Nonparametric measures of local causality and tests of local non-causality in time series

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Abstract

The study of the causal relationships in a process $(Y_t, Z_t)_{t\in\mathbb{Z}}$ is a subject of a particular interest in finance and economy. A widely-used approach is to consider the notion of Granger causality, which in the case of first order Markovian processes is based on the joint distribution function of (Y_t, Z_{t-1}) given Y_{t-1}. The Granger causality measures proposed so far are global in the sense that if the relationship between Y_t and Z_{t-1} changes with the value taken by Y_{t-1}, this will not be captured. To circumvent this limitation, this paper proposes *local* Granger causality indices based on the conditional copula of (Y_t, Z_{t-1}) given $Y_{t-1} = x$. Exploiting the asymptotic behavior of two kernel-based conditional copula estimators for α-mixing processes, the asymptotic normality of nonparametric estimators of these local Granger indices is deduced and confidence intervals are built. Tests of local non-causality are developed as well. The efficiency of the proposed methods is investigated via simulations and their usefulness is illustrated on the bivariate time series of Standard & Poor's 500 prices and trading volumes.

Keywords: α-mixing processes, conditional copula, Kendall's tau, kernel estimation, Spearman's rho, weak convergence

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1. Introduction

The concept of causality as originally introduced by Wiener (1956) and Granger (1969) is helpful for studying the dynamic relationships in multivariate time series. This notion is defined in terms of predictability at horizon one of a random variable (or random vector) Y from its past and the past of another random variable (or vector) Z. Specifically, assume that data are available for the process $(Y_t, Z_t)_{t\in\mathbb{Z}}$, and let $\mathbf{Y}_{t-1}$, $\mathbf{Z}_{t-1}$ be the observations up to time $t-1$ on Y and Z, respectively. According to Granger (1969), the causality from Z to Y *one period ahead* is defined as follows: Z is said to cause Y if $\mathbf{Z}_{t-1}$ can help predict Y_t, conditional on $\mathbf{Y}_{t-1}$.

Many works considered testing the null hypothesis of non-causality. For example, testing causality has been investigated for multivariate ARMA models by Boudjellaba et al. (1992) and Boudjellaba et al. (1994). Because the Granger non-causality is a form of conditional independence, tests can be deduced from standard conditional independence tests; see Florens & Fougere (1996), for instance. In the context of i.i.d. data, such procedures were derived by Song (2009), Huang (2010), Bergsma (2013), Su & Spindler (2013) and Linton & Gozalo (2014), among others. Generalizations to the case of time series have been investigated by Su & White (2008) and Su & White (2012) under α-mixing and by de Matos & Fernandes (2007) and Su & White (2008) under β-mixing. See also the recent contributions by Bouezmarni et al. (2012), Wang & Hong (2013) and Bouezmarni & Taamouti (2014).

When the hypothesis of non-causality is rejected, one may be interested in measuring the strength of this causal relationship. The first causality measures were proposed by Geweke (1982) and Geweke (1984) using the mean-squared forecast errors, and by Gouriéroux et al. (1987) based on the Kullback–Leibler information. Causality indices under parametric models were later investigated by Polasek (1994) and Polasek (2002), and by Dufour & Taamouti (2010) under ARMA models, where measures for short and long run were proposed. Mainly inspired by the fact that these measures suffer from model misspecification, nonparametric indices where proposed by Taamouti et al. (2014) using the Kullback–Leibler information and nonparametric density copula estimators. Recently, Zhang et al. (2016) investigated causality measures at multiple horizons for exchange rate and commodity prices.

It is worth mentioning that all of the above-cited papers focus on characterizing the global relationship between Y_t and $\mathbf{Z}_{t-1}$, conditional on $\mathbf{Y}_{t-1}$.

Unfortunately, if the nature of the link between Y_t and $\mathbf{Z}_{t-1}$ changes with the value taken by $\mathbf{Y}_{t-1}$, this feature will not necessarily be captured by global measures. A possible solution to this issue is to compute the partial correlation coefficient. However, doing so implicitly assumes a linear relationship and the measure depends on the marginal behavior. In other words, such an approach would suffer from the same drawbacks as the classical Pearson correlation coefficient.

To circumvent these limitations, this paper proposes nonparametric local Granger causality indices for measuring the strength of the relationship in $(Y_t, \mathbf{Z}_{t-1})$ given a particular value taken by Y_{t-1}. In order to simplify the presentation, a focus is put on Markovian models of order one. In that particular case, one considers the dependence structure of (Y_t, Z_{t-1}) given $Y_{t-1} = x$ as captured by its associated conditional copula. This approach allows for the definition of nonparametric measures of local causality that do not suffer from the drawbacks that arise when using partial correlations. Specifically, let $\{(Y_t, Z_t)\}_{t\in\mathbb{Z}}$ be a stationary process and define the local causality distribution function

$$H_x^{Z\to Y}(y,z) = \mathbb{P}\left(Y_t \leq y, Z_{t-1} \leq z | Y_{t-1} = x\right). \tag{1}$$

Then if the conditional marginal distributions $F_{1x}(y) = \mathbb{P}\left(Y_t \leq y | Y_{t-1} = x\right)$ and $F_{2x}(z) = \mathbb{P}\left(Z_{t-1} \leq z | Y_{t-1} = x\right)$ are continuous, Sklar's Theorem guarantees the existence of a unique copula $C_x^{Z\to Y} : [0,1]^2 \to [0,1]$ such that

$$H_x^{Z\to Y}\left(y,z\right) = C_x^{Z\to Y}\left\{F_{1x}(y), F_{2x}(z)\right\}.$$

The bivariate function $C_x^{Z\to Y}$ will be called the *local causality copula* and corresponds to the dependence structure of (Y_t, Z_{t-1}) given $Y_{t-1} = x$.

The first goal of this paper is to describe nonparametric estimators of $C_x^{Z\to Y}$ in a general framework of serially dependent bivariate data. The weak convergence of suitably standardized versions of these conditional copula estimators is deduced from general results by Bouezmarni et al. (2016) under some conditions on the α-mixing coefficients of $\{(Y_t, Z_t)\}_{t\in\mathbb{Z}}$. Then, local causality indices are defined for measuring the strength of the causal relationship in a bivariate time series and nonparametric estimators based on the empirical conditional copulas are proposed. Their asymptotic normality is established from the functional delta method. Tests for the null hypothesis of local non-causality are developed as well.

The paper is organized as follows. In Section 2, two estimators of the local causality copula are described and their asymptotic behavior is obtained in the light of results by Bouezmarni et al. (2016). In Section 3, general local causality indices are defined and the large-sample behavior of nonparametric estimators is investigated. A consistent estimator of the asymptotic variance is also proposed, leading to interval estimations of the local causality measures. In Section 4, tests for the null hypothesis of local non-causality are developed. Section 5 investigates the sampling properties of point and interval estimators of causality indices based on the Spearman and Kendall measures of association. The efficiency of tests of local non-causality is studied as well. An illustration on financial data is provided in Section 6. Technical details and the assumptions required in Section 2 and 3 are relegated to the Appendix.

2. Estimation of the local causality copula

2.1. Two estimators of $C_x^{Z\to Y}$

Let $(Y_1, Z_1), \ldots, (Y_{n+1}, Z_{n+1})$ be a realization of the stationary process $\{(Y_t, Z_t)\}_{t\in\mathbb{Z}}$. In that context, an estimator of the joint conditional distribution in (1) is

$$H_{xh}^{Z\to Y}(y,z) = \sum_{i=2}^{n+1} \mathcal{K}\left(\frac{Y_{i-1}-x}{h}\right) \mathbb{I}\left(Y_i \le y, Z_{i-1} \le z\right),$$

where $\mathcal{K} = \mathcal{K}_n$ is a non-negative kernel-based weight that may depend on $Y_1, \ldots, Y_n$ and $h = h_n$ is a bandwidth parameter. Hereafter, it is assumed that

$$\sum_{i=1}^{n} \mathcal{K}\left(\frac{Y_i - x}{h}\right) = 1.$$

A first estimator of the local causality copula arises upon noting that $C_x^{Z\to Y}$ can be extracted from $H_x^{Z\to Y}$ via

$$C_x^{Z\to Y}(u,v) = H_x^{Z\to Y}\left\{F_{1x}^{-1}(u), F_{2x}^{-1}(v)\right\}.$$

From this representation, a natural plug-in estimator of $C_x^{Z\to Y}$ is given by

$$C_{xh}^{Z\to Y}(u,v) = H_{xh}^{Z\to Y}\left\{F_{1xh}^{-1}(u), F_{2xh}^{-1}(v)\right\}, \tag{2}$$

where F_{1xh}^{-1} and F_{2xh}^{-1} are the left-continuous generalized inverses of $F_{1xh}(y) = \lim_{z\to\infty} H_{xh}^{Z\to Y}(y,z)$ and $F_{2xh}(z) = \lim_{y\to\infty} H_{xh}^{Z\to Y}(y,z)$, respectively.

As noted by Veraverbeke et al. (2011) and Gijbels et al. (2011) in the i.i.d. case, the plug-in estimator $C_{xh}^{Z\to Y}$ may be severely biased, especially when the conditional marginal distributions strongly depend on the covariate. For that reason, a second estimator that aims at reducing this possible effect of the covariate on the margins is proposed. To this end, define for each $i \in \{1,\ldots,n\}$ the *pseudo-uniformized* observations $(\widetilde{U}_i, \widetilde{V}_i) = (F_{1Y_ih_1}(Y_{i+1}), F_{2Y_ih_2}(Z_i))$, where h_1 and h_2 are bandwidth parameters that may differ from h. Then, let

$$G_{xh}^{Z\to Y}(y,z) = \sum_{i=1}^{n} \mathcal{K}\left(\frac{Y_i - x}{h}\right) \mathbb{I}\left(\widetilde{U}_i \le y, \widetilde{V}_i \le z\right).$$

An alternative estimator of $C_x^{Z\to Y}$ is then given by

$$\widetilde{C}_{xh}^{Z\to Y}(u,v) = G_{xh}^{Z\to Y}\left\{G_{1xh}^{-1}(u), G_{2xh}^{-1}(v)\right\}, \tag{3}$$

where $G_{1xh}(y) = \lim_{z\to\infty} G_{xh}^{Z\to Y}(y,z)$ and $G_{2xh}(z) = \lim_{y\to\infty} G_{xh}^{Z\to Y}(y,z)$.

2.2. Weak convergence

This section describes the asymptotic behavior of the processes

$$\mathbb{C}_{xh}^{Z\to Y} = \sqrt{nh}\left(C_{xh}^{Z\to Y} - C_x^{Z\to Y}\right) \quad \text{and} \quad \widetilde{\mathbb{C}}_{xh}^{Z\to Y} = \sqrt{nh}\left(\widetilde{C}_{xh}^{Z\to Y} - C_x^{Z\to Y}\right).$$

These large-sample results are derived under the assumption that the stationary process $\{(Y_t, Z_t)\}_{t\in\mathbb{Z}}$ is α-mixing. Specifically, for each $r \in \mathbb{N}\cup\{0\}$, define its associated α-mixing coefficient of lag r by

$$\alpha(r) = \sup_{k\in\mathbb{Z}} \alpha\left(\mathcal{F}_{-\infty}^{k}, \mathcal{F}_{k+r}^{\infty}\right),$$

where $\mathcal{F}_a^b$ is the σ-field generated by $\{(Y_t, Z_t)\}_{a\le t\le b}$ and

$$\alpha(\mathcal{A},\mathcal{B}) = \sup_{(A,B)\in\mathcal{A}\times\mathcal{B}} \left|\mathbb{P}(A\cap B) - \mathbb{P}(A)\,\mathbb{P}(B)\right|.$$

Then $\alpha(r)\to 0$ as $r\to\infty$, which means that $\{(Y_t, Z_t)\}_{t\in\mathbb{Z}}$ is α-mixing.

