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OPTIMAL SHARING OF A WATER RESOURCE
BY AN AGRICULTURAL AND AN OIL SECTOR

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The Alberta Dilemma : Optimal Sharing of a Water Resource by an Agricultural and an Oil Sector

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Abstract

The purpose of this paper is to characterize the optimal time paths of production and water usage by an agricultural and an oil sector that have to share a limited water resource. We show that for any given water stock, if the oil stock is sufficiently large, it will become optimal to have a phase during which the agricultural sector is inactive. This may mean having an initial phase during which the two sectors are active, then a phase during which the water is reserved for the oil sector and the agricultural sector is inactive, followed by a phase during which both sectors are active again. The agricultural sector will always be active in the end as the oil stock is depleted and the demand for water from the oil sector decreases. In the case where agriculture is not constrained by the given natural inflow of water once there is no more oil, we show that oil extraction will always end with a phase during which oil production follows a pure Hotelling path, with the implicit price of oil net of extraction cost growing at the rate of interest. If the natural inflow of water does constitute a constraint for agriculture, then oil production never follows a pure Hotelling path, because its full marginal cost must *always* reflect not only the imputed rent on the finite oil stock, but also the positive opportunity cost of water.

Keywords: nonrenewable natural resources, renewable natural resources, order of use, water resource, oil

JEL classification: Q1, Q2, Q3

Résumé

Nous arrivons à caractériser complètement les sentiers optimaux de production et d'utilisation de l'eau par un secteur agricole et un secteur pétrolier qui partagent une même ressource en eau. Nous montrons que pour un stock d'eau donné, si le secteur pétrolier est suffisamment important il deviendra optimal d'avoir une phase durant laquelle le secteur agricole sera inactif. Ceci peut signifier une phase initiale durant laquelle les deux secteurs sont actifs, suivie d'une phase durant laquelle seul le secteur pétrolier est actif et enfin une phase durant les deux secteurs sont à nouveau actifs. Le secteur agricole sera toujours actif à la fin, puisque la demande d'eau du secteur pétrolier décroît au fur et à mesure que le stock de pétrole s'épuise. Dans le cas où l'agriculture n'est pas contrainte par le flux d'apport naturel en eau une fois le stock de pétrole épuisé, nous montrons que le sentier d'extraction du pétrole se terminera toujours par une phase à la Hotelling durant laquelle le prix implicite du pétrole net du coût d'extraction croîtra au taux d'intérêt. Si au contraire le flux d'apport naturel est contraignant pour le secteur agricole, alors le taux d'extraction du pétrole ne suivra jamais un sentier purement à la Hotelling, car le plein coût marginal du pétrole devra *toujours* refléter non seulement la rente imputable au stock fixe de pétrole, mais également un coût implicite positif pour l'eau.

Mots clés : ressources naturelles non renouvelables, ressources naturelles renouvelables, ordre d'exploitation, eau, pétrole.

Classification JEL : Q1, Q2, Q3

1 Introduction

Several years of drought have recently exacerbated a dilemma faced by the province of Alberta concerning the sustainability of water usage by the various sectors of its economy. The dilemma comes from the choices that must be made between conflicting uses of a limited common water resource by important sectors of its economy. This is particularly true of the agricultural and oil sectors, two of the mainstays of the Alberta economy and two large water users.[1] Water is an essential input for the agricultural sector, for irrigation and other purposes. Water is also used intensively by Alberta's important and growing oil sector in order to enhance oil recovery.[2] The optimal allocation of the scarce water resource between those alternative uses poses a problem of intertemporal choice, given that both water and oil are subject to dynamic constraints.

The purpose of this paper is to characterize the optimal time paths of production and water usage of the two sectors. We show that for any given initial water stock, these time paths will take different configurations depending on the size of the initial oil stock and on whether or not the natural water recharge imposes a long-run constraint on the agricultural sector. We are able to identify critical values of the oil stock that determine the specific phases of the optimal paths. *Ceteris paribus*, the larger the oil stock, the greater the pressure on the scarce water resource. We show that for sufficiently large oil stocks, it will become optimal to have a phase during which the agricultural sector is inactive. This may mean having a first phase during which the two sectors are active, then a phase during which the water is reserved for the oil sector and the agricultural sector is inactive, followed by a phase during which both sectors are active again. The agricultural sector will always be active in the end as the oil stock is depleted and the demand for water from the oil sector decreases. Agriculture becomes the only water user once the oil stock is exhausted. It then may or may not be constrained by the natural inflow of water. In the case where it is not, we show that oil extraction will always end with a phase during which the oil production path

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[1]See Griffiths and Woynillowicz (2003) for an overview of the consequences of the demand for water by Alberta's oil industry on the management of the province's water resources.

[2]For a description of the different ways in which the use of water enters the oil recovery processes in Alberta and for some summary data on water use by that industry, see Canadian Association of Petroleum Producers (2002) and Alberta Environment (2004).

follows a pure Hotelling path (Hotelling, 1931), with the implicit price of oil net of extraction cost growing at the rate of interest. Otherwise the oil production path never follows a pure Hotelling path, because its full marginal cost must always reflect not only the imputed rent on the finite oil stock, but also the positive opportunity cost of water.

The problem analyzed here concerns the optimal order of use of the common water resource as an input by a renewable and a nonrenewable sector. It is related to the literature on the optimal order of use over time of multiple pools of a natural resource to serve a single market (Herfindahl (1967), Kemp and Long (1980), Lewis (1982), Kemp and Long (1984), Hartwick, Kemp and Long (1986), Amigues *et al.* (1998), Favard (2002), Holland (2003)). One particularity however is that the decision concerns the order of use of a single common resource pool by multiple sectors of the economy, rather than multiple resource pools by a single user. As such it is more closely related to Gaudet, Moreaux and Salant (2001), who analyze the optimal order of use of many nonrenewable resource pools to serve multiple markets, or to Chakravorty and Krulce (1994), Chakravorty, Roumasset and Kinping (1997) and Chakravorty, Krulce and Roumasset (forthcoming), where the analysis concerns the optimal order of use of many differentiated resources for different purposes. However none of those analyses can be applied directly to the problem studied in this paper, since another one of its particularities is that the common resource is renewable and one of the sectors using it as an input exploits a nonrenewable resource.

The two sectors are called agriculture and oil and they share a water resource, but the model could be adapted to similar situations where two economic activities face a common constraint on the use of an essential input. One might think for example of the common resource as the absorption capacity of the environment, being shared by two polluting industries, one of which exploits a nonrenewable resource.

In the next section we present the model and derive some general propositions concerning the rates of production of the two sectors. The optimal paths for the case where the natural inflow of water constitutes a long-run constraint on agriculture are derived in Section 3. In Section 4 we show how these paths are modified when the agricultural sector is not constrained by the natural inflow of water. We then briefly conclude in Section 5.

2 The model

Consider an economy that produces an agricultural product and oil, both of which use water as an input, drawn from a common source. The agricultural product can be produced indefinitely, as long as the essential water input is available. Oil is a nonrenewable resource, whose initial stock is fixed and therefore subject to exhaustion.

Let $y_a(t)$ denote agricultural production and $y_m(t)$ oil production at time t. The unit cost of production in sector i, $i = a, m$, is $c_i > 0$, excluding any imputed rents on water and oil stocks. The gross social benefit derived from the production of sector i is $u_i(y_i)$, which is assumed to satisfy:

$$u_i'(y_i) > 0, \quad u_i''(y_i) < 0 \quad \text{for all} \quad y_i \geq 0 \quad \text{and} \quad u_i(0) = 0, \quad c_i < u_i'(0) < +\infty. \tag{1}$$

The purpose of these assumptions will become clear in due course.

Sector i consumes net k_i units of water per unit of production.[3] Total net consumption of water by sector i is therefore $k_i y_i$. The total stock of water available at time t is $X(t) \geq 0$ and the given initial stock is $X_0 > 0$. The stock of water is recharged by a natural inflow $\bar{x}$. The dynamics of the water stock, after withdrawal, is therefore given by:

$$\dot{X}(t) = \bar{x} - k_a y_a(t) - k_m y_m(t). \tag{2}$$

The oil stock to which the oil sector has access at time t is $S(t)$ and its fixed initial stock is $S_0 > 0$. The oil stock dynamics is given by:

$$\dot{S}(t) = -y_m(t). \tag{3}$$

When the water stock is drawn down to zero, the aggregate water consumption is constrained by the natural water inflow : $k_a y_a + k_m y_m \leq \bar{x}$. Each sector then faces an upper

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[3]The net consumption of water by a sector may differ from the gross consumption to the extent that a fraction of the water used is returned to the water cycle. So if gross withdrawal is h_i and a fraction α_i is returned to the cycle, then $k_i = (1 - \alpha_i)h_i$. Typically α_m is relatively low and $\alpha_a > \alpha_m$ (See Griffiths and Woynillowicz (2003)).

bound to its production, given by $\bar{y}_i = \bar{x}/k_i$, which is the maximum output that can be achieved in that situation when the other sector is inactive.

