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Strategy-Proof and Envyfree Random Assignment*

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Abstract

We study the random assignment of indivisible objects among a set of agents with strict preferences. We show that there exists no mechanism which is strategy-proof, envyfree and unanimous. Then we weaken the latter requirement to q-unanimity: when each agent ranks a different object at the top, then any agent shall receive his most preferred object with probability of at least q. We show that if a mechanism satisfies strategy-proofness, envyfreeness, ex-post weak non-wastefulness, ex-post weak efficiency and q-unanimity, then q must be smaller than or equal to $\frac{2}{|N|}$ (where $|N|$ is the number of agents). We introduce a new mechanism called random careless dictator (RCD) and show that RCD achieves this maximal bound. In addition, for three agents, RCD is characterized by the first four properties.

JEL Classification: D63, D70.

Keywords: random assignment, strategy-proofness, envyfreeness, q-unanimity.

1 Introduction

We consider the assignment of indivisible objects without monetary transfers among a set of agents with strict preferences. Each agent desires exactly one object, and even though there are no transfers, one may allow objects' characteristics to include a fixed monetary payment. Problems like this arise in many real-life applications such as on-campus housing (where rents are fixed), organ allocation, school choice with ties, etc. Now if several agents

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would like to consume the same object, then any deterministic assignment is unfair. This is the main reason of implementing random assignments in such contexts.

Since agents report their preferences, incentive properties are important (as otherwise the assignment is based on false preferences). Strategy-proofness ensures that each agent truthfully reveals his ordinal preferences over objects for any utility representation of his preferences. The literature on random allocation mechanisms contains several impossibility results when strategy-proofness and equal treatment of equals are married with different ex-ante/ex-post notions of efficiency.[1] In some sense, these notions are hence "too strong". In addition, equal treatment of equals may be considered "too weak" a notion of fairness as it only constrains random assignments in rare cases where agents' preferences are identical, which seems contrived given that fairness concerns were the principal reason to consider random assignments in the first place. Our paper keeps the strategy-proofness requirement, strengthens equal treatment of equals to envyfreeness and explores the efficiency frontier given these two constraints.

First, we marry these properties with the weakest efficiency requirement, namely unanimity. It requires that if all agents rank different objects first, then each agent shall receive his most preferred object with probability one – in other words, whenever there exists a unique efficient assignment than this assignment is chosen for sure. Unfortunately, this results again in an impossibility (together with strategy-proofness and envyfreeness). Given this we introduce a measure by how much unanimity is respected: q-unanimity means that in any such situation any agent receives with probability of at least q his most preferred object. Of course, 0-unanimity is satisfied by any mechanism and by lowering q from one we obtain a possibility together with strategy-proofness and envyfreeness. The important question is to determine the exact bound. We show that for three agents this bound is equal to $\frac{2}{3}$.

In order to achieve this bound, we introduce a new mechanism, called random careless dictator mechanism (RCD). In this mechanism, any agent is chosen with equal probability to be the dictator, then this agent receives his most preferred object and any other object is equally divided (in terms of probabilities) among the remaining agents. We show for three agents that RCD is the unique mechanism which is ex-post weak non-wasteful, ex-post weak efficient, strategy-proof and envyfree. Hence, RCD is characterized by a natural set of properties for three agents. RCD satisfies $\frac{2}{3}$-unanimity for three agents (as any agent is chosen with probability $\frac{1}{3}$ to be the dictator and when another agent is the dictator, the agent receives his most preferred object with probability $\frac{1}{2}$). For an arbitrary number $|N|$

[1]Throughout 'ex-ante' is to be understood as before realizing the final deterministic assignment; this corresponds to the term 'interim' used in mechanism design outside of the literature on random assignments.

of agents, RCD satisfies $\frac{2}{|N|}$-unanimity. Most importantly, we show that this bound cannot be increased by any mechanism satisfying the properties in the characterization of RCD: for an arbitrary set of agents N, if a mechanism is ex-post weak non-wasteful, ex-post weak efficient, strategy-proof and envyfree, then it only satisfies q-unanimity for $q \leq \frac{2}{|N|}$. In other words, in the class of mechanisms satisfying these properties, RCD achieves the maximal bound for q-unanimity (and any greater q results in an impossibility).

Below we discuss in detail the related literature. The main starting point is the impossibility of strategy-proofness, envyfreeness and ex-ante efficiency. Bogomolnaia and Moulin (2001) show that this remains unchanged when envyfreeness is weakened to equal treatment of equals. Furthermore, they introduce the probabilistic serial (PS) mechanism and show that it is envyfree and ex-ante efficient.[2] Nesterov (2017) shows that the impossibility remains unchanged when ex-ante efficiency is replaced with ex-post efficiency.[3]

It is known that the random serial dictatorship (RSD) mechanism satisfies strategy-proofness, equal treatment of equals and ex-post efficiency, i.e., weakening both envyfreeness and ex-ante efficiency results in a possibility (and RSD violates those two properties). Our contribution is the first one to keep envyfreeness (since fairness understood as equity is the principal reason for implementing a random assignment) and strategy-proofness and to explore by how much exactly we have to weaken ex-post efficiency to arrive at a possibility result.

Most of our results focus on the pure assignment problem where all agents find all objects acceptable. If agents may rank objects as unacceptable and possibly receive no object, notions of efficiency have to take into account the set of (un)assigned objects:[4] a deterministic assignment is (weakly) non-wasteful if no (unassigned) agent prefers an unassigned object to his assignment. Then ex-ante non-wastefulness says that if an agent prefers an object to remaining unassigned or to being assigned some other object (either of which occurs with positive probability), then the preferred object is assigned with probability one. Martini (2016) shows that there is no mechanism satisfying strategy-proofness, equal treatment of equals and ex-ante non-wastefulness, i.e. another principal impossibility result on the full domain. Erdil (2014) studies the ex-ante waste of strategy-proof mechanisms, and shows that RSD is dominated by a strategy-proof mechanism which is less wasteful. It is an open problem to establish the minimal waste in the class of mechanisms satisfying strategy-proofness and equal treatment of equals. We show that by allowing waste, one can increase the bound for q-unanimity beyond $\frac{2}{3}$ for three agents. More precisely, we construct

[2]Hashimoto et al. (2014) establish a characterization of the PS mechanism.

[3]Zhang (2019) establishes a strong manipulability result by adding several auxiliary axioms.

[4]Bogomolnaia and Moulin (2015) show PS to achieve the maximal size guarantee among all envyfree mechanisms.

a mechanism for three agents which (i) assigns any agent no object with probability $\frac{1}{6}$ and (ii) satisfies strategy-proofness, envyfreeness and $\frac{5}{6}$-unanimity.