Because the estimation of the local causality copula is based on the trivariate process $(Y_t, Z_{t-1}, Y_{t-1})_{t\in\mathbb{Z}}$, where the third component is the conditioning variable, or covariate, the setup in this paper is a special case of that investigated by Bouezmarni et al. (2016). In this latter work, the estimation of conditional copulas from general three-dimensional α-mixing processes is considered. In particular, one can deduce from their Proposition 2.1 the weak convergence of the process

$$\mathbb{H}_{xh}^{Z\to Y} = \sqrt{nh}\left(H_{xh}^{Z\to Y} - H_x^{Z\to Y}\right).$$

This weak convergence takes place in the space $\ell^\infty(\mathbb{R}^2)$ of bounded functions in $\mathbb{R}^2$. The result is formally stated next. In the sequel, $\dot{H}_x^{Z\to Y}$ and $\ddot{H}_x^{Z\to Y}$ denote the first and second derivatives of $H_x^{Z\to Y}$ with respect to x, *i.e.*

$$\dot{H}_x^{Z\to Y}(y,z) = \frac{\partial}{\partial x}\, H_x^{Z\to Y}(y,z) \quad \text{and} \quad \ddot{H}_x^{Z\to Y}(y,z) = \frac{\partial^2}{\partial x^2}\, H_x^{Z\to Y}(y,z).$$

Proposition 2.1. *Suppose that Assumptions $\mathcal{A}_1$–$\mathcal{A}_2$, W_1–W_5 and W_{11}–W_{13} are satisfied. If $nh\to\infty$ and $nh^5 \to K^2 < \infty$, the empirical process $\mathbb{H}_{xh}^{Z\to Y}$ converges weakly in the space $\ell^\infty(\mathbb{R}^2)$ to a Gaussian limit $\mathbb{H}_x^{Z\to Y}$ such that for K_2–K_4 defined in Assumptions W_2–W_4,*

$$\mathrm{E}\left\{\mathbb{H}_x^{Z\to Y}(y,z)\right\} = K\left\{K_2\,\dot{H}_x^{Z\to Y}(y,z) + K_3\,\ddot{H}_x^{Z\to Y}(y,z)\right\}$$

and

$$\begin{aligned}\mathrm{Cov}\left\{\mathbb{H}_x^{Z\to Y}(y,z), \mathbb{H}_x^{Z\to Y}(y',z')\right\} &= K_4\, H_x^{Z\to Y}\left\{\min(y,y'), \min(z,z')\right\} \\ &\quad - K_4\, H_x^{Z\to Y}(y,z)\, H_x^{Z\to Y}(y',z').\end{aligned}$$

Because the estimator $C_{xh}^{Z\to Y}$ can be expressed in terms of a Hadamard differentiable functional of $H_{xh}^{Z\to Y}$ (see Bücher & Volgushev (2013)), the weak convergence of $\mathbb{C}_{xh}^{Z\to Y}$ in the space $\ell^\infty([0,1]^2)$ of bounded functions in $[0,1]^2$ can be deduced from an application of the functional delta method (see van der Vaart & Wellner (1996) for more details). Before stating the result, define

$$C_x^{[1]}(u,v) = \frac{\partial}{\partial u}\, C_x^{Z\to Y}(u,v) \quad \text{and} \quad C_x^{[2]}(u,v) = \frac{\partial}{\partial v}\, C_x^{Z\to Y}(u,v).$$

Corollary 2.2. *Under the conditions of Proposition 2.1 and if in addition Assumption $\mathcal{A}_3$ holds, $\mathbb{C}_{xh}^{Z\to Y}$ converges weakly in the space $\ell^\infty([0,1]^2)$ to*

$$\mathbb{C}_x^{Z\to Y}(u,v) = \alpha_x^{Z\to Y}(u,v) - C_x^{[1]}(u,v)\,\alpha_x^{Z\to Y}(u,1) - C_x^{[2]}(u,v)\,\alpha_x^{Z\to Y}(1,v),$$

where in terms of the process $\mathbb{H}_x^{Z\to Y}$ defined in Proposition 2.1, $\alpha_x^{Z\to Y}(u,v) = \mathbb{H}_x^{Z\to Y}\{F_{1x}^{-1}(u), F_{2x}^{-1}(v)\}$.

In view of the bias function given in Proposition 2.1, a consequence of Corollary 2.2 is that the asymptotic bias of $\mathbb{C}_{xh}^{Z\to Y}$ is

$$\begin{aligned}\mathbb{B}_x(u,v) &= K\Big[K_2\,\dot{C}_x^{Z\to Y}(u,v) + K_3\,\ddot{H}_x^{Z\to Y}\left\{F_{1x}^{-1}(u), F_{2x}^{-1}(v)\right\} \\ &\qquad - K_3\,C_x^{[1]}(u,v)\ddot{F}_{1x}\left\{F_{1x}^{-1}(u)\right\} - K_3\,C_x^{[2]}(u,v)\ddot{F}_{2x}\left\{F_{2x}^{-1}(v)\right\}\Big].\end{aligned}$$

The covariance function can be derived as well, but its expression is cumbersome.

As noted by Bouezmarni et al. (2016) in the general case, the asymptotic behavior of $\mathbb{C}_{xh}^{Z\to Y}$ under α-mixing is the same as that under serial independence. In other words, the impact of time-dependency is asymptotically negligible. This behavior is a consequence of Assumption $\mathcal{A}_1$ on the α-mixing coefficients, combined with the use of a kernel function that smooths the covariate space in a shrinking neighborhood of x as n goes to infinity. Note however that compared to the i.i.d. setting, Assumptions W_1 and W_{11}–W_{13} on the weight functions are needed in order to tackle moments of order six entailed by time-dependency.

Before stating the result on the weak convergence of $\widetilde{\mathbb{C}}_{xh}^{Z\to Y}$, introduce the Gaussian process $\mathbb{G}_x^{Z\to Y}$ that arises as the limit of $\mathbb{H}_{xh}^{Z\to Y}$ when (Y_1, Z_1), $\ldots$, (Y_{n+1}, Z_{n+1}) is replaced by $(U_1, V_1), \ldots, (U_n, V_n)$, where $U_i = F_{1Y_i}(Y_{i+1})$ and $V_i = F_{2Y_i}(Z_i)$. This situation corresponds to a case where the marginal conditional distributions are known. One then deduces from Proposition 2.1 with $H_x^{Z\to Y} = C_x^{Z\to Y}$ that the bias of $\mathbb{G}_x^{Z\to Y}$ is given for $(u,v)\in[0,1]^2$ by

$$\widetilde{\mathbb{B}}_x(u,v) = K\left\{K_2\,\dot{C}_x^{Z\to Y}(u,v) + K_3\,\ddot{C}_x^{Z\to Y}(u,v)\right\}$$

and for $(u,v),(u',v')\in[0,1]^2$, its covariance function is

$$\begin{aligned}\mathrm{Cov}\left\{\mathbb{G}_x^{Z\to Y}(u,v), \mathbb{G}_x^{Z\to Y}(u',v')\right\} &= K_4\,C_x^{Z\to Y}\left\{\min(u,u'), \min(v,v')\right\} \\ &\quad - K_4\,C_x^{Z\to Y}(u,v)\,C_x^{Z\to Y}(u',v').\end{aligned}$$

The next proposition is deduced from Proposition 3.1 of Bouezmarni et al. (2016).

Proposition 2.3. *Suppose that Assumptions $\mathcal{A}_1$, $\mathcal{A}_3$–$\mathcal{A}_5$ and W_1–W_{13} are satisfied. If $n\min(h_1,h_2)\to\infty$, $n\max(h_1^5,h_2^5)<\infty$ and $h/\min(h_1,h_2)<\infty$, $\widetilde{\mathbb{C}}_{xh}$ converges weakly in the space $\ell^\infty([0,1]^2)$ to the Gaussian process*

$$\widetilde{\mathbb{C}}_x^{Z\to Y}(u,v) = \mathbb{G}_x^{Z\to Y}(u,v) - C_x^{[1]}(u,v)\,\mathbb{G}_x^{Z\to Y}(u,1) - C_x^{[2]}(u,v)\,\mathbb{G}_x^{Z\to Y}(1,v).$$

One can show that the asymptotic bias of $\widetilde{\mathbb{C}}_{xh}$ is $\widetilde{\mathbb{B}}_x$. It can also be seen from Proposition 2.1 and Proposition 2.3 that $\mathbb{C}_{xh}$ and $\widetilde{\mathbb{C}}_{xh}$ share the same covariance structure. However, they have a different bias function in general.

3. Measuring local causality

3.1. Theoretical measures of local causality

Measuring the strength of the causal relationship from Z to Y can be done using functionals of the local causality copula. Specifically, let $\Lambda : \ell^\infty([0,1]^2) \to \mathbb{R}$ be such that $\Lambda(\Pi) = 0$, $\Lambda(M) = 1$ and $\Lambda(W) = -1$, where $\Pi(u,v) = uv$, $M(u,v) = \min(u,v)$ and $W(u,v) = \max(u+v-1,0)$ are respectively the independence, perfect positive dependence and perfect negative dependence copulas. A measure of local causality from Z to Y at x based on Λ is then

$$\theta^{Z\to Y}_{\Lambda,x} = \Lambda\left(C^{Z\to Y}_x\right). \tag{4}$$

This measure has the desirable property of being marginal-free. Among the possibilities, one can define local causality indices based on the popular Spearman and Kendall functionals given respectively for $\delta \in \ell^\infty([0,1]^2)$ by

$$\Lambda_\rho(\delta) = 12 \int_{[0,1]^2} \delta(u,v)\,\mathrm{d}u\mathrm{d}v - 3 \quad \text{and} \quad \Lambda_\tau(\delta) = 4 \int_{[0,1]^2} \delta(u,v)\,\mathrm{d}\delta(u,v) - 1.$$

The corresponding measures of local causality will be referred to $\rho^{Z\to Y}_x$ and $\tau^{Z\to Y}_x$ in the sequel.

