity and additive decomposability are redundant if $n = 1$. The set of individuals other than the person under consideration is $N = \{1, \dots, n\}$.

An income distribution is an $(n + 1)$ -dimensional vector $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$ where $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}_+^n$ is the income vector of the members of society other than individual $n + 1$ and $x \in \mathbb{R}_+$ is the income of this person $n + 1$. For $n \in \mathbb{N}$, $\mathbf{1}_n$ is the vector consisting of n ones. For $\mathbf{y}, \mathbf{y}' \in \mathbb{R}_+^n$ and a subset M of N , the vector $\mathbf{z} = (\mathbf{y}|_M, \mathbf{y}'|_{N \setminus M})$ is defined as follows. For all $i \in N$,

$$z_i = \begin{cases} y_i & \text{if } i \in M, \\ y'_i & \text{if } i \in N \setminus M. \end{cases}$$

An individual measure of deprivation is a function $D: \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_+$. $D(\mathbf{y}; x)$ is interpreted as the deprivation suffered by the person under consideration with income x when the remaining members of the society have the incomes described by \mathbf{y} in the distribution $(\mathbf{y}; x)$. We let $B(\mathbf{y}; x) = \{j \in N \mid y_j > x\}$ denote the set of those with a higher income than person $n + 1$.

The *linear income shortfall deprivation measure* D^L proposed by Hey and Lambert (1980) and inspired by Yitzhaki (1979) is defined as follows. For all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D^L(\mathbf{y}; x) = \begin{cases} 0 & \text{if } B(\mathbf{y}; x) = \emptyset, \\ \sum_{j \in B(\mathbf{y}; x)} \frac{1}{n} (y_j - x) & \text{if } B(\mathbf{y}; x) \neq \emptyset. \end{cases}$$

This index can be generalized in an intuitive manner. For any increasing function $F: \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$, the corresponding individual deprivation index D^F is defined by letting, for all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D^F(\mathbf{y}; x) = \begin{cases} 0 & \text{if } B(\mathbf{y}; x) = \emptyset, \\ \sum_{j \in B(\mathbf{y}; x)} F(y_j - x) & \text{if } B(\mathbf{y}; x) \neq \emptyset. \end{cases}$$

These indices generalize D^L , which is the special case obtained by choosing $F(t) = t/n$ for all $t \in \mathbb{R}_{++}$ in the definition of D^F . Consequently, we refer to the members of this class as *generalized income shortfall deprivation measures*.

Paul (1991) proposes an individual index that is sensitive to transfers among richer individuals. It is obtained by considering specific increasing transformations of the *ratios* x/y_j as opposed to the income shortfalls $y_j - x$. His class uses a parameter $\beta \in (1, \infty)$ that reflects the degree of sensitivity to income transfers among the better-off. For any $\beta \in (1, \infty)$, the index D^β is defined by letting, for all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D^\beta(\mathbf{y}; x) = \begin{cases} 0 & \text{if } B(\mathbf{y}; x) = \emptyset, \\ \sum_{j \in B(\mathbf{y}; x)} \frac{1}{n} \left(\frac{x}{y_j}\right)^\beta - \frac{1}{n} |B(\mathbf{y}; x)| & \text{if } B(\mathbf{y}; x) \neq \emptyset. \end{cases}$$

The indices of Chakravarty and Chattopadhyay (1994) and of Podder (1996) are special cases of the class D^U , where $U: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing and strictly concave utility

function. This class of measures was suggested by Hey and Lambert (1980) and the individual index corresponding to any such U is defined by letting, for all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D^U(\mathbf{y}; x) = \begin{cases} 0 & \text{if } B(\mathbf{y}; x) = \emptyset, \\ \sum_{j \in B(\mathbf{y}; x)} [U(y_j) - U(x)] & \text{if } B(\mathbf{y}; x) \neq \emptyset. \end{cases}$$

Chakravarty and Chattopadhyay (1994) employ a power function U with a positive power, whereas Podder (1996) suggests to use a logarithmic utility function U .

