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Cahier 02-2005

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Dépôt légal, Bibliothèque nationale du Canada, 2005, ISSN 0821-4441
Dépôt légal, Bibliothèque nationale du Québec, 2005, ISBN 2-89382-496-X

The double curse of a common property productive asset oligopoly

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December 29, 2004

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Acknowledgements: I would like to thank Gérard Gaudet and Ngo Van Long for helpful comments and the SSHRC and FQRSC for financial support.

Abstract:

We build a subgame perfect Nash equilibrium of a common property productive asset oligopoly. We derive two surprising results. First, the steady state level of asset can be a decreasing function of the asset's implicit growth rate. This phenomenon arises when the initial stock of asset is *below a certain threshold*. It represents a double curse for a common property productive asset where the well-known tragedy of the commons due to a lack of property rights is exacerbated by an increase in the productivity of the asset. Second, we show that a reduction in the number of firms exploiting the asset can, in the short run, result in an increase of the industry's exploitation and a decrease of the level of the asset's stock. *Journal of Economic Literature* Classification Numbers: D43, L13, Q34, C73.

Keywords: Productive Asset, Oligopoly, Tragedy of the commons, Dynamic Games.

1. Introduction

We consider a common property productive asset that is exploited by a fixed number of firms. The equilibrium path of the stock of asset depends on several parameters that characterize the nature of the asset as well as the market conditions. In this paper we study the sensitivity of the equilibrium path of the stock of asset to the implicit growth rate of the asset and to the total number of firms that share access to the asset. We use an oligopoly model, where firms exploit the asset and compete in the market of output by choosing quantities.

An important parameter that impacts the long-run stock of asset is the asset's implicit growth rate which can be considered as an indicator of its fertility rate. It is intuitive to expect that, at any given date, the larger the implicit growth rate of the asset the larger the stock of asset. We show that this intuition may be wrong: the steady state level of asset can be a decreasing function of the asset's growth rate. A productive asset with a given implicit growth rate can converge to a smaller steady state level than a productive asset with a smaller implicit growth rate. This possibility arises when the initial stock of asset is *below a certain threshold*. This phenomenon represents a double curse for a common property productive asset. When firms fail to cooperate, it is well known that the harvesting of a resource is more aggressive in the absence of property rights than it is when property rights are clearly defined: the asset lives *the tragedy of the commons*. In addition to the well-known tragedy of the commons due to the lack of property rights, an asset's stock with a given implicit growth would, in the long-run, converge to an even smaller level if its growth rate increased. The additional productivity gives an incentive to firms to increase their exploitation. If the stock is below a certain threshold, the impact of the overall production expansion outweighs the impact of an increase in fertility and the asset converges to a stock level that is smaller than the level it would have reached under a smaller implicit growth rate.

In the second part of the paper, we study the impact of a change in the total number of firms that share access to the resource. We show that reducing the number of firms can have, in the short-run, a negative impact on the asset's stock. Contrary to static oligopoly theory we show that in the case of a productive asset oligopoly, the industry's total production path when the number of firms is M can (initially) exceed the industry's production path when the number of firms is $N > M$. In the long run however, reducing the number firms that share access to the resource results in a larger asset's stock and larger industry's production rate. The industry's production path when the number of firms is N and the industry's production path when the number of firms is $M < N$ can intersect twice.

To conduct our study we use a differential game framework (see Dockner, Jorgensen, Long and Sorger [5]) and focus on closed-loop strategies: the strategy of a firm is a production rule that can depend on time *and* on the asset's stock. The differential game framework has been a useful tool to study the exploitation of a productive asset by a fixed number of agents. In a continuous time framework[1], Dockner and Long [6] showed that in a transboundary pollution game between two countries there exist a continuum of closed-loop Nash equilibria and that as the interest rate tends to zero the equilibrium long-run stock of pollution converges to the equilibrium stock of pollution under cooperation. Dockner and Sorger [7] studied the joint exploitation of a productive asset by two agents and constructed a continuum of closed-loop Nash equilibria all of which result in the long-run in over-exploitation of the asset. Moreover, they showed that as the interest rate tends to zero the steady states generated by the closed-loop Nash equilibria converge to the first-best long-run asset's stock. Benchekroun [2] studies the impact of unilateral production restrictions in a duopoly jointly exploiting a productive asset. It is shown that a unilateral production restriction can result in a decrease of the stock of asset in the long-run.

In this paper we use a continuous time framework where the time horizon is infinite, future utility is discounted and the strategies considered do not depend on the history of the game and do not allow for the use of credible threats. Unlike in Dockner and Long [6] and Dockner and Sorger [7], in this model, each agent's instantaneous utility depends on all agents exploitation rates. This is due to the fact that firms are oligopolists in the market of output. The model we use extends the duopoly model in Benchekroun [2] to an oligopoly. Benchekroun [2] determines the impact of *a marginal* unilateral production restriction. Starting at a closed-loop Nash equilibrium the production strategy of a single firm is marginally modified. It is shown that this unilateral reduction (exogenously imposed) can decrease the long-run level of the asset's stock, and can increase the long-run profits of

[1] Discrete time differential games are often referred to as difference games. For discrete time frameworks, see for example Levhari and Mirman [12], Benhabib and Radner [3], Dutta and Sundaram [8,9].

the firm that is subject to the production restriction[2]. In this paper we study the impact of a change in the number of firms that constitute an oligopoly. The elimination of $N-M$ firms in an N firms oligopoly can be viewed as an instance of a unilateral reduction of production of $N-M$ firms. However, the change in production of the $N-M$ firms that are eliminated is clearly not a *marginal* reduction of their production[3]. We show that a decrease in the number of firms that share access to the resource always results in a larger long-run stock of asset. However, the short-run impact of a decrease in the total number of firms turns out to be ambiguous. Moreover in this paper we determine the impact of a change in the implicit growth rate on the asset's stock and show the possibility of the *double curse of a common property productive asset* whereby the stock of an asset with a given implicit growth rate can converge to a smaller level than an asset with a smaller implicit growth rate.

We construct a Subgame Perfect Nash Equilibrium (SPNE) of the game. It is common in the literature to choose a specific functional form of the instantaneous objective function to obtain an explicit form of the equilibrium (see Benhabib and Radner [3], Dockner and Long [6], Dockner and Sorger [7] or Benchekroun [2]). Benhabib and Radner [3] use a linear utility, and Dockner and Sorger [7] use the square root function. In this paper, the instantaneous objective function of an agent is a quadratic function of all agents' exploitation rates. We then use this equilibrium as a benchmark from which we analyze the effect of a change in the implicit growth rate, as well as the impact of reduction in the number of firms. The benchmark equilibrium constructed exhibits a very regular and intuitive behavior. The equilibrium production of a firm is a non decreasing piecewise linear function of the stock. When the stock of asset is large enough the equilibrium production rule is constant and the production rate corresponds to the production rate of a static oligopoly: when the resource is abundant, the level of the resource does not affect the actions of the players. When the asset's stock is below a certain threshold all firms cease production. Our results are thus not due to the complexity or an irregular behavior of the benchmark equilibrium used.

We present the model in section 2. In section 3 we establish the double curse of the common property productive asset. In section 4 we determine the impact of reducing the

[2] See Gaudet and Salant [11] for a study of unilateral production reductions in a static oligopoly.

[3] In our model, changing the number of firms from N to $M < N$ can also be interpreted as a merger of $N-M+1$ firms.

number of firms that share access to the asset on the industry's production rule and the equilibrium path of the asset's stock.

2. The Model

Let S denote the stock of a productive asset. Access to the asset is shared by N firms. Let $u_i(t)$ denote the exploitation rate of the asset by firm i at time t. Firms exploit the asset to produce an output. Firms constitute an oligopoly in the output market and are assumed to have constant and identical average costs. For simplicity assume that production cost is zero and that the rate of transformation of the asset into the output is one unit of output per unit of asset exploited: $q_i(t) = u_i(t)$, where $q_i(t)$ is the quantity of output produced by firm i at t . The inverse demand function for output $P(q)$ is given by

$$P(q) = a - bq$$

where $q = \sum_{i=1}^{N} q_i$.

