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TOP-CYCLE RATIONALIZABILITY

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Abstract.   We identify necessary and sufficient conditions for the choice
set from every subset A of a (finite) universal set X to coincide with the top
cycle in A of some fixed tournament on X.

1. Introduction

The standard economic interpretation of rationality is preference maximization: the decision maker is assumed to possess a preference relation over a relevant universe of alternatives; he chooses from any feasible set he faces the maximal elements of that relation in that set.

This hypothesis implies strong behavioral regularities, two of which are of special interest to us here. First, choices are *context-free*: if the decision maker chooses x, but not y, from a set A containing both alternatives, then he cannot choose y, and not x, from any other set B containing them. Second, choices are *acyclic*: if the decision maker chooses only x from the pair $\{x, y\}$ and only y from the pair $\{y, z\}$, then he cannot choose only z from the pair $\{x, z\}$.

Both context-dependent and cyclic choices are persistently observed: see, e.g., Loomes, Starmer and Sugden (1991) and Camerer (1994). These observations call for non-standard interpretations of rationality.

One interesting approach postulates that choices are based on several preference relations. Kalai, Rubinstein and Spiegler (2002) assume multiple relations and use each of them to choose from some, but typically not all, feasible sets. Manzini and Mariotti (2005) apply sequentially a fixed ordered list of (incomplete) relations to all feasible sets. Xu and Zhou (2004) analyze choices that arise as subgame-perfect equilibria of games with perfect information. These theories are able to account for *both* context-dependent and cyclic choices. To the extent that they postulate more than one underlying preference relation, they constitute rather major departures from the standard view of rationality.

We explore here an alternative that remains closer to it. While *context-dependent* choices clash directly with the hypothesis of a single underlying preference, we argue that a coherent explanation of *cyclic* choices need not invoke multiple preferences.

To see this, let us take a second look at the basic cyclic pattern of choice described earlier. Any binary relation that would explain it would have to deem x better than y, y better than z, and z better than x. This cyclic (strict) preference is not a difficulty in itself; it only becomes one when we consider the feasible set $\{x, y, z\}$ because no maximal element of the revealed preference exists in that set.

If we weaken the requirement of maximality in a suitable way, however, the observed cycle need no longer be inconsistent with a single underlying preference. One weak notion of "maximality" is the *top cycle*. The top cycle of a (complete and asymmetric) binary relation in a set (Moon, 1968) is the smallest nonempty subset of it having the property that each alternative in the subset is better than every alternative outside it. Schwartz (1972, 1986) defended the concept as an expression of rational choice and proposed two extensions of it to binary relations that need not be asymmetric. See also Deb (1977) and Duggan (2004). In our example, the top cycle of the cyclic revealed preference in the set $\{x, y, z\}$ is that set itself.

Suppose now that the decision maker is endowed with a complete, asymmetric, but possibly cyclic relation on some finite universal set X. In every feasible subset, he chooses the top cycle of that "tournament" in that subset. As we just saw, this may produce cyclic choices. But it cannot yield context-dependent choices: if x, but not y, belongs to the top cycle in some subset A containing y, then y cannot belong to the top cycle in a set B containing x unless x belongs to the top cycle in B as well.

In what follows, we identify two sets of necessary and sufficient conditions for the choice set from each subset A of X to coincide with the top cycle in A of some fixed tournament on X.

2.   Strong and weak forms of choice consistency

Following Arrow (1959), we model the choice behavior of a decision maker by means of a choice rule defined over all subsets of a finite set. Let X be a finite set of cardinality $|X| \geq 3$ and let $\mathcal{X}$ be the set of nonempty subsets of X. A *choice rule* is a mapping $f : \mathcal{X} \to \mathcal{X}$ such that $f(A) \subseteq A$ for every $A \in \mathcal{X}$. Note that, by definition, $f(A)$ is nonempty for every $A \in \mathcal{X}$.

In principle, choice rules are observable objects; the purpose of choice theory is to identify which properties of these rules are characteristic of various rationality hypotheses. The standard form of rationality –maximization of a preference preordering– is characterized by either of the following two well-known properties.