Denote by $\hat{y}_i$ the level of output that would maximize the net benefit generated by sector i if both water and oil were abundant, thus not justifying any scarcity rent. It is given by $u_i'(\hat{y}_i) = c_i$. The assumptions on $u_i(y_i)$ in (1) imply that $\hat{y}_i > 0$.

The planner's problem can be formulated as that of choosing the time paths of $y_a(t)$ and $y_m(t)$, for all $t \geq 0$, so as to maximize:

$$\int_0^\infty e^{-rt}[u_a(y_a(t)) - c_a y_a(t) + u_m(y_m(t)) - c_m y_m(t)]dt$$

subject to

$$\dot{X}(t) = \bar{x} - k_a y_a(t) - k_m y_m(t), \ X(t) \geq 0, \ X(0) = X_0, \text{given} \tag{4}$$

$$\dot{S}(t) = -y_m(t), \ \lim_{t \to \infty} S(t) \geq 0, \ S(0) = S_0, \text{given} \tag{5}$$

$$y_a(t) \geq 0, \ y_m(t) \geq 0. \tag{6}$$

where r is the rate of discount. Notice that contrary to the stock of oil, the stock of water may be replenished by withdrawing less than the constant natural inflow. This explains why it is necessary to impose explicitly that $X(t) \geq 0$ for all $t > 0$ and not only at $t = \infty$, as for $S(t)$.

In order to take into account the pure state constraint $X(t) \geq 0$, define the Lagrangian function:

$$L(X, S, y_a, y_m, \lambda_m, \lambda_w, \mu, t) = H + \mu(t)X(t)$$

where the Hamiltonian H is given by:

$$H(X, S, y_a, y_m, \lambda_m, \lambda_w, t) = e^{-rt}[u_a(y_a) - c_a y_a + u_m(y_m) - c_m y_m] - \lambda_m y_m + \lambda_w[\bar{x} - k_a y_a - k_m y_m].$$

Then the following conditions, along with (4), (5) and (6), are necessary:[4]

$$
u'_a(y_a(t)) \begin{cases} = c_a + e^{rt}\lambda_w(t)k_a & \text{if } y_a(t) > 0 \\ \\ \leq c_a + e^{rt}\lambda_w(t)k_a & \text{otherwise.} \end{cases} \tag{7}
$$

$$
u'_m(y_m(t)) \begin{cases} = c_m + e^{rt}[\lambda_m(t) + \lambda_w(t)k_m] & \text{if } y_m(t) > 0 \\ \\ \leq c_m + e^{rt}[\lambda_m(t) + \lambda_w(t)k_m] & \text{otherwise.} \end{cases} \tag{8}
$$

$$
\dot{\lambda}_w(t) \begin{cases} = 0 & \text{if } X(t) > 0 \\ \\ = -\mu(t) \leq 0 & \text{otherwise.} \end{cases} \tag{9}
$$

$$
\dot{\lambda}_m(t) = 0 \tag{10}
$$

$$
\lim_{t\to\infty} \lambda_w(t) \geq 0, \quad \lim_{t\to\infty} \lambda_w(t)X(t) = 0, \lim_{t\to\infty} X(t) \geq 0 \tag{11}
$$

$$
\lim_{t\to\infty} \lambda_m(t) \geq 0, \quad \lim_{t\to\infty} \lambda_m(t)S(t) = 0, \lim_{t\to\infty} S(t) \geq 0 \tag{12}
$$

In view of the assumptions on $u(y_m)$ in (1), condition (12) will be satisfied only with $\lim_{t\to\infty} S(t) = 0$ and the date of exhaustion of the oil stock — denote it by T_m — will be finite. Furthermore, we must have $y_m(T_m) = 0$, since the implicit oil price $(c_a + e^{rt}[\lambda_m(t) + \lambda_w(t)k_m])$ must reach the choke price $(u'_m(0))$ at the exact moment of exhaustion of the oil stock. Otherwise there would be a jump in the implicit price of oil and it would always pay to delay exhaustion in order to benefit from that jump.

From condition (10), we know that $\lambda_m(t)$, the shadow value of oil, is constant over time. Henceforth we will simply write it λ_m, to signify this. As for $\lambda_w(t)$, the shadow value of water, we know that it is constant while the stock of water is positive and decreasing over time while the stock of water is zero. Henceforth, we will denote it simply λ_w over intervals of time where the stock of water is known to be positive and explicitly as $\lambda_w(t)$ otherwise.

From the above set of conditions, we can immediately derive the following propositions:

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[4]See Seierstad and Sydsaeter (1987), Theorem 16, page 244, on the necessity of the transversality conditions.

Proposition 1 *Over any interval of time such that $S(t) > 0$, we must have $y_m(t) > 0$.*

Proof. Since the oil stock must be fully depleted, $y_m(t)$ will necessarily become positive at some point in time. Suppose $y_m(t) = 0$ for $t \in [t_1, t_2)$ and $y_m(t_2) > 0$. From (8) and the assumption that $u''(y_m) < 0$ for all $y_m > 0$, it follows that $\lambda_w(t_2) < \lambda_w(t_1)$. Hence, by condition (9), there must be a nondegenerate subinterval $[\theta, t_2)$ of $[t_1, t_2)$ along which $X(t) = 0$. But then the initial conditions at t_2 are the same as at θ, since $S(t_2) = S(\theta)$ and $X(t_2) = X(\theta) = 0$. Therefore, if $y_m(\theta) = 0$ was optimal, so must be $y_m(t_2) = 0$, a contradiction. ∎

Proposition 2 *Along any interval of time where $X(t) > 0$ and $S(t) > 0$, (i) if both sectors are active, then $\dot{y}_a(t) < 0$ and $\dot{y}_m(t) < 0$ over that interval; (ii) if only the oil sector is active, then $\dot{y}_a(t) = 0$ and $\dot{y}_m(t) < 0$.*

Proof. When $X(t) > 0$ and $S(t) > 0$, from (9) and (10), $\dot{\lambda}_w(t) = \dot{\lambda}_m(t) = 0$ and hence, differentiating (7) and (8) with respect to time, we get:

$$\dot{y}_a(t) = \frac{r e^{rt} k_a \lambda_w}{u''_a(y_a(t))} < 0 \tag{13}$$

$$\dot{y}_m(t) = \frac{r e^{rt} [\lambda_m + k_m \lambda_w]}{u''_m(y_m(t))} < 0, \tag{14}$$

which proves part (i) of the proposition. Part (ii) follows immediately from (14) and the fact that if $y_a(t) = 0$ over the interval in question, then $\dot{y}_a(t) = 0$ over that interval. ∎

Proposition 3 *Along any interval of time where $X(t) = 0$ and $S(t) > 0$, (i) if both sectors are active, then $\dot{y}_a(t) > 0$ and $\dot{y}_m(t) < 0$; (ii) if only the oil sector is active, then $\dot{y}_a(t) = 0$ and $\dot{y}_m(t) = 0$.*

Proof. If $X(t) = 0$ over some interval of time, then $\dot{X}(t) = 0$ over that interval. This means that $k_a y_a(t) + k_m y_m(t) = \bar{x}$ and therefore:

$$k_a \dot{y}_a(t) + k_m \dot{y}_m(t) = 0. \tag{15}$$

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Differentiating (7) and (8) with respect to time and using (10), we find that:

$$\dot{y}_a(t) = \frac{e^{rt} k_a [r\lambda_w(t) + \dot{\lambda}_w(t)]}{u_a''(y_a(t))} \tag{16}$$

$$\dot{y}_m(t) = \frac{e^{rt} \{r\lambda_m + k_m [r\lambda_w(t) + \dot{\lambda}_w(t)]\}}{u_m''(y_m(t))}. \tag{17}$$

Substituting into (15), we find:

$$r\lambda_w(t) + \dot{\lambda}_w(t) = \frac{\dfrac{-rk_m\lambda_m}{u_m''(y_m(t))}}{\dfrac{k_a^2}{u_a''(y_a(t))} + \dfrac{k_m^2}{u_m''(y_m(t))}} < 0. \tag{18}$$

Therefore $\dot{y}_a(t) > 0$, from (16), and $\dot{y}_m(t) < 0$, from (15), which proves part (i) of the proposition. The proof of part (ii) follows immediately from the fact that if $y_a(t) = 0$ over the interval in question, then $y_m(t) = \bar{y}_m$ over that interval. ∎

It will be useful to distinguish between the case where $\hat{y}_a > \bar{y}_a$ and that where $\hat{y}_a < \bar{y}_a$. In the first case, discussed in next Section 3, the water availability poses a long-run constraint on agriculture, since, even in the absence of the oil sector, a water use of $k_a\hat{y}_a$ cannot be sustained. In the second case, discussed in Section 4, a water use of $k_a\hat{y}_a$ can be sustained indefinitely after the stock of oil has been depleted.