The dynamics of the asset's stock or its production function in the absence of any exploitation is given by:

$$\overset{\bullet}{S} = F(S)\,, \; S(0) = S_0 \tag{1}$$

where

$$F(S) = \begin{cases} \delta S & \text{for} \quad S \leq S_y \\ \delta S_y \left(\frac{S_{\max} - S}{S_{\max} - S_y} \right) & \text{for} \quad S > S_y \end{cases} \tag{2}$$

This production function can for example describe the dynamics of a fish population:

- δ is a positive parameter referred to as *the implicit growth rate*. When the size of the population is very small the population grows at an exponential rate : there are no habitat constraints to the asset's growth (see for example Clark [4]).

- the maximum sustainable yield of the asset is δS_y

- beyond S_y the asset grows at a decreasing rate

- $S_{\max}$ represents the habitat carrying capacity beyond which the asset's growth rate is negative: growth is limited by the availability of food and space.

For simplicity, we normalize $S_{\max}$ to 1. Let r denote the interest rate assumed positive and identical for all firms and let

$$\delta_0 \equiv Max\{\frac{r\left(1+N^2\right)}{2}, \frac{a\left(1+N^2\right)}{S_y b\left(1+N\right)^2}\}. \tag{3}$$

Assumption A1: $\delta > \delta_0$.

The assumption that the rate of growth is sufficiently large relative to the rate of discount r ensures that an interior positive stable steady state stock exists[4].

The objective of each firm i $(i = 1, .., N)$ is to maximize the discounted sum of instantaneous profits,

$$J^i = \int_0^\infty P\left(\sum_{k=1}^N q_k\left(t\right)\right) q_i\left(t\right) e^{-rt} dt \tag{4}$$

subject to

$$\overset{\bullet}{S} = F\left(S\right) - \sum_{k=1}^N q_k\left(t\right),\ S\left(0\right) = S_0. \tag{5}$$

We use a differential game framework to analyze the initial value problem (4). We consider the set of closed-loop strategies. A closed-loop strategy is a decision rule ϕ_i that gives an output rate for each moment and each level of the asset's stock. Firms can adjust their production to the stock of the asset:

$$q_i\left(t\right) = \phi_i\left(t, S\left(t\right)\right).$$

The restrictions imposed on the strategies are given in the following definition.

Definition: A N-tuple of strategies $(\phi_1, .., \phi_N)$ is said to be admissible if

(a) $q_i\left(t\right) = \phi_i\left(t, S\left(t\right)\right)$ is well defined for $i = 1, .., N$ and all $t \geq 0$.

(b) the function $t \mapsto q_i\left(t\right) = \phi_i\left(t, S\left(t\right)\right)$ is measurable for $i = 1, .., N$

(c) $\phi_i\left(t, 0\right) = 0$ for $i = 1, .., N$, and

(d) the asset's stock trajectory in (5) has a unique solution.

This definition is borrowed from Dockner and Sorger [7] and includes the minimal requirements for the asset's stock trajectory in (5) and the objective functionals in (4) to be well defined.

[4]The assumption $\delta > \frac{a\left(1+N^2\right)}{S_y b\left(1+N\right)^2}$ is justified in Remark 1 (see also Appendix A).

Let $(\phi_1^*, .., \phi_N^*)$ be an N-tuple of admissible closed-loop strategies and let $\phi_{-i}^* \equiv \left(\phi_1^*, ..\phi_{i-1}^*, \phi_{i+1}^* .., \phi_N^*\right)$. We say that $(\phi_1^*, .., \phi_N^*)$ is a Subgame Perfect Nash Equilibrium (SPNE) if for every possible initial condition (S_0, t_0) :

$$J^i\left(\phi_i^*, \phi_{-i}^*\right) \geq J^i\left(\phi_i, \phi_{-i}^*\right) \text{ with } i = 1, .., N \tag{6}$$

for any closed-loop strategy ϕ_i such that $\left(\phi_i, \phi_{-i}^*\right)$ is an admissible N-tuple of closed-loop strategies.

3. A SPNE and *the double curse of a common property productive asset*

We start by characterizing a SPNE of the oligopolistic competition amongst the firms that share access to the asset.

Proposition 1 :

Let ϕ^* denote the following production strategy

$$\phi^*(S) = \begin{cases} 0 & \text{for} \quad 0 \leq S \leq S_{1,N} \\ \frac{a-D-KS}{(1+N)b} & \text{for} \quad S_{1,N} < S \leq S_{2,N} \\ q^c & \text{for} \quad S_{2,N} < S \end{cases} \tag{7}$$

where $K = \frac{b(N+1)^2(r-2\delta)}{2N^2}$, $D = \frac{a\left(1+N^2\right)(2\delta-r)}{2N^2\delta}$, $q^c = \frac{a}{(1+N)b}$, $S_{1,N} = \frac{a-D}{K}$, $S_{2,N} = -\frac{D}{K}$.

The vector $(\phi^*, .., \phi^*)$ constitutes a symmetric SPNE.

Proof: see Appendix A.

Note when the asset is "abundant" $(S \geq S_{2,N})$ firms can simply adopt the production they would adopt under a static Cournot game where the inverse demand function is $P = a - bq$ and the production costs are zero.

Remark 1: The assumption $\delta > \frac{a\left(1+N^2\right)}{S_y b(1+N)^2}$ from assumption A1 ensures that $S_y > S_{2,N}$.

Although all firms have free access to the asset and form an oligopoly in the market of output, each firm decides *voluntarily* to abstain from exploitation if the asset's stock is *too small* $(S \leq S_{1,N})$. Depletion of the asset is then avoided. Benhabib and Radner [3] also exhibit equilibria where both agents *voluntarily* cease consumption if the asset's stock is below a given threshold[5]. We refer to the threshold level of stock $S_{1,N}$ as the

[5] This is unlike the equilibria constructed in Levhari and Mirman [15] and Dockner and Sorger [9] under which exploitation is positive for any positive level of the asset's stock. In their models the marginal utility of consumption tends to infinity when consumption tends to zero.

maturity threshold. Note that in this context firms have rents accruing from their market power in the market of output and also from having access to the productive asset. The value to a firm of an additional unit of stock is $\frac{\partial V}{\partial S}$ where V denotes the value function of a firm, that is the discounted sum of a firm's profits along the equilibrium path, when the stock of asset is S. The function V is given by

$$V(S) = \begin{cases} \left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) & \text{if} \quad 0 \leq S < S_{1,N} \\ W(S) & \text{if} \quad S_{1,N} \leq S < S_{2,N} \\ \frac{a^2}{(1+N)^2 br} & \text{if} \quad S_{2,N} \leq S \end{cases} \tag{8}$$

with W given by (see Proposition 1 and the proof in Appendix A)

$$W(S) = \frac{1}{2}KS^2 + DS + G.$$

We can determine the value to a firm of an additional unit of stock, which we refer to as the asset's rent

$$\frac{\partial V}{\partial S}(S) = \begin{cases} \frac{r}{\delta}\frac{1}{S_1}\left(\frac{S}{S_1}\right)^{\frac{r}{\delta}-1} W(S_1) & \text{if} \quad 0 \leq S < S_{1,N} \\ \frac{\partial W}{\partial S}(S) & \text{if} \quad S_{1,N} \leq S < S_{2,N} \\ 0 & \text{if} \quad S_{2,N} \leq S \end{cases} \tag{9}$$

The rent associated with an additional unit of stock is decreasing with the stock of the asset. The rent tends to infinity when the stock approaches zero. Firms prefer to leave the asset grow and refrain from any exploitation as long as the stock has not reached its maturity threshold ($S \leq S_{1,N}$). It is interesting to note that the maturity threshold $S_{1,N}$ is not necessarily a monotone function of the implicit growth rate of the asset δ. This is possible when S_y is sufficiently large. The precise result is formulated in this corollary:

Corollary 1:
Let $\delta_1 \equiv \frac{1}{2}r\left((1+N^2) + N\sqrt{1+N^2}\right)$. When $S_y > \frac{a\left(1+N^2\right)}{\delta_1 b(1+N)^2}$ we have
(i) $\delta_0 < \delta_1$ and
(ii)

$$\frac{d(S_{1,N})}{d\delta} > 0 \text{ for } \delta_0 < \delta < \delta_1 \text{ and } \frac{d(S_{1,N})}{d\delta} < 0 \text{ for all } \delta > \delta_1.$$

Proof: see Appendix B.