Finally, we extend RCD to the full domain where any agent's random assignment can be seen as his choice from a fixed budget determined by the dictators. This has some flavor as Gibbard (1977, 1978), who characterized convex combinations of dictatorships and duples, and Barberà and Jackson (1995) of trading along fixed prices.

The paper is organized as follows. Section 2 introduces random assignment, properties and several popular mechanisms. Section 3 contains the impossibility with unanimity and establishes the bound of $\frac{2}{3}$ for three agents and q-unanimity. Section 4 introduces RCD and establishes its characterization for three agents. Furthermore, it is shown that RCD achieves the maximal bound of q-unanimity in the class of ex-post weak non-wasteful, ex-post weak efficient, strategy-proof and envyfree mechanisms. Section 5 discusses the extension of RCD to the full preference domain, another variation of our impossibility on this domain, and the possible waste of mechanisms.

2 Model

Let $N = \{1, \dots, n\}$ denote the set of agents and $O = \{o_1, \dots, o_n\}$ denote the finite set of objects. Throughout we suppose $|N| = |O|$ and $|N| \geq 3$. Each agent i has a strict preference over $O \cup \{i\}$ where i stands for being unassigned; let R_i denote the corresponding linear order.[5] Let $\mathcal{R}^i$ denote the set of all strict preferences of agent i over $O \cup \{i\}$. Let $\mathcal{R}^N = \times_{i \in N} \mathcal{R}^i$ denote the set of all preference profiles $R = (R_1, \dots, R_n)$. Let $\underline{\mathcal{R}}^i$ denote the set of all strict preferences of agent i over $O \cup \{i\}$ such that $oR_i i$ for all $o \in O$, i.e. all objects are acceptable. Let $\underline{\mathcal{R}}^N = \times_{i \in N} \underline{\mathcal{R}}^i$ denote the pure assignment domain. We call $\mathcal{R}^N$ the full domain.

An assignment is a mapping $\mu : N \to O \cup N$ such that $\mu_i \in O \cup \{i\}$ for all $i \in N$ and $\mu_i \neq \mu_j$ for all $i \neq j$. Let $\mathcal{M}$ denote the set of all assignments.

An assignment μ is efficient under R if there exists no $\mu' \in \mathcal{M}$ such that $\mu'_i R_i \mu_i$ for all $i \in N$ and $\mu'_j P_j \mu_j$ for some $j \in N$. Let $\mathcal{PO}(R)$ denote the set of all efficient assignments under R.

An assignment μ is weakly efficient under R if there exists no $\mu' \in \mathcal{M}$ such that $\mu'_i P_i \mu_i$ for all $i \in N$. Let $\mathcal{WPO}(R)$ denote the set of all weakly efficient assignments under R.

An assignment μ is non-wasteful under R if for all $i \in N$ and all $x \in O \cup \{i\}$, $xR_i\mu_i$ implies there exists $j \in N$ with $\mu_j = x$. Note that this implies $\mu_i R_i i$. Let $\mathcal{NW}(R)$ denote the set of all non-wasteful assignments under R.

[5]This means R_i is (i) complete, (ii) transitive and (iii) antisymmetric (xR_iy and yR_ix implies $x = y$).

An assignment μ is weakly non-wasteful under R if for all $i \in N$ and all $x \in O \cup \{i\}$, $xR_i\mu_i$ and $iR_i\mu_i$ together imply that there exists $j \in N$ with $\mu_j = x$. Again this implies $\mu_i R_i i$. Hence, we only consider it waste if an unassigned object is desired by an *unassigned* agent or if an agent is assigned when they would prefer to remain unassigned. Let $\mathcal{WNW}(R)$ denote the set of all weakly non-wasteful assignments under R. Note that weak non-wastefulness only requires the knowledge of each agent's acceptable objects but not the exact ranking among them (which is necessary to determine non-wastefulness or efficiency of a deterministic allocation).

For any profile R, we have $\mathcal{PO}(R) \subseteq \mathcal{NW}(R) \subseteq \mathcal{WNW}(R)$, and there is no relation between (weak) non-wastefulness and weak efficiency.

Let $\Delta(\mathcal{M})$ denote the set of all probability distributions over $\mathcal{M}$. Given $p \in \Delta(\mathcal{M})$, let p_{ia} denote the associated probability of i being assigned a. Let $supp(p)$ denote the support of p. Then (i) p is ex-post efficient under R if $supp(p) \subseteq \mathcal{PO}(R)$, (ii) p is ex-post weakly efficient under R if $supp(p) \subseteq \mathcal{WPO}(R)$, (iii) p is ex-post non-wasteful under R if $supp(p) \subseteq \mathcal{NW}(R)$, and (iv) p is ex-post weakly non-wasteful under R if $supp(p) \subseteq \mathcal{WNW}(R)$.

For all $i \in N$, all $R_i \in \mathcal{R}^i$ and all $x \in O \cup \{i\}$, let $B(x, R_i) = \{y \in O \cup \{i\} : yR_ix\}$. Then given any $p, q \in \Delta(\mathcal{M})$, p_i stochastically R_i-dominates q_i if for all $x \in O \cup \{i\}$,

$$\sum_{y \in B(x,R_i)} p_{iy} \geq \sum_{y \in B(x,R_i)} q_{iy}.$$

Then p is sd-efficient under R if there exists no $q \in \Delta(\mathcal{M})$ such that q_i stochastically R_i-dominates p_i for all $i \in N$ with strict dominance holding for some agent. Note that sd-efficiency means that p is efficient in terms of expected utility for any vNM-representations of agents' ordinal preferences. This is an ex-ante efficiency notion before realizing the deterministic assignment to be implemented.

A mechanism is a mapping $\varphi : \mathcal{R}^N \to \Delta(\mathcal{M})$. Then $\varphi(R)$ denotes the random assignment chosen for R, and $\varphi_{ia}(R)$ denotes the probability of agent i being assigned object a. Then φ is *sd-efficient* if for all $R \in \mathcal{R}^N$, $\varphi(R)$ is sd-efficient under R. Similarly, we define ex-post (weak) efficiency and ex-post (weak) non-wastefulness for a mechanism.

Then φ is *strategy-proof* if for all $R \in \mathcal{R}^N$, all $i \in N$ and all $R'_i \in \mathcal{R}^i$, $\varphi_i(R)$ stochastically R_i-dominates $\varphi_i(R'_i, R_{-i})$. Furthermore, φ is *envyfree* if for all $R \in \mathcal{R}^N$ and all $i \in N$, $\varphi_i(R)$ stochastically R_i-dominates $\varphi_j(R)$ (where in $\varphi_j(R)$ the outside option j is replaced by i). Finally, φ is *symmetric* (respectively, treats equals equally) if for all $R \in \mathcal{R}^N$ and all $i, j \in N$, $R_i = R_j$ implies $\varphi_i(R) = \varphi_j(R)$.