Weak Axiom of Revealed Preference (WARP). If $x, y \in X$ and there is $A \in \mathcal{X}$ such that $x \in f(A)$ and $y \in A \backslash f(A)$, then there is no $B \in \mathcal{X}$ such that $y \in f(B)$ and $x \in B$.

Choice Axiom (CA). If $A, B \in \mathcal{X}$, $B \subseteq A$, and $f(A) \cap B \neq \emptyset$, then $f(B) = f(A) \cap B$.

In different ways, these axioms express the idea that choices should be fully context-free. Arrow (1959) showed that a choice rule f satisfies CA (or, equivalently, WARP) if and only if there exists a complete, reflexive, and transitive binary relation R on X such that $f(A)$ coincides with the maximal elements of R in A for every $A \in \mathcal{X}$. This means that cyclic choices are ruled out as well. In order to allow for possibly cyclic choices, we weaken the above conditions as follows.

Weakened Weak Axiom of Revealed Preference (WWARP). If $x, y \in X$ and there is $A \in \mathcal{X}$ such that $x \in f(A)$ and $y \in A \backslash f(A)$, then there is no $B \in \mathcal{X}$ such that $y \in f(B)$ and $x \in B \backslash f(B)$.

Weak Choice Axiom (WCA). If $A, B \in \mathcal{X}$, $B \subseteq A$, and $f(A) \cap B \neq \emptyset$, then $f(B) \subseteq f(A)$.

The Weakened Weak Axiom of Revealed Preference only rules out choices that are *unambiguously* context-dependent: if the decision maker selects an alternative x and rejects another alternative y in some context, he cannot select y *and reject* x in another context. We are not aware of any discussion of WWARP in the literature.

The Weak Choice Axiom states that if the feasible set shrinks but some chosen alternatives remain feasible, then nothing new should appear in the choice set. This condition appears in Bordes (1976) and is sometimes called the "dual Chernoff condition".

It is easy to see that WCA implies WWARP. Suppose the former is satisfied and the later is violated. Let $x, y \in X$ and $A, B \in \mathcal{X}$ be such that $x \in f(A)$, $y \in A \backslash f(A)$, $y \in f(B)$, and $x \in B \backslash f(B)$. Since $\{x, y\} \subseteq A$ and $f(A) \cap \{x, y\} \neq \emptyset$, WCA requires $f(\{x, y\}) \subseteq f(A)$, hence $y \notin f(\{x, y\})$. Similarly, $\{x, y\} \subseteq B$ and $f(B) \cap \{x, y\} \neq \emptyset$ imply $f(\{x, y\}) \subseteq f(B)$, hence $x \notin f(\{x, y\})$. Thus $f(\{x, y\}) = \emptyset$, a contradiction.

We will see later that WWARP does not imply WCA and that both axioms are consistent with cyclic choices. Because they are mild requirements, we complement them with two further plausible conditions. These conditions are direct consequences of WARP or CA but they are not implied by WWARP or WCA.

Binary Dominance Consistency (BDC). If $A \in \mathcal{X}$, $x \in A$, and $f(\{x, y\}) = \{x\}$ for all $y \in A \backslash \{x\}$, then $f(A) = \{x\}$.

Weak Contraction Consistency (WCC). If $A \in \mathcal{X}$ and $|A| \geq 2$, then $f(A) \subseteq \bigcup_{x \in A} f(A \backslash \{x\})$.

Binary Dominance Consistency says that an alternative that is chosen against every single other alternative in a set should be the choice from that set as a whole. It is the choice-rule formulation of what is sometimes called the "Condorcet winner principle".

Weak Contraction Consistency says that nothing is chosen in a set of cardinality k unless it is chosen in some of its subsets of cardinality $k-1$. This condition appears in Lahiri (2001). It is much weaker than Chernoff's (1954) Postulate 4, the classic contraction consistency requirement that if $B \subseteq A$, then $f(A) \cap B \subseteq f(B)$. Note that Weak Contraction Consistency implies the "Condorcet loser principle": if $f(\{x, y\}) = \{y\}$ for all $y \in A\backslash\{x\}$, then $f(A) \subseteq A\backslash\{x\}$.

3. Choice rules based on tournaments

In this section, we define a class of choice rules meeting all the weak consistency requirements defined in Section 2, that is, WWARP, WCA, BDC, and WCC.