We note that for a given $S_y > 0$, the inequality $S_y > \frac{a\left(1+N^2\right)}{\delta_1 b(1+N)^2}$ is always satisfied when the number of firms is large enough. Then, there exists a range of δ such that an increase in the implicit growth rate results in an increase in the maturity threshold. If the implicit growth rate increases, firms are willing to let the stock grow to a higher level before they start harvesting. This behavior is possible only if S_y is sufficiently large.

Remark 2: When $S_y < \frac{a\left(1+N^2\right)}{\delta_1 b(1+N)^2}$ we have $\delta_0 = \frac{a\left(1+N^2\right)}{S_y b(1+N)^2} > \delta_1$ and thus $\frac{d\left(S_{1,N}\right)}{d\delta} < 0$ for all $\delta > \delta_0$. If S_y is not sufficiently large then an increase in the implicit growth rate unambiguously results in a smaller maturity threshold. If the implicit growth rate increases, firms start harvesting at smaller stock levels.

Let $\{S_N^*(t)\}$ denote the equilibrium time path of the asset when the number of firms is N and let $\Phi_N^*(S)$ denote the industry's production when the number of firms is N and the stock of asset is S: $\Phi_N^*(S) \equiv \sum_{i=1}^N \phi_i^*(S)$.

Corollary 2:

a) Φ_N^* is a non-decreasing function of the asset's stock.
b) Let $S_{1,N}^{\infty} \equiv \frac{a}{b\delta(N+1)} \frac{2\delta - r\left(1+N^2\right)}{2\delta - r(1+N)}$. For $\delta S_y < \frac{aN}{b(1+N)}$, we have (see Figure 1)

$$\lim_{t\to\infty} S_N^*(t) = S_{1,N}^{\infty} \text{ for all } S_0 > 0$$

c) For $\Phi_N^*(S_{2,N}) = \frac{aN}{b(1+N)} < \delta S_y$, there are three positive stationary asset's stocks denoted $S_{1,N}^{\infty}, S_{2,N}^{\infty}$ and $S_{3,N}^{\infty}$ with

$$S_{1,N}^{\infty} < S_{2,N}^{\infty} = \frac{aN}{\delta b(N+1)} < S_{3,N}^{\infty} = 1 - \frac{(1-S_y)\,aN}{\delta S_y b(N+1)}. \tag{10}$$

The asset's stock converges to $S_{1,N}^{\infty}$ when $S_0 \in (0, S_{2,N}^{\infty})$ and to $S_{3,N}^{\infty}$ when $S_0 > S_{2,N}^{\infty}$ (see Figure 2).

Proof: see Appendix C**.**

When the asset's stock and implicit growth rate are high enough (i.e. $\frac{aN}{b(1+N)S_y} < \delta$ and $S_0 > S_{2,N}$), exploiting the asset at a rate corresponding to the equilibrium of a pure static Cournot game $(q^c, .., q^c)$ is sustainable as a SPNE. Firms can play the equilibrium of a static Cournot game endlessly. If the implicit growth rate is not sufficiently high ($\delta < \frac{aN}{b(1+N)S_y}$), playing the static Cournot equilibrium is not sustainable. Firms can still adopt the production rate of the static Cournot game if the asset's stock is above $S_{2,N}$. However such a production rate lasts only for a finite period of time after which firms eventually reduce their production rate as the asset's stock falls below $S_{2,N}$ and converges to $S_{1,N}^{\infty} \in (S_{1,N}, S_{2,N})$.

The sensitivity analysis of $S_{1,N}^{\infty}$ to a change in the implicit growth rate of the asset reveals a surprising result. It is intuitive to expect that, ceteris paribus, in particular given an initial asset's stock, an asset that has a given implicit growth rate should reach a higher steady state equilibrium level than an asset with a smaller implicit growth rate. The following proposition shows that this very intuitive prediction can be wrong.

Proposition 2: The double curse of a common property productive asset *Let* $\bar{\delta} \equiv \frac{1}{2} r \left(1 + N^2 + \sqrt{N(N-1)(N^2+1)}\right)$,

if $S_y < \frac{a\left(1+N^2\right)}{\bar{\delta} b (1+N)^2}$ *(i.e.* $\delta_0 > \bar{\delta}$*) then*

$$\frac{dS_{1,N}^{\infty}}{d\delta} < 0 \text{ for all } \delta > \delta_0 > \bar{\delta}$$

if $S_y > \frac{a\left(1+N^2\right)}{\bar{\delta} b (1+N)^2}$ *then*

$$\frac{dS_{1,N}^{\infty}}{d\delta} > 0 \text{ for } \delta_0 < \delta < \bar{\delta} \text{ and } \frac{dS_{1,N}^{\infty}}{d\delta} < 0 \text{ for all } \delta > \bar{\delta}.$$

Proof: See appendix D.

This corollary has an interesting implication. Let δ'' and δ' be such that $\delta'' > \delta' > \bar{\delta}$ then, from Proposition 2, we have $S_{1,N}^{\infty}(\delta'') < S_{1,N}^{\infty}(\delta')$ where $S_{i,N}^{\infty}(\delta)$ denotes the steady state level i ($i \in \{1,2,3\}$) of asset, when the number of firms is N and the implicit growth rate is δ. For any initial asset's stock level S_0 such that $S_0 \in \left(0, S_{2,N}^{\infty}(\delta'')\right)$ the equilibrium path of the stock of asset converges to a level $S_{1,N}^{\infty}(\delta'')$ when the implicit growth rate is δ'' that is smaller than the level it converges to when the implicit growth is $\delta' < \delta''$! An increase in the implicit growth rate induces each firm to hasten its exploitation. When the asset's stock is below a certain threshold, the effect of the increase in productivity of the resource is not enough to compensate the increased harvesting and a decrease in the long-run stock of asset occurs (see Figure 3). It can be shown that the industry's production in the long-run is an increasing function of the implicit growth rate. However, in the short-run, the impact of a change in the implicit growth rate on the industry's production is ambiguous. We can first note that $\bar{\delta} < \delta_1$ therefore for $\delta \in \left(\bar{\delta}, \delta_1\right)$ we have $\frac{dS_{1,N}^{\infty}}{d\delta} < 0$ whereas $\frac{dS_{1,N}}{d\delta} > 0$: this case is presented in Figure 4. We clearly see that there exists a range of initial levels of the stock of asset such that, in the short-run, the production of the industry when the implicit growth rate is δ' is larger then the industry's production when the implicit growth rate is $\delta'' > \delta'$ with $\delta_1 > \delta'' > \delta' > \bar{\delta}$.

To better understand the intuition behind this result we study the impact of a change in the implicit growth rate on the resource rent $\left(\frac{\partial V}{\partial S}\right)$. For $S \in (S_{1,N}, S_{2,N})$ we have

$$\frac{\partial}{\partial \delta}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial K}{\partial \delta} S + \frac{\partial D}{\partial \delta}$$

where

$$\frac{\partial K}{\partial \delta} = -\frac{b\,(N+1)^2}{N^2} \text{ and } \frac{\partial D}{\partial \delta} = \frac{a\,(1+N^2)}{N^2 \delta}\frac{r}{2\delta}$$

Therefore

$$\frac{\partial}{\partial \delta}\left(\frac{\partial V}{\partial S}\right) = -\frac{b\,(N+1)^2}{N^2} S + \frac{a\,(1+N^2)}{N^2 \delta}\frac{r}{2\delta} > 0 \text{ iff } S < S_\delta \equiv \frac{r}{\delta^2}\frac{a}{2b}$$

When $S > S_\delta$ we have $\frac{\partial^2 V}{\partial \delta \partial S} < 0$: an increase in the implicit growth rate reduces the resource rent. A reduced rent induces firms to increase their exploitation rate of the resource and results in the long-run in a smaller stock of asset. We can show that for $\delta > \bar{\delta}$ we have $S_{1,N}^{\infty} > S_\delta$ and therefore, if the industry is initially at the steady state, an increase in the implicit growth rate translates into a decrease in the resource rent.