3.2. Nonparametric estimators

The estimation of the local causality index $\theta^{Z\to Y}_{\Lambda,x}$ defined in Equation (4) can be based on the empirical local causality copulas $C^{Z\to Y}_{xh}$ and $\widetilde{C}^{Z\to Y}_{xh}$. Specifically, two estimators of $\theta^{Z\to Y}_{\Lambda,x}$ are given by

$$\theta^{Z\to Y}_{\Lambda,xh} = \Lambda\left(C^{Z\to Y}_{xh}\right) \quad \text{and} \quad \widetilde{\theta}^{Z\to Y}_{\Lambda,xh} = \Lambda\left(\widetilde{C}^{Z\to Y}_{xh}\right).$$

The next result establishes the asymptotic normality of

$$\Theta^{Z\to Y}_{\Lambda,xh} = \sqrt{nh}\left(\theta^{Z\to Y}_{\Lambda,xh} - \theta^{Z\to Y}_{\Lambda,x}\right) \quad \text{and} \quad \widetilde{\Theta}^{Z\to Y}_{\Lambda,xh} = \sqrt{nh}\left(\widetilde{\theta}^{Z\to Y}_{\Lambda,xh} - \theta^{Z\to Y}_{\Lambda,x}\right).$$

Proposition 3.1. *Assume that Λ is Hadamard differentiable with derivative at g given by Λ'_g. Then, let*

$$\sigma^2_{\Lambda,x} = \mathrm{Var}\left\{\Lambda'_{C_x^{Z\to Y}}\left(\mathbb{C}_x^{Z\to Y}\right)\right\} = \mathrm{Var}\left\{\Lambda'_{C_x^{Z\to Y}}\left(\widetilde{\mathbb{C}}_x^{Z\to Y}\right)\right\}. \qquad (5)$$

(i) Under the conditions of Corollary 2.2, $\Theta^{Z\to Y}_{\Lambda,xh}$ converges in law to the Normal distribution with mean $\mu_{\Lambda,x} = \Lambda'_{C_x^{Z\to Y}}(\mathbb{B}_x)$ and variance $\sigma^2_{\Lambda,x}$.

(ii) Under the conditions of Proposition 2.3, $\widetilde{\Theta}^{Z\to Y}_{\Lambda,xh}$ converges in law to the Normal distribution with mean $\widetilde{\mu}_{\Lambda,x} = \Lambda'_{C_x^{Z\to Y}}(\widetilde{\mathbb{B}}_x)$ and variance $\sigma^2_{\Lambda,x}$.

Note that the functionals associated to the Kendall and Spearman causality measures are Hadamard differentiable with derivatives given respectively by

$$\begin{aligned}
(\Lambda_\rho)'_g(\delta) &= 12\int_{[0,1]^2} \delta(u,v)\,\mathrm{d}u\mathrm{d}v,\\
(\Lambda_\tau)'_g(\delta) &= 4\int_{[0,1]^2} \{\delta(u,v)\,\mathrm{d}g(u,v) + g(u,v)\,\mathrm{d}\delta(u,v)\}.
\end{aligned}$$

Hence, the conclusions of Proposition 3.1 apply in these cases.

3.3. Estimation of the asymptotic variance

If the goal is to build a confidence interval for a local causality measure, Proposition 3.1 cannot be used directly since the asymptotic variance $\sigma^2_{\Lambda,x}$ in Equation (5) is unknown. In order to motivate the form of the estimator of $\sigma^2_{\Lambda,x}$ that will be introduced, first consider a context where one wants to estimate a conditional mean $\mu_x = \mathrm{E}(Y|X=x)$. Based on observations $(Y_1,X_1),\ldots,(Y_n,X_n)$, a natural estimator is given by the weighted mean

$$\mu_{xh} = \sum_{i=1}^n \mathcal{K}\left(\frac{X_i - x}{h}\right) Y_i.$$

Then, under Assumptions W_1–W_5, it can be shown that $\sqrt{nh}(\mu_{xh} - \mu_x)$ is asymptotically Normal with variance $K_4\sigma^2_x$, where $\sigma^2_x = \mathrm{Var}(Y|X=x)$. It can also be shown that a consistent estimator of σ^2_x is given by

$$\sigma^2_{xh} = \sum_{i=1}^n \mathcal{K}\left(\frac{X_i - x}{h}\right)(Y_i - \mu_{xh})^2.$$

Hence, a consistent estimator of the asymptotic variance of μ_{xh} is given by $\widehat{K}_4 \sigma^2_{xh}$, where in view of Assumption W_4,

$$\widehat{K}_4 = nh \sum_{i=1}^{n} \left\{ \mathcal{K}\left(\frac{X_i - x}{h} \right) \right\}^2 .$$

Now in order to adapt the idea to the context of local causality indices, recall that $\sigma^2_{\Lambda,x}$ is the variance of $\Lambda'_{C_x^{Z\to Y}}(\mathbb{C}_x^{Z\to Y})$. Then, observe that the process $\mathbb{C}_x^{Z\to Y}$ can be seen as the weak limit of

$$\sqrt{nh} \sum_{i=2}^{n+1} \mathcal{K}\left(\frac{Y_{i-1} - x}{h} \right) L_{x,i},$$

where for each $(u,v) \in [0,1]^2$,

$$\begin{aligned} L_{x,i}(u,v) &= \mathbb{I}\left\{ Y_i \le F_{1x}^{-1}(u), Z_{i-1} \le F_{2x}^{-1}(v) \right\} - C_x^{Z\to Y}(u,v) \\ &\quad - C_x^{[1]}(u,v) \left[\mathbb{I}\left\{ Y_i \le F_{1x}^{-1}(u) \right\} - u \right] \\ &\quad - C_x^{[2]}(u,v) \left[\mathbb{I}\left\{ Z_{i-1} \le F_{2x}^{-1}(v) - v \right\} \right] . \end{aligned}$$

Hence, since Hadamard derivatives are linear functionals (see van der Vaart & Wellner (1996)), $\Lambda'_{C_x^{Z\to Y}}(\mathbb{C}_x^{Z\to Y})$ can be taken as the limit of $\sqrt{nh}\,\lambda_{xh}$, where in terms of $\lambda_{x,i} = \Lambda'_{C_x^{Z\to Y}}(L_{x,i})$,

$$\lambda_{xh} = \sum_{i=2}^{n+1} \mathcal{K}\left(\frac{Y_{i-1} - x}{h} \right) \lambda_{x,i}.$$

In view of the above discussion, an estimator of $\sigma^2_{\Lambda,x}$ could therefore be based on $\lambda_{x,2}, \ldots, \lambda_{x,n+1}$. However, since the marginal conditional distributions F_{1x}, F_{2x} and the partial derivatives $C_x^{[1]}$, $C_x^{[2]}$ are unknown, consider instead the version

$$\begin{aligned} \widehat{L}_{x,i}(u,v) &= \mathbb{I}\left\{ Y_i \le F_{1xh}^{-1}(u), Z_{i-1} \le F_{2xh}^{-1}(v) \right\} \\ &\quad - \widehat{C}_x^{[1]}(u,v)\, \mathbb{I}\left\{ Y_i \le F_{1xh}^{-1}(u) \right\} \\ &\quad - \widehat{C}_x^{[2]}(u,v)\, \mathbb{I}\left\{ Z_{i-1} \le F_{2xh}^{-1}(v) \right\} , \end{aligned}$$

where $\widehat{C}_x^{[1]}$ and $\widehat{C}_x^{[2]}$ are estimators of the partial derivatives of $C_x^{\mathrm{Z\to Y}}$ that are uniformly consistent in the sense that for any $\varepsilon > 0$,

$$\sup_{\substack{u\in[\varepsilon,1-\varepsilon] \\ v\in[0,1]}} \left| \widehat{C}_x^{[1]}(u,v) - C_x^{[1]}(u,v) \right| \quad \text{and} \quad \sup_{\substack{v\in[\varepsilon,1-\varepsilon] \\ u\in[0,1]}} \left| \widehat{C}_x^{[2]}(u,v) - C_x^{[2]}(u,v) \right|$$

converge in probability to zero. Then, letting $\widehat{\lambda}_{x,i} = \Lambda'_{C_x^{Z\to Y}}(\widehat{L}_{x,i})$, the proposed estimator of $\sigma^2_{\Lambda,x}$ is

$$\widehat{\sigma}^2_{\Lambda,x} = \widehat{K}_4 \sum_{i=2}^{n+1} \mathcal{K}\left(\frac{Y_{i-1}-x}{h}\right)\left(\widehat{\lambda}_{x,i} - \widehat{\lambda}_{xh}\right)^2,$$

where

$$\widehat{\lambda}_{xh} = \sum_{i=2}^{n+1} \mathcal{K}\left(\frac{Y_{i-1}-x}{h}\right)\widehat{\lambda}_{x,i}.$$

The consistency $\widehat{\sigma}^2_{\Lambda,x}$ is stated next.

Proposition 3.2. *Assume that Λ is Hadamard differentiable with derivative at g given by Λ'_g. Moreover, assume there exists a constant $D > 0$ such that*

$$\mathbb{P}\left(\max_{j=1,2}\sup_{u,v\in[0,1]}\left|\widehat{C}_x^{[j]}(u,v)\right| > D\right) \to 0.$$

Under the conditions of Corollary 2.2, $\widehat{\sigma}^2_{\Lambda,x}$ is a consistent estimator of $\sigma^2_{\Lambda,x}$.

When the functional Λ is linear, its Hadamard derivative is free of g, *i.e.* $\Lambda'_g = \Lambda'$ for all g; this happens in particular for the Spearman functional. In most cases, however, Λ'_g needs to be estimated. One can then replace $\Lambda'_{C_x^{Z\to Y}}$ by an estimator $\widehat{\Lambda'_{C_x^{Z\to Y}}}$ in the above procedure as long as for any $\delta \in \ell^\infty([0,1]^2)$,

$$\left|\widehat{\Lambda'_{C_x^{Z\to Y}}}(\delta) - \Lambda'_{C_x^{Z\to Y}}(\delta)\right| = o_{\mathbb{P}}(1).$$

This modification has no impact on the conclusion of Proposition 3.2. For example, in the case of the Kendall functional, $(\Lambda_\tau)'_{C_x^{Z\to Y}}$ is estimated by

$$\widehat{(\Lambda_\tau)'}_{C_x^{Z\to Y}}(\delta) = 4\int_{[0,1]^2}\left\{\delta(u,v)\,\mathrm{d}C_{xh}(u,v) + C_{xh}(u,v)\,\mathrm{d}\delta(u,v)\right\}.$$

3.4. Confidence intervals

Proposition 3.1 and Proposition 3.2 can now be combined to build confidence intervals. Neglecting the possible bias, an approximate confidence interval of level $1-\alpha$ for $\theta^{Z\to Y}_{\Lambda,x}$ based on $\theta^{Z\to Y}_{\Lambda,xh}$ is given by

$$\mathcal{CI}_\alpha(\theta^{Z\to Y}_{\Lambda,x}) = \theta^{Z\to Y}_{\Lambda,xh} \pm \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\frac{\widehat{\sigma}_{\Lambda,x}}{\sqrt{nh}},$$

where Φ is the cdf of the standard Normal distribution. The confidence interval is similar when $\theta^{Z\to Y}_{\Lambda,x}$ is estimated by $\widetilde{\theta}^{Z\to Y}_{\Lambda,xh}$.

Strictly speaking, this confidence interval is asymptotically of level $1-\alpha$ if and only if the large-sample bias of $\theta^{Z\to Y}_{\Lambda,xh}$ vanishes. Since this bias term is generally difficult to estimate, a strategy would be to choose a bandwidth h such that $nh^5 \to 0$ in order that the biases tend to zero asymptotically. However, based on many numerical experiments, it is usually safer to simply neglect it, since it is often close to zero.