A (binary) relation P on X is a *tournament* if it is both complete (for all distinct $x, y \in X$, $(x, y) \in P$ or $(y, x) \in P$) and asymmetric (for all $x, y \in X$, $(x, y) \in P \Rightarrow (y, x) \notin P$). As usual, the notation xPy means that $(x, y) \in P$.

A P-cycle (or just a *cycle*) in X is a sequence $(x_1, ..., x_n)$ of elements of X, all distinct, such that $x_i P x_{i+1}$ for $i = 1, ..., n-1$ and $x_n P x_1$. When no confusion arises, we call the set $\{x_1, ..., x_n\}$ itself a cycle. The number n is its length.

Given $A \in \mathcal{X}$, define $P|A = P \cap (A \times A)$: this restriction of P to the subset A is itself a tournament on A. The transitive closure of $P|A$, denoted $\overline{P|A}$, is defined as usual: for all $x, y \in A$, $(x, y) \in \overline{P|A}$ if and only if there exist a positive integer n and $x_0, ..., x_n \in A$ such that $x = x_0$, $x_n = y$, and $x_{i-1} P x_i$ for $i = 1, ..., n$. This is a complete and transitive relation.

Definition 1. The *top cycle* of P in A, denoted $t(P|A)$, is the set of maximal elements of $\overline{P|A}$ in A: $x \in t(P|A)$ if and only if $x\overline{P|A}y$ for all $y \in A\backslash\{x\}$.

Selecting from every feasible set the top cycle, in that set, of an arbitrary but fixed tournament on the universal set defines a choice rule. Any such rule will be called a top-cycle rule.

Definition 2. A choice rule f is a *top-cycle rule* if there exists a tournament P on X such that $f(A) = t(P|A)$ for all $A \in \mathcal{X}$.

It is important to note that $\overline{P|A}$ does *not* coincide with $\overline{P}|A$: therefore $t(P|A)$ is generally not the set of maximal elements of $\overline{P}$ in A, unless $A = X$. For that reason, top-cycle choice rules need not satisfy the Weak Axiom of Revealed Preference or Arrow's Choice Axiom. They reflect a weaker form of rational choice.

Before we turn to our results, we record a few basic facts about the top cycle that will be used in our proofs. For an in-depth study of the top cycle, we refer to Laslier (1997).

Lemma. *Let P be a tournament on X. Then,*
(a) $t(P|X)$ is either a singleton or a P-cycle;
(b) $t(P|X)$ is the unique (inclusion) minimal set $Y \in \mathcal{X}$ possessing the property that

xPy for all $x \in Y$ and all $y \in X \backslash Y$;
(c) the following statements are equivalent:
    (i) $t(P|X) = X$;
    (ii) there is a P-cycle of length $|X|$ in X;
    (iii) for each $n = 3, ..., |X|$, there is a P-cycle of length n in X;
    (iv) for each $x \in X$ and $n = 3, ..., |X|$, x belongs to a P-cycle of length n in X.

Proof. Statements (a) and (b) are well known. The equivalence of the first three statements in (c) is also known: see, e.g., Harary (1969), Theorem 16.11 and Corollary 16.11(a). Since (iv) implies (iii), it suffices to prove that (ii) implies (iv). The proof mimics that of Theorem 16.11 in Harary (1969). Fix a P-cycle of length $|X|$ in X: $x_1 P x_2 P ... P x_{|X|} P x_1$ and let $x \in X$, say, $x = x_1$.

First, observe that x belongs to a cycle of length 3: if t^* denotes the smallest $t \in \{3, ..., |X|\}$ such that $x_t P x_1$, then $x_1 P x_{t^*-1} P x_{t^*} P x_1$ is such a cycle.

Next, proceed by induction. Fix n, $3 \le n \le |X| - 1$, and suppose x belongs to a cycle of length n in X, say $x = y_1 P y_2 P ... P y_n P y_1 = x$. Distinguish two cases.

Case 1. There exist $z \in X \backslash \{y_1, ..., y_n\}$ and $t, t' \in \{1, 2, ..., n\}$ such that $y_t P z P y_{t'}$.