3 The natural water inflow poses a long-run constraint on agriculture

Let us now consider the case where $\hat{y}_a > \bar{y}_a$. It is useful to first characterize the two extreme situations where there is only either an agricultural or an oil sector in operation. After having done this, we turn to the analysis of the situation where the two sectors coexist. We treat the initial oil stock as a pivotal parameter and define a number of critical values of this stock that are important in determining the shapes of the optimal paths. These critical values are then used to fully characterize the optimal paths.

3.1 Only one of the two sectors is active

If there were no oil sector then, when $\hat{y}_a > \bar{y}_a$, two phases can be distinguished. The first phase ends at T_w, which denotes the date at which the water stock is exhausted. During that phase, the water stock is positive, so that λ_w is a constant, and condition (7) is satisfied with equality, meaning that:

$$u'_a(y_a(t)) = c_a + e^{rt}\lambda_w k_a.$$

Agricultural production exceeds $\bar{y}_a$ and is decreasing towards $\bar{y}_a$, with $\dot{y}_a(t)$ given by (13). The values of T_w and λ_w are obtained from:

$$u'_a(\bar{y}_a) = c_a + e^{rT_w}\lambda_w k_a \quad \text{and} \quad \int_0^{T_w} (k_a y_a(t) - \bar{x})dt = X_0.$$

The second phase begins at T_w and has $y_a(t) = \bar{y}_a$ for all $t > T_w$. Therefore $X(t) = 0$ for all $t > T_w$.

Once the existence of the oil sector is taken into account, these two phases will characterize the agricultural production path after the oil stock is exhausted, provided it is exhausted before the water stock. If the water stock is exhausted before the oil stock, then agricultural production enters the second phase as soon as the oil stock is exhausted. Since the oil stock is always exhausted in finite time, it follows that if $\hat{y}_a > \bar{y}_a$, the optimal path always ends with a final phase during which $y_a = \bar{y}_a$ and $X(t) = 0$.[5]

If on the other hand there were no agricultural sector, then two cases need to be distinguished, according to whether the initial stock of water is abundant relative to the initial stock of oil or not. In the first case, the stock of oil is exhausted before the stock of water and therefore $\lambda_w = 0$, since by assumption there is no other use for water. We would therefore have a pure Hotelling-type path, with the rate of extraction given by condition (8) satisfied with equality, so that:

$$u'_m(y_m(t)) = c_m + e^{rt}\lambda_m, \tag{19}$$

with λ_m a constant from condition (10). Oil extraction decreases towards zero, with $\dot{y}_m(t)$

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[5]This assures that the transversality condition (11) is satisfied.

given by (14). The date of exhaustion of the oil stock, T_m, and λ_m are determined by:

$$u'_m(0) = c_a + e^{rT_m}\lambda_m \quad \text{and} \quad \int_0^{T_m} y_m(t)dt = S_0.$$

This first case occurs if, for $y_m(t)$ given by (19) and the values of T_m and λ_m just determined, we have:

$$\int_0^{T_m} (k_m y_m(t) - \bar{x})dt \leq X_0. \tag{20}$$

Otherwise we have the second case, which is characterized by three phases. In a first phase, the water stock is being exhausted and, from condition (8):

$$u'_m(y_m(t)) = c_m + e^{rt}[\lambda_m + \lambda_w k_m], \tag{21}$$

with λ_m and λ_w both positive constants, by (10) and (9). The rate of oil extraction is decreasing towards $\bar{y}_m$, with $\dot{y}_m(t)$ given by (14), until the exhaustion of the water stock at T_w. Then follows a second phase during which the oil extraction rate is constrained by the natural inflow to $\bar{y}_m$. This phase ends at some date $\widetilde{T} \geq T_w$ such that:

$$u'_m(\bar{y}_m) = c_m + e^{r\widetilde{T}}\lambda_m.$$

From that date on, there follows a Hotelling-type path like the one just described in the first case. Notice that if (20) happened to be satisfied with strict inequality, then we are left with just the Hotelling-type path of the first case: the second phase collapses, since then $\lambda_w = 0$, and $T_m = T_w$.

3.2 Both sectors are active

Consider now the situation where both sectors are present from the outset. We can immediately prove the following:

Proposition 4 *If $\hat{y}_a > \bar{y}_a$, then once the stock of water is exhausted, it will never be replenished.*

Proof. As just shown above, if $\hat{y}_a > \bar{y}_a$, the optimal path always ends with a phase during which $X(t) = 0$. Therefore, if an interval of time during which $X(t) = 0$ is followed by an interval of time during which $X(t) > 0$, there must follow a third interval of time during which $X(t) = 0$. Suppose this were the case. Then it must be that $S(t) > 0$ at the beginning of the second interval, for otherwise it is optimal to keep $X(t) = 0$ forever. By Proposition 2, neither $y_a(t)$ nor $y_m(t)$ can be increasing during an interval where $S(t) > 0$ and $X(t) > 0$. But the assumed sequence of intervals necessitates that $\dot{X}(t)$ be at first positive and then negative during the second interval, which means that total water usage must increase from a level lower than $\bar{x}$ to eventually a level higher than $\bar{x}$. Therefore the assumed sequence cannot be optimal. ∎

In order to pursue the case where both sectors are present, it will now be useful to define a number of threshold levels on S_0, the initial stock of oil. These critical values of S_0 will determine whether, for any given initial water stock, X_0: i) the water stock is exhausted before the oil stock or not; ii) there is a period of inactivity of the agricultural sector or not; iii) there is initially a period of inactivity of the agricultural or not.

These critical values are defined as follows:

i) Consider a situation where both the stock of water and the stock of oil were to be exhausted at exactly the same instant of time, so that $T_w = T_m$ and:

$$\int_0^{T_m} y_m(t)dt = S_0 \tag{22}$$

$$\int_0^{T_m} [k_a y_a(t) - \bar{x}]dt + k_m S_0 = X_0. \tag{23}$$

From (9) and (10), we know that λ_m and λ_w are constant for all $t \in [0, T_m]$ and from conditions (7) and (8), we must have:

$$u_a'(y_a(t)) = c_a + e^{rt}\lambda_w k_a, \quad t \in [0, T_m] \tag{24}$$

$$u_m'(y_m(t)) = c_m + e^{rt}[\lambda_m + \lambda_w k_m], \quad t \in [0, T_m]. \tag{25}$$

Furthermore, $y_a(t) = \bar{y}_a$ for all $t \in [T_m, \infty)$, as demonstrated above, and $y_m(T_m) = 0$,

since the oil price $(c_a + e^{rt}[\lambda_m + \lambda_w k_m])$ must reach the choke price $(u'_m(0))$ at the moment of exhaustion of the oil stock. This means that:

$$u'_a(\bar{y}_a) = c_a + e^{rT_m}\lambda_w k_a \tag{26}$$

$$u'_m(0) = c_m + e^{rT_m}[\lambda_m + \lambda_w k_m]. \tag{27}$$

From (22), (25) and (27) we can uniquely determine T_m, $\lambda_m + \lambda_w k_m$ and the entire path of $y_m(t)$. Then λ_w and the path of $y_a(t)$ for $t \in [0, T_m]$ follow from (24) and (26). Finally (23) determines, for any X_0, the level of S_0 such that the simultaneous activity of both sectors just solved for exactly exhausts X_0 at T_m. Call this threshold level $\hat{S}_0(X_0)$. It is monotonically increasing in X_0 and it must go through the origin, since otherwise we could not have $T_m = T_w$ at $X_0 = 0$. For any $S_0 < (>) \hat{S}_0(X_0)$, we will have $T_w > (<) T_m$.