4. Testing for local non-causality

Saying that there is no local causality relationship from Z to Y at x in the process $\{(Y_t, Z_t)\}_{t\in\mathbb{Z}}$ means that Y_t and Z_{t-1} are conditionally independent with respect to $Y_{t-1} = x$. In other words, the local causality copula corresponds to the independence copula in that case, *i.e.* $C^{Z\to Y}_x(u,v) = \Pi(u,v) = u\,v$. The following result is a consequence of Corollary 2.2 and of Proposition 2.3. Before stating it, let $\Omega_\Lambda : \ell^\infty([0,1]^4) \to \mathbb{R}$ be the functional such that for $\eta \in \ell^\infty([0,1]^4)$,

$$\Omega_\Lambda(\eta) = \Lambda'_{C^{Z\to Y}_x}\left[\Lambda'_{C^{Z\to Y}_x}\left\{\eta(\cdot,\cdot,u',v')\right\}\right].$$

In other words, the operator inside the brackets is computed with respect to the first two arguments of η.

Proposition 4.1. *Suppose that the conditions in Corollary 2.2 and in Proposition 2.3 are satisfied respectively for $\mathbb{C}^{Z\to Y}_{xh}$ and $\widetilde{\mathbb{C}}^{Z\to Y}_{xh}$. Then under the null hypothesis of non-causality from Z to Y, $\sqrt{nh}\,\theta^{Z\to Y}_{\Lambda,xh}$ and $\sqrt{nh}\,\widetilde{\theta}^{Z\to Y}_{\Lambda,xh}$ are asymptotically Normal with variance $\sigma^2_{\Lambda,x} = K_4\,\Omega_\Lambda(\gamma)$, where for $(u,v),(u',v') \in [0,1]^2$,*

$$\gamma(u,u',v,v') = \{\min(u,u') - uu'\}\{\min(v,v') - vv'\}.$$

Proposition 4.1 can be exploited to test the null hypothesis of local non-causality from Z to Y. In that case, the null and alternative hypotheses are

$$\mathbb{H}_0 : \theta^{Z\to Y}_{\Lambda,x} = 0 \quad \text{and} \quad \mathbb{H}_1 : \theta^{Z\to Y}_{\Lambda,x} \neq 0.$$

Neglecting the asymptotic bias, a test based on the statistic $\theta^{Z\to Y}_{\Lambda,xh}$ will reject the null hypothesis of local non-causality whenever

$$\sqrt{nh}\,\left|\theta^{Z\to Y}_{\Lambda,xh}\right| > \widehat{K}_4\,\Omega_\Lambda(\gamma)\,\Phi^{-1}\left(1-\frac{\alpha}{2}\right).$$

The test based on $\widetilde{\theta}_{\Lambda,xh}^{Z\to Y}$ is similar. For the Spearman and Kendall functionals, $\Omega_{\Lambda_\rho}(\gamma) = 1$ and $\Omega_{\Lambda_\tau}(\gamma) = 4/9$. This is a consequence of the fact that

$$\int_{[0,1]^4} \{\min(u,u') - uu'\}\{\min(v,v') - vv'\}\, \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}u'\, \mathrm{d}v' = 1.$$

5. Simulation study

5.1. Preliminaries

The aim of this section is to investigate how well the method introduced in this work perform. A particular attention will be given to the procedures based on the Spearman and Kendall functionals. By straightforward computations, explicit expressions for their empirical versions are given by

$$\begin{aligned}
\rho_{xh}^{Z\to Y} &= 12 \sum_{i=2}^{n+1} \mathcal{K}\left(\frac{Y_{i-1}-x}{h}\right) \{1 - F_{1xh}(Y_i)\}\{1 - F_{2xh}(Z_{i-1})\} - 3, \\
\widetilde{\rho}_{xh}^{Z\to Y} &= 12 \sum_{i=1}^{n} \mathcal{K}\left(\frac{Y_i - x}{h}\right) \left\{1 - G_{1xh_1}(\widetilde{U}_i)\right\} \left\{1 - G_{2xh_2}(\widetilde{V}_i)\right\} - 3, \\
\tau_{xh}^{Z\to Y} &= 4 \sum_{i,j=2}^{n+1} \mathcal{K}\left(\frac{Y_{i-1}-x}{h}\right) \mathcal{K}\left(\frac{Y_{j-1}-x}{h}\right) \mathbb{I}\left(Y_j \le Y_i, Z_{j-1} \le Z_{i-1}\right) - 1, \\
\widetilde{\tau}_{xh}^{Z\to Y} &= 4 \sum_{i,j=1}^{n} \mathcal{K}\left(\frac{Y_i - x}{h}\right) \mathcal{K}\left(\frac{Y_j - x}{h}\right) \mathbb{I}\left(\widetilde{U}_j \le \widetilde{U}_i, \widetilde{V}_i \le \widetilde{V}_j\right) - 1.
\end{aligned}$$

Following Gijbels et al. (2011), mildly modified versions of Kendall's measure were used, where the summation is taken over all $i \neq j$ and then standardized with

$$1 - \sum_{i=1}^{n} \left\{\mathcal{K}\left(\frac{Y_i - x}{h}\right)\right\}^2.$$

The interval estimation of causality measures requires the estimation of the partial derivatives $C_x^{[1]}$ and $C_x^{[2]}$. In the upcoming simulations, one considers the finite difference estimator given by

$$\widehat{C}_x^{[1]}(u,v) = \sqrt{nh}\left\{C_{xh}^{Z\to Y}(u_h^\star, v) - C_{xh}^{Z\to Y}(u,v)\right\},$$

where $u_h^\star = u + \min\{1/\sqrt{nh}, 1-u\}$. The estimator $\widehat{C}_x^{[2]}$ is defined similarly. This particular choice fulfills the assumption stated in Proposition 3.2. The results that will be reported have been obtained using the triweight function $L(y) = 35(1-y^2)^3\,\mathbb{I}(|y| \le 1)/32$, leading to the local linear kernel

$$\mathcal{K}(y) = L(y) \left(\frac{S_{n,2} - y\, S_{n,1}}{S_{n,0}\, S_{n,2} - S_{n,1}^2} \right),$$

where for $a \in \{h, h_1, h_2\}$,

$$S_{n,\ell} = \sum_{i=1}^{n} \left(\frac{Y_i - x}{a} \right)^{\ell} L\left(\frac{Y_i - x}{a} \right), \quad \ell \in \{0,1,2\}.$$

Negative weights are taken to be zero and the remaining weights are rescaled in order that they sum to one. Using similar arguments as those in Li & Racine (2007), one can show that Assumptions W_1–W_{13} are satisfied whenever the alpha-mixing coefficients are of order $O(r^{-a})$ for $a > 6$. Other experiments with the Nadaraya–Watson kernel show very similar results, so they are not presented here. For the selection of the bandwidth parameters h, h_1 and h_2, several methods have been considered in the case $h = h_1 = h_2$, namely

(1) the plug-in bandwidth selection rule of Gijbels et al. (2011);

(2) the minimization of the estimated integrated squared error;

(3) setting $h = S(Y_1, \ldots, Y_n) \times n^{-1/5}$, where $S(\cdot)$ is either the variance, the inter-quartile range or the range between the 5th and 95th percentiles.

Based on many experiments, the third method based on the inter-quartile range performs better for the tests based on Kendall's functional, while for Spearman's functional, it is the third method using the 5th and 95th percentiles that is the best. Also, it is worth mentioning that the performance of Spearman's functional is more sensible to the choice of h compared to the Kendall functional, at least for the scenarios that were considered.

5.2. Accuracy of the local causality estimators

The performance of the local causality measures $\rho_{xh}^{Z\to Y}$, $\widetilde{\rho}_{xh}^{Z\to Y}$, $\tau_{xh}^{Z\to Y}$ and $\widetilde{\tau}_{xh}^{Z\to Y}$ will be investigated in the light of their bias and variance. To this end, a vector autoregressive model of order one has been considered. Specifically, time series have been simulated based on the stationary process

$$\begin{pmatrix} Y_t \\ Z_t \end{pmatrix} = \Sigma \begin{pmatrix} Y_{t-1} \\ Z_{t-1} \end{pmatrix} + \boldsymbol{\epsilon}_t,$$

where $\Sigma \in \mathbb{R}^{2\times 2}$ is such that $\Sigma_{11} = \Sigma_{22} = (\theta_2 - \theta_1\theta_3)/(1-\theta_3^2)$ and $\Sigma_{21} = \Sigma_{12} = (\theta_1 - \theta_2\theta_3)/(1-\theta_3^2)$ and $\boldsymbol{\epsilon}_t$ is distributed as a centered and symmetric bivariate Normal distribution with variance $\sigma_\epsilon^2 = 1 - \Sigma_{11}^2 - \Sigma_{21}^2 - 2\theta_3\Sigma_{11}\Sigma_{21}$ and correlation $\rho_\epsilon = \theta_3(1-\Sigma_{11}^2 - \Sigma_{21}^2) - 2\Sigma_{11}\Sigma_{21}$. With this particular choice of parameters, $(Y_t, Z_t, Y_{t-1}, Z_{t-1})$ is centered Normal with covariance matrix

$$\Upsilon = \begin{bmatrix} 1 & \theta_3 & \theta_1 & \theta_2 \\ \theta_3 & 1 & \theta_2 & \theta_1 \\ \theta_1 & \theta_2 & 1 & \theta_3 \\ \theta_2 & \theta_1 & \theta_3 & 1 \end{bmatrix}.$$

In that case, the local causality is controlled by a Normal copula with parameter $\varrho = (\theta_1 - \theta_2\,\theta_3)/(\sqrt{1-\theta_2^2}\sqrt{1-\theta_3^2})$, *i.e.* $C_x^{\mathrm{Z}\to\mathrm{Y}}(u,v) = \Phi_\varrho\{\Phi^{-1}(u), \Phi^{-1}(v)\}$, where Φ_ϱ is the cdf of the bivariate standard Normal distribution with correlation $\varrho \in [-1,1]$. The results reported in Figure 1 and Figure 2 are based on 1 000 replicates of this model when $x = 1/2$.

First observe that $\widetilde{\rho}_{xh}^{Z\to Y}$ outperforms $\rho_{xh}^{Z\to Y}$ both in terms of bias and variance when $(\theta_1, \theta_2, \theta_3) \in \{(.4, -.25, .3), (-.4, .25, .3), (.2, .44, .44)\}$; their performance are however similar when $(\theta_1, \theta_2, \theta_3) = (.0, .3, .0)$. This might be explained by the influence of the conditional marginal distributions F_{1x} and F_{2x} on $\mathrm{E}(\rho_{xh}^{Z\to Y})$, which is quite low for the latter model. The same comment can be made about $\tau_{xh}^{Z\to Y}$ and $\widetilde{\tau}_{xh}^{Z\to Y}$. Finally, it is worth noting that the bias and variance of $\widetilde{\rho}_{xh}^{Z\to Y}$ and $\widetilde{\tau}_{xh}^{Z\to Y}$ stabilize as the bandwidth parameter h increases. However, the biases of $\rho_{xh}^{Z\to Y}$ and $\tau_{xh}^{Z\to Y}$ are more sensible to the values of h.