Remark 1 *(i) The following is straightforward but useful: on the domain $\underline{\mathcal{R}}^N$, if a mechanism φ satisfies strategy-proofness and envyfreeness, then for all $R, R' \in \underline{\mathcal{R}}^N$ and all*

$i, j \in N$,

$$\sum_{x \in O} \varphi_{ix}(R) = \sum_{x \in O} \varphi_{jx}(R') \equiv Q_\varphi.$$

Thus, every agent receives the probability share Q_φ of objects (in O) at any preference profile.

(ii) Furthermore, it is immediate that on the domain $\underline{\mathcal{R}}^N$, a mechanism φ satisfies ex-post (weak) non-wastefulness if and only if $Q_\varphi = 1$.

Below we introduce some of the well-known mechanisms on the pure assignment domain.

The uniform assignment (UA) mechanism[6] divides all objects equally irrespective of preferences: for all $R \in \underline{\mathcal{R}}^N$, $UA_{io}(R) = \frac{1}{n}$ for all $i \in N$ and $o \in O$.

A strict priority ranking over N is denoted by $\succ$. Let $\mathcal{L}$ denote the set of all strict priority rankings. Given $\succ \in \mathcal{L}$, let $f^{\succ}$ denote the (deterministic) serial dictatorship (SD-)mechanism.[7] Then the random serial dictatorship (RSD) mechanism is defined by $RSD(R) = \frac{1}{n!} \sum_{\succ \in \mathcal{L}} f^{\succ}(R)$ for all $R \in \underline{\mathcal{R}}^N$.

We omit the formal definition of the probabilistic serial (PS) mechanism[8] and only give an intuitive formulation and refer the reader for the formal definition to Bogomolnaia and Moulin (2001): each agent starts eating with uniform speed from his most preferred object; once an object is finished, each agent eats with uniform speed from his most preferred object among the remaining ones, and so on until all objects are finished. The random assignment of any agent in PS is simply the shares of objects the agent has eaten during this process.

3 Unanimity

For the following requirement we do not have to distinguish between the ex-post notion and the ex-ante notion as they are identical. A mechanism φ satisfies unanimity if for any profile R where there exists $\mu \in \mathcal{M}$ such that for all $i \in N$, $\mu_i R_i \mu'_i$ for all $\mu' \in \mathcal{M}$, then $\varphi_{i\mu_i}(R) = 1$ for all $i \in N$. Hence, $\varphi(R)$ attaches probability one to μ. Note that as preferences are strict the assignment μ has to be unique.

Theorem 1 *On the domain $\underline{\mathcal{R}}^N$ for $|N| \geq 3$, there exists no mechanism which is strategy-proof, envyfree, and unanimous.*

[6] Chambers (2004) characterizes UA via consistency.

[7] For any $R \in \underline{\mathcal{R}}^N$ and $i_1 \succ i_2 \succ \cdots \succ i_n$, i_1 receives his most R_{i_1}-preferred object in O (denoted by $f^{\succ}_{i_1}(R)$), and for $l = 2, \ldots, n$, i_l receives his most R_{i_l}-preferred object in $O \backslash \{f^{\succ}_{i_l}(R), \ldots, f^{\succ}_{i_{l-l}}(R)\}$ (denoted by $f^{\succ}_{i_l}(R)$).

[8] Bogomolnaia (2015) offers an alternative definition of PS, and Katta and Sethuraman (2006) extend PS to the domain where indifferences are allowed.

The proof rests on the following lemma.

Lemma 1 *On the domain $\underline{\mathcal{R}}^N$ for $|N| = 3$, there exists no mechanism which is strategy-proof, envyfree, and satisfies the following Property $(*)$: an object that is ranked first by one agent and last by all others is assigned to the agent ranking it first with probability one.*

Proof. Towards a contradiction, let φ be a mechanism satisfying strategy-proofness, envyfreeness and Property $(*)$. Then Property $(*)$ implies that all objects are assigned with probability one. Hence, $Q_\varphi = 1$ (see Remark 1). Next, consider

R''_1	R''_2	R''_3
a	c	a
b	a	b
c	b	c

By Property $(*)$, $\varphi_{2c}(R'') = 1$. Next, consider

R'_1	R'_2	R'_3
a	a	a
b	c	b
c	b	c

i.e. the profile where 2 changes the relative ranking of the two most-preferred objects. Then by strategy-proofness (and envyfreeness) we have $\varphi_{2a}(R') + \varphi_{2c}(R') = 1$ (and $\varphi_{ia}(R') = \frac{1}{3}$ for all $i \in N$). Hence, $\varphi_{2c}(R') = \frac{2}{3}$ and, by envyfreeness, $\varphi_{1c}(R') = \varphi_{3c}(R') = \frac{1}{6}$. Next, consider

R_1	R_2	R_3
a	a	b
b	c	a
c	b	c

i.e. the profile where 3 changes the relative ranking of the two most-preferred objects. By strategy-proofness (and envyfreeness) we have $\varphi_{3a}(R) \leq \frac{1}{3}$and $\varphi_{3c}(R) = \frac{1}{6}$ (and $\varphi_{1c}(R) = \frac{1}{6}$). Hence $\varphi_{2c}(R) = \frac{2}{3}$ and thus $\varphi_{2a}(R) \leq \frac{1}{3}$. By envyfreeness, $\varphi_{1a}(R) = \varphi_{2a}(R) \leq \frac{1}{3}$. Since the assignment probabilities for a sum to one, we have $\varphi_{ia}(R) = \frac{1}{3}$ for all $i \in N$. Solving for the

b's residual assignment probabilities, we find $\varphi(R)$ as given below:

R_1	R_2	R_3
$\frac{1}{3}: a$	$\frac{1}{3}: a$	$\frac{1}{2}: b$
$\frac{1}{2}: b$	$\frac{2}{3}: c$	$\frac{1}{3}: a$
$\frac{1}{6}: c$	$0: b$	$\frac{1}{6}: c$

Towards a contradiction we now show $\varphi_{3a}(R) = 0$ and $\varphi_{1a}(R) = \varphi_{2a}(R) = \frac{1}{2}$. For that, consider