If $y_1 P z$, let t^* be the smallest $i \in \{1, ..., n\}$ such that $z P y_i$. Then $y_1 P ... P y_{t^*-1} P z P y_{t^*} P ... P y_n P y_1$ is a P-cycle of length $n+1$ in X containing y_1. If $z P y_1$, the argument is similar: let t^* be the largest $i \in \{1, ..., n\}$ such that $y_i P z$. Then $y_1 P ... P y_{t^*} P z P y_{t^*+1} P ... P y_n P y_1$ (with the understanding that $y_{n+1} = y_1$) is again a P-cycle of length $n+1$ in X containing y_1.

Case 2. There does not exist $z \in X \backslash \{y_1, ..., y_n\}$ and $t, t' \in \{1, 2, ..., n\}$ such that $y_t P z P y_{t'}$.

Then $X \backslash \{y_1, ..., y_n\}$ is partitioned into two subsets U and W, where

$$u \ \in \ U \Leftrightarrow u P y_i \text{ for } i = 1, ..., n,$$
$$w \ \in \ W \Leftrightarrow y_i P w \text{ for } i = 1, ..., n.$$

These sets are obviously disjoint. Because there is a P-cycle of length $|X|$ in X, neither set is empty and there exist $u \in U$ and $w \in W$ such that $w P u$. Then $u P y_1 P ... P y_{n-1} P w P u$ is a P-cycle of length $n+1$ in X containing y_1. ∎

4. Equivalence results

There is a tight connection between the weak forms of choice consistency described in Section 2 and the top-cycle rules defined in Section 3. We begin by showing this in the simpler case where all binary choices are clear-cut. Say that a choice rule f is *resolute* if $|f(A)| = 1$ whenever $|A| = 2$.

Theorem 1. *For any choice function f, the following statements are equivalent:*

i) f is resolute and satisfies WWARP, BDC, and WCC;

ii) f is resolute and satisfies WCA, BDC, and WCC;

iii) f is a top-cycle rule.

Proof. *Step 1: Every top-cycle rule is resolute and satisfies WWARP, WCA, BDC, and WCC.*

Let P be a tournament on X. It is obvious that the top-cycle choice rule $t(P|.)$ is resolute and satisfies BDC. Because WWARP follows from WCA, we only need to check WCA and WCC.

To check WCA, suppose $B \subseteq A$ and $t(P|A) \cap B \neq \emptyset$. Let $x \in t(P|B)$ and $z \in A$. We must show $x\overline{P|A}z$. Pick $y \in t(P|A) \cap B$. By definition of $t(P|B)$, $x\overline{P|B}y$. Hence, a fortiori, $x\overline{P|A}y$. By definition of $t(P|A)$, $y\overline{P|A}z$. By transitivity of $\overline{P|A}$, $x\overline{P|A}z$.

To check WCC, we first prove that $t(P|.)$ satisfies the so-called Strong Superset Property (Bordes, 1976): if $A, B \in \mathcal{X}$, $B \subseteq A$, and $t(P|A) \subseteq B$, then $t(P|A) = t(P|B)$.

Fix A, B satisfying the premises of that property. By statement (b) of the Lemma, $t(P|A)$ is the minimal nonempty subset C of A with the property

$$xPy \text{ for all } x \in C \text{ and all } y \in A\backslash C. \tag{1}$$

A fortiori, $t(P|A)$ is a nonempty subset C of B with the property

$$xPy \text{ for all } x \in C \text{ and all } y \in B\backslash C. \tag{2}$$

Moreover, if a nonempty set $C \subsetneq t(P|A)$ possessed property (2), it would also possess property (1), a contradiction. Thus $t(P|A)$ is a minimal nonempty subset C of B with property (2). But such a minimal set is unique and equal to $t(P|B)$. Thus $t(P|A) = t(P|B)$, as claimed.

We are now ready to establish WCC. Fix $A \in \mathcal{X}$, $|A| \geq 2$. The case $|A| = 2$ being trivial, suppose $|A| \geq 3$. Let $x \in t(P|A)$. We show that there exists $y \in A$ such that $x \in t(P|A\backslash\{y\})$.