5.3. Coverage probability of interval estimations

A general D-Vine structure for bivariate processes was recently suggested by Beare & Seo (2015). Specifically, let $C_1, \ldots, C_5$ be such that C_1 is the copula of (Y_t, Z_t), C_2 is the copula of (Y_{t-1}, Y_t), C_3 is the conditional copula of (Y_{t-1}, Z_t) given $Y_t = x$, C_4 is the conditional copula of (Y_t, Z_{t-1}) given $Y_{t-1} = x$ and C_5 is the conditional copula of (Z_t, Z_{t-1}) given $Y_{t-1} = x, Y_t = x'$. In this setup, C_4 plays the role of the local causality copula.

Here, the coverage probabilities of interval estimations based on $\rho_{xh}^{Z\to Y}$, $\widetilde{\rho}_{xh}^{Z\to Y}$, $\tau_{xh}^{Z\to Y}$ and $\widetilde{\tau}_{xh}^{Z\to Y}$ will be estimated in the case when C_4 is the Normal copula parameterized in such a way that $\tau_x^{Z\to Y} \in \{0, .1, .2, .3, .4\}$; this is done easily using the relationship $\varrho = \sin(\pi\,\tau_x^{Z\to Y}/2)$. The copulas C_1, C_2, C_3 and C_5 are also Normal and are parameterized in terms of their respective values

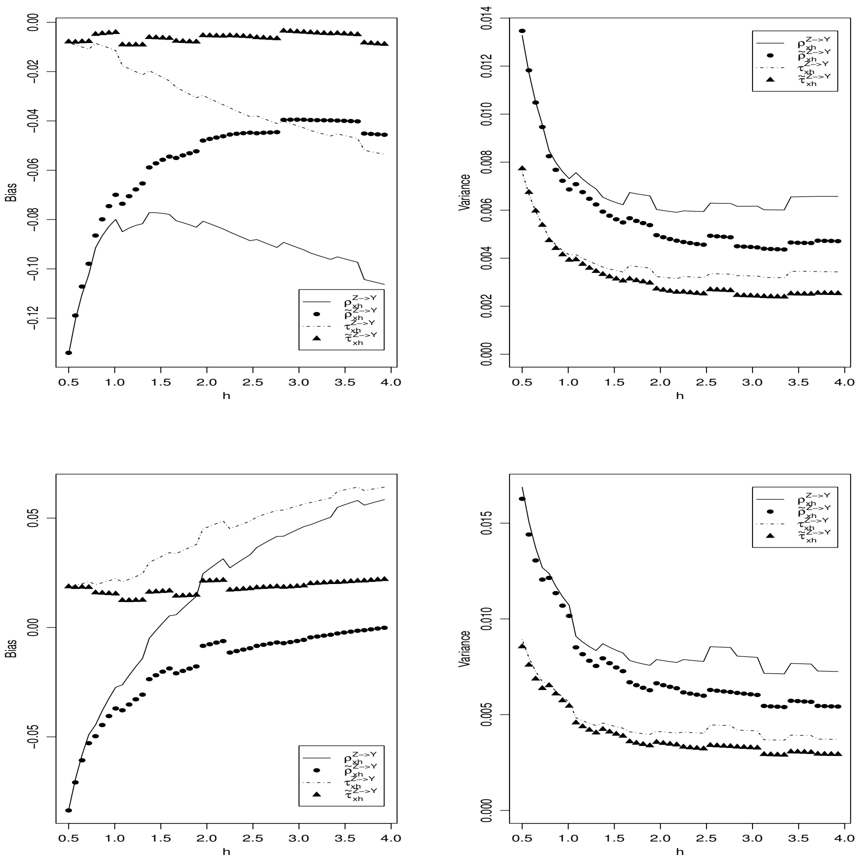

Figure 1: Estimated Bias (left panels) and Variance (right panels) of $\rho_{xh}^{Z\to Y}$, $\widetilde{\rho}_{xh}^{Z\to Y}$, $\tau_{xh}^{Z\to Y}$ and $\widetilde{\tau}_{xh}^{Z\to Y}$ as a function of the bandwidth parameter h under Gaussian vector autoregressive processes when $n = 250$. Upper panels: $(\theta_1, \theta_2, \theta_3) = (.4, -.25, .3)$; bottom panels: $(\theta_1, \theta_2, \theta_3) = (-.4, .25, .3)$.

τ_1, τ_2, τ_3 and τ_5 of Kendall's tau. The results on the coverage probability in the case $x = 0.5$ are to be found in Table B.1.

Generally speaking, the coverage probabilities tend to be closer to their 95% nominal level as n increases. An exception occurs for $\rho_{xh}^{Z\to Y}$ when $(\tau_1, \tau_2, \tau_3, \tau_5) = (.5, .3, .75, .05)$. In that case, since the coverage probabilities of $\widetilde{\rho}_{xh}^{Z\to Y}$ behave well, this may be due to the influence of the conditional marginal distributions on the asymptotic bias. Finally note that the coverage probabilities are very similar for all levels of conditional dependence as

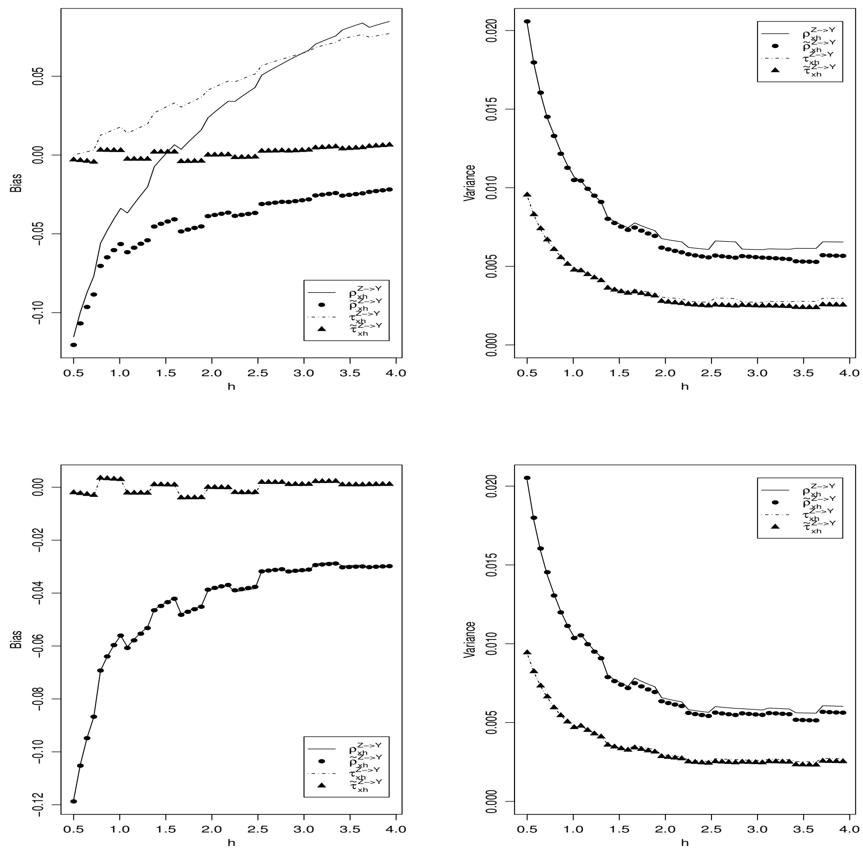

Figure 2: Estimated Bias (left panels) and Variance (right panels) of $\rho_{xh}^{Z\to Y}$, $\widetilde{\rho}_{xh}^{Z\to Y}$, $\tau_{xh}^{Z\to Y}$ and $\widetilde{\tau}_{xh}^{Z\to Y}$ as a function of the bandwidth parameter h under Gaussian vector autoregressive processes when $n = 250$. Upper panels: $(\theta_1, \theta_2, \theta_3) = (.2, .44, .44)$; bottom panels: $(\theta_1, \theta_2, \theta_3) = (.0, .3, .0)$.

measured by $\tau_x^{Z\to Y}$.

5.4. Power of the tests of local non-causality

Consider testing the null hypothesis $\mathbb{H}_0$ of the local non-causality from Z to Y at x, *i.e.* the conditional independence between Y_t and Z_{t-1} given $Y_{t-1} = x$. To this end, one considers again the D-Vine structure for $(Y_t, Z_t)_{t\in\mathbb{Z}}$ described in Subsection 5.3. Here, C_4 is taken to be either the Normal or the Clayton copula; the latter is defined by $C_\theta^{\rm CL}(u, v) = (u^{-\theta}+v^{-\theta}-1)^{-1/\theta}$, $\theta > 0$. Both the Normal and the Clayton copulas are parameterized in such a way that $\tau_x^{Z\to Y} = (3/2)\{\Phi(x) - 1/2\}^2$. From well-known relationships between Kendall's tau and the parameters of the Normal and Clayton copulas, this corresponds to

$$\varrho = \sin\left[\frac{3\left\{\Phi(x) - 1/2\right\}^2}{2\pi}\right] \quad \text{and} \quad \theta = \frac{6\left\{\Phi(x) - 1/2\right\}^2}{2 - 3\left\{\Phi(x) - 1/2\right\}^2}.$$

The values of x are chosen in order that $\tau_x^{Z\to Y} \in \{0, .1, .25\}$. Here again, C_1, C_2, C_3 and C_5 are Normal copulas parameterized in terms of their respective values τ_1, τ_2, τ_3 and τ_5 of Kendall's tau; the results in Table B.2 concerns the case when C_4 is Normal, while Table B.3 concerns the Clayton copula.

Looking at Tables B.2–B.3, one can say that generally speaking, the four tests are good at maintaining their nominal level under the null hypothesis of non-causality. An exception occurs for the test based on $\rho_{xh}^{Z\to Y}$ when $(\tau_1, \tau_2, \tau_3, \tau_5) \in \{(.5, .3, .3, .05), (5, .3, .75, .05)\}$, when $C_x^{Z\to Y}$ is the Normal or the Clayton copula. It can be seen that the nominal levels in these situations are rather far from 5%. This might be due to the fact that the procedure neglects the bias. Since $\widetilde{\rho}_{xh}^{Z\to Y}$ performs well under these models, it is an indication that this bias is due to the conditional marginal distributions.

The ability of the tests to reject the null hypothesis under departures from non-causality is good and increases with the sample size, as expected. Interestingly, when $n = 250$, the two tests based on Kendall's tau are more powerful than those based on Spearman's rho, while the latter are better for larger sample sizes. It is especially true when $\tau_x^{Z\to Y} = .1$ and $n = 1\,000$. Note however that these conclusions for $\rho_{xh}^{Z\to Y}$ might be influenced by the fact that the latter hardly keeps its nominal level when $(\tau_1, \tau_2, \tau_3, \tau_5) \in \{(.5, .3, .3, .05), (5, .3, .75, .05)\}$.

6. Illustration on financial data

The following illustration is based on the bivariate time series of the 1 512 daily observations taken between January 2010 and January 2016 for the compounded changes in prices (returns) and trading volume of the Standard and Poor's 500 (S&P500) Index. The relationship between these two indices has been extensively studied, both from a theoretical and from an empirical perspective. According to the tests of stationarity reported in Bouezmarni et al. (2012), one will work instead with the first difference in logarithmic returns (Y) and with the first difference in logarithmic volume (Z). As a consequence, the upcoming conclusions will have to be interpreted in terms of growth rates.