Note that when $S_{1,N} < S_\delta$ then, for $\delta > \bar{\delta}$ and for $S_0 \in (S_{1,N}, S_\delta)$, we have $\frac{\partial^2 V}{\partial \delta \partial S}(S_0) > 0$: an increase in the implicit growth rate increases the short-run resource rent and therefore induces a more conservative exploitation in the short-run[6]. This follows from the fact that for $S_{1,N} < S \leq S_{2,N}$ the equilibrium production strategy is given by $\phi^*(S) = \frac{a - \frac{\partial V}{\partial S}}{(1+N)b}$ and therefore $\frac{\partial^2 V}{\partial \delta \partial S}(S_0) > 0$ implies $\frac{\partial \phi^*(S_0)}{\partial \delta} < 0$: a larger implicit growth rate will result in a smaller production rate at a given $S_0 \in (S_{1,N}, S_\delta)$. The short-run impact of an increase in the implicit growth rate on the production of the industry is thus ambiguous and depends on the initial stock of asset.

[6]It can be checked that the set of parameters (δ, r, N) such that $S_{1,N} < S_\delta$ and $\delta > \bar{\delta}$ is not empty.

4. Reducing access to the resource

We determine below the impact of a change in the number of firms on the equilibrium path of the asset's stock and the industry's production rule. More precisely we want to check if reducing access to the shared asset results in a more *conservationist* exploitation of the asset. We compare the outcomes (asset's stock at the steady state and the industry's production) of the SPNE when the number of firms is reduced from N to $M < N$.

4.1. The impact on the stock of asset

We show below that regardless of the initial stock of asset, the long-run level of the asset when the number of firms is M is larger than the long-run asset's stock when the number of firms is $N > M$.

Lemma 1: For any $0 < M < N$ we have

$$S_{1,N}^{\infty} < S_{1,M}^{\infty}$$

Proof: See appendix E.

As shown in Corollary 2, the steady state level of asset can depend on the initial stock of asset. We distinguish three cases:

- If $\delta S_y < \frac{aM}{b(1+M)} < \frac{aN}{b(1+N)}$ then there is one positive stationary asset's stock given by $S_{1,M}^{\infty}$ when the number of firms is M and one positive stationary asset's stock $S_{1,N}^{\infty}$ when the number of firms is N. Since $S_{1,N}^{\infty}$ is strictly decreasing with respect to N we have

$$S_{1,N}^{\infty} < S_{1,M}^{\infty} \text{ for } M < N$$

- If $\frac{aM}{b(1+M)} < \delta S_y < \frac{aN}{b(1+N)}$ then there are three positive steady states when the number of firms is M given by $S_{1,M}^{\infty}, S_{2,M}^{\infty}$ and $S_{3,M}^{\infty}$ in (10) and one positive stationary asset's stock $S_{1,N}^{\infty}$ when the number of firms is N. Using Lemma 1 we have

$$S_{1,N}^{\infty} < S_{1,M}^{\infty} < S_{2,M}^{\infty} < S_{3,M}^{\infty} \text{ for all } M < N$$

Regardless of the initial asset's stock, the asset's stock converges to a larger steady state level when the number of firms is reduced.

- If $\frac{aM}{b(1+M)} < \frac{aN}{b(1+N)} < \delta S_y$ then there are three positive steady states when the number of firms is M given by $S^{\infty}_{1,M}$, $S^{\infty}_{2,M}$ and $S^{\infty}_{3,M}$ and there are three possible steady states when the number of firms is N given by $S^{\infty}_{1,N}$, $S^{\infty}_{2,N}$ and $S^{\infty}_{3,N}$. From Lemma 1 we have $S^{\infty}_{1,N} < S^{\infty}_{1,M}$ and it is straightforward to show that $S^{\infty}_{2,M} < S^{\infty}_{2,N}$ and $S^{\infty}_{3,M} > S^{\infty}_{3,N}$.

For $S_0 \in \left(0, S^{\infty}_{2,M}\right)$ we have

$$\lim_{t\to\infty} S^*_N(t) = S^{\infty}_{1,N} < \lim_{t\to\infty} S^*_M(t) = S^{\infty}_{1,M}$$

For $S_0 \in \left(S^{\infty}_{2,M}, S^{\infty}_{2,N}\right)$ we have

$$\lim_{t\to\infty} S^*_N(t) = S^{\infty}_{1,N} < \lim_{t\to\infty} S^*_M(t) = S^{\infty}_{3,M}$$

since $S^{\infty}_{3,M} > S^{\infty}_{1,M} > S^{\infty}_{1,N}$.

For $S_0 \in \left(S^{\infty}_{2,N}, \infty\right)$ we have

$$\lim_{t\to\infty} S^*_N(t) = S^{\infty}_{3,N} < \lim_{t\to\infty} S^*_M(t) = S^{\infty}_{3,M}$$

When $S_0 = S^{\infty}_{2,M}$, we have $S^*_M(t) = S^{\infty}_{2,M}$ for all $t > 0$ with $S^{\infty}_{2,M} > S^{\infty}_{1,M} > \lim_{t\to\infty} S^*_N(t) = S^{\infty}_{1,N}$. When $S_0 = S^{\infty}_{2,N}$, we have $S^*_N(t) = S^{\infty}_{2,N}$ for all $t > 0$ with $S^{\infty}_{2,N} < S^{\infty}_{3,N} < \lim_{t\to\infty} S^*_M(t) = S^{\infty}_{3,M}$.

We can therefore state that in the three possible cases, regardless of the initial asset's stock, the asset's stock when the number of firms is M converges to a larger steady state level than the level it converges to when the number of firms is $N > M$.

4.2. The impact on the production strategies

In static oligopoly theory, when firms sell a homogenous product, a decrease in the total number of firms results in a decrease of the industry's total production and an increase in the price of the product provided the inverse demand is downward sloping and the cost function is not *too* concave[7] (see for example Amir and Lambson [1]). In the case of a linear demand function and constant marginal cost a decrease in the total number of firms unambiguously results in a decrease of industry's output. In our context

[7] More precisely Amir and Lambson [1] shows, in a Cournot competition, that if $-P'(z) + C''(x) > 0$ for all z and x, where P denotes the market's inverse demand, z and x aggregate and single firm outputs and C the cost function, then total industry's output is an increasing function of the total number of firms.

where an oligopoly exploits a productive asset, when the initial stock is beyond a certain threshold, the game simply replicates the outcome of a repeated Cournot game. Then, the results from static oligopoly theory apply. In particular, a decrease in the total number of firms induces a decrease of the industry's production. The case of interest to us is when the asset's stock plays a role in the choice of actions of the players, i.e. $S_0 < S_{2,N}^{\infty}$. When $S_0 < S_{2,N}^{\infty}$, the outcome of a reduction in the number of firms on the *long-run* production of the industry can be opposite to the outcome of a reduction in the number of firms in a static context. However this difference does not come as a surprise. If the short-run industry's production decreases, the asset's stock can grow to a higher level in the long-run and could result in an increase in the long-run production of the industry. A more surprising result that is established below is the possibility that a reduction in the number of firms that have access to the resource can also lead to an increase in the *short-run* industry's production.

Lemma 2:

Assume the number of firms that share access is decreased from N to $M < N$, we have

$$S_{1,N} < S_{1,M} \text{ and } S_{2,M} < S_{2,N}$$

Proof: See Appendix F where we show that $\frac{dS_{1,N}}{dN} < 0$ and $\frac{dS_{2,N}}{dN} > 0$.

Proposition 3:

For any $N > 3$ and $1 < M < N$ there exists $\hat{S}_1, \hat{S}_2 > 0$ with $S_{1,M} < \hat{S}_1 < S_{2,M} < \hat{S}_2 < S_{2,N}$ and such that

$$\Phi_M^*(S) > \Phi_N^*(S) \text{ for all } S \in \left(\hat{S}_1, \hat{S}_2\right)$$

Proof: See Appendix G.