R_1^*	R_2^*	R_3^*
a	a	b
c	c	a
b	b	c

By Property $(*)$, we have $\varphi_{3b}(R^*) = 1$, and hence, $\varphi_{3a}(R^*) + \varphi_{3c}(R^*) = 0$. By envyfreeness $\varphi_{1a}(R^*) = \varphi_{1c}(R^*) = \varphi_{2a}(R^*) = \varphi_{2c}(R^*) = \frac{1}{2}$. Finally, consider again profile R, where compared to R^*, 1 changes the relative ranking of the two least-preferred objects. By strategy-proofness (and envyfreeness) we have $\varphi_{1a}(R) = \frac{1}{2}$ (and $\varphi_{2a}(R) = \frac{1}{2}$). Hence, $\varphi_{3a}(R) = 0$ and we arrive at the contradiction described above. ■

Proof of Theorem 1. Towards a contradiction, suppose that φ is a mechanism satisfying strategy-proofness, envyfreeness and unanimity. First, let $|N| = 3$. By Lemma 1, it suffices to show that φ satisfies Property $(*)$ – a contradiction as we have seen. So consider a profile R where one agent, say 1, ranks an object, say a, first while both other agents, 2 and 3, rank it last. By envyfreeness, $\varphi_{2a}(R) = \varphi_{3a}(R)$. Now either 2 and 3 differ in their first ranked objects b and c – then each agent receives their first ranked object by unanimity and $\varphi_{2a}(R) = \varphi_{3a}(R) = 0$ – or both rank the same object, say b, first. In the latter case, a change in R_3 to $cR_3'bR_3'aR_3'3$ and for $R' = (R_3', R_{-3})$ lets 3 receive a with zero probability (by unanimity) and with the same probability as before (by strategy-proofness). Hence, $0 = \varphi_{3a}(R') = \varphi_{3a}(R) = \varphi_{2a}(R)$. In either case $\varphi_{1a}(R) = 1$, which establishes Property $(*)$.

Second, let $|N| > 3$. Suppose that ϕ is a mechanism which is unanimous, strategy-proof and envyfree. We show that then there exists a mechanism, which is unanimous, strategy-proof and envyfree, on the domain $\underline{\mathcal{R}}^{N\setminus\{n\}}$ and set of objects $O\setminus\{o_n\}$. Let $\hat{R}$ be a profile where agent i ranks o_i first ($i \in N$) and all agents except n rank o_n last. By unanimity, we have $\phi_{io_i}(R) = 1$ for all $i \in N$. Let $\underline{\mathcal{R}}'$ denote the set of all profiles $R \in \underline{\mathcal{R}}^N$ such that (i) $R_n = \hat{R}_n$ and (ii) all agents in $N\setminus\{n\}$ rank o_n last. Then starting from $\hat{R}$ by strategy-proofness and envyfreeness we obtain for all $i \in N\backslash\{n\}$, $\sum_{l=1}^{n-1} \phi_{io_l}(R) = 1$ and $\phi_{io_n}(R) = 0$. Now define φ on domain $\underline{\mathcal{R}}^{N\backslash\{n\}}$ and set of objects $O\backslash\{o_n\}$ as follows: for all $R \in \underline{\mathcal{R}}'$, all $i \in N\backslash\{n\}$ and all $o \in O\backslash\{o_n\}$, $\varphi_{io}((R_l|(O \cup \{l\})\backslash\{o_n\})_{l\in N\backslash\{n\}}) = \phi_{io}(R)$ (where $R_l|(O \cup \{l\})\backslash\{o_n\}$ denotes the restriction of R_l to the set $(O \cup \{l\})\backslash\{o_n\}$). It is straightforward that φ is a well-defined mechanism on the domain $\underline{\mathcal{R}}^{N\backslash\{n\}}$ and set of objects $O\backslash\{o_n\}$ satisfying unanimity, strategy-proofness and envyfreeness. Now by induction and by the fact that for $|N| = 3$ such a mechanism does not exist, this is a contradiction. ■

Note that (i) RSD is unanimous, strategyproof and symmetric, (ii) UA is strategyproof and envyfree and (iii) PS is sd-efficient (and hence in particular unanimous) and envyfree.

Instead of requiring full unanimity, one may only require q-unanimity – where for a "unanimous profile"[9] each agent receives his top ranked object with probability at least q. [10]

Formally, given $q \in [0, 1]$, a mechanism φ satisfies q-unanimity if for any profile R where there exists $\mu \in \mathcal{M}$ such that for all $i \in N$, $\mu_i R_i \mu'_i$ for all $\mu' \in \mathcal{M}$, then $\varphi_{i\mu_i}(R) \geq q$ for all $i \in N$.[11]

For three agents, it will turn out that the random careless dictator mechanism (defined in the next section) satisfies $\frac{2}{3}$-unanimity. By the following result this is the maximal bound that can be attained as long as we want to ensure weak non-wastefulness, strategy-proofness and envyfreeness.

Proposition 1 *On the domain $\underline{\mathcal{R}}^N$ for $|N| = 3$, if a mechanism satisfies ex-post weak non-wastefulness, strategy-proofness, envyfreeness, and q-unanimity, then $q \leq \frac{2}{3}$.*

Note that this result strengthens Theorem 1 for three agents since strategy-proofness, envyfreeness and unanimity imply ex-post weak non-wastefulness – see Remark 1.

Proof. Let $N = \{1, 2, 3\}$ and $O = \{a, b, c\}$. Let φ be a mechanism satisfying the properties. By Remark 1, $Q_\varphi = 1$.

First, we show that $q \leq \frac{2}{3}$ under the auxiliary assumption that φ is neutral[12] (with

[9]At such a profile any two agents rank different objects first.

[10]I had taken our "unanimous profile" and the footnote – 'where for a profile where each agent ranks a different object first each agent receives ...' reads more smoothly, no?

[11]This is weaker than requiring μ to be chosen with probability at least q.

[12]Given a permutation $\sigma : O \to O$ and $R \in \mathcal{R}^N$, let R_i^σ be such that (i) for all $a, b \in O$, aR_ib iff $\sigma(a)R_i^\sigma\sigma(b)$ and (ii) for all $a \in O$, aR_ii iff $\sigma(a)R_ii$, and $R^\sigma = (R_i^\sigma)_{i\in N}$. A mechanism φ is neutral if for any permutation $\sigma : O \to O$ and $R \in \mathcal{R}^N$, we have $\varphi_{io}(R) = \varphi_{i\sigma(o)}(R^\sigma)$ for all $i \in N$ and all $o \in O$.

respect to renaming objects). Consider the profile

R_1	R_2	R_3
$\frac{1}{2}-\delta : a$	$\frac{1}{2}-\delta : a$	$1-2\varepsilon-2\delta : c$
$\frac{1}{2}-\varepsilon : b$	$\frac{1}{2}-\varepsilon : b$	$2\delta : a$
$\varepsilon+\delta : c$	$\varepsilon+\delta : c$	$2\varepsilon : b$

and set $\varphi_{3a}(R) =: 2\delta$ and $\varphi_{3b}(R) =: 2\varepsilon$. The remaining probabilities follow as residuals given envyfreeness and that probabilities sum to one (by $Q_\varphi = 1$). In particular. $\varphi_{1c}(R) = \varphi_{2c}(R) = \varepsilon + \delta$.