The causality from Z to Y is then investigated from the sample (Y_2, Z_1, Y_1), ..., $(Y_{1511}, Z_{1510}, Y_{1510})$. For these data, the value of the partial correlation coefficient of (Y_t, Z_{t-1}) given Y_{t-1} is -0.024×10^{-4}, leading to the conclusion of a global non-causality (p-value = 0.36). However, as mentioned in the introduction, such a conclusion can be misleading when the relationship between Y_t and Z_{t-1} changes according to the value taken by Y_{t-1}. This is exactly what happens here. For example, if one considers the sub-sample for which $Y_{t-1} > 0$, then the partial correlation coefficient is 0.072, which is significantly different from zero (p-value = 0.039). On the other hand, the subsample for which $Y_{t-1} < 0$ leads to a partial correlation coefficient of -0.095 (p-value = 0.01).

In order to take into account the levels of Y_{t-1}, a solution is to rely on local causality indices as introduced in Section 3. Figure 3 reports the values of $\rho_{xh}^{Z\to Y}$, $\widetilde{\rho}_{xh}^{Z\to Y}$, $\tau_{xh}^{Z\to Y}$ and $\widetilde{\tau}_{xh}^{Z\to Y}$ as a function of x, for x ranging between the 10^{th} and the 90^{th} percentile of the Y. In addition, the 95% point-wise confidence intervals as computed from the method in Section 3 are given. Clearly, the values taken by the local causality indices depend on x.

In Figure 4, the same curves are given, this time with the 95% point-wise critical values of the test of local non-causality are given. It can be seen that the curves based on $\rho_{xh}^{Z\to Y}$ and $\widetilde{\rho}_{xh}^{Z\to Y}$ are below the lower bound when Y_{t-1} is less than 0.002 while it exceeds the upper bound for $Y_{t-1} > 0.004$. This is in accordance with the conclusions of the tests of non-causality based on the partial correlation coefficient.

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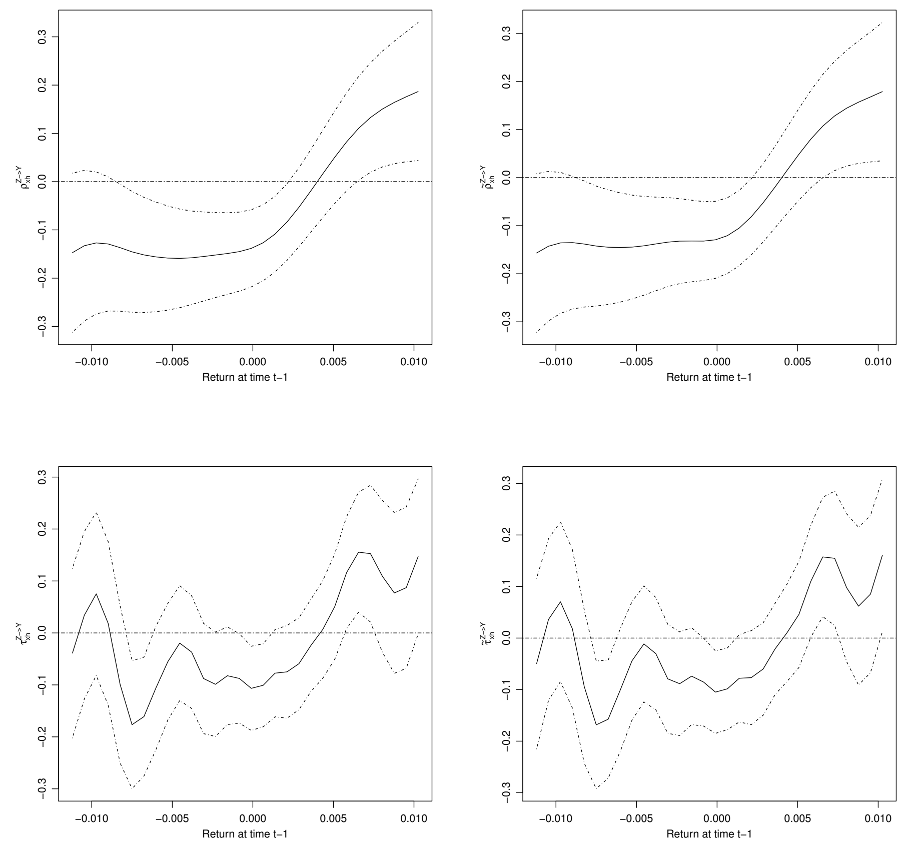

Figure 3: Curves of $\rho_{xh}^{Z\to Y}$ (upper left), $\widetilde{\rho}_{xh}^{Z\to Y}$ (upper right), $\widetilde{\tau}_{xh}^{Z\to Y}$ (bottom left) and $\widetilde{\tau}_{xh}^{Y\to Z}$ (bottom right) as a function of x, together with 95% point-wise confidence bands, for the Standard and Poor's 500 bivariate time series

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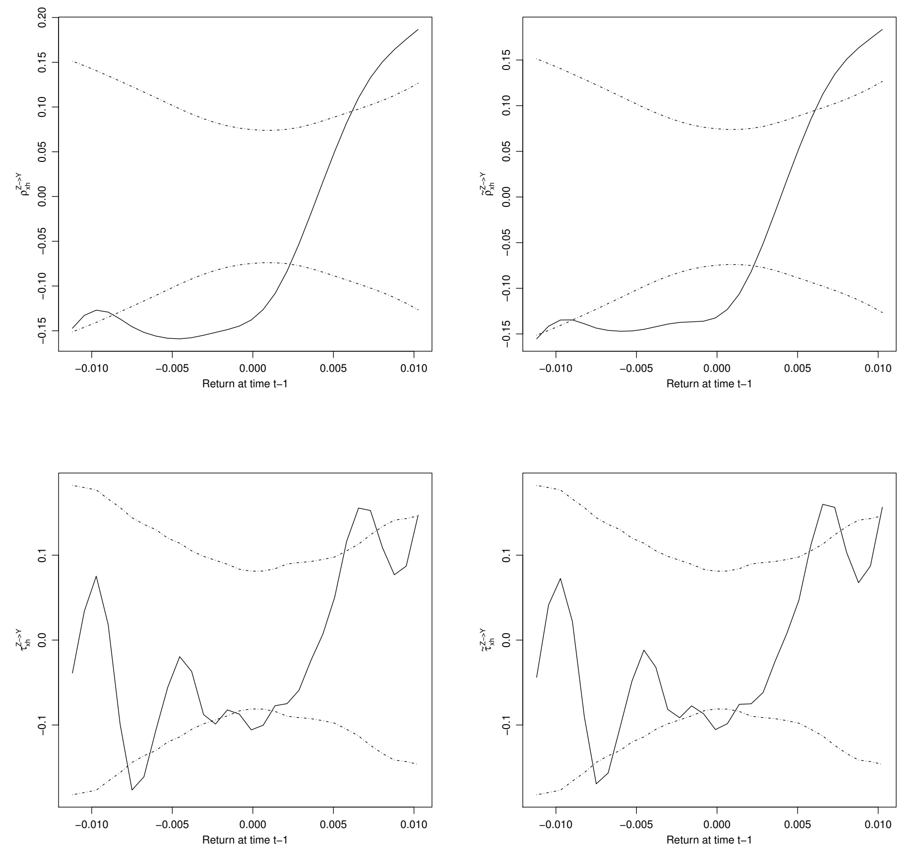

Figure 4: Curves of $\rho_{xh}^{Z\to Y}$ (upper left), $\widetilde{\rho}_{xh}^{Z\to Y}$ (upper right), $\widetilde{\tau}_{xh}^{Z\to Y}$ (bottom left) and $\widetilde{\tau}_{xh}^{Y\to Z}$ (bottom right) as a function of x, together with the 95% point-wise critical values of the test of local non-causality, for the Standard and Poor's 500 bivariate time series

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Appendix A. Assumptions needed in Proposition 2.1, Corollary 2.2 and Proposition 2.3

Appendix A.1. Conditions on the weights

For simplicity, one uses in the sequel the notation

$$w_{ni}(x,h) = \mathcal{K}\left(\frac{Y_i - x}{h}\right) \quad \text{and} \quad w'_{ni}(x,h) = \frac{\partial}{\partial x} w_{ni}(x,h). \tag{A.1}$$

In addition, let

$$I_{xn} = \{j \in \{1,\ldots,n\} : w_{nj}(x,h) > 0\} \quad \text{and} \quad J_{xn} = \left\{ \min_{i \in I_{nx}} Y_i,\ \max_{i \in I_{nx}} Y_i \right\}.$$

The following assumptions are needed to establish Proposition 2.1, Corollary 2.2 and Proposition 2.3.

W_1. $\max_{1\le i\le n} |w_{ni}(x,h)| = O_{\mathbb{P}}\left((nh)^{-1}\right)$;

W_2. $\sum_{i=1}^{n} w_{ni}(x,h)(Y_i - x) = h^2 K_2 + o_{\mathbb{P}}\left((nh)^{-1/2}\right)$ for some $K_2 < \infty$;

W_3. $\sum_{i=1}^{n} w_{ni}(x,h)(Y_i - x)^2/2 = h^2 K_3 + o_{\mathbb{P}}\left((nh)^{-1/2}\right)$ for some $K_3 < \infty$;

W_4. $nh \sum_{i=1}^{n} \{w_{ni}(x,h)\}^2 = K_4 + o_{\mathbb{P}}(1)$ for some $K_4 > 0$;

W_5. $\max_{i\in I_{xn}} Y_i - \min_{i\in I_{xn}} Y_i = o_{\mathbb{P}}(1)$;

W_6. $\sup_{\xi\in J_{xn}} \sum_{i=1}^{n} |w'_{ni}(\xi,h)| = O_{\mathbb{P}}(h^{-1})$;

W_7. $\sup_{\xi\in J_{xn}} \sum_{i=1}^{n} \{w'_{ni}(\xi,h)\}^2 = O_{\mathbb{P}}(n^{-1}h^{-3})$;

W_8. For some finite constant C,

$$\mathbb{P}\left(\sup_{\xi\in J_{xn}} \max_{1\le i\le n} |w_{ni}(\xi,h)\, \mathbb{I}\left(|Y_i - x| > Ch\right)| > 0\right) = o_{\mathbb{P}}(1);$$

W_9. There exists $D_K < \infty$ such that for all a_n,

$$\sup_{\xi\in J_{xn}} \left| \sum_{i=1}^{n} w_{ni}(\xi,a_n)(Y_i - \xi) - a_n^2\, D_K \right| = o_{\mathbb{P}}(a_n^2);$$

W_{10}. There exists $E_K < \infty$ such that for all a_n,

$$\sup_{\xi\in J_{xn}} \left| \sum_{i=1}^{n} w_{ni}(\xi,a_n)(Y_i - z)^2 - a_n^2\, E_K \right| = o_{\mathbb{P}}(a_n^2).$$

W_{11}. $\sup_{\xi \in J_{xn}} \sum_{i=1}^{n} \{w_{ni}(\xi, h)\}^{L_1} = O_{\mathbb{P}}\left((nh)^{-L_1+1}\right)$;

W_{12}. For any $2 \leq k \leq 6$, define $V_k = \{1 \leq \ell_2 < \cdots < \ell_k\}$ and $\boldsymbol{\ell} = (\ell_2, \ldots, \ell_k)$. Then, for any positive integers $L_1 + \cdots + L_k = r \leq 6$:

$$\sup_{\xi \in J_{xn}} \max_{\boldsymbol{\ell} \in V_k} \sum_{i=1}^{n-\ell_k} \{w_{ni}(\xi, h)\}^{L_1} \prod_{j=2}^{k} \{w_{n,i+\ell_j}(\xi, h)\}^{L_j} = O_{\mathbb{P}}\left(\frac{h^{k-1}}{(nh)^{r-1}}\right)$$

and

$$\sup_{\xi \in J_{xn}} \max_{\boldsymbol{\ell} \in V_k} \sum_{i=1}^{n-\ell_k} (Y_i - \xi) \{w_{ni}(\xi, h)\}^{L_1} \prod_{j=2}^{k} \{w_{n,i+\ell_j}(\xi, h)\}^{L_j} = O_{\mathbb{P}}\left(\frac{h^{k+1}}{(nh)^{r-1}}\right).$$

Appendix A.2. Additional conditions

$\mathcal{A}_1$. The α-mixing coefficients of $\{(Y_t, Z_t)\}_{t \in \mathbb{Z}}$ are such that $\alpha(r) = O(r^{-a})$ for some $a > 6$.