The decrease in the number of firms has an ambiguous effect on the production of the industry. For $N > 3$ there always exists a range of S for which the overall production increases following a reduction in the number of firms (see Figure 5).

More precisely, for any $S_0 \in \left(\hat{S}_1, \hat{S}_2\right)$ the industry's production path when the number of firms is N is initially below the industry's production path when the number of firms is $M < N$. This implies that the equilibrium path of the asset's stock when the number of firms is N is above the equilibrium path of the asset's stock when the number of firms is reduced to M. Moreover in section 4.1 we established that the steady state level of the asset's stock when the number of firms is M is larger than steady state level of the asset's stock when the number of firms is N. Therefore the equilibrium path of the asset's stock when the number of firms is N and the equilibrium path of the asset's stock when the number of firms is reduced to M must intersect. Following the same argument we can also state that the industry's production path when the number of firms is N and the industry's production path when the number of firms is M can intersect twice. We sum up our findings in the following proposition

Proposition 4:

For any $N > 3$ *and* $1 < M < N$ *there exists* $\hat{S}_1, \hat{S}_2 > 0$ *with* $S_{1,M} < \hat{S}_1 < S_{2,M} < \hat{S}_2 < S_{2,N}$ *and such that for any* $S_0 \in \left(\hat{S}_1, \hat{S}_2\right)$ *there exist* $\tilde{t}_1$, $\tilde{t}_2 > 0$ *and* $\tilde{t}_3 > \tilde{t}_2$ *such that*

(*i*) $S_N^*(t) > S_M^*(t)$ *for* $0 \leq t < \tilde{t}_1$ *and* $S_N^*(t) < S_M^*(t)$ *for* $t > \tilde{t}_1$

(*ii*) $\Phi_M^*(S_M^*(t)) > \Phi_N^*(S_N^*(t))$ *for* $t \in [0, \tilde{t}_2) \cup (\tilde{t}_3, \infty)$ *and*

(*iii*) $\Phi_M^*(S_M^*(t)) < \Phi_N^*(S_N^*(t))$ *for* $t \in \left(\tilde{t}_2, \tilde{t}_3\right)$

Remark 3: We note in the proof of Proposition 3 that we have $\Phi_M^*(S) \leq \Phi_N^*(S)$ for all $S \geq 0$ only when $M = 1$.

5. Concluding remarks

We have presented a model of a common property productive asset oligopoly. We built a SPNE and used it to study the sensitivity of the equilibrium path of the stock of asset to the implicit growth rate of the asset and to the total number of firms that share access to the asset.

We showed that the steady state level of asset can be a decreasing function of the asset's growth rate. When the initial stock of asset is *below a certain threshold* an increase in the implicit growth rate of the asset may result in a lower level of the asset's stock in the long-run. This has interesting policy implications. Suppose for example a regulation agency seeks to increase the long-run stock of asset by promoting research to increase the productivity of the asset. This intervention could have the exact opposite effect to the one intended.

We also determined the impact of a change in the total number of firms that share access to the resource. We show that reducing the number of firms can have in the short-run a negative impact on the asset's stock. This also has an interesting implication. Suppose that a regulation agency seeks to achieve a more conservationist exploitation policy than the equilibrium exploitation achieved under an oligopoly of N firms. A tempting intervention, when possible, is to further restrict access to the resource. We show that in the short run such policy can result in a more intensive exploitation of the asset.

In our model, a decrease in the number of firms by for example one firm can also be interpreted as a merger of two firms or the formation of a coalition of two firms. The results of this paper suggest that coalition formations of firms that share access to the resource deserve a specific attention and the results of the theory of coalition formations and mergers in static frameworks (e.g. Salant, Switzer and Reynolds [13] or Gaudet and Salant [11]) cannot be directly extended to this context. This is a direction of future research.

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Appendix A: Proof of Proposition 1

The vector $(\phi^*, .., \phi^*)$ constitutes a symmetric SPNE if there exist N value functions $V_1, .., V_N$ continuously differentiable such that the function ϕ^* is solution to the problem:

$$rV_i(S) = \underset{\phi_i.}{Max}\{(a - b(\phi_i + (N-1)\phi^*))\phi_i + V_i'(S)(F(S) - (\phi_i + (N-1)\phi^*))\} \text{ with } i = 1, .., N \quad (11)$$

We use "the undetermined coefficients technique" (see Fershtman and Kamien [10] or Dockner et al. [5]) to determine the value functions $V_1, .., V_N$. The originality of the equilibrium we exhibit is that it combines the two solutions that we obtain from the standard application of "the undetermined coefficients technique". The transition from one solution to another is determined by requiring that the value function is continuously differential at the level of stock where the transition occurs.

Consider the following value function $V(S)$:

$$V(S) = \begin{cases} \left(\frac{S}{S_{1,N}}\right)^{\frac{r}{\delta}} W(S_{1,N}) & \text{if} \quad 0 \leq S < S_{1,N} \\ W(S) & \text{if} \quad S_{1,N} \leq S < S_{2,N} \\ \frac{a^2}{(1+N)^2 br} & \text{if} \quad S_{2,N} \leq S \end{cases} \tag{12}$$

where

$$W(S) = \frac{K}{2}S^2 + DS + G$$

with

$$K = \left(\frac{1}{2}r - \delta\right)\frac{b(N+1)^2}{N^2}, \; D = -\frac{a(1+N^2)}{\delta b(1+N)^2}K,$$

$$G = \frac{a^2}{br(N+1)^2}\left(1 - \frac{(1+N^2)^2}{2N^2}\left(2 - \frac{r}{\delta}\right)\frac{r}{2\delta}\right) \text{ and } S_{1,N} = \frac{a-D}{K}, \; S_{2,N} = -\frac{D}{K}$$

After substitution of K we have

$$D = \frac{a(1+N^2)}{2N^2\delta}(2\delta - r).$$

The rest of the proof consists of showing that (i) the value function above, $V(S)$, is continuously differentiable and that (ii) the function ϕ^* given by (7) is solution of the problem (11) where $V_1(S) = .. = V_N(S) = V(S)$.

(i) Proof that $V(S)$ is continuously differentiable:
The function $V(S)$ is clearly continuously differentiable over $[0, S_{1,N}), (S_{1,N}, S_{2,N})$, and $(S_{2,N}, \infty)$ respectively with:

$$V'(S) = \begin{cases} \frac{r}{\delta S_{1,N}}\left(\frac{S}{S_{1,N}}\right)^{\frac{r}{\delta}-1} W(S_{1,N}) & \text{if} \quad 0 \leq S < S_{1,N} \\ W'(S) & \text{if} \quad S_{1,N} \leq S < S_{2,N} \\ 0 & \text{if} \quad S_{2,N} \leq S \end{cases} \tag{13}$$

We need to check that the function $V(S)$ is continuously differentiable at $S_{1,N}$ and at $S_{2,N}$.

- We first check that $V(S)$ is continuous at $S_{1,N}$ and at $S_{2,N}$. We have

$$\underset{S \to S_{1,N},\, S < S_{1,N}}{lim} V(S) = W(S_1) = \underset{S \to S_{1,N},\, S > S_{1,N}}{lim} V(S)$$

and

$$\lim_{S \to S_2,\, S<S_2} V(S) = W(S_2) = \frac{a^2}{(1+N)^2 br} = \lim_{S \to S_2,\, S>S_2} V(S).$$

Therefore the function V is continuous at $S_{1,N}$ and $S_{2,N}$.

- We now check that V' is continuous at $S_1$and S_2. We have

$$\lim_{S \to S_{1,N},\, S<S_{1,N}} V'(S) = \frac{rW(S_{1,N})}{\delta S_{1,N}}$$

It can be checked that

$$\frac{rW(S_{1,N})}{\delta S_{1,N}} = W'(S_{1,N}) = a$$

and thus

$$\lim_{S \to S_{1,N},\, S<S_{1,N}} V'(S) = \lim_{S \to S_{1,N},\, S>S_{1,N}} V'(S) \text{ , i.e. } V' \text{ is continuous at } S_{1,N}$$

Moreover

$$\lim_{S \to S_{2,N},\, S<S_{2,N}} V'(S) = KS_2 + D = 0 \text{ i.e. } V' \text{ is continuous at } S_{2,N}$$

and thus V' is continuous at $S_{1,N}$ and $S_{2,N}$.