Esposito (2010), following these earlier contributions, proposes and characterizes a class based on relative utility gaps, employing a power function with a positive power α , defined by $U(t) = t^\alpha$ for all $t \in \mathbb{R}_+$. The class of individual deprivation measures of Esposito (2010) is D^α with a parameter $\alpha \in \mathbb{R}_{++}$ defined by letting, for all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D^\alpha(\mathbf{y}; x) = \begin{cases} 0 & \text{if } B(\mathbf{y}; x) = \emptyset, \\ \sum_{j \in B(\mathbf{y}; x)} \frac{1}{n} \frac{(y_j^\alpha - x^\alpha)}{y_j^\alpha} & \text{if } B(\mathbf{y}; x) \neq \emptyset. \end{cases}$$

What is common to the classes D^β , D^U and D^α is that they cannot be expressed as functions of the income shortfalls $y_j - x$. Thus, these measures accommodate the feasibility aspect of individual deprivation by deviating from the linearity exhibited in D^L and from the notion that individual deprivation is based on income shortfalls. In contrast, the measures we advocate—the subclass of D^F corresponding to increasing and strictly concave functions F —retain the traditional reliance on income shortfalls, thereby illustrating that the desire to incorporate feasibility issues does not require the income shortfall approach to be abandoned altogether.

3 Generalized income shortfall deprivation measures

The linear index of individual deprivation D^L is characterized by Ebert and Moyes (2000) and by Bossert and D'Ambrosio (2006). There are several significant differences that distinguish our axiomatization of the class D^F from these earlier approaches. We will discuss these differences in more detail after introducing our list of axioms.

Normalization requires individual deprivation to be equal to zero whenever there is no one in society with a higher income than the individual under consideration.

Normalization. For all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$, if $B(\mathbf{y}; x) = \emptyset$, then

$$D(\mathbf{y}; x) = 0.$$

The next axiom is a focus axiom, requiring that the income levels of those who are at or below person $(n + 1)$'s income level are irrelevant. This property parallels Sen's (1976) focus axiom for poverty measures and it formalizes the idea that the comparison group in measuring deprivation consists of all members of society who are richer than the individual under consideration.

Focus. For all $\mathbf{y}, \mathbf{y}' \in \mathbb{R}_+^n$ and for all $x \in \mathbb{R}_+$, if $B(\mathbf{y}; x) = B(\mathbf{y}'; x)$ and $y_j = y'_j$ for all $j \in B(\mathbf{y}; x)$, then

$$D(\mathbf{y}; x) = D(\mathbf{y}'; x).$$

Anonymity demands that the index treat all individuals other than the person whose deprivation is being measured equally, paying no attention to their identities.

Anonymity. For all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$ and for all permutations $\pi: N \rightarrow N$,

$$D((y_{\pi(1)}, \dots, y_{\pi(n)}); x) = D(\mathbf{y}; x).$$

We require D to be increasing in the incomes of those with higher incomes than the individual under consideration. The axiom only applies in situations such that the set $B(\mathbf{y}; x)$ is non-empty; if this set is empty, the property is vacuously satisfied.

Increasingness. For all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$, for all $j \in B(\mathbf{y}; x)$ and for all $\varepsilon \in \mathbb{R}_{++}$,

$$D(((\mathbf{y} + \varepsilon \mathbf{1}_n)|_{\{j\}}, \mathbf{y}|_{N \setminus \{j\}}); x) > D(\mathbf{y}; x).$$

Translation invariance requires the index to be absolute, that is, invariant with respect to equal absolute changes in all incomes.

Translation invariance. For all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$ and for all $\delta \in \mathbb{R}$ such that $(\mathbf{y} + \delta \mathbf{1}_n; x + \delta) \in \mathbb{R}_+^{n+1}$,

$$D(\mathbf{y} + \delta \mathbf{1}_n; x + \delta) = D(\mathbf{y}; x).$$

Additive decomposability is a separability property. The version of Ebert and Moyes (2000) which they call additive decomposition postulates that, for any income distribution, deprivation for that distribution and any reference group is equal to the sum of the levels of deprivation that result if the reference group is divided into two subgroups, keeping the income distribution unchanged (the case where one of the subgroups is empty is covered by the axiom). Clearly, if the reference group is fixed and given by the entire society, the axiom does not apply except in degenerate cases. However, a natural analogue is

obtained by considering distributions where the individuals in each of two subgroups of the comparison group have the same income as individual $n + 1$ (and, therefore, do not contribute to that person's deprivation) and then apply the additivity requirement using these distributions. The following axiom is a weakening of the version employed by Bossert and D'Ambrosio (2006).