If $t(P|A) \neq A$, let $y \in A\backslash t(P|A)$. Then $t(P|A) \subseteq A\backslash y \subseteq A$ and since $t(P|.)$ satisfies the Strong Superset Property, $t(P|A\backslash\{y\}) = t(P|A)$. So $x \in t(P|A\backslash\{y\})$.

If $t(P|A) = A$, we know from the Lemma (more precisely, from the fact that (i) implies (iv)) that x belongs to a P- cycle of length $|A|-1$ in A. That cycle is in $A\backslash\{y\}$ for some $y \in A\backslash\{x\}$. Since (ii) implies (i), that cycle coincides with $t(P|A\backslash\{y\})$. Thus $x \in t(P|A\backslash\{y\})$.

Step 2: If a choice rule is resolute and satisfies WWARP, BDC, and WCC, then it is a top-cycle rule.

Let f satisfy the stated properties. Define $P = \{(x, y) \in X \times X | x \neq y$ and $f(\{x, y\}) = \{x\}\}$. Because f is defined on $\mathcal{X}$ (hence in particular on all subsets of cardinality two) and is resolute, P is complete and asymmetric: it is a tournament on X. We claim that

$$f(A) = t(P|A) \tag{3}$$

for all $A \in \mathcal{X}$.

By construction of P, (3) holds for all $A \in \mathcal{X}$ such that $|A| \leq 2$. Next, proceed by induction. Fix $k \geq 3$ and assume that (3) holds for all $A \in \mathcal{X}$ such that $|A| \leq k - 1$. Let now $A \in \mathcal{X}$ be a set of cardinality $|A| = k$.

If $|t(P|A)| = 1$, $t(P|A)$ consists of a single $x \in A$ such that xPy, hence, $f(\{x, y\}) = \{x\}$, for all $y \in A \setminus \{x\}$. By BDC, $f(A) = \{x\} = t(P|A)$.

If $|t(P|A)| \geq 2$, we first show that

$$t(P|A) \subseteq f(A). \tag{4}$$

Suppose, by contradiction, that there exists $x \in t(P|A) \setminus f(A)$. This implies

$$f(A) \subseteq t(P|A) \tag{5}$$

because if there exists $y \in f(A) \setminus t(P|A)$, then xPy by definition of $t(P|A)$, and therefore

$$\begin{aligned} x &\in f(\{x, y\}), \ y \in \{x, y\} \setminus f(\{x, y\}), \\ y &\in f(A), \ x \in A \setminus f(A), \end{aligned}$$

contradicting WWARP.

Because of (5), $t(P|A) \neq \{x\}$. Thus $t(P|A)$ is a cycle, necessarily of length $n \geq 3$, containing x: say, $x = x_1 P x_2 P...P x_n P x_1 = x$. We claim that

$$f(A) \cap t(P|A) = \emptyset. \tag{6}$$

To prove this, note first that $x = x_1 \notin f(A)$ implies $x_2 \notin f(A)$ since otherwise

$$\begin{aligned} x_1 &\in f(\{x_1, x_2\}), \ x_2 \in \{x_1, x_2\} \setminus f(\{x_1, x_2\}), \\ x_2 &\in f(A), \ x_1 \in A \setminus f(A), \end{aligned}$$

contradicting WWARP.

Repeating this argument yields successively $x_3 \notin f(A)$, ..., $x_n \notin f(A)$, proving (6). But (6) and (5) imply that $f(A) = \emptyset$, a contradiction. We have proved (4).

To complete the proof of (3), suppose, by contradiction, that $f(A) \supsetneq t(P|A)$. Let $x \in f(A) \setminus t(P|A)$. By WCC, there exists $y \in A$ such that $x \in f(A \setminus \{y\})$.

Because $|t(P|A)| \geq 2$, $t(P|A) \cap (A\backslash\{y\}) \neq \emptyset$. Because $A\backslash\{y\} \subseteq A$ and $t(P|.)$ satisfies WCA, $t(P|A\backslash\{y\}) \subseteq t(P|A)$. Hence $x \notin t(P|A)$ implies $x \notin t(P|A\backslash\{y\})$. But by the induction hypothesis, $f(A\backslash\{y\}) = t(P|A\backslash\{y\})$. Thus $x \notin f(A\backslash\{y\})$, a contradiction.