$\mathcal{A}_2$. $H_\xi^{Z\to Y}(y, z)$, $\dot{H}_\xi^{Z\to Y}(y, z)$ and $\ddot{H}_\xi^{Z\to Y}(y, z)$ exist and are continuous for $(\xi, y, z) \in J_{xn} \times \mathbb{R}^2$.

$\mathcal{A}_3$. The partial derivatives $C_x^{[1]}(u, v)$ and $C_x^{[2]}(u, v)$ exist and are continuous respectively on the sets $(0, 1) \times [0, 1]$ and $[0, 1] \times (0, 1)$.

$\mathcal{A}_4$. For $j = 1, 2$, $F_{j\xi}\{F_{j\xi}^{-1}(u)\}$, $\dot{F}_{j\xi}\{F_{j\xi}^{-1}(u)\}$ and $\ddot{F}_{j\xi}\{F_{j\xi}^{-1}(u)\}$ exist and are continuous for $(\xi, u) \in J_{xn} \times [0, 1]$.

$\mathcal{A}_5$. $C_\xi^{Z\to Y}(u, v)$, $\dot{C}_\xi^{Z\to Y}(u, v)$ and $\ddot{C}_\xi^{Z\to Y}(u, v)$ exist and are continuous for $(\xi, u, v) \in J_{xn} \times [0, 1]^2$.

Appendix B. Proof of proposition 3.2

First note that $\sigma_{\Lambda,x}^2 = \mathrm{Var}\{\Lambda'_{C_x^{\mathrm{Z\to Y}}}(\mathbb{C}_x^{\mathrm{Z\to Y}})\}$ can be written in terms of the covariance function of $\mathbb{C}_x^{\mathrm{Z\to Y}}$. Specifically, letting $\Omega = \Lambda'_{C_x^{\mathrm{Z\to Y}}} \circ \Lambda'_{C_x^{\mathrm{Z\to Y}}}$, one obtains from the linearity of $\Lambda'_{C_x{}^{\mathrm{Z\to Y}}}$ that

$$\begin{aligned} \sigma_{\Lambda,x}^2 &= \mathrm{Cov}\left\{\Lambda'_{C_x^{\mathrm{Z\to Y}}}(\mathbb{C}_x^{\mathrm{Z\to Y}}), \Lambda'_{C_x^{\mathrm{Z\to Y}}}(\mathbb{C}_x^{\mathrm{Z\to Y}})\right\} \\ &= \Omega\left[\mathrm{Cov}\left\{\mathbb{C}_x^{Z\to Y}(u, v), \mathbb{C}_x^{Z\to Y}(u', v')\right\}\right]. \end{aligned}$$

Moreover, one retrieves from the work of Bouezmarni et al. (2016) that the covariance function of $\mathbb{C}_x^{\mathrm{Z}\to\mathrm{Y}}$ can itself be expressed as

$$K_4\,\Upsilon\circ\Upsilon\left\{C_x^{\mathrm{Z}\to\mathrm{Y}}(u\wedge u', v\wedge v') - C_x^{\mathrm{Z}\to\mathrm{Y}}(u,v)C_x^{\mathrm{Z}\to\mathrm{Y}}(u',v')\right\},$$

where for any $\delta\in\ell^\infty([0,1]^2)$ and $\Delta\in\ell^\infty([0,1]^4)$,

$$\begin{aligned}\Upsilon(\delta)(u,v) &= \delta(u,v) - C_x^{[1]}(u,v)\delta(u,1) - C_x^{[2]}(u,v)\delta(1,v),\\ \Upsilon\circ\Upsilon(\Delta) &= \Upsilon\left[\Upsilon\left\{\Delta(\cdot,\cdot),u',v'\right\}\right].\end{aligned}$$

The functional Υ is a linear functional that corresponds to the Hadamard derivative of the copula mapping $H_x^{\mathrm{Z}\to\mathrm{Y}}\mapsto H_x^{\mathrm{Z}\to\mathrm{Y}}\circ\{F_{1x}^{-1},F_{2x}^{-1}\}$ (more details can be found in Bouezmarni et al. (2016)). Proceeding as in Appendix Appendix A.1, first define $w_{ni}(x,h)=\mathcal{K}\{(Y_i-x)/h\}$ for each $i\in\{2,\ldots,n+1\}$. From the conditions on the estimators of the partial derivatives of $C_x^{\mathrm{Z}\to\mathrm{Y}}$, one can readily show that for each $i\in\{1,\ldots,n\}$,

$$\sup_{(u,v)\in[0,1]^2}\left|\widehat{L}_{x,i}(u,v) - L'_{x,i}(u,v)\right| = o_{\mathbb{P}}(1),$$

where

$$L'_{x,i}(u,v) = \Upsilon\left[\mathbb{I}\left\{Y_i\le F_{1xh}^{-1}(u), Z_{i-1}\le F_{2xh}^{-1}(v)\right\}\right].$$

Hence, by the definition of $\widehat{\sigma}^2_{\Lambda,x}$, one can write

$$\begin{aligned}\frac{\widehat{\sigma}^2_{\Lambda,x}}{\widehat{K}_4} &= \sum_{i=2}^{n+1} w_{ni}(x,h)\left\{\Upsilon\left(\widehat{L}_{x,i}\right) - \sum_{j=2}^{n+1} w_{nj}(x,h)\,\Upsilon\left(\widehat{L}_{x,j}\right)\right\}^2\\
&= \sum_{i=2}^{n+1} w_{ni}(x,h)\left\{\Upsilon\left(L'_{x,i}\right) - \sum_{j=2}^{n+1} w_{nj}(x,h)\,\Upsilon\left(L'_{x,j}\right)\right\}^2 + o_{\mathbb{P}}(1)\\
&= \sum_{i=2}^{n+1} w_{ni}(x,h)\left\{\Upsilon\left(L'_{x,i}\right)\right\}^2 - \left\{\sum_{j=2}^{n+1} w_{nj}(x,h)\,\Upsilon\left(L'_{x,j}\right)\right\}^2 + o_{\mathbb{P}}(1)\\
&= \Omega\left\{\Upsilon\circ\Upsilon\left\{\sum_{i=2}^{n+1} w_{ni}(x,h)\,\mathbb{I}\left\{Y_i\le F_{1xh}^{-1}(u\wedge u'), Z_{i-1}\le F_{2xh}^{-1}(v\wedge v')\right\}\right\}\right]\\
&\quad -\Omega\left\{\Upsilon\circ\Upsilon\left\{\sum_{j=2}^{n+1} w_{nj}(x,h)\,\mathbb{I}\left\{Y_j\le F_{1xh}^{-1}(u), Z_{j-1}\le F_{2xh}^{-1}(v)\right\}\right\}^2\right] + o_{\mathbb{P}}(1)\\
&= \Omega\left[\Upsilon\circ\Upsilon\left\{C_{xh}^{\mathrm{Z}\to\mathrm{Y}}(u\wedge u', v\wedge v') - C_{xh}^{\mathrm{Z}\to\mathrm{Y}}(u,v)C_{xh}^{\mathrm{Z}\to\mathrm{Y}}(u',v')\right\}\right] + o_{\mathbb{P}}(1).\end{aligned}$$

Since $C_{xh}^{\mathrm{Z}\to\mathrm{Y}}$ is consistent for $C_x^{\mathrm{Z}\to\mathrm{Y}}$ and the fact that $\widehat{K}_4 = K_4 + o_{\mathbb{P}}(1)$, one can conclude from the Continuous Mapping Theorem that $\widehat{\sigma}^2_{\Lambda,x} \to \sigma^2_{\Lambda,x} = K_4\,\Omega(\gamma)$.

Table B.1: Coverage probabilities, as estimated from 1 000 replicates, of 95% confidence intervals for the local causality measures $\rho_x^{Z\to Y}$ and $\tau_x^{Z\to Y}$ under bivariate D-vine time series in which $C_x^{Z\to Y}$ is the Normal copula