Additive decomposability. For all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$ such that $B(\mathbf{y}; x) \neq \emptyset$ and for all $B^1, B^2 \subseteq B(\mathbf{y}; x)$ such that $B^1 \cap B^2 = \emptyset$ and $B^1 \cup B^2 = B(\mathbf{y}; x)$,

$$D(\mathbf{y}; x) = D((\mathbf{y}|_{B^1}, x\mathbf{1}_n|_{N \setminus B^1}); x) + D((\mathbf{y}|_{B^2}, x\mathbf{1}_n|_{N \setminus B^2}); x).$$

A thorough comparison between our set of axioms and those employed by Ebert and Moyes (2000) and by Bossert and D'Ambrosio (2006) is in order.

As is the case for the present contribution, Ebert and Moyes (2000) and Bossert and D'Ambrosio (2006) employ the focus axiom and translation invariance.

Moreover, Ebert and Moyes (2000) and Bossert and D'Ambrosio (2006) use linear homogeneity which is crucial, along with their remaining axioms, to obtain the linear structure of F that leads to the index D^L . Clearly, this axiom is not required in the present approach.

The normalization properties in these earlier contributions are considerably stronger than ours because they impose specific positive values of individual deprivation for specific distributions. This move is required to obtain the result that the sum of income shortfalls from those with higher incomes is divided by the total number of individuals in the population, as is the case for D^L . In contrast, our normalization condition merely states that individual deprivation is zero if no one has a higher income than the person under consideration. This represents a quite pronounced departure from these earlier approaches.

Ebert and Moyes (2000) employ an anonymity condition just like ours. This requirement is not imposed by Bossert and D'Ambrosio (2006) because the normalization condition used in that contribution (in conjunction with the remaining axioms) is sufficient to ensure that the resulting index is anonymous.

Both Ebert and Moyes (2000) and Bossert and D'Ambrosio (2006) impose an additive decomposability property. The two versions differ in that the additive-decomposition axiom in Ebert and Moyes (2000) can only be used in their specific setting where the reference group is permitted to vary independently from the income distribution under consideration. Thus, this axiom cannot be formulated in the framework considered in

Bossert and D’Ambrosio (2006) and in the present contribution. A more detailed discussion of this crucial issue can be found in Bossert and D’Ambrosio (2006). Additive decomposability as defined here is a weakening of the axiom that is used by Bossert and D’Ambrosio (2006); the version employed here only applies to situations in which the set of individuals $B(\mathbf{y}; x)$ is non-empty.

The independence condition of Ebert and Moyes (2000) cannot be formulated in the setting used here and in Bossert and D’Ambrosio (2006) because it rests on the assumption made by Ebert and Moyes (2000) that the composition and the size of the reference group may vary in a way that is independent of the income distribution. Consequently, the axiom does not appear in the present contribution and neither is it used in Bossert and D’Ambrosio (2006).

The only property we use that does not play a role in these earlier articles is increasingness. It is not required in Ebert and Moyes (2000) and in Bossert and D’Ambrosio (2006) because degenerate measures such as that assigning an individual deprivation value of zero to all income distributions are ruled out by their other axioms, notably their normalization properties that are also responsible for features other than the assignment of a zero value to situations with no individual deprivation. Because our normalization property is more modest in its scope, the additional monotonicity property has to be imposed explicitly.

A final observation that is of some interest is that the transformation of the relevant income shortfalls in the index D^L is linear—as mentioned earlier, it is given by the function F such that $F(t) = t/n$ for all $t \in \mathbb{R}_{++}$. Clearly, this transformation F is continuous and, thus, this continuity property cannot but be implied by the axiom systems used by Ebert and Moyes (2000) and by Bossert and D’Ambrosio (2006). In contrast, no continuity properties are implied by the system of axioms employed in the first characterization of the present paper—the only requirement imposed on F is increasingness. This represents another quite fundamental departure from the earlier literature and illustrates that the characterization theorem is not merely a minor variation on existing results.