ii) Now consider a hypothetical situation where $X_0 = 0$, $y_a(0) = 0$, with (7) just satisfied with equality at $t = 0$, and $y_a(t) > 0$ for all $t > 0$. It is then optimal to maintain $X_0 = 0$ indefinitely (Proposition 4). Therefore, from (2), we would have:

$$k_a y_a(t) + k_m y_m(t) = \bar{x}, \quad t \in [0, \infty), \tag{28}$$

from which it follows that $y_m(0) = \bar{y}_m$. We also know that along an optimal oil extraction path, $y_m(t) > 0$ for $t \in [0, T_m)$, from Proposition 1, and that $y_m(T_m) = 0$. The solution being interior in both sectors, we must therefore have, from (7), (8) and (28):

$$u'_a\left(\bar{y}_a - \frac{k_m}{k_a}y_m(t)\right) = c_a + e^{rt}\lambda_w(t)k_a, \quad t \in [0, T_m] \tag{29}$$

$$u'_m(y_m(t)) = c_m + e^{rt}[\lambda_m + \lambda_w(t)k_m], \quad t \in [0, T_m], \tag{30}$$

and hence:

$$k_a u'_m(y_m(t)) - k_m u'_a\left(\bar{y}_a - \frac{k_m}{k_a}y_m(t)\right) = k_a c_m + e^{rt}k_a\lambda_m - k_m c_a \quad t \in [0, T_m]. \tag{31}$$

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Knowing that $y_a(0) = 0$, $y_m(0) = \bar{y}_m$ and $y_m(T_m) = 0$, we therefore have:

$$k_a u'_m(\bar{y}_m) - k_m u'_a(0) = k_a c_m + k_a \lambda_m - k_m c_a \tag{32}$$

$$k_a u'_m(0) - k_m u'_a(\bar{y}_a) = k_a c_m + e^{rT_m} k_a \lambda_m - k_m c_a. \tag{33}$$

Conditions (32) and (33) determine λ_m and T_m. The entire path of $y_m(t)$ then follows from (31) and that of $y_a(t)$ from (28).

The resulting cumulative oil extraction during $[0, T_m]$ is:

$$\int_0^{T_m} y_m(t) dt = \widetilde{S}.$$

The paths thus derived are optimal if and only if $S_0 = \widetilde{S}$. If we had $S_0 > \widetilde{S}$, the pressure on water demand from the oil sector and hence on the shadow value of water would be such that the nonnegativity constraint on $y_a(t)$ is strictly binding and the right hand-side of (29) is strictly greater than the left-hand side at $t = 0$, with $y_a(0) = 0$. As a consequence, the agricultural sector will initially be inactive over some positive interval of time, until the equality is reestablished in (29). On the other hand, if $S_0 < \widetilde{S}$, then the right hand-side of (29) is strictly smaller than the left-hand side at $t = 0$ with $y_a(0) = 0$ and the optimal solution requires $y_a(0) > 0$.

Next, assume $X_0 > 0$ and $S_0 > \hat{S}_0(X_0)$, and therefore $T_w < T_m$, and consider a scenario where the agricultural sector is active throughout the interval of time over which the water stock is being exhausted, just becomes inactive at the exact moment that the water stock is exhausted and immediately becomes active again. Hence $y_a(t) > 0$ for $t \in [0, T_w)$, $y_a(T_w) = 0$ and $y_a(t) > 0$ for $t \in (T_w, \infty)$.

If this scenario is to constitute an optimal solution, it must be the case that $S(T_w) = \widetilde{S}$, with the optimal paths over the interval $[T_w, T_m]$ being characterized as in the hypothetical situation just described with $X_0 = 0$ and $S_0 = \widetilde{S}$. Therefore, in order for the oil stock to be exhausted over the interval $[0, T_m]$ and for the water stock to be

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exhausted over the interval $[0, T_w]$, it is necessary that:

$$\int_0^{T_w} y_m(t)dt = S_0 - \widetilde{S} \tag{34}$$

and

$$\int_0^{T_w} [k_a y_a(t) - \bar{x}]dt + k_m[S_0 - \widetilde{S}] = X_0. \tag{35}$$

Over the interval $[0, T_w]$, λ_m and λ_w are constant and the solution for both sectors is interior, so that:

$$u_a'(y_a(t)) = c_a + e^{rt}\lambda_w k_a, \quad t \in [0, T_w] \tag{36}$$

$$u_m'(y_m(t)) = c_m + e^{rt}[\lambda_m + \lambda_w k_m], \quad t \in [0, T_w]. \tag{37}$$

At $t = T_w$, we must have $y_m(T_w) = \bar{y}_m$, since $y_a(T_w) = 0$ by assumption. Hence:

$$u_a'(0) = c_a + e^{rT_w}\lambda_w k_a \tag{38}$$

$$u_m'(\bar{y}_m) = c_m + e^{rT_w}[\lambda_m + \lambda_w k_m]. \tag{39}$$

Conditions (34), (37) and (39) uniquely determine T_w, $\lambda_m + \lambda_w k_m$ and the path of $y_m(t)$ over the interval of time $[0, T_w]$. Then λ_w and the path of $y_a(t)$ over the same interval are determined from (38) and (36).

In order for this scenario to constitute an optimal solution, the constraint (35) must also be satisfied. This determines, for any X_0, the level of S_0 that will exactly exhaust the water stock at T_w, determined above. We will denote it $\widetilde{S}_0(X_0)$. It is monotonically increasing in X_0, with $\widetilde{S}_0(0) = \widetilde{S}$.

The scenario just described will be optimal if and only $S_0 = \widetilde{S}_0(X_0)$. If $S_0 < \widetilde{S}_0(X_0)$, then there is relatively less pressure on water demand from the oil sector and the left-hand side of (38) exceeds the right-hand side: $u_a'(0) > c_a + e^{rT_w}\lambda_w k_a$. Optimality then requires $y_a(T_w) > 0$. If on the other hand $S_0 > \widetilde{S}_0(X_0)$, then the demand for water from the oil sector pushes the shadow value of water up to a level such that the nonnegativity constraint on $y_a(t)$ becomes strictly binding at T_w and $u_a'(0) < c_a + e^{rT_w}\lambda_w k_a$. The

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agricultural sector will therefore be inactive over a positive interval of time instead of just at $t = T_w$.

iii) Imagine now a hypothetical situation where, starting with a stock of water $X_0 = 0$, we have $y_a(t) = 0$ for $t \in [0, \widetilde{T}]$ and $y_a(t) > 0$ for $t \in (\widetilde{T}, \infty)$, where $\widetilde{T}$ is such that $S(\widetilde{T}) = \widetilde{S}$. For $t > \widetilde{T}$, the optimal path is then the one determined above in the hypothetical situation where $X_0 = 0$ and $S_0 = \widetilde{S}$. From (7), the shadow value of the water stock, $\lambda_w(t)$, over the interval $[0, \widetilde{T}]$ will satisfy:

$$u'_a(0) = c_a + e^{rt} \lambda_w(t) k_a,$$

and hence $\dot{\lambda}_w(t) = -r\lambda_w(t)$. From (7) and (8) evaluated at $t = 0$, the shadow value associated to the oil stock will be given by:

$$k_a u'_m(\bar{y}_m) - k_m u'_a(0) = k_a c_m + k_a \lambda_m - k_m c_a.$$

The cumulative oil extraction over the interval $[0, \widetilde{T}]$ is:

$$\overline{S} := \widetilde{T}\bar{y}_m + \widetilde{S}. \tag{40}$$

The hypothetical path just described will be optimal for all $t \in [0, \infty)$ if and only if $S_0 = \overline{S}$. If $S_0 > \overline{S}$, then $\widetilde{T}$ must be greater than the one determined in (40). If $\widetilde{S} < S_0 < \overline{S}$, then the pressure on the demand for water from the oil sector will be lower than if $S_0 = \overline{S}$ and it becomes optimal to begin with $y_a(t) > 0$ over some interval $[0, \tau]$, with $\tau < \widetilde{T}$.