If we compare this to the unanimous profile R^* where, relative to R, 2 swaps the ranking of a and b, we still have $\varphi_{2c}(R^*) = \varepsilon + \delta$ (by strategy-proofness) as well as $\varphi_{1c}(R^*) = \varepsilon + \delta$ (by envyfreeness), so that $\varphi_{3c}(R^*) = 1 - 2\varepsilon - 2\delta$ ($Q_\varphi = 1$). Hence, $q \leq 1 - 2\varepsilon - 2\delta$.

Next, consider the following four profiles:

$\tilde{R}'_1$	$\tilde{R}'_2$	$\tilde{R}'_3$
b	b	a
a	a	$\frac{2}{3}-2\varepsilon : c$
$\frac{1}{6}+\varepsilon : c$	$\frac{1}{6}+\varepsilon : c$	b

R'_1	R'_2	R'_3
$\frac{1}{3} : a$	$\frac{1}{3} : a$	$\frac{1}{3} : a$
$\frac{1}{2}-\varepsilon : b$	$\frac{1}{2}-\varepsilon : b$	$\frac{2}{3}-2\varepsilon : c$
$\frac{1}{6}+\varepsilon : c$	$\frac{1}{6}+\varepsilon : c$	$2\varepsilon : b$

$\tilde{R}_1$	$\tilde{R}_2$	$\tilde{R}_3$
a	a	$\frac{2}{3}-2\delta : c$
$\frac{1}{6}+\delta : c$	$\frac{1}{6}+\delta : c$	b
b	b	a

R''_1	R''_2	R''_3
$\frac{1}{2}-\delta : a$	$\frac{1}{2}-\delta : a$	$\frac{2}{3}-2\delta : c$
$\frac{1}{6}+\delta : c$	$\frac{1}{6}+\delta : c$	$2\delta : a$
$\frac{1}{3} : b$	$\frac{1}{3} : b$	$\frac{1}{3} : b$

In R'' agents 1 and 2 have swapped the ranking of c and b, relative to R. By strategy-proofness and envyfreeness we still have $\varphi_{ia}(R'') = \varphi_{ia}(R) = \frac{1}{2} - \delta$ for $i = 1, 2$ and hence $\varphi_{3a}(R'') = 2\delta$. Moreover, by envyfreeness, $\varphi_{3b}(R'') = \frac{1}{3}$ and hence $\varphi_{3c}(R'') = \frac{2}{3} - 2\delta$. In $\tilde{R}$, where relative to R'' agent 3 has swapped the ranking of a and b, we still have $\varphi_{3c}(\tilde{R}) = \frac{2}{3} - 2\delta$ (by strategy-proofness) and hence $\varphi_{ic}(\tilde{R}) = \varphi_{ic}(R'') = \frac{1}{6} + \delta$ for $i = 1, 2$ (by envyfreeness and $Q_\varphi = 1$).

In R' agent 3 has changed the ranking of a and c, relative to R. By strategy-proofness we still have $\varphi_{3b}(R') = \varphi_{3b}(R) = 2\varepsilon$ and hence $\varphi_{1b}(R') = \varphi_{2b}(R') = \frac{1}{2} - \varepsilon$. Moreover by envyfreenes, $\varphi_{ia}(R') = \frac{1}{3}$ for $i \in N$, and hence $\varphi_{1c}(R') = \varphi_{2c}(R') = \frac{1}{6} + \varepsilon$. In $\tilde{R}'$ relative to R', 1 and 2 have changed the ranking of a and b, and by strategy-proofness and envyfreeness we obtain $\varphi_{1c}(\tilde{R}') = \varphi_{2c}(\tilde{R}') = \frac{1}{6} + \varepsilon$. Note that profile $\tilde{R}$ is obtained from $\tilde{R}'$ by renaming objects as follows: b becomes a, a becomes c, and c becomes b. Thus, by neutrality, we then have $\varphi_{1c}(\tilde{R}') = \varphi_{1b}(\tilde{R}) = \frac{1}{6} + \varepsilon$. Hence, as a residual we have $\varphi_{1a}(\tilde{R}) = \frac{2}{3} - \varepsilon - \delta$, and likewise

$\varphi_{2a}(\tilde{R}) = \frac{2}{3} - \varepsilon - \delta$. Since the sum of assignment probabilities for a cannot exceed one, we have $2\varepsilon + 2\delta \geq \frac{1}{3}$. Hence, $q \leq 1 - 2\varepsilon - 2\delta \leq \frac{2}{3}$.

Second, assume there exists a non-neutral mechanism which is strategy-proof, envyfree and q-unanimous with $q > \frac{2}{3}$. Then any mechanism derived from it by permuting objects would likewise satisfy the three latter properties – but a uniform mixture over all these permuted mechanisms would restore neutrality and satisfy q-unanimity with $q > \frac{2}{3}$ – a contradiction to the first part. ■

4 The Random Careless Dictator Mechanism

For the pure assignment domain with $|N| = |O| = n$ denote by $top(R_i)$ denote the most preferred object under R_i and define the *random careless dictator mechanism (RCD)* ϕ as follows. Given $i \in N$, for any $R \in \underline{\mathcal{R}}^N$ and any $o \in O$, set $\phi^i_{io}(R) = 1$ if $o = top(R_i)$ and otherwise $\phi^i_{io}(R) = 0$; moreover for all $j \in N \backslash \{i\}$, $\phi^i_{jo}(R) = 0$ if $o = top(R_i)$ and otherwise $\phi^i_{jo}(R) = \frac{1}{n-1}$. In words, the careless dictator mechanism (with dictator i) ϕ^i lets i pick her most preferred object before dividing all remaining objects equally among the other agents. Finally, for all $j \in N$, all $a \in O$ and all $R \in \underline{\mathcal{R}}^N$, we define $\phi_{ja}(R) = \frac{1}{n}\sum_{i\in N}\phi^i_{ja}(R)$. In words, ϕ is defined as the uniform mixture over all ϕ^i – it selects some agent i uniformly at random and then applies the careless dictator mechanism ϕ^i with agent i as dictator and ignoring others' preferences. We extend RCD from the pure assignment domain to the full domain in Section 5.

When all objects are acceptable, the following observations are immediate:

- ϕ is neutral, anonymous[13] (with respect to renaming agents), strategy-proof and envyfree.

- ϕ Pareto dominates UA – but itself is Pareto dominated by RSD.