The above axioms characterize the class D^F of individual deprivation measures.

Theorem 1 *An individual deprivation index D satisfies normalization, focus, anonymity, increasingness, translation invariance and additive decomposability if and only if there exists an increasing function $F: \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ such that $D = D^F$.*

Proof. That D^F satisfies the requisite axioms for any increasing function $F: \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ is straightforward to verify.

Conversely, suppose D is an individual deprivation index satisfying the axioms. Consider first all distributions of the form $((y_j \mathbf{1}_n|_{\{j\}}, x \mathbf{1}_n|_{N \setminus \{j\}}); x)$ with $y_j > x$. Thus, for each of these distributions, there exists an individual $j \in N$ such that

$$B((y_j \mathbf{1}_n|_{\{j\}}, x \mathbf{1}_n|_{N \setminus \{j\}}); x) = \{j\} \neq \emptyset$$

and everyone in N other than j has the same income as person $n + 1$. Using translation invariance with $\delta = -x$ implies

$$D((y_j \mathbf{1}_n|_{\{j\}}, x \mathbf{1}_n|_{N \setminus \{j\}}); x) = D(((y_j - x) \mathbf{1}_n|_{\{j\}}, 0 \mathbf{1}_n|_{N \setminus \{j\}}); 0). \quad (1)$$

Let, for all $t \in \mathbb{R}_{++}$,

$$F(t) = D((t \mathbf{1}_n|_{\{j\}}, 0 \mathbf{1}_n|_{N \setminus \{j\}}); 0). \quad (2)$$

The function F thus defined clearly has the domain \mathbb{R}_{++} . Increasingness and the definition of F together imply that $F(y_j - x) > F((y_j + x)/2 - x)$ and the term on the right side of this inequality is non-negative because D cannot assume negative values. Thus, F maps into \mathbb{R}_{++} . Using increasingness and the definition of F again, it follows that F must be an increasing function. By anonymity, F can be chosen to be independent of j .

Now let $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$ be arbitrary.

If $B(\mathbf{y}; x) = \emptyset$, normalization implies that $D(\mathbf{y}; x) = 0$.

If $B(\mathbf{y}; x) \neq \emptyset$, the focus axiom allows us to assume, without loss of generality, that $y_i = x$ for all $i \in N \setminus B(\mathbf{y}; x)$. Focus and repeated application of additive decomposability together imply

$$D(\mathbf{y}; x) = \sum_{j \in B(\mathbf{y}; x)} D((y_j \mathbf{1}_n|_{\{j\}}, x \mathbf{1}_n|_{N \setminus \{j\}}); x) = \sum_{j \in B(\mathbf{y}; x)} F(y_j - x)$$

because of (1) and (2). ■

The axioms used in the above characterization result are independent, as illustrated by means of the following six examples.

Example 1 Let $F: \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ be increasing and define, for all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D(\mathbf{y}; x) = \begin{cases} 1 & \text{if } B(\mathbf{y}; x) = \emptyset, \\ \sum_{j \in B(\mathbf{y}; x)} F(y_j - x) & \text{if } B(\mathbf{y}; x) \neq \emptyset. \end{cases}$$

This index satisfies all of the axioms except for normalization.

Example 2 Let $F: \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ be increasing and define, for all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D(\mathbf{y}; x) = \begin{cases} 0 & \text{if } B(\mathbf{y}; x) = \emptyset, \\ \sum_{j \in B(\mathbf{y}; x)} F(y_j - x) + \sum_{j \in N \setminus B(\mathbf{y}; x)} F(|y_j - x|) & \text{if } B(\mathbf{y}; x) \neq \emptyset. \end{cases}$$

This index satisfies all of the axioms except for focus.

Example 3 Let, for all $j \in \{1, \dots, n-1\}$, $F_j: \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ be increasing and let $F_n(t) = 2F_1(t)$ for all $t \in \mathbb{R}_{++}$. Now define, for all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D(\mathbf{y}; x) = \begin{cases} 0 & \text{if } B(\mathbf{y}; x) = \emptyset, \\ \sum_{j \in B(\mathbf{y}; x)} F_j(y_j - x) & \text{if } B(\mathbf{y}; x) \neq \emptyset. \end{cases}$$

This index satisfies all of the axioms except for anonymity.