Finally, assume $X_0 > 0$ and $S_0 > \widetilde{S}(X_0)$ and consider a scenario where $y_a(t) = 0$ for all $t \in [0, T_w]$, with (7) just satisfied with equality at $t = 0$. This scenario cannot be optimal unless $S(T_w) = \overline{S}$, with the optimal path over the interval $[T_w, \infty]$ being given by the one found above for the hypothetical situation where $X_0 > 0$ with $S_0 = \overline{S}$. Furthermore, since the oil stock is exhausted over the interval $[0, T_m]$ and the water

stock is exhausted over the interval $[0, T_w]$, we must have:

$$\int_0^{T_w} y_m(t)dt = S_0 - \overline{S} \tag{41}$$

and

$$k_m[S_0 - \overline{S}] = X_0. \tag{42}$$

Since, by assumption, condition (7) is just satisfied with equality at $t = 0$, we have:

$$u_a'(0) = c_a + \lambda_w k_a. \tag{43}$$

This determines the constant value of λ_w that holds for all $t \in [0, T_w]$. It follows that $u_a'(0) < c_a + e^{rt}\lambda_w k_a$ for all $t \in (0, T_w]$, assuring that the agricultural sector remains inactive while the water stock is being depleted.

From (43) and (8), we must have:

$$k_a u_m'(y_m(t)) = k_a c_m + e^{rt}[k_a \lambda_m + k_m u_a'(0) - k_m c_a], \quad t \in [0, T_w]. \tag{44}$$

Knowing that $y_m(T_w) = \bar{y}_m$, λ_m and T_w can be determined from (41) and (44) evaluated at T_w. The entire path of $y_m(t)$ then follows from (44).

This will constitute the optimal solution under the assumed scenario provided that S_0 is such that condition (42) is satisfied, which implies that $\overline{S}_0(X_0) = \overline{S} + \dfrac{X_0}{k_m}$. The function $\overline{S}_0(X_0)$ is monotonically increasing in X_0, with $\overline{S}_0(0) = \overline{S}$.

If $\widetilde{S}_0(X_0) < S_0 < \overline{S}_0(X_0)$, the water demand from the oil sector puts relatively less pressure on the value of water than when $S_0 = \overline{S}_0(X_0)$. As a result $u_a'(0) > c_a + \lambda_w k_a$. It therefore becomes optimal for the agricultural sector to be active during an interval of time $[0, \tau]$, where $\tau < \widetilde{T}$ denotes the time at which

$$u_a'(0) = c_a + e^{r\tau}\lambda_w k_a. \tag{45}$$

The agricultural sector then becomes inactive and remains so until the water stock

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reaches $\widetilde{S}$, at time $\widetilde{T}$. On the other hand, if $S_0 > \overline{S}_0(X_0)$, then $u_a'(0) < c_a + \lambda_w k_a$ and the agricultural sector is inactive from the start and remains so until $\widetilde{T}$.

3.3 The optimal paths

The threshold values $\widehat{S}_0(X_0)$, $\widetilde{S}_0(X_0)$, $\widetilde{S}$, $\overline{S}_0(X_0)$ and $\overline{S}$ just defined now allow us to characterize the optimal paths in $(X(t), S(t))$-space.

For any given $X_0 > 0$, the optimal paths of the agricultural sector and of the oil sector have the following properties, where $y_a^*(t)$ and $y_m^*(t)$ denote the interior solution to (7) and (8) respectively:

If $S_0 \geq \overline{S}_0(X_0)$:

$$
y_a(t) = \begin{cases}
0 & \text{for } t \in [0, \widetilde{T}]; \\
\bar{y}_a - \frac{k_m}{k_a} y_m(t) > 0 & \text{for } t \in (\widetilde{T}, T_m); \\
\bar{y}_a & \text{for } t \in [T_m, \infty).
\end{cases}
$$

$$
y_m(t) = \begin{cases}
y_m^*(t) > 0 & \text{for } t \in [0, T_w); \\
\bar{y}_m & \text{for } t \in [T_w, \widetilde{T}]; \\
y_m^*(t) > 0 & \text{for } t \in (\widetilde{T}, T_m); \\
0 & \text{for } t \in [T_m, \infty).
\end{cases}
$$

If $\overline{S}_0(X_0) > S_0 > \widetilde{S}_0(X_0)$:

$$
y_a(t) = \begin{cases}
y_a^*(t) > 0 & \text{for } t \in [0, \tau); \\
0 & \text{for } t \in [\tau, \widetilde{T}]; \\
\bar{y}_a - \frac{k_m}{k_a} y_m(t) > 0 & \text{for } t \in (\widetilde{T}, T_m); \\
\bar{y}_a & \text{for } t \in [T_m, \infty).
\end{cases}
$$

$$
y_m(t) = \begin{cases}
y_m^*(t) > 0 & \text{for } t \in [0, T_w); \\
\bar{y}_m & \text{for } t \in [T_w, \widetilde{T}]; \\
y_m^*(t) > 0 & \text{for } t \in (\widetilde{T}, T_m); \\
0 & \text{for } t \in [T_m, \infty).
\end{cases}
$$

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If $S_0 = \widetilde{S}_0(X_0)$:

$$y_a(t) = \begin{cases} y_a^*(t) > 0 & \text{for } t \in [0, T_w); \\ 0 & \text{for } t = T_w; \\ \bar{y}_a - \frac{k_m}{k_a} y_m(t) > 0 & \text{for } t \in (T_w, T_m); \\ \bar{y}_a & \text{for } t \in [T_m, \infty). \end{cases}$$

$$y_m(t) = \begin{cases} y_m^*(t) > 0 & \text{for } t \in [0, T_m); \\ 0 & \text{for } t \in [T_m, \infty). \end{cases}$$

If $\widetilde{S}_0(X_0) > S_0 > \widehat{S}_0(X_0)$:

$$y_a(t) = \begin{cases} y_a^*(t) > 0 & \text{for } t \in [0, T_w); \\ \bar{y}_a - \frac{k_m}{k_a} y_m(t) > 0 & \text{for } t \in (T_w, T_m); \\ \bar{y}_a & \text{for } t \in [T_m, \infty). \end{cases}$$

$$y_m(t) = \begin{cases} y_m^*(t) > 0 & \text{for } t \in [0, T_m); \\ 0 & \text{for } t \in [T_m, \infty). \end{cases}$$

If $\widehat{S}_0(X_0) \geq S_0$:

$$y_a(t) = \begin{cases} y_a^*(t) > 0 & \text{for } t \in [0, T_w); \\ \bar{y}_a & \text{for } t \in [T_w, \infty). \end{cases}$$

$$y_m(t) = \begin{cases} y_m^*(t) > 0 & \text{for } t \in [0, T_m); \\ 0 & \text{for } t \in [T_m, \infty). \end{cases}$$

Figure 1 illustrates the optimal paths in $(X(t), S(t))$-space for different values of S_0 and a given X_0.

The case of $\overline{S}_0(X_0) > S_0 > \widetilde{S}_0(X_0)$ is well suited to illustrate the time paths of the different implicit prices. In that case, there are five distinct phases, as depicted in Figure 2.

In the first phase, during the interval $[0, \tau)$, the stock of water is positive and both sectors are active, with $(y_a(t), y_m(t)) = (y_a^*(t), y_m^*(t))$. During this phase, the full marginal cost of production of the agricultural sector, $c_a + k_a e^{rt} \lambda_w$, is increasing. It reaches the agricultural choke price, $u_a(0)$, at $t = \tau$, at which time the agricultural sector stops producing.