- An object, which is never ranked first or ranked first by all, is assigned uniformly at random (to each agent with probability $\frac{1}{n}$). In particular, if there is no heterogeneity in first preferences then we arrive at the uniform assignment, foregoing any gains from trade in lower ranked objects.

- An object that is ranked first only by i, is assigned to i with probability $\frac{1}{n} + \frac{n-1}{n}\frac{1}{n-1} = \frac{2}{n}$ – this is the maximal probability with which an object can be assigned to an individual agent under ϕ. Hence, ϕ satisfies $\frac{2}{n}$-unanimity.

[13]Given a permutation $\tau : N \to N$ and $R \in \mathcal{R}^N$, let $\tau(R)$ be the profile such that for all $i \in N$, $\tau(R)_i = R_{\tau(i)}$. A mechanism φ is anonymous if for any permutation $\tau : N \to N$ and $R \in \mathcal{R}^N$, we have $\varphi_i(\tau(R)) = \varphi_{\tau(i)}(R)$ for all $i \in N$.

- An object that is ranked first by all agents but i, is never assigned to i and to each $j \neq i$ with equal probability $\frac{1}{n-1}$.

Moreover, since RCD assigns all objects with probability one, it is ex-post weakly non-wasteful, and since there is always at least one individual who receives his top object, RCD is (à fortiori) ex-post weak efficient. For three agents these two properties together with strategy-proofness and envyfreeness characterize RCD.[14]

Theorem 2 *On the domain $\underline{\mathcal{R}}^N$ for $|N| = 3$, the random careless dictator mechanism is the unique mechanism satisfying ex-post weak non wastefulness, ex-post weak efficiency, strategy-proofness and envyfreeness.*

Proof. As shown above, RCD satisfies the properties of Theorem 2. For the other direction, let $N = \{1, 2, 3\}$ and $O = \{a, b, c\}$. Let φ denote an arbitrary mechanism that satisfies the properties above. We will show below that for any profile φ and RCD choose the same random assignment.

1. First, we show that whenever some object, say a, is ranked first twice, φ assigns it to both agents ranking it first with probability $\frac{1}{2}$ (and to the third with probability zero). For that, assume w.l.o.g. that 3 is the agent not ranking a first and that 3 ranks c above b. Now, consider the following profile $\tilde{R}$:

$\tilde{R}_1$	$\tilde{R}_2$	$\tilde{R}_3$
a	a	c
b	b	a
c	c	b

By ex-post weak efficiency, 3 cannot receive a with positive probability – in that case, 1 or 2 (say 1) would receive b and the other (say 2) c. But if instead 1 received a, 2 received b and 3 received c, everyone would be strictly better off. So by ex-post weak efficiency, 3 receives a with probability zero and by envyfreeness both 1 and 2 receive a with probability $\frac{1}{2}$ each. Moreover, by strategy-proofness and envyfreeness, the same assignment probabilities hold independently from the ordering in which 1 and 2 rank b and c. Finally, even as 3 demotes a and ranks it last, strategy-proofness demands that it receives none of it (while the other two still receive it with probability $\frac{1}{2}$ by envyfreeness).

2. Next, consider the case where some object, say a, is ranked first by all agents. As we will see, under φ all objects are assigned uniformly. For the case where all agents have the

[14]Here by characterization we mean with respect to marginal distributions, i.e. for any mechanism φ satisfying the properties below, we have $\varphi_i(R) = \phi_i(R)$ for all $i \in N$ and all $R \in \underline{\mathcal{R}}^N$.

same preferences, this is immediate. If there is some disagreement on the second and third ranked objects, consider w.l.o.g. profile R below along with two other profiles:

R_1	R_2	R_3
a	a	a
b	b	c
c	c	b

R'_1	R'_2	R'_3
b	b	a
a	a	c
c	c	b

R''_1	R''_2	R''_3
b	b	a
a	a	b
c	c	c

In profile R'', we have $\varphi_{3c}(R'') = \frac{1}{3}$ (by envyfreeness) and $\varphi_{3b}(R'') = 0$ (since b is ranked first twice, see above). By strategy-proofness we thus have $\varphi_{3c}(R') = \frac{1}{3}$ and $\varphi_{3b}(R') = 0$, and by envyfreeness $\varphi_{1c}(R') = \varphi_{2c}(R') = \frac{1}{3}$. By strategy proofness and envyfreeness, 1 and 2 are still assigned c with probability $\frac{1}{3}$ as we first replace R'_1 by R_1 and then R'_2 by R_2. Hence c is assigned uniformly in profile R. Of course the same is true for a (by envyfreeness) and, as a residual, for b.

3. Next, we will show that if two agents rank the same object, say b, first while a third, say 3, ranks a different object, say c, first, then 3 will receive his most preferred object (c) with probability $\frac{2}{3}$. For that, note that the probability with which 3 receives c is independent of the order in which he ranks a and b (by strategy-proofness). Assume he ranks b second, i.e. consider the following profile Q:

Q_1	Q_2	Q_3
b	b	c
$\vdots$	$\vdots$	b
		a

Then we know by the above (**1.**) that $\varphi_{3b}(Q) = 0$. Moreover for $Q'_3 : bQ'_3cQ'_3a$ we know that $\varphi(Q'_3, Q_{-3})$ is the uniform assignment (**2.**), so by strategy-proofness, $\varphi_{3c}(Q) + \varphi_{3b}(Q) = \frac{2}{3}$. Hence, as claimed, $\varphi_{3c}(Q) = \frac{2}{3}$.

4. Together, **1.**, **3.** and envyfreeness pin down the assignment for most profiles where two distinct objects are ranked first. For these profiles, two agents, say 1 and 2, rank the same object, say b, first. If they also agree on the ranking of a and c, then by envyfreeness, each receives $\frac{1}{6}$ of the object that 3 ranks first, say c.

If they do not agree on the ranking of a and c, there are two possible profiles (up to a permutation of 1 and 2 and objects a and c). Consider first

$\tilde{P}_1$	$\tilde{P}_2$	$\tilde{P}_3$
$\frac{1}{2} : b$	$\frac{1}{2}b$	$\frac{2}{3} : c$
$\frac{1}{6} + \tilde{\epsilon} : c$	$\frac{1}{3} + \tilde{\epsilon} : a$	$0b$
$\frac{1}{3} - \tilde{\epsilon} : a$	$\frac{1}{6} - \tilde{\epsilon} : c$	$\frac{1}{3}a$

To see that in $\tilde{P}$, $\tilde{\varepsilon} = 0$, consider a switch by 1 in their ranking of b and c – then they (and 3) receive c with probability $\frac{1}{2}$ (by **1.**) and 2 receives b with probability $\frac{2}{3}$ (by **3.**). By envyfreeness, 1 (and 3) receive b with probability $\frac{1}{6}$. As a residual 1 receives a with probability $\frac{1}{3}$ – and does so even before the switch (by strategy-proofness).