Example 4 Define, for all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D(\mathbf{y}; x) = 0.$$

This index satisfies all of the axioms except for increasingness.

Example 5 Let $F: \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ be increasing and define, for all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D(\mathbf{y}; x) = \begin{cases} 0 & \text{if } B(\mathbf{y}; x) = \emptyset, \\ \sum_{j \in B(\mathbf{y}; x)} F(y_j) & \text{if } B(\mathbf{y}; x) \neq \emptyset. \end{cases}$$

This index satisfies all of the axioms except for translation invariance.

Example 6 Define, for all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D(\mathbf{y}; x) = \begin{cases} 0 & \text{if } B(\mathbf{y}; x) = \emptyset, \\ \left(\sum_{j \in B(\mathbf{y}; x)} \sqrt{y_j - x} \right)^2 & \text{if } B(\mathbf{y}; x) \neq \emptyset. \end{cases}$$

This index satisfies all of the axioms except for additive decomposability.

4 Proximity-sensitive deprivation measures

The following axiom expresses what we think is a plausible aspect of individual deprivation. Loosely speaking, it requires that, *ceteris paribus*, an individual's deprivation is more affected by increases in higher incomes that occur closer to her own.

Proximity sensitivity. For all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$, for all $j \in B(\mathbf{y}; x)$ and for all $y'_j, y''_j \in \mathbb{R}_{++}$ such that $y_j < y'_j < y''_j$,

$$\begin{aligned} & \frac{D((y'_j \mathbf{1}_n|_{\{j\}}, \mathbf{y}|_{N \setminus \{j\}}); x) - D((y_j \mathbf{1}_n|_{\{j\}}, \mathbf{y}|_{N \setminus \{j\}}); x)}{y'_j - y_j} \\ > & \frac{D((y''_j \mathbf{1}_n|_{\{j\}}, \mathbf{y}|_{N \setminus \{j\}}); x) - D((y'_j \mathbf{1}_n|_{\{j\}}, \mathbf{y}|_{N \setminus \{j\}}); x)}{y''_j - y'_j}. \end{aligned}$$

If proximity sensitivity is added to the list of axioms in Theorem 1, the subclass of our measures that are associated with a function F that is strictly concave in addition to being increasing is characterized. In the presence of proximity sensitivity, the axiom increasingness is redundant and can therefore be dropped from the list.

Theorem 2 *An individual deprivation index D satisfies normalization, focus, anonymity, translation invariance, additive decomposability and proximity sensitivity if and only if there exists an increasing and strictly concave function $F: \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ such that $D = D^F$.*

Proof. Suppose $F: \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ is increasing and strictly concave. For future reference, observe first that F is strictly concave if and only if

$$\frac{F(v) - F(u)}{v - u} > \frac{F(w) - F(v)}{w - v} \quad (3)$$

for all $u, v, w \in \mathbb{R}_{++}$ such that $u < v < w$.

That D^F satisfies normalization, focus, anonymity, translation invariance and additive decomposability follows immediately from Theorem 1. We now prove that D^F also satisfies proximity sensitivity, provided that F is strictly concave. By definition of D^F ,

$$\frac{D^F((y'_j \mathbf{1}_n|_{\{j\}}, \mathbf{y}|_{N \setminus \{j\}}); x) - D^F((y_j \mathbf{1}_n|_{\{j\}}, \mathbf{y}|_{N \setminus \{j\}}); x)}{y'_j - y_j} = \frac{F(y'_j - x) - F(y_j - x)}{y'_j - y_j}$$

and

$$\frac{D^F((y''_j \mathbf{1}_n|_{\{j\}}, \mathbf{y}|_{N \setminus \{j\}}); x) - D^F((y'_j \mathbf{1}_n|_{\{j\}}, \mathbf{y}|_{N \setminus \{j\}}); x)}{y''_j - y'_j} = \frac{F(y''_j - x) - F(y'_j - x)}{y''_j - y'_j}$$

for all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$, for all $j \in B(\mathbf{y}; x)$ and for all $y'_j, y''_j \in \mathbb{R}_{++}$ such that $y_j < y'_j < y''_j$. Thus, proximity sensitivity is satisfied if and only if

$$\frac{F(y'_j - x) - F(y_j - x)}{y'_j - y_j} > \frac{F(y''_j - x) - F(y'_j - x)}{y''_j - y'_j}$$

for all $x \in \mathbb{R}_+$ and for all $y_j, y'_j, y''_j \in \mathbb{R}_{++}$ such that $x < y_j < y'_j < y''_j$. This inequality must be true, as can be seen by setting $u = y_j - x$, $v = y'_j - x$ and $w = y''_j - x$ in (3).