Then begins the second phase, which lasts throughout the interval $[\tau, T_w)$. Since the full marginal cost of agriculture continues to increase over that interval, the agricultural sector
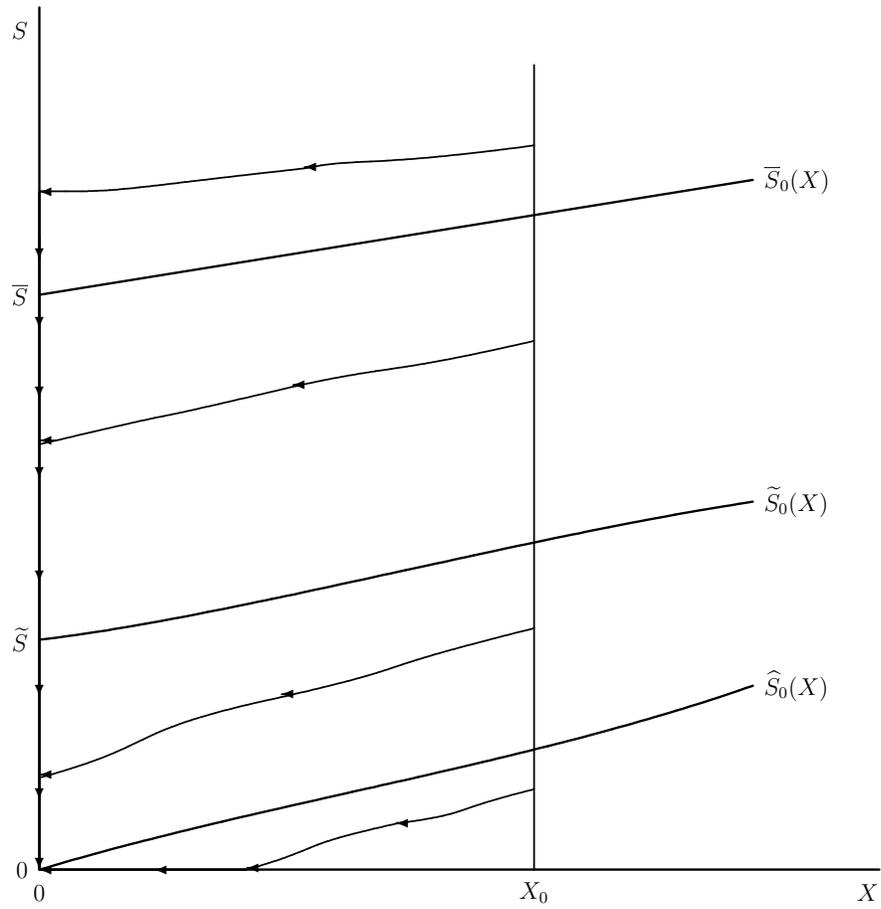
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Figure 1 : The optimal paths in (X, S)-space

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$u'_m(y_m(t))$

$u'_m(0)$

$u'_m(\bar{y}_m)$

$c_m + \lambda_m + k_m\lambda_w$

$c_m + \lambda_m$

c_m

$c_m + e^{rt}[\lambda_m + k_m\lambda_w(t)]$

$\tau \qquad T_w \qquad \widetilde{T} \qquad\qquad\qquad T_m \qquad t$

$u'_a(y_a(t))$

$u'_a(0)$

$c_a + k_a\lambda_w$

$c_a + k_a\lambda_w(t)$

$u'_a(\bar{y}_a)$

c_a

$\tau \qquad T_w \qquad \widetilde{T} \qquad\qquad\qquad T_m \qquad t$
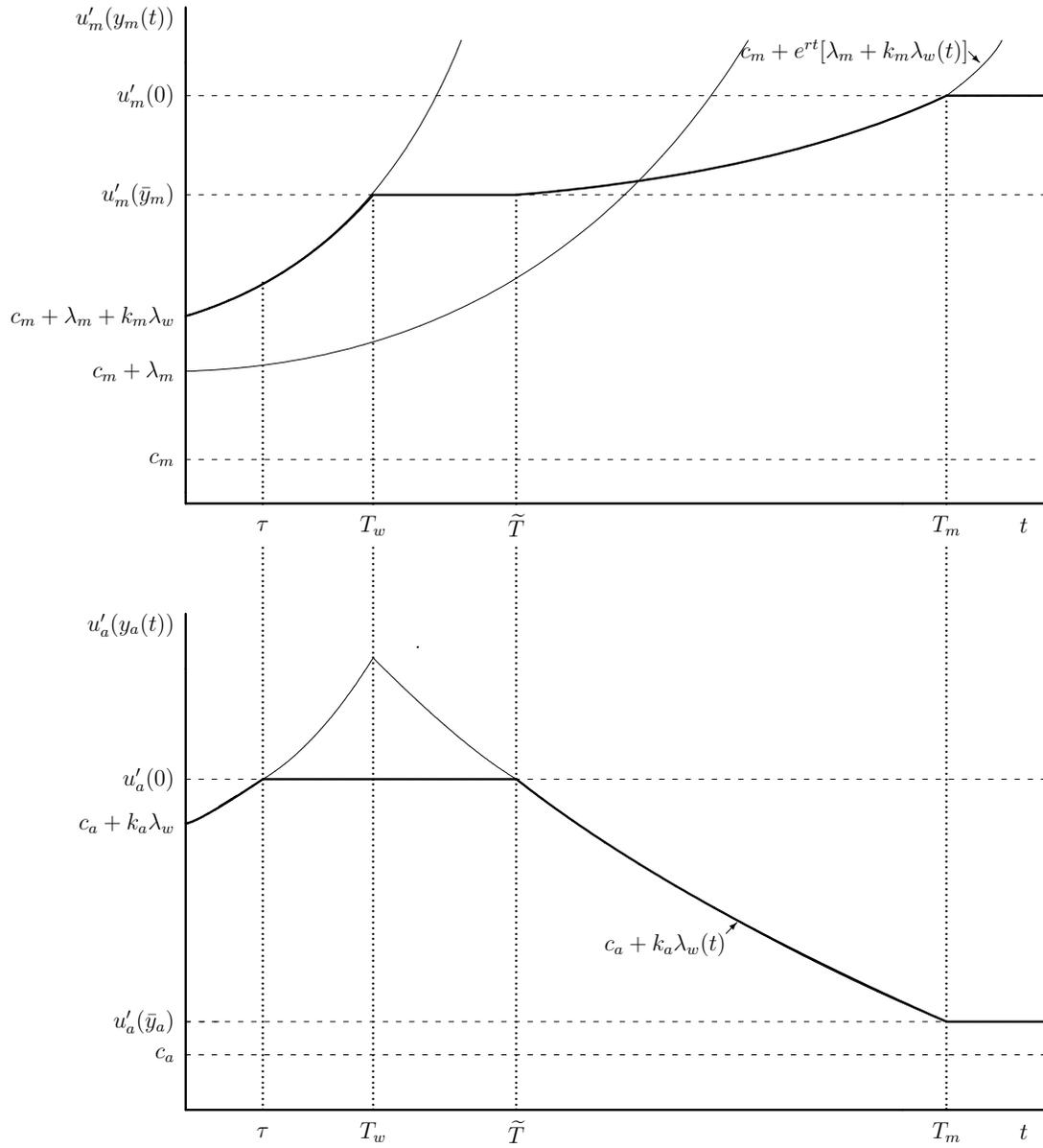
Figure $2 : \hat{y}_a > \bar{y}_a$ and $\overline{S}_0(X_0) > S_0 > \widetilde{S}_0(X_0)$

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remains inactive and we have $(y_a(t), y_m(t)) = (0, y_m^*(t))$.

At time T_w, the water stock is exhausted. From that point on, the water stock will remain at zero (Proposition 4) and total water consumption becomes constrained by $\bar{x}$, the natural water inflow. Although the shadow value of water then begins decreasing, the full marginal cost of agricultural production is higher than the choke price and will remain so for some time.

We therefore have a third phase, over the interval $[T_w, \widetilde{T}]$, during which $(y_a(t), y_m(t)) = (0, \bar{y}_m)$. The implicit price of oil remains constant over that interval, at $u_m'(\bar{y}_m) = c_m + e^{rt}[\lambda_m + \lambda_w(t)k_m]$, since water consumption is constrained to $\bar{x}$ and hence oil production is constrained to $\bar{y}_m$. Note that since $y_a(\tau) = y_a(\widetilde{T}) = 0$, it must be the case that $\lambda_w(\widetilde{T}) = e^{-r(\widetilde{T}-\tau)}\lambda_w(\tau)$, with $\lambda_w(\tau) = \lambda_w$, the constant shadow value of water over the interval $[0, T_w]$. The new shadow value of water is decreasing during that third phase, because, as the oil stock decreases, so does the pressure on water demand. At time $\widetilde{T}$, the full marginal cost of agriculture becomes just low enough for agricultural production to resume.

Then begins a fourth phase, during which $(y_a(t), y_m(t)) = (\bar{y}_a - \frac{k_m}{k_a}y_m^*(t), y_m^*(t))$ until the oil stock is exhausted, at T_m. Over the interval $(\widetilde{T}, T_m)$, the full marginal cost of oil production is increasing and eventually reaches the choke price for oil at T_m, when $u_m'(0) = c_m + e^{rT_m}[\lambda_m + \lambda_w(T_m)k_m]$. The full cost of agriculture is decreasing during this phase, until at T_m we have $u_a'(\bar{y}_a) = c_a + e^{rT_m}\lambda_w(T_m)k_a$.