Step 3: If a choice rule is resolute and satisfies WCA, BDC, and WCC, then it is a top-cycle rule.

This follows from Step 2 and the fact that WCA implies WWARP. ∎

5. Discussion

We briefly comment on our theorem and the related literature.

1) The properties in statement i) of Theorem 1 are independent, and so are those in statement ii). The rule $f(A) = A$ for all $A \in \mathcal{X}$ satisfies all properties in Theorem 1 except the resoluteness condition. For an example violating only BDC, fix $X = \{x, y, z\}$ and let $f(\{x, y\}) = f(\{x, z\}) = \{x\}$, $f(\{y, z\}) = \{y\}$, and $f(X) = \{x, y\}$. For an example violating only WWARP and WCA, let $X = \{x, y, z\}$, $f(\{x, y\}) = \{x\}$, $f(\{x, z\}) = \{z\}$, $f(\{y, z\}) = \{y\}$, and $f(X) = \{x\}$. For an example violating only WCC, let $X = \{w, x, y, z\}$, define the tournament $P = \{(x, w), (y, w), (z, w), (x, y), (y, z), (z, x)\}$, and let $f(A) = t(P|A)$ for all $A \in \mathcal{X}\backslash\{X\}$ and $f(X) = X$.

2) Tournaments have been extensively studied in graph theory (see in particular Moon's (1968) book) and in social choice theory (see Moulin's (1984) survey and Laslier's (1997) book). Most contributions in social choice define "choice procedures" as mappings selecting, for every $A \in \mathcal{X}$ *and every possible tournament on X*, a set of alternatives in A. This is perfectly in line with the intended interpretation of a tournament as the observable outcome of some vote. It allows one to define and use properties relating choices across tournaments: an example is the neutrality property stating that if two tournaments may be obtained from each other by relabeling the alternatives, then the corresponding choice sets can be obtained from each other by the same relabeling. Moulin (1986), for instance, characterizes both the top-cycle and the so-called uncovered set choice procedures by combining axioms of that nature with properties relating choices across feasible sets. Dutta (1988) offers a corresponding characterization of the minimal covering set choice procedure. These authors do not obtain "exact" axiomatizations: they show that the choice procedures they consider are the minimal ones satisfying the axioms they impose.

In contrast to the social choice literature, we focus on choices made by a decision maker whose preferences are not observable. This constrains us to use only properties relating choices across feasible sets. In that spirit, Duggan (1997) offers a characterization of the class of choice rules satisfying WCA alone: he shows that the axiom is equivalent to rationalizability by a particular type of *pseudo-relation* called a proper weak pseudo-order. A pseudo-relation is a *collection* of binary relations, one for each $A \in \mathcal{X}$. Our theorem is about choices based on a single binary relation.

3) A variant of Theorem 1 holds without the resoluteness assumption. Let R be a complete, but not necessarily asymmetric, relation and let $P(R) = \{(x, y) \in X \times X | (x, y) \in R$ and $(y, x) \notin R\}$. For any $A \in \mathcal{X}$, the *weak top cycle of R in A*, denoted $wt(R|A)$, is the set of maximal elements of $\overline{R|A}$ in A. The concept is due to Schwartz (1972, 1986), who calls it the GETCHA set. Schwartz shows that it is the minimal nonempty subset B of A with the property that $xP(R)y$ for all $x \in B$ and all $y \in A \backslash B$. Call a choice rule f a *weak top-cycle rule* if there is a complete relation R on X such that $f(A) = wt(R|A)$ for all $A \in \mathcal{X}$.

Theorem 2. A *choice rule f satisfies WCA, BDC, and WCC if and only if it is a weak top-cycle rule.*

Proof. A straightforward modification of Step 1 of the proof of Theorem 1 shows that any weak top-cycle rule meets WCA, BDC, and WCC.

Conversely, let f satisfy the stated properties. Define $R = \{(x, y) \in X \times X | x \in f(\{x, y\})\}$. By construction

$$f(A) = wt(R|A) \tag{7}$$

whenever $|A| \leq 2$. Proceeding by induction, fix $k \geq 3$, assume (7) holds for all $A \in \mathcal{X}$ such that $|A| \leq k - 1$, and let $A \in \mathcal{X}$ be a set of cardinality $|A| = k$.