5. The only type of profile with two distinct first-ranked objects that remains is represented by profile P below where relative to $\tilde{P}$, 3 has changed the ranking of a and b, now ranking last the object that is ranked first by the other two agents. Consider P alongside the following two profiles P' and P'':

P_1	P_2	P_3
$\frac{1}{2} : b$	$\frac{1}{2} : b$	$\frac{2}{3} : c$
$\frac{1}{6} + \epsilon : c$	$\frac{1}{3} + \epsilon : a$	$\frac{1}{3} : a$
$\frac{1}{3} - \epsilon : a$	$\frac{1}{6} - \epsilon : c$	$0 : b$

P'_1	P'_2	P'_3
b	a	c
c	b	a
a	c	b

P''_1	P''_2	P''_3
b	a	c
c	b	b
a	c	a

P'''_1	P'''_2	P'''_3
b	a	b
c	b	c
a	c	a

Note that P is just one switch away from the 'Condorcet-cycle' profile P': if 2 switches a and b we are there. Hence, by strategy-proofness, $\varphi_{2c}(P') = \varphi_{2c}(P) = \frac{1}{6} - \epsilon$.

At P', the set of weakly efficient assignments is equal to

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ b & a & c \end{pmatrix}, \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}, \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ c & a & b \end{pmatrix} \text{ and } \mu_4 = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}.$$

Note that μ_2 is the only weakly efficient assignment at P' where agent 2 receives c, and μ_2 is the only weakly efficient assignment where agent 3 receives a. Hence, by ex-post weak efficiency and $\varphi(P')$ attaches probability $\frac{1}{6} - \epsilon$ to μ_2, we have $\varphi_{2c}(P') = \varphi_{3a}(P')$. From P'' to P' agent 3 switches a and b and by strategy-proofness, $\varphi_{3a}(P') \geq \varphi_{3a}(P'')$. Finally, at P''', $\varphi_{2a} = \frac{2}{3}$ (by **3.**) and hence $\varphi_{3a}(P''') = \varphi_{1a}(P''') = \frac{1}{6}$. As 3 switches b and c from P'' to P''', by strategy-proofness, $\varphi_{3a}(P'') = \varphi_{3a}(P''') = \frac{1}{6}$ and (by transitivity of the $\geq$-relation) we arrive at $\varphi_{2c}(P) \geq \frac{1}{6}$ so that $\varepsilon = 0$.

6. Finally, up to relabelling of objects and agents, there are two types of profiles where three distinct objects are ranked first – either there are three distinct 2nd ranked alternatives, so that we are in a 'condorcet cycle' profile, or there is an object that is ranked second twice.

For a profile of the first type, consider once more P'. Looking at P, strategy-proofness yields $\varphi_{2c}(P') = \varphi_{2c}(P) = \frac{1}{6}$. Moreover by ex-post weak efficiency, $\varphi_{3a}(P') = \varphi_{2c}(P')$. In the same way, one can show that the other probabilities of receiving a second or third ranked objects are $\frac{1}{6}$ which leaves each agent with a probability of receiving their top ranked object with probability $\frac{2}{3}$. Analogously, the same can be shown for any 'Condorcet-cycle profile'.

We are left with P''. By strategy-proofness, $\varphi_{3c}(P'') = \varphi_{3c}(P') = \frac{2}{3}$. Also $\varphi_{3c}(P'') + \varphi_{3b}(P'') = \varphi_{3c}(P''') + \varphi_{3b}(P''') = \frac{5}{6}$, and hence, $\varphi_{3b}(P'') = \frac{1}{6}$. Then as residual $\varphi_{3a}(P'') = \frac{1}{6}$ and by envyfreeness, $\varphi_{1a}(P'') = \frac{1}{6}$. Hence, $\varphi_{2a}(P'') = \frac{2}{3}$. By strategy-proofness, $\varphi_{2c}(P'') = \varphi_{2c}(\tilde{P}) = \frac{1}{6}$. All remaining probabilities follow as residual. ■

To see that our axioms are independent, observe that if we dropped ex-post weak non-wastefulness, a random dictatorship where only the dictator is assigned her most preferred object would satisfy the remaining axioms. PS satisfies all axioms but strategy-proofness, RSD all but envyfreeness and UA all but ex-post weak efficiency.

While it is not clear whether Theorem 2 extends to more than three agents, we show that the space for efficiency improvements beyond RCD is limited. In particular, at a profile where agent i ranks an object first that is ranked last by all others, i will only receive it with probability $\frac{2}{n}$. This implies that the bound $\frac{2}{n}$ is the best scenario for which q-unanimity can be attained.

Theorem 3 *On the domain $\underline{\mathcal{R}}^N$ for $|N| \geq 3$,*

(i) the random careless dictator mechanism satisfies ex-post weak non wastefulness, ex-post weak efficiency, strategy-proofness, envyfreeness and $\frac{2}{|N|}$-unanimity; and

(ii) if a mechanism satisfies ex-post weak non wastefulness, ex-post weak efficiency, strategy-proofness, envyfreeness and q-unanimity, then $q \leq \frac{2}{|N|}$. In particular, any such mechanism assigns an object x – ranked first by i and last by all others – with probability smaller than or equal to $\frac{2}{|N|}$ to i.

Proof. Part (i) is obvious. We show (ii). Let $N = \{1, \ldots, n\}$.

1. As before, by ex-post weak efficiency, if all but i rank an object first, i will receive it with probability zero, while others receive it with probability $\frac{1}{n-1}$: Suppose all $j \in N \backslash \{i\}$ rank object o_1 first while i ranks o_n first. Then complete the rankings as follows: $o_1 \succ_j o_2 \succ_j \cdots \succ_j o_n$ for all j and $o_n \succ_i o_1 \succ_i o_{n-1} \succ_i o_{n-2} \succ_i \cdots \succ_i o_2$ for i. If i were assigned o_1 everyone could be made better off by the following trade: i gives up o_1 for o_n, whoever had o_n gets o_{n-1} instead... whoever had o_2 gets o_1 instead.

By strategy-proofness and envyfreeness, the same holds if all but i change the order in which the rank objects below o_1, or if i demotes o_1 in his ranking.