Now suppose that D satisfies the axioms of the theorem statement. As in the proof of Theorem 1, consider first all distributions of the form $((y_j \mathbf{1}_n |_{\{j\}}, x \mathbf{1}_n |_{N \setminus \{j\}}); x)$ with $y_j > x$. Translation invariance implies (1) and we can define the function F by (2).

Again, it is immediate that the domain of F is given by \mathbb{R}_{++} .

Proximity sensitivity and the definition of F together imply that we must have

$$\frac{F(y'_j - x) - F(y_j - x)}{y'_j - y_j} > \frac{F(y''_j - x) - F(y'_j - x)}{y''_j - y'_j}$$

for all $x \in \mathbb{R}_+$ and for all $y_j, y'_j, y''_j \in \mathbb{R}_{++}$ such that $x < y_j < y'_j < y''_j$. Using (3) again, this inequality implies that F is strictly concave.

By way of contradiction, suppose that F is not increasing. Then there exist $u, v \in \mathbb{R}_{++}$ such that $u < v$ and $F(u) \geq F(v)$. Because F is strictly concave, this implies that F is decreasing on (v, ∞) . Invoking the strict concavity of F again, it follows that F must assume negative values for sufficiently large values of its argument, a contradiction to the assumption that D maps into \mathbb{R}_+ . Thus, F must be increasing.

The remainder of the proof is identical to that of Theorem 1. ■

The axioms used in Theorem 2 are independent. To see this, the six examples of the previous section can be employed, provided that the requisite transformations F and F_1, \dots, F_n are chosen to be strictly concave.

An interesting observation is that the presence of proximity sensitivity rather than increasingness in the list of axioms implies that the function F must be continuous; this is an immediate consequence of the well-known result that (strict) concavity implies continuity for functions whose domain is an open subset of \mathbb{R} . Note that no such implication is obtained in Theorem 1.

Clearly, any strictly concave function $F: \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ can be used to generate an individual deprivation index that belongs to the subclass characterized in Theorem 2. Prominent examples include functions of the form $F(t) = t^\alpha/n$ for all $t \in \mathbb{R}_{++}$ where the power α is in the interval $(0, 1)$ to ensure that the resulting function is strictly concave. For instance, the square root multiplied by $1/n$ is obtained for $\alpha = 1/2$.

5 Concluding remarks

In this paper we propose and characterize a family of individual deprivation indices with the intention of better capturing the *intensity* of deprivation felt by an individual in the comparison to those who are better off. These indices are obtained by enriching measures that are based on income shortfalls while retaining their fundamental structure. The key feature addressed is that of *feasibility* of the object of deprivation.

Capturing the intensity of individual sentiments also is a key task in other disciplines such as psychology. In that area, relative deprivation is seen as the determinant of envy, a negative sentiment based on undeserved inferiority. In discussing this issue, Ben-Ze'ev (2000, p.311) supports our view that income shortfalls should enter in a strictly concave manner: in Ben-Ze'ev's (2000) contribution, the curve expressing envy towards the good fortune of others indeed turns out to be *bell-shaped*.

An interesting task for future research consists of extending the analysis by incorporating *intertemporal* aspects. Allowing past experiences and future expectations to matter may enhance our understanding of the impact of including feasibility considerations when assessing individual deprivation.