Therefore the function $V(S)$ is continuously differentiable over $[0, \infty)$.

(ii) We now show that the function ϕ^* given by (7) is solution of the problem (11) where $V_1(S) = V_2(S) = .. = V(S)$.

Let $V_i(S) = V(S)$ for $i = 1, ..N$.

- For $S \geq S_{1,N}$ problem (11) admits an interior solution. The function ϕ_i^* is then given by the following first-order condition of the problem (11):

$$(a - b(N-1)\phi^*) - 2b\phi_i - V_i'(S) = 0 \text{ with } i = 1, .., N. \tag{14}$$

For a symmetric equilibrium we have

$$\phi_i^*(S) = \frac{a - V'(S)}{(N+1)b} \text{ for all } i = 1, .., N \tag{15}$$

Substituting into (11) gives

$$rV = \left(a - bN\frac{a - V'}{(N+1)b}\right)\frac{a - V'}{(N+1)b} + V'\left(F(S) - N\frac{a - V'}{(N+1)b}\right) \tag{16}$$

or after simplification

$$rV = \frac{(a - N^2 V')(a - V')}{(N+1)^2 b} + V' F(S) \tag{17}$$

It can be checked that the value function V satisfies the differential equation above for all $S \geq S_{1,N}$.

Substituting V' from (13) into (15) yields exactly ϕ^*. The level of stock $S_{2,N}$ is determined such that V is continuously differentiable in the neighborhood of $S_{2,N}$.

- For $S < S_{1,N}$ problem (11) has the corner solution: $\phi_i^*(S) = 0$. It can be checked that the function $V(S)$ given by (12) satisfies the differential equation obtained after substitution of (15) into (11) ■

Appendix B: Proof of Corollary 1

(i) Let $\delta_1 \equiv \frac{1}{2} r \left((1+N^2) + N\sqrt{1+N^2}\right)$. From assumption A1 we have two possibilities: either $\delta_0 = \frac{1}{2} r (1+N^2)$ or $\delta_0 = \frac{a\left(1+N^2\right)}{S_y b (1+N)^2}$. When $\delta_0 = \frac{1}{2} r (1+N^2)$ we clearly have $\delta_0 < \delta_1$.When $\delta_0 = \frac{a\left(1+N^2\right)}{S_y b(1+N)^2}$ then $S_y > \frac{a\left(1+N^2\right)}{\delta_1 b(1+N)^2}$ is equivalent to $\delta_1 > \delta_0$.

(ii) We have $S_{1,N} = \frac{a-D}{K}$ which gives after substitution and simplification

$$S_{1,N} = \frac{(2\delta - r(1+N^2))\, a}{(2\delta - r)(N+1)^2 b\delta}$$

Taking the derivative with respect to δ gives

$$\frac{dS_{1,N}}{d\delta} = \frac{(4\delta - r)\, r(1+N^2) - 4\delta^2}{(2\delta - r)^2 (N+1)^2 b\delta^2} a$$

Let $n(\delta) \equiv (4\delta - r)\, r(1+N^2) - 4\delta^2$ we have $n(\delta) = -4(\delta - \delta_1)(\delta - \delta_2)$ where $\delta_1 \equiv \frac{1}{2} r \left((1+N^2) + N\sqrt{1+N^2}\right)$ and $\delta_2 \equiv \frac{1}{2} r \left((1+N^2) - N\sqrt{1+N^2}\right)$.

From assumption A1 we have $\delta > \delta_0$ implying $\delta > \delta_2$ and from (i) we have $\delta_0 < \delta_1$ when $S_y > \frac{a\left(1+N^2\right)}{\delta_1 b(1+N)^2}$. Therefore, when $S_y > \frac{a\left(1+N^2\right)}{\delta_1 b(1+N)^2}$ we have $n(\delta) > 0$ for $\delta \in (\delta_0, \delta_1)$ and $n(\delta) < 0$ for $\delta > \delta_1$. When $S_y < \frac{a\left(1+N^2\right)}{\delta_1 b(1+N)^2}$ we have $n(\delta) < 0$ for all $\delta > \delta_0$ ■

Appendix C: Proof of Corollary 2

a) For a symmetric equilibrium this amounts to show that ϕ^* is a non-decreasing function of the stock of asset. We have for $S \in (S_{1,N}, S_{2,N})$

$$\phi^*(S) = \frac{a - V'(S)}{(N+1)\,b}$$

and thus

$$\phi^{*\prime}(S) = \frac{-V''(S)}{(N+1)\,b} = -\frac{K}{2b\,(1+N)}$$

with

$$K = \left(\frac{1}{2}r - \delta\right)\frac{b\,(N+1)^2}{N^2}$$

From assumption A1, it is straightforward that $K < 0$ and therefore $\phi^*(S)$ and $\Phi^*(S)$ are non-decreasing functions of S, strictly increasing over $(S_{1,N}, S_{2,N})$.

b) Stationary asset's stock levels are characterized by

$$\dot{S} = F(S) - N\phi^*(S) = 0 \tag{18}$$

For $\delta S_y < \phi^*(S_{2,N}) < \phi^*(S_y)$, or $S_{2,N} < S_y < \frac{\phi^*(S_{2,N})}{\delta}$ then (18) has one and only one positive root given by

$$S_{1,N}^{\infty} = \frac{N\,(a-D)}{NK + (1+N)\,\delta b}$$

after substitution of D and K we obtain

$$S_{1,N}^{\infty} = \frac{a}{b\delta\,(N+1)}\frac{2\delta - r\,(1+N^2)}{2\delta - r\,(1+N)}$$

Moreover, we have

$$\dot{S} = F(S) - N\phi^*(S) < 0 \text{ for all } S > S_{1,N}^{\infty}$$

and

$$\dot{S} = F(S) - N\phi^*(S) > 0 \text{ for all } S < S_{1,N}^{\infty}$$

therefore for any initial asset's stock S_0 the asset's stock equilibrium path converges to $S_{1,N}^{\infty}$: $S_{1,N}^{\infty}$ is globally asymptotically stable.

c) For $\phi^*(S_{2,N}) < \delta S_y$, (18) has three positive roots given by

$$S_{2,N}^{\infty} = \frac{aN}{\delta b\,(N+1)} \tag{19}$$

and

$$S_{3,N}^{\infty} = S_{\max} - \frac{(S_{\max} - S_y)\,aN}{\delta S_y b\,(N+1)}$$

$$S_{1,N}^{\infty} < S_{2,N}^{\infty} = \frac{aN}{\delta b\left(N+1\right)} < S_{3,N}^{\infty} = S_{\max} - \frac{\left(S_{\max} - S_y\right) aN}{\delta S_y b\left(N+1\right)}$$

Moreover, we have

$$\dot{S} = F\left(S\right) - N\phi^*\left(S\right) > 0 \text{ for all } S < S_{1,N}^{\infty}$$

$$\dot{S} = F\left(S\right) - N\phi^*\left(S\right) < 0 \text{ for all } S_{1,N}^{\infty} < S < S_{2,N}^{\infty}$$

Therefore, for any initial asset's stock S_0 within $\left(0, S_{2,N}^{\infty}\right)$ the asset's stock equilibrium path converges monotonically to $S_{1,N}^{\infty}$.

$$\dot{S} = F\left(S\right) - N\phi^*\left(S\right) > 0 \text{ for all } S_{2,N}^{\infty} < S < S_{3,N}^{\infty}$$

and

$$\dot{S} = F\left(S\right) - N\phi^*\left(S\right) < 0 \text{ for all } S > S_{3,N}^{\infty}.$$

For any initial asset's stock S_0 within $\left(S_{2,N}^{\infty}, \infty\right)$ the asset's stock equilibrium path converges monotonically to $S_{3,N}^{\infty}$. The stationary asset's stock $S_{2,N}^{\infty}$ is unstable ■