$\tau_x^{Z\to Y}$	Estimator	$(\tau_1,\tau_2,\tau_3,\tau_5) = (.3,.1,.1,.1)$ $n=250$	$n=500$	$n=1\,000$	$(\tau_1,\tau_2,\tau_3,\tau_5) = (.05,.05,.05,.05)$ $n=250$	$n=500$	$n=1\,000$
0	$\rho_{xh}^{Z\to Y}$	90.6	94.0	95.2	89.6	92.7	93.5
	$\widetilde{\rho}_{xh}^{Z\to Y}$	88.7	92.3	93.2	88.9	92.3	93.9
	$\tau_{xh}^{Z\to Y}$	94.0	94.0	95.0	93.1	94.6	94.6
	$\widetilde{\tau}_{xh}^{Z\to Y}$	93.8	94.2	94.6	93.3	94.4	94.3
0.1	$\rho_{xh}^{Z\to Y}$	90.4	94.0	95.2	89.9	93.8	94.7
	$\widetilde{\rho}_{xh}^{Z\to Y}$	90.1	93.5	94.5	89.6	93.4	94.7
	$\tau_{xh}^{Z\to Y}$	92.9	93.8	94.4	93.0	94.1	94.9
	$\widetilde{\tau}_{xh}^{Z\to Y}$	92.3	93.9	94.5	92.1	94.1	94.9
0.2	$\rho_{xh}^{Z\to Y}$	90.5	93.6	94.5	90.1	93.4	93.9
	$\widetilde{\rho}_{xh}^{Z\to Y}$	89.6	93.2	93.9	90.0	93.0	93.6
	$\tau_{xh}^{Z\to Y}$	93.5	94.3	95.0	93.2	95.1	95.2
	$\widetilde{\tau}_{xh}^{Z\to Y}$	93.2	94.1	95.4	93.9	95.1	95.0
0.3	$\rho_{xh}^{Z\to Y}$	90.8	93.4	94.4	91.1	92.5	93.7
	$\widetilde{\rho}_{xh}^{Z\to Y}$	90.5	92.7	94.2	90.7	92.2	93.9
	$\tau_{xh}^{Z\to Y}$	93.6	93.7	94.4	93.9	93.6	93.9
	$\widetilde{\tau}_{xh}^{Z\to Y}$	94.3	93.7	94.0	93.9	93.1	93.8
0.4	$\rho_{xh}^{Z\to Y}$	89.4	92.7	93.9	90.1	92.0	94.2
	$\widetilde{\rho}_{xh}^{Z\to Y}$	89.8	92.0	94.0	90.2	92.1	93.9
	$\tau_{xh}^{Z\to Y}$	94.2	94.7	94.8	93.5	94.4	94.8
	$\widetilde{\tau}_{xh}^{Z\to Y}$	94.3	94.4	95.4	93.9	94.6	95.0
		$(\tau_1,\tau_2,\tau_3,\tau_5) = (.5,.3,.3,.05)$ $n=250$	$n=500$	$n=1\,000$	$(\tau_1,\tau_2,\tau_3,\tau_5) = (5,.3,.75,.05)$ $n=250$	$n=500$	$n=1\,000$
0	$\rho_{xh}^{Z\to Y}$	94.0	94.3	93.7	77.3	82.1	84.3
	$\widetilde{\rho}_{xh}^{Z\to Y}$	89.8	92.4	93.2	89.6	92.4	93.6
	$\tau_{xh}^{Z\to Y}$	93.5	94.3	94.5	93.6	94.7	95.4
	$\widetilde{\tau}_{xh}^{Z\to Y}$	93.7	94.2	94.1	93.3	93.7	95.4
0.1	$\rho_{xh}^{Z\to Y}$	94.7	93.7	93.2	75.2	81.1	83.2
	$\widetilde{\rho}_{xh}^{Z\to Y}$	90.6	91.4	94.0	89.3	92.2	93.9
	$\tau_{xh}^{Z\to Y}$	95.0	93.9	95.7	92.3	94.3	93.5
	$\widetilde{\tau}_{xh}^{Z\to Y}$	94.5	93.5	95.8	92.6	93.9	94.1
0.2	$\rho_{xh}^{Z\to Y}$	93.7	94.3	94.4	71.1	75.5	77.0
	$\widetilde{\rho}_{xh}^{Z\to Y}$	89.8	92.6	92.8	90.8	92.6	93.7
	$\tau_{xh}^{Z\to Y}$	93.7	92.7	94.2	93.6	92.9	94.3
	$\widetilde{\tau}_{xh}^{Z\to Y}$	93.6	93.0	93.9	92.8	92.4	94.5
0.3	$\rho_{xh}^{Z\to Y}$	93.1	93.9	94.6	65.8	67.2	68.8
	$\widetilde{\rho}_{xh}^{Z\to Y}$	90.0	92.2	93.9	90.2	92.7	94.2
	$\tau_{xh}^{Z\to Y}$	91.6	93.4	94.3	93.7	92.7	93.5
	$\widetilde{\tau}_{xh}^{Z\to Y}$	91.5	94.3	94.8	93.3	92.1	92.7
0.4	$\rho_{xh}^{Z\to Y}$	91.6	94.1	95.1	50.3	54.8	55.2
	$\widetilde{\rho}_{xh}^{Z\to Y}$	88.6	91.9	93.5	90.3	91.8	93.3
	$\tau_{xh}^{Z\to Y}$	92.4	93.8	93.0	94.4	94.2	94.9
	$\widetilde{\tau}_{xh}^{Z\to Y}$	92.0	93.9	93.7	93.0	92.9	93.4

Table B.2: Percentages of rejection of the null hypothesis of local non-causality, as estimated from 1 000 replicates, for the tests at level $\alpha = 0.05$ based on $\rho_{xh}^{Z\to Y}$, $\widetilde{\rho}_{xh}^{Z\to Y}$, $\tau_{xh}^{Z\to Y}$ and $\widetilde{\tau}_{xh}^{Z\to Y}$ under bivariate D-vine time series in which $C_x^{Z\to Y}$ is the Normal copula

$\tau_x^{Z\to Y}$	Estimator	$(\tau_1,\tau_2,\tau_3,\tau_5) = (.3, .1, .1, .1)$ $n = 250$	$n = 500$	$n = 1\ 000$	$(\tau_1,\tau_2,\tau_3,\tau_5) = (.05, .05, .05, .05)$ $n = 250$	$n = 500$	$n = 1\ 000$
0	$\rho_{xh}^{Z\to Y}$	5.5	6.4	4.6	5.7	5.2	4.0
	$\widetilde{\rho}_{xh}^{Z\to Y}$	6.0	5.8	4.3	6.1	5.0	4.4
	$\tau_{xh}^{Z\to Y}$	4.3	4.9	4.4	5.0	4.7	5.0
	$\widetilde{\tau}_{xh}^{Z\to Y}$	5.3	5.2	4.9	4.8	4.9	5.2
0.1	$\rho_{xh}^{Z\to Y}$	17.9	36.8	69.1	16.1	35.2	65.9
	$\widetilde{\rho}_{xh}^{Z\to Y}$	17.0	35.9	65.8	16.8	34.5	65.5
	$\tau_{xh}^{Z\to Y}$	20.2	25.2	41.3	19.7	26.6	41.9
	$\widetilde{\tau}_{xh}^{Z\to Y}$	18.8	25.2	41.2	18.9	26.0	41.1
0.25	$\rho_{xh}^{Z\to Y}$	27.9	71.8	98.5	27.0	72.6	98.0
	$\widetilde{\rho}_{xh}^{Z\to Y}$	26.2	71.0	98.4	25.0	70.9	98.1
	$\tau_{xh}^{Z\to Y}$	40.3	63.0	86.5	39.3	64.6	87.2
	$\widetilde{\tau}_{xh}^{Z\to Y}$	38.9	62.4	85.2	37.8	61.5	86.1

		$(\tau_1,\tau_2,\tau_3,\tau_5) = (.5, .3, .3, .05)$ $n = 250$	$n = 500$	$n = 1\ 000$	$(\tau_1,\tau_2,\tau_3,\tau_5) = (.5, .3, .75, .05)$ $n = 250$	$n = 500$	$n = 1\ 000$
0	$\rho_{xh}^{Z\to Y}$	5.4	8.3	10.1	12.5	9.8	8.2
	$\widetilde{\rho}_{xh}^{Z\to Y}$	5.7	4.2	4.6	5.3	6.0	5.0
	$\tau_{xh}^{Z\to Y}$	5.0	6.6	5.6	5.4	6.5	5.4
	$\widetilde{\tau}_{xh}^{Z\to Y}$	5.2	7.0	6.7	5.7	5.5	4.8
0.1	$\rho_{xh}^{Z\to Y}$	26.6	49.8	81.9	4.5	14.2	38.6
	$\widetilde{\rho}_{xh}^{Z\to Y}$	14.3	33.6	66.7	15.9	35.4	67.7
	$\tau_{xh}^{Z\to Y}$	20.1	28.4	41.7	17.9	22.9	39.8
	$\widetilde{\tau}_{xh}^{Z\to Y}$	19.0	27.3	40.3	20.6	25.9	43.8
0.25	$\rho_{xh}^{Z\to Y}$	34.0	77.5	98.2	15.4	51.6	92.9
	$\widetilde{\rho}_{xh}^{Z\to Y}$	26.1	69.9	96.7	29.1	70.2	98.3
	$\tau_{xh}^{Z\to Y}$	41.3	61.8	85.0	38.3	60.6	85.0
	$\widetilde{\tau}_{xh}^{Z\to Y}$	38.2	61.4	83.5	39.2	63.2	86.7

Table B.3: Percentages of rejection of the null hypothesis of local non-causality, as estimated from 1 000 replicates, for the tests at level $\alpha = 0.05$ based on $\rho_{xh}^{Z\to Y}$, $\widetilde{\rho}_{xh}^{Z\to Y}$, $\tau_{xh}^{Z\to Y}$ and $\widetilde{\tau}_{xh}^{Z\to Y}$ under bivariate D-vine time series in which $C_x^{Z\to Y}$ is the Clayton copula

$\tau_x^{Z\to Y}$	Estimator	$(\tau_1,\tau_2,\tau_3,\tau_5) = (.3,.1,.1,.1)$ $n=250$	$n=500$	$n=1\,000$	$(\tau_1,\tau_2,\tau_3,\tau_5) = (.05,.05,.05,.05)$ $n=250$	$n=500$	$n=1\,000$
0	$\rho_{xh}^{Z\to Y}$	5.3	5.7	4.3	5.4	5.0	4.0
	$\widetilde{\rho}_{xh}^{Z\to Y}$	5.6	5.3	3.9	6.0	4.9	4.0
	$\tau_{xh}^{Z\to Y}$	4.6	5.0	4.6	4.4	5.3	4.6
	$\widetilde{\tau}_{xh}^{Z\to Y}$	5.3	5.9	4.9	4.7	5.1	4.9
0.1	$\rho_{xh}^{Z\to Y}$	18.5	37.9	72.1	16.3	35.1	66.5
	$\widetilde{\rho}_{xh}^{Z\to Y}$	16.5	34.6	68.3	16.1	34.2	65.7
	$\tau_{xh}^{Z\to Y}$	20.6	25.2	41.6	21.5	25.3	40.3
	$\widetilde{\tau}_{xh}^{Z\to Y}$	20.7	25.7	41.2	20.3	26.4	39.5
0.25	$\rho_{xh}^{Z\to Y}$	28.4	73.4	98.1	26.6	73.9	97.6
	$\widetilde{\rho}_{xh}^{Z\to Y}$	25.5	71.6	97.8	25.6	72.6	97.8
	$\tau_{xh}^{Z\to Y}$	41.1	62.9	84.8	40.4	62.8	87.0
	$\widetilde{\tau}_{xh}^{Z\to Y}$	40.0	61.7	84.4	40.4	62.6	86.7

		$(\tau_1,\tau_2,\tau_3,\tau_5) = (.5,.3,.3,.05)$ $n=250$	$n=500$	$n=1\,000$	$(\tau_1,\tau_2,\tau_3,\tau_5) = (.5,.3,.75,.05)$ $n=250$	$n=500$	$n=1\,000$
0	$\rho_{xh}^{Z\to Y}$	5.5	8.5	10.3	12.9	10.3	8.5
	$\widetilde{\rho}_{xh}^{Z\to Y}$	5.9	4.3	4.9	4.8	6.3	4.4
	$\tau_{xh}^{Z\to Y}$	5.0	6.3	5.0	5.4	6.7	5.2
	$\widetilde{\tau}_{xh}^{Z\to Y}$	4.7	5.9	4.8	5.5	6.6	5.5
0.1	$\rho_{xh}^{Z\to Y}$	27.5	53.0	83.6	5.8	14.5	38.0
	$\widetilde{\rho}_{xh}^{Z\to Y}$	14.7	35.4	66.4	16.3	34.4	67.0
	$\tau_{xh}^{Z\to Y}$	21.9	29.7	43.8	19.3	21.9	41.3
	$\widetilde{\tau}_{xh}^{Z\to Y}$	19.3	26.7	41.4	21.8	24.4	42.7
0.25	$\rho_{xh}^{Z\to Y}$	34.2	78.4	98.5	14.6	52.4	90.9
	$\widetilde{\rho}_{xh}^{Z\to Y}$	25.3	69.1	97.3	27.5	70.6	97.4
	$\tau_{xh}^{Z\to Y}$	40.6	64.0	86.6	36.0	58.6	84.3
	$\widetilde{\tau}_{xh}^{Z\to Y}$	39.1	62.4	84.7	38.7	59.3	85.7