References

- BEN-ZE'EV, A., 2000, *The subtlety of emotions*, MIT Press, Cambridge (MA).
- BERREBI, Z.M. AND J. SILBER, 1985, "Income inequality indices and deprivation: a generalization," *Quarterly Journal of Economics*, **100**, 807–810.
- BOSSERT, W. AND C. D'AMBROSIO, 2006, "Reference groups and individual deprivation," *Economics Letters*, **90**, 421–426.
- CHAKRAVARTY, S.R., 1997, "Relative deprivation and satisfaction orderings," *Keio Economic Studies*, **34**, 17–31.
- CHAKRAVARTY, S.R. AND A.B. CHAKRABORTY, 1984, "On indices of relative deprivation," *Economics Letters*, **14**, 283–287.
- CHAKRAVARTY, S.R. AND N. CHATTOPADHYAY, 1994, "An ethical index of relative deprivation," *Research on Economic Inequality*, **5**, 231–240.
- CHAKRAVARTY, S.R., N. CHATTOPADHYAY AND A. MAJUMDER, 1995, "Income inequality and relative deprivation," *Keio Economic Studies*, **32**, 1–15.

- CHAKRAVARTY, S.R. AND P. MOYES, 2003, "Individual welfare, social deprivation and income taxation," *Economic Theory*, **21**, 834–869.
- DONALDSON, D. AND J.A. WEYMARK, 1980, "A single-parameter generalization of the Gini indices of inequality," *Journal of Economic Theory*, **22**, 67–86.
- DUCLOS, J.-Y., 2000, "Gini indices and the redistribution of income," *International Tax and Public Finance*, **7**, 141–162.
- EBERT, U. AND P. MOYES, 2000, "An axiomatic characterization of Yitzhaki's index of individual deprivation," *Economics Letters*, **68**, 263–270.
- ESPOSITO, L., 2010, "Upper boundedness for the measurement of relative deprivation," *Review of Income and Wealth*, **56**, 632–639.
- FREY, B.S. AND A. STUTZER, 2002, "What can economists learn from happiness research?," *Journal of Economic Literature*, **40**, 402–435.
- GURR, T., 1968, "A causal model of civil strife: a comparative analysis using new indices," *American Political Science Review*, **62**, 1104–1124.
- HEY, J.D. AND P.J. LAMBERT, 1980, "Relative deprivation and the Gini coefficient: comment," *Quarterly Journal of Economics*, **95**, 567–573.
- IMEDIO-OLMEDO, L.J., E.M. PARRADO-GALLARDO AND E. BÁRCENA-MARTÍN, 2012, "Income inequality indices interpreted as measures of relative deprivation/satisfaction," *Social Indicators Research*, **109**, 471–491.
- JONES, A.M. AND J. WILDMAN, 2008, "Health, income and relative deprivation: evidence from the BHPS," *Journal of Health Economics*, **27**, 308–324.
- KAKWANI, N., 1984, "The relative deprivation curve and its applications," *Journal of Business and Economic Statistics*, **2**, 384–394.
- PAUL, S., 1991, "An index of relative deprivation," *Economics Letters*, **36**, 337–341.
- PODDER, N., 1996, "Relative deprivation, envy and economic inequality," *Kyklos*, **49**, 353–376.
- RUNCIMAN, W., 1966, *Relative deprivation and social justice*, Routledge, London.
- SALTI, N., 2010, "Relative deprivation and mortality in South Africa," *Social Science & Medicine*, **70**, 720–728.
- SEN, A., 1976, "Poverty: an ordinal approach to measurement," *Econometrica*, **44**, 219–231.

- STACK, S., 1984, "Income inequality and property crime," *Criminology*, **22**, 229–257.
- STARK, O. AND J.E. TAYLOR, 1989, "Relative deprivation and international migration," *Demography*, **26**, 1–14.
- VERME, P., 2011, "Two classes of generalized deprivation indexes," *Economics Bulletin*, **31**, 2021–2019.
- WAGSTAFF, A. AND E. VAN DOORSLAER, 2000, "Income inequality and health: what does the literature tell us?," *Annual Review of Public Health*, **21**, 543–567.
- WEYMARK, J.A., 1981, "Generalized Gini inequality indices," *Mathematical Social Sciences*, **1**, 409–430.
- YITZHAKI, S., 1979, "Relative deprivation and the Gini coefficient," *Quarterly Journal of Economics*, **93**, 321–324.
- YITZHAKI, S., 1980, "Relative deprivation and the Gini coefficient: reply," *Quarterly Journal of Economics*, **95**, 575–576.