Appendix D: proof of Proposition 2

From Corollary 2 where we determined $S_{1,N}^{\infty}$ we obtain

$$\frac{d\left(S_{1,N}^{\infty}\right)}{d\delta} = \frac{d\left(\frac{a}{b\delta(N+1)}\frac{2\delta - r\left(1+N^2\right)}{2\delta - r(1+N)}\right)}{d\delta} = -\frac{m\left(\delta\right)a}{\left(r - 2\delta + Nr\right)^2\left(N+1\right)b\delta^2}$$

where $m\left(\delta\right) \equiv \left(4\delta^2 - 4r\delta\left(1+N^2\right) + \left(1+N+N^2+N^3\right)r^2\right)$.
We have $m\left(\delta\right) = 4\left(\delta - \bar{\delta}\right)\left(\delta - \check{\delta}\right)$ where $\bar{\delta} \equiv \frac{1}{2}r\left(1+N^2+\sqrt{N\left(N-1\right)\left(N^2+1\right)}\right) > 0$ and $\check{\delta} \equiv \frac{1}{2}r\left(1+N^2-\sqrt{N\left(N-1\right)\left(N^2+1\right)}\right) < 0$. From assumption A1 we have $\delta > \check{\delta}$ for all $\delta > \delta_0$.

If $\frac{a\left(1+N^2\right)}{\bar{\delta}b(1+N)^2} > S_y$ then, given assumption A1, $\delta_0 > \bar{\delta}$ and therefore $\frac{dS_{1,N}^{\infty}}{d\delta} < 0$ for all $\delta > \delta_0$. However if $\frac{a\left(1+N^2\right)}{\bar{\delta}b(1+N)^2} < S_y$ then given assumption A1, $\delta_0 < \bar{\delta}$ and we have : $\frac{dS_{1,N}^{\infty}}{d\delta} > 0$ for $\delta_0 < \delta < \bar{\delta}$ and $\frac{dS_{1,N}^{\infty}}{d\delta} < 0$ for $\delta > \bar{\delta}$ ■

Appendix E: Proof of Lemma 1

The steady level of asset $S_{1,N}^{\infty}$ the number of firms is N (given in Corollary 2) can be written in the following form:

$$S_{1,N}^{\infty} = \frac{a}{\delta b}f\left(N\right)$$

where

$$f(x) = \frac{\varrho - x^2}{(1+x)(\varrho - x)} \text{ and where } \varrho \equiv \frac{2\delta}{r} - 1$$

Using assumption A1, we have $\varrho - 1 > 0$ for all $N > 1$ and thus

$$f'(x) = -\frac{(\varrho + x^2)(\varrho - 1)}{(1+x)^2 (\varrho - x)^2} < 0$$

Since f is a monotone strictly decreasing function of x and so is $S_{1,N}^{\infty}$ with respect to N ■

Appendix F: Proof of Lemma 2

We have by definition of $S_{1,N} = \frac{1}{K}(a - D)$ after substitution of the constants D and K and simplification:

$$S_{1,N} = \frac{(2\delta - r(1 + N^2))\, a}{(2\delta - r)(N+1)^2 b\delta}$$

and thus

$$\frac{dS_{1,N}}{dN} = -2\frac{(2\delta - r(1+N))\, a}{(2\delta - r)(N+1)^3}$$

From assumption A1 we have $2\frac{\delta}{r} - (1 + N^2) > 0$ and therefore $\frac{dS_{1,N}}{dN} < 0$.

Similarly, from the definition of $S_{2,N} = -\frac{D}{K}$ we have after substitution of the constants D and K and simplification

$$S_{2,N} = \frac{a(1+N^2)}{\delta b (N+1)^2}$$

and thus

$$\frac{dS_{2,N}}{dN} = 2\frac{(N-1)\, a}{(N+1)^3 \delta b} > 0 \; ■$$

Appendix G:

From Lemma 2 we have $\Phi_M^*(S) = 0 < \Phi_N^*(S)$ for all $S_{1,N} < S \leq S_{1,M}$. Moreover, for $S > S_{2,N}$ we have

$$\Phi_M^*(S) = \frac{aM}{b(1+M)} < \Phi_N^*(S) = \frac{aN}{b(N+1)}$$

To prove our claim it is sufficient to exhibit a level of $S_{1,M} \leq S \leq S_{2,N}$ such that $\Phi_M^*(S) > \Phi_N^*(S)$. We show below that at $S = S_{2,M} < S_{2,N}$ we have

$$\Phi_M^*(S_{2,M}) = \frac{aM}{b(1+M)} > \Phi_N^*(S_{2,M})$$

$$\begin{aligned}\Phi_M^*(S_{2,M}) - \Phi_N^*(S_{2,M}) &= \frac{aM}{b(1+M)} - \\ &N\left(\frac{a - \frac{a\left(1+N^2\right)}{2N^2\delta}(2\delta - r)}{(1+N)\,b} + \left(\delta - \frac{1}{2}r\right)\frac{(N+1)}{N^2}\frac{a(1+M^2)}{\delta b(1+M)^2}\right)\end{aligned}$$

After simplification we have

$$\Phi_M^*(S_{2,M}) - \Phi_N^*(S_{2,M}) = a(N-M)\frac{(MN-N-2)\,\delta + r(1-MN)}{N\delta(N+1)\,b(1+M)^2}$$

The sign of $\Phi_M^*(S_{2,M}) - \Phi_N^*(S_{2,M})$ is the same as the sign of $\Delta \equiv (MN-N-2)\,\delta + r(1-MN)$. We now note that for all $M < N$, Δ is an increasing function of δ. From assumption A1, $\delta > \frac{r}{2}(1+N^2)$ and thus

$$\Delta > (MN-N-2)\frac{r}{2}\left(1+N^2\right) + r(1-MN)$$

which after simplification yields

$$\Delta > \frac{1}{2}(N+1)\,rN(NM-N-M-1)$$

We argue now that $\Lambda \equiv (NM-N-M-1) > 0$ for $N > 3$ and all $1 < M < N$. Indeed $\Lambda = ((N-1)(M-1)-2) > 0$ for all $N > 3$ and $M > 1$. Therefore for $N > 3$ and $1 < M < N$ we have $\Delta > 0$ and thus

$$\Phi_M^*(S_{2,M}) - \Phi_N^*(S_{2,M}) > 0.$$

Therefore, $\Phi_M^*(S)$ and $\Phi_N^*(S)$ must intersect once in $(S_{1,M}, S_{2,M})$: denote the intersection point $\hat{S}_1$. Similarly $\Phi_M^*(S)$ and $\Phi_N^*(S)$ must intersect once in $(S_{2,M}, S_{2,N})$: denote the intersection point $\hat{S}_2$. Note that for $M = 1$ we have $\Delta = (-2)\,\delta + r(1-N) < 0$ and thus $\Phi_1^*(S) \leq \Phi_N^*(S)$ for all $S \geq 0$ ■