In the final phase there is no more oil, so there remains only the agricultural sector. Therefore $(y_a(t), y_m(t)) = (\bar{y}_a, 0)$ for all $t \in [T_m, \infty)$ and the implicit price of agriculture is constant at $u_a'(\bar{y}_a)$.

The other cases are now easily characterized. If $S_0 \geq \overline{S}_0(X_0)$, the price paths have exactly the same configuration as in Figure 2. Only now the pressure on water demand from the oil sector is so high that $\tau = 0$ and the first phase collapses: the agricultural sector is inactive from the beginning and remains inactive until time $\widetilde{T}$.

If $S_0 = \widetilde{S}_0(X_0)$, then $\tau = T_w = \widetilde{T}$, which means that the second and third phases collapse. The agricultural sector is active throughout except for an instant, at T_w. We therefore have a phase ending at T_w during which the water stock is being exhausted, with both sectors active and the full marginal cost of production increasing in both sectors. This is followed by

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a phase ending at T_m during which the remaining oil stock is being exhausted, still with both sectors active, but now with the full marginal cost of agriculture decreasing and that of oil still increasing, although at a slower rate due to the fact that $\lambda_w(t)$ is now decreasing. The final phase has the agricultural sector producing indefinitely at the full capacity permitted by the natural water inflow and the price of agriculture constant. This case is a borderline case. It separates the cases where, given the initial water stock, the size of the initial oil stock dictates that the agricultural sector should remain inactive during some period of time, from those cases where it does not.

When $S_0 < \widetilde{S}_0(X_0)$, then the initial oil stock is not sufficiently large, relative to the water stock, for it to be optimal to interrupt agricultural production in order to favor oil production. Therefore the agricultural sector will always be active, $\tau = T_w = \widetilde{T}$, and there are only three phases, as in the case when $S_0 = \widetilde{S}_0(X_0)$.

Two subcases of $S_0 < \widetilde{S}_0(X_0)$ need to be distinguished. If $\widetilde{S}_0(X_0) > S_0 > \widehat{S}_0(X_0)$, then the water stock will be exhausted before the oil stock. The three phases are characterized, on the production side, by: $(y_a(t), y_m(t)) = (y_a^*(t), y_m^*(t))$ during the interval $[0, T_w)$; $(y_a(t), y_m(t)) = (\bar{y}_a - \frac{k_m}{k_a} y_m^*(t), y_m^*(t))$ during the interval $[T_w, T_m)$; $(y_a(t), y_m(t)) = (\bar{y}_a, 0)$ during the interval $[T_m, \infty)$. As for the implicit price paths, both are increasing during the interval $[0, T_w)$, while the water stock is being depleted, but decreasing for agriculture and increasing for oil during the interval $[T_w, T_m)$, at which point begins the final phase, with the implicit price of agriculture given by $u_a'(\bar{y})$ for all $t \geq T_m$.

On the other hand, if $S_0 < \widehat{S}_0(X_0)$, the initial oil stock is small enough that it is optimal to exhaust it before the water stock. Then the three phases are characterized on the production side by: $(y_a(t), y_m(t)) = (y_a^*(t), y_m^*(t))$ during the interval $[0, T_m)$; $(y_a(t), y_m(t)) = (y_a^*(t), 0)$ during the interval $[T_m, T_w)$; $(y_a(t), y_m(t)) = (\bar{y}_a, 0)$ during the interval $[T_w, \infty)$. During the first of those phases, the full marginal costs and hence the implicit prices are increasing in both sectors, until there is no more oil. Since the water stock is still positive at that point, the shadow value of water remains constant at λ_w and therefore the implicit price of agriculture keeps increasing, until the water stock is exhausted. This occurs at T_w, when $u_a'(\bar{y}_a) = c_a + e^{rT_w}\lambda_w k_a$. Then follows the usual final phase, with the price of agriculture constant at $u_a'(\bar{y}_a)$ for all $t \geq T_w$.

4 The natural water inflow poses no constraint on agriculture

Consider now the case where $\hat{y}_a < \bar{y}_a$. In this case water availability poses no constraint on the agricultural sector and, if there were no oil sector, the shadow value of water would be zero. From condition (7) we then have $u'_a(y_a(t)) = c_a$ and hence $y_a(t) = \hat{y}_a$ for all $t \geq 0$. This will obviously be the case for all $t \geq T_m$, once the existence of an oil sector is taken into account.[6]

If there were no agricultural sector, then exactly the same two cases as in Section 3 need to be distinguished. In one case, water is abundant, $\lambda_w = 0$, and we have a pure Hotelling-type path for the oil sector. In the other case, water is scarce and the optimal path would be characterized by the same three phases derived in Section 3.

Now let the two sectors be present from the outset. All the threshold levels of Section 3 remain pertinent and can be similarly defined. Clearly, if $S_0 < \hat{S}_0(X_0)$, so that $T_w > T_m$, then water availability is never a constraint for either sector and $\lambda_w = 0$ for all $t > 0$. We then have $y_a(t) = \hat{y}_a$ and oil production follows the same Hotelling-type path as if there were no agricultural sector.

It not necessary however that $T_m < T_w$ in order for water to have no value. Indeed, assume $S_0 > \hat{S}_0(X_0)$, so that $T_m > T_w$, and consider a hypothetical situation where $y_a(t) = \hat{y}_a$ for all $t \in [0, T_m]$ and λ_m, T_m and $y_m^*(t)$ solve:

$$u'(y_m(t)) = c_m + e^{rt}\lambda_m, \quad t \in [0, T_m],$$

$$u'() = c_m + e^{rT_m}\lambda_m,$$

and

$$\int_0^{T_m} y_m(t)dt = S_0.$$

---

[6]Since $\hat{y}_a < \bar{y}_a$, this means that the water stock will be replenished once the oil stock is exhausted. It would be natural to impose an upper bound on the stock of water. We have chosen to ignore this issue here, since, if any excess can simply be wasted or freely disposed of, the existence of this upper bound will have no impact on the nature of the optimal paths. Note that in this case, since the stock of water is positive in the end, the transversality condition (11) will be satisfied with the shadow value of water becoming zero.

For this to constitute the optimal solution, S_0 must be such that it also satisfies:

$$T_m[k_a \hat{y}_a - \bar{x}] + k_m S_0 = X_0. \tag{46}$$

Denote the level of S_0 required to satisfy (46) by $S_0^H(X_0)$. Then for any initial oil stock $S_0 \leq S_0^H(X_0)$, $\lambda_w = 0$, the optimal oil production path is a pure Hotelling-type path and $y_a(t) = \hat{y}_a$ for all $t > 0$. On the other hand, if $S_0 > S_0^H(X_0)$, then water is scarce and $\lambda_w > 0$.

Since the oil stock is continuously decreasing over the interval $[0, T_m)$ (Proposition 1) and $y_m(T_m) = 0$, for any $S_0 > S_0^H(X_0)$, the stock of oil must eventually reach $S_0^H(X_0)$ at some date $T_H < T_m$. When the oil stock reaches $S_0^H(X_0)$, water becomes abundant and $\lambda_w(t)$ becomes zero and remains at zero for all $t \geq T_H$. This means that the final phase, during which agriculture is the only active sector, with $y_a(t) = \hat{y}_a$ for all $t \in [T_m, \infty)$, is necessarily preceded by a phase during which $y_a(t) = \hat{y}_a$ and oil production follows a pure Hotelling-type path, with $y_m(t) = y_m^*(t) < \bar{x} - \frac{k_a}{k_m} \hat{y}_a$.

Figure 3 depicts the implicit price paths for the case where $\overline{S}_0(X_0) > S_0 > \widetilde{S}_0(X_0)$. The first three phases are exactly the same as in Section 3. The first phase, for $t \in [0, \tau)$, has $(y_a(t), y_m(t)) = (y_a^*(t), y_m^*(t))$, with the full marginal cost of both oil and agricultural production increasing. At $t = \tau$, the full marginal cost of agricultural production reaches the choke price from below and the agricultural sector ceases to produce. The second phase, for $t \in [\tau, T_w)$, has $(y_a(t), y_m(t)) = (0, y_m^*(t))$. Oil production becomes constrained by the natural inflow of water just as the water stock becomes exhausted, $t = T_w$. The third phase, for $t \in [T_w, \widetilde{T})$, has $(y_a(t), y_m(t)) = (0, \bar{y}_m)$. The full marginal cost of water is decreasing during that phase and reaches the agricultural choke price from above at $t = \widetilde{T}$, after which point agricultural production resumes.