Suppose first that $|f(A)| = 1$, say, $f(A) = \{x\}$. For any $y \in A \backslash \{x\}$, $\{x, y\} \subseteq A$ and $f(A) \cap \{x, y\} \neq \emptyset$. By WCA, $f(\{x, y\}) \subseteq f(A)$, that is, $f(\{x, y\}) = \{x\}$. Since this is true for all $y \in A \backslash \{x\}$, we obtain $wt(R|A) = \{x\} = f(A)$.

Suppose next that $|f(A)| > 1$. In this case we must have $|wt(R|A)| > 1$ as well because if $wt(P|A)$ consists of a single $x \in A$, then $f(\{x, y\}) = \{x\}$ for all $y \in A \backslash \{x\}$ and BDC implies $f(A) = \{x\}$.

Then, for all $x \in A$, $A \backslash \{x\} \subseteq A$ and $f(A) \cap (A \backslash \{x\}) \neq \emptyset$. By WCA, $f(A \backslash \{x\}) \subseteq f(A)$. Thus, $\cup_{x \in A} f(A \backslash \{x\}) \subseteq f(A)$. Combining this with WCC,

$$\cup_{x \in A} f(A \backslash \{x\}) = f(A).$$

Applying the same argument to $wt(R|.)$ instead of f, we obtain

$$\cup_{x \in A} wt(R|A \backslash \{x\}) = wt(R|A).$$

By the induction hypothesis, $f(A \backslash \{x\}) = wt(R|A \backslash \{x\})$ for every $x \in A$. Therefore $f(A) = \cup_{x \in A} f(A \backslash \{x\}) = \cup_{x \in A} wt(R|(A \backslash \{x\})) = wt(R|A)$. ∎

Lahiri (2001) proves a variant of Theorem 2. Instead of WCA, he uses a more complex condition dubbed Expansion Independence: if $A \in \mathcal{X}$, $x \in f(A)$, $z \in X \backslash A$, and there exists $y \in A$ such that $y \in f(\{y, z\})$, then $x \in f(A \cup \{z\})$. We find this requirement somewhat tailored to the weak top-cycle rules. In particular, it is easy to

see that they are the minimal rules satisfying it: if f meets Expansion Independence, there is a complete relation R on X such that $f(A) \supseteq wt(R|A)$ for all $A \in \mathcal{X}$.[1]

Interestingly, WWARP cannot replace WCA in Theorem 2 (and is thus a strictly weaker axiom): there exist choice rules meeting WWARP, BDC, and WCC that are not weak top-cycle rules. For a simple example, let $X = \{x, y, z\}$, $f(\{x, y\}) = \{x, y\}$, $f(\{x, z\}) = \{x\}$, $f(\{y, z\}) = \{y\}$, and $f(X) = \{x\}$. This choice rule meets WWARP, BDC, and WCC. If R is a complete relation such that $f = wt(R|.)$, the choices from pairs force $R = \{(x, y), (y, x), (x, z), (y, z)\}$. But then $wt(R|X) = \{x, y\} \neq f(X)$: f is not a weak top-cycle rule.

WWARP is of particular interest as the weakest form of the requirement that choices be context-independent. Its implications for non-resolute choice rules remain unclear and deserve further study.

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[1] Let f satisfy Expansion Independence, define R as in the proof of Theorem 2, and let $B \in \mathcal{X}$. Unless $wt(R|B)$ is a singleton, it is an R cycle, say, $x_1 R x_2 R...R x_K$: with the convention that $K + 1 = 1$, $x_i \in f(\{x_i, x_{i+1}\})$ for $i = 1, ..., K$. Repeated application of Expansion Independence yields $x_i \in f(\{x_1, ..., x_K\})$ for $i = 1, ..., K$, that is, $wt(R|B) = f(wt(R|B))$. This is obviously also true if $wt(R|B)$ is a singleton. Next, applying Expansion Independence to $A = wt(R|B)$ and all $z \in B \backslash wt(R|B)$ gives $wt(R|B) \subseteq f(B)$.

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