Figure 1

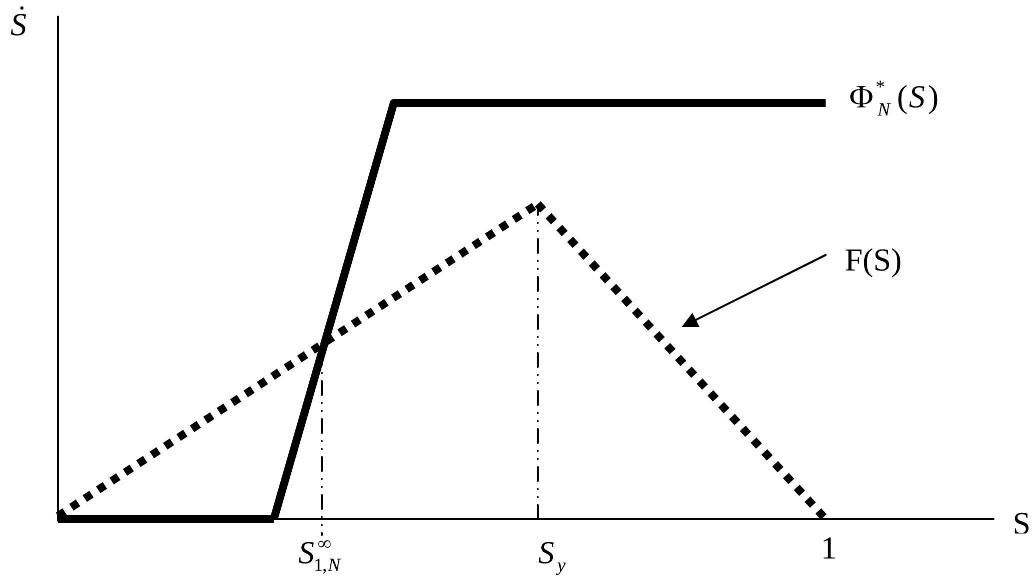

Figure 2

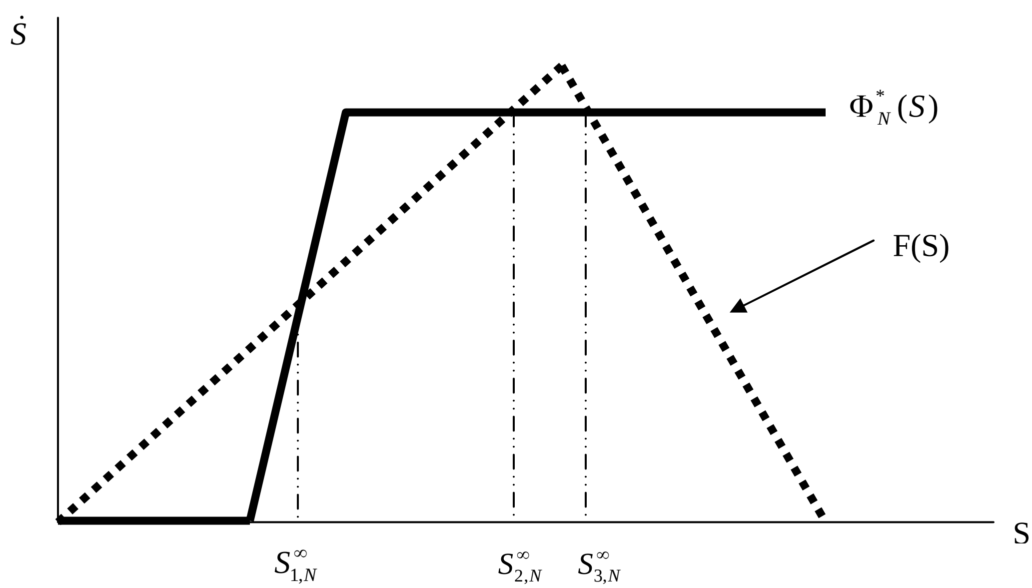

Figure 3

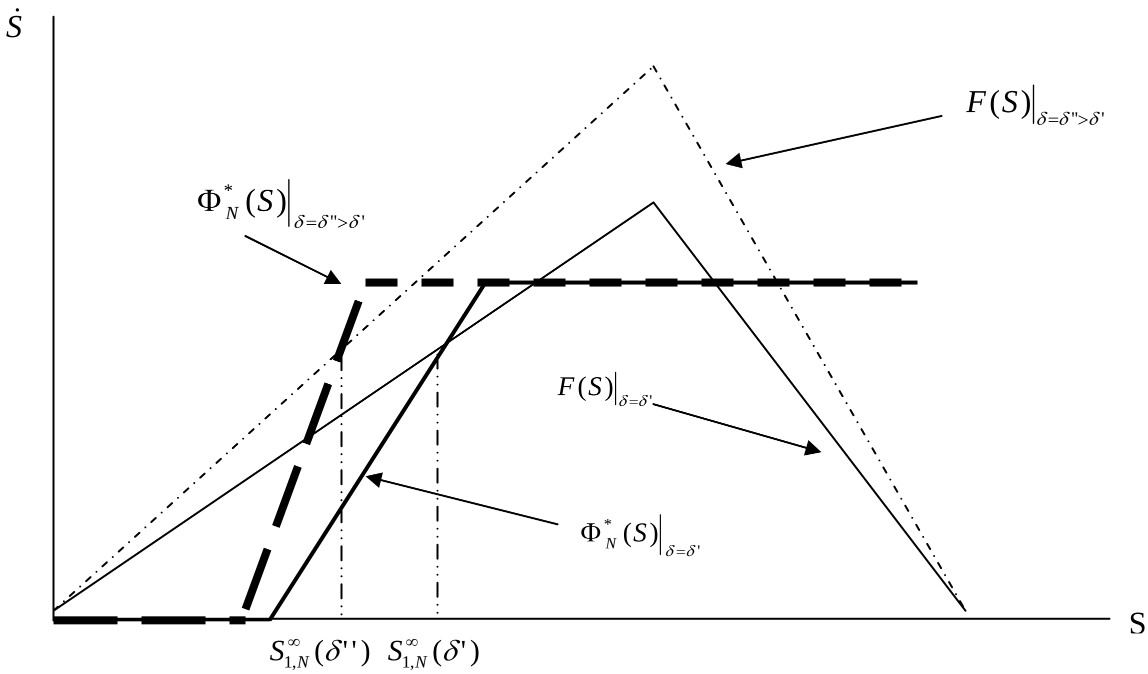

Figure 4

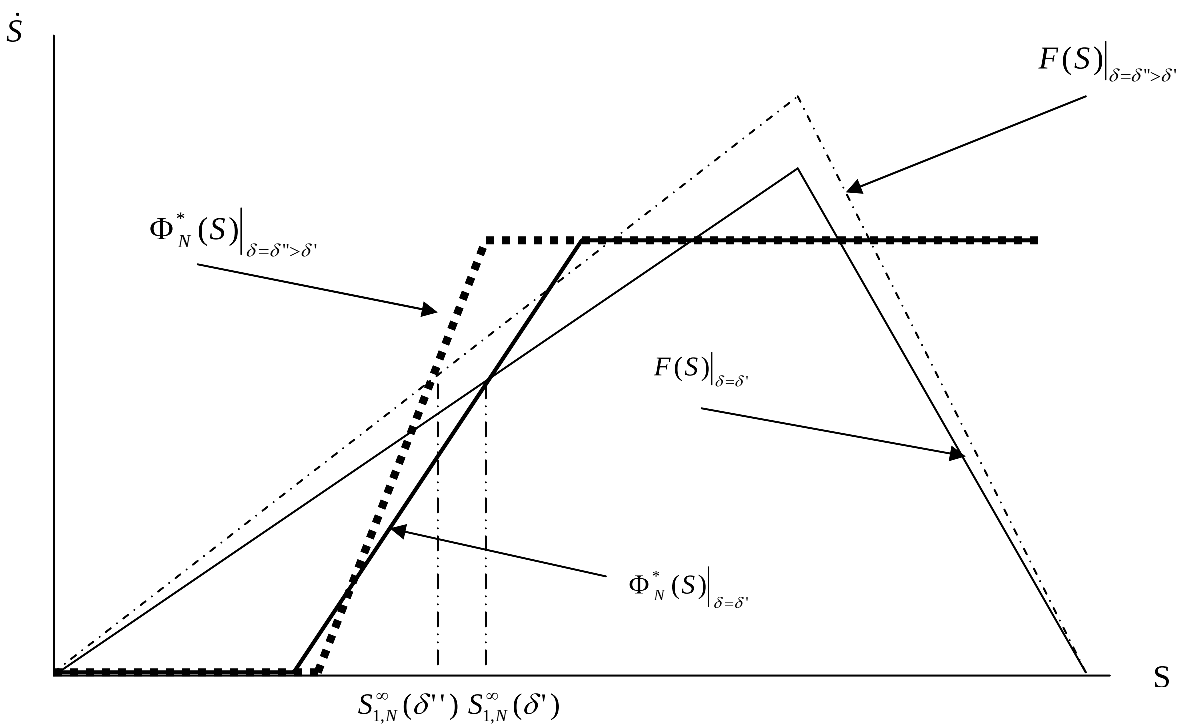

Figure 5