During the fourth phase, for $t \in (\widetilde{T}, T_m)$, both sectors are active. This phase can now be divided into two sub-phases. The first sub-phase occurs during the interval $(\widetilde{T}, T_H)$, when the natural water inflow constitutes a binding constraint on total water consumption. The optimal production paths are $(y_a(t), y_m(t)) = (y_a^*(t), \bar{y}_m - \frac{k_a}{k_m} y_a^*(t))$. By Proposition 3, oil production is decreasing and agricultural production is increasing towards $\hat{y}_a$. The second sub-phase occurs during the interval $[T_H, T_m)$. Total water consumption is not constrained
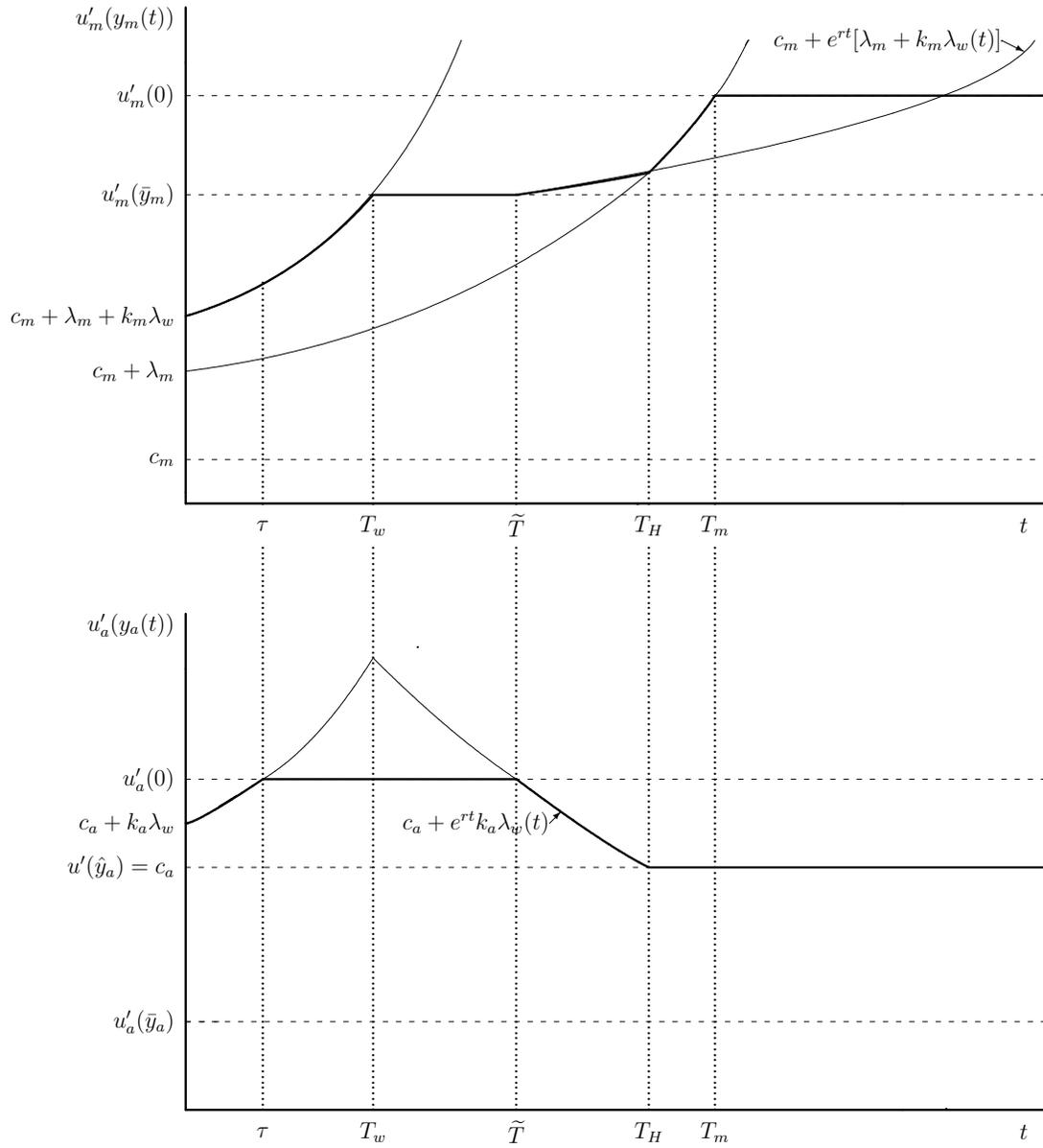
Figure 3 : $\hat{y}_a < \bar{y}_a$ and $\overline{S}_0(X_0) > S_0 > \widetilde{S}_0(X_0)$

by the natural water inflow, $\lambda_w(t) = 0$ and the optimal production paths are given by $(y_a(t), y_m(t)) = (\hat{y}_a, y_m^*(t))$, with $y_m^*(t) < \bar{y}_m - \frac{k_a}{k_m}\hat{y}_a)$. Thus oil production follows a pure Hotelling-type path during that sub-phase. The fifth phase is the final phase, with $y_a(t) = \hat{y}$ for all $t \in [T_m, \infty)$.

As with the paths depicted in Figure 2 of Section 3, for any given X_0 the paths depicted in Figure 3 contain all the other possible path configurations as special cases, depending on S_0. If $S_0 > \overline{S}_0(X_0)$, then $\tau = 0$ and the agricultural sector is inactive from the beginning and remains inactive until $t = \widetilde{T}$. If $S_0 < \widetilde{S}_0(X_0)$, the five phases corresponding to the case where $\overline{S}_0(X_0) > S_0 > \widetilde{S}_0(X_0)$ described in Figure 3 collapse into three phases, since then $\tau = T_w = \widetilde{T}$ and the agricultural sector is always active. The optimal paths during those three phases are exactly as in the case where $\hat{y}_a > \bar{y}_a$, except for the fact that now the next to last phase will always be composed of the two sub-phases described above. The second of those two sub-phases is always characterized by a pure Hotelling-type path, due to the fact that water availability does not constitute a constraint beyond T_H when $\hat{y}_a < \bar{y}_a$.

5   Conclusion

We have analyzed the problem faced by an economy in which a nonrenewable resource sector, such as oil, and a reproducible good sector, such as agriculture, must share as an essential input some renewable resource, such as water. The optimal allocation over time of the scarce resource between the two sectors poses a dynamic optimization problem involving two state variables: the stock of oil and the stock of water. We have been able to fully characterize the solution to this problem in order to show how, for a given initial stock of water, the production paths and the water usage of the two sectors depend on the size of the initial stock of oil and on whether or not the natural inflow of water constitutes a constraint on the agricultural sector in the long run, when there is no more oil left.

A striking result is that the optimal paths may involve abandoning agriculture after some time, in order to reserve the water for the oil sector during an interval of time, at the end of which agricultural activity resumes. This can occur whether the water resource constitutes a long-run constraint on agriculture or not. It will occur when the demand pressure on the value of water from the oil sector is such that the full marginal cost of agriculture reaches the

agricultural choke price from below before the water stock is exhausted. We have identified, for any given initial stock of water, the critical range inside which the initial oil stock must fall in order for this to be a characteristic of the optimal paths. If the initial oil stock is above that critical range, then the full marginal cost of agriculture is initially higher than the agricultural choke price and the agricultural sector is inactive from the outset. If the initial oil stock is below that critical range, then both sectors are always active, as long as the oil is not fully depleted. Once the oil stock is depleted, the agricultural sector produces indefinitely at the level that equates gross marginal benefit to marginal cost of production, as in a static equilibrium, unless its production is constrained by the natural inflow of water.

Another feature of the solution is that the optimal path of the oil sector does not generally follow a pure Hotelling-type path, with the implicit price of oil net of extraction cost growing at the rate of interest. This is because the full marginal cost of oil production must account not only for the rent imputed on the finite oil stock but also that imputed on the stock of water. Only in the case where the natural inflow of water does not pose a long-run constraint on agricultural production will there be a phase during which oil production follows a pure Hotelling path. In that case, this will occur once the oil stock falls below a certain critical value, beyond which water becomes abundant, being a constraint neither for the oil nor for the agricultural sector.

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