2. Next, we show that many profiles where all $j \neq i$ agree in their ranking require a uniform assignment. Consider the following profile (where j stands for all $n-1$ agents other than i)

P_j	P_i	P'_j	P'_i	P''_j	P''_i	P'''_j	P'''_i	P''''_j	P''''_i
o_1	o_1	o_1	$\frac{2}{n} : o_2$	o_1	$\frac{2}{n} : o_2$	$\frac{1}{n} : o_2$	o_2	o_2	o_2
o_2	o_2	o_2	$0 : o_1$	o_2	o_3	$\frac{1}{n} : o_1$	o_3	o_3	o_3
o_3	o_3	o_3	o_3	o_3	o_4	o_3	o_4	o_4	o_4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
o_{n-1}	o_{n-1}	o_{n-1}	o_{n-1}	o_{n-1}	o_n	o_{n-1}	o_n	o_1	o_n
o_n	o_n	o_n	o_n	o_n	$0 : o_1$	o_n	o_1	o_n	o_1

In P, we have the uniform assignment (by envyfreeness and ex-post weak non-wastefulness). In P', i changes the ranking of her first two objects – by **1.** she now never receives o_1 and hence o_2 with probability $\frac{2}{n}$. The same holds as she further demotes o_1 in P''. Note that she still receives $\frac{1}{n}$ of each intermediate object (think of making one swap at a time). Now in P''' there is a change by the j's (one at a time) – by strategy-proofness each j that swaps o_1 and o_2, still receives $\frac{1}{n}$ of o_3 and below and by envyfreeness all others do so as well. Moreover, eventually once we are at P''', all agents receive o_2 with probability $\frac{1}{n}$. Finally, let all j demote o_1 to the penultimate position. By strategy-proofness and envyfreeness, they all still receive o_n with probability $\frac{1}{n}$ and by envyfreeness all agents receive objects o_2 to o_{n-1} with probability $\frac{1}{n}$. Hence we are at a uniform assignment.

In the same way that P and its uniform assignment served as a starting point, we now proceed from $P'''' =: \tilde{P}$ and arrive at $\tilde{P}''''$:

$\tilde{P}''''_j$	P''''_i
o_3	o_3
o_4	o_4
$\vdots$	$\vdots$
o_2	o_n
o_1	o_2
o_n	o_1

and find the associated assignment to be uniform. (Were we pushed o_1 to the last and second-to-last position, we now push object o_2 to the second to last and third-to-last position – other than that it's analogous.) Continuing in this way, we eventually arrive at

P_j^*	P_i^*
o_{n-1}	o_{n-1}
o_{n-2}	o_n
o_{n-3}	o_{n-2}
$\vdots$	$\vdots$
o_2	o_3
o_1	o_2
o_n	o_1

for which we again find that the assignment is uniform. In a final swap by i of o_n and o_{n-1}, the probability of o_{n-1} drops to zero (by **1.**) and thus i is assigned o_n with probability $\frac{2}{n}$ – all others with probability $\frac{n-2}{n(n-1)}$. By strategy-proofness and envyfreeness, this remains the case as i changes the order in which she ranks objects below o_n or the other agents change the order in which they rank objects above (now they may also differ in their rankings). In particular, agents in $N \backslash \{i\}$ may each rank a different object first. Then we still have i being assigned n with probability $\frac{2}{n}$ which serves as an upper bound on q-unanimity. ■

Remark 2 *Instead of q-unanimity, we may require exact q-unanimity – where for a "unanimous profile" each agent receives his top ranked object with (exact) probability q.*

Formally, given $q \in [0,1]$, a mechanism φ satisfies exact q-unanimity if for any profile R where there exists $\mu \in \mathcal{M}$ such that for all $i \in N$, $\mu_i R_i \mu'_i$ for all $\mu' \in \mathcal{M}$, then $\varphi_{i\mu_i}(R) = q$ for all $i \in N$.

Note that 1-unanimity and exact 1-unanimity are equivalent and exact q-unanimity implies q-unanimity. Hence, Proposition 1 and Theorem 3 remain unchanged when q-unanimity is replaced with exact q-unanimity (as RCD satisfies exact $\frac{2}{|N|}$-unanimity).

5 Discussion

5.1 Extensions of the Random Careless Dictator Mechanism

Below we provide an extension of RCD from the pure assignment domain to the full domain. This extension has the following two key features: (i) instead of choosing each agent uniformly as a dictator we choose the dictator uniformly among the agents who rank at least one real object acceptable and (ii) instead of dividing any remaining object equally among all other agents we divide any remaining object equally among all other agents who rank this object acceptable. This creates a certain budget of probability shares possibly exceeding one, and then any agent chooses his stochastically most preferred random assignment from it.

For any R_i, let $top(R_i)$ denote the most R_i-preferred element in $O \cup \{i\}$ and $A(R_i) = \{o \in O : oP_i i\}$ denote the set of acceptable objects under R_i.

Given $R \in \mathcal{R}^N$, let $N_+(R) = \{i \in N : A(R_i) \neq \emptyset\}$ denote set of agents who rank at least one real object acceptable.

Now for any $i \in N$ and $R \in \mathcal{R}^N$ such that $A(R_i) \neq \emptyset$, we define for dictator i the "budget" $\beta^i(R)$ on O as follows: (i) $\beta^i_{itop(R_i)}(R) = 1$ and $\beta^i_{ix}(R) = 0$ for all $x \in O \backslash \{top(R_i)\}$; and (ii) for all $j \in N \backslash \{i\}$, $\beta^i_{jx}(R) = 0$ for all $x \in O \backslash (A(R_j) \backslash \{top(R_i)\})$, and $\beta^i_{jx}(R) = \frac{1}{|\{l \in N \backslash \{i\} : x \in A(R_l)\}|}$ for all $x \in A(R_j) \backslash \{top(R_i)\}$. In the RCD ϕ on the full domain agent j's "total budget" for profile R is given by $\beta_j(R) \equiv \frac{1}{|N_+(R)|} \sum_{i \in N_+(R)} \beta^i_j(R)$ and any agent j receives his most preferred random assignment from his "total budget": if $o_1 P_j o_2 P_j o_3 \cdots o_k P_j j$ and $A(R_j) = \{o_1, \ldots, o_k\}$, then choose the largest index $l \in \{1, \ldots, k\}$ such that $\sum_{t=1}^{l} \beta_{jo_t}(R) \leq 1$, and then set $\phi_{jo_t}(R) = \beta_{jo_t}(R)$ for all $t = 1, \ldots, l$, and $\phi_{jo_{l+1}}(R) = 1 - \sum_{t=1}^{l} \beta_{jo_t}(R)$ (if $l+1 \leq k$), and, respectively, $\phi_{jj}(R) = 1 - \sum_{t=1}^{l} \beta_{jo_t}(R)$ (if $l + 1 > k$).

Importantly, if there are only two individuals who find at least one object acceptable, ϕ coincides with RSD. Note that RCD is strategy-proof, envyfree and ex-post weak efficient. Furthermore, as dictators determine the budget, any agent's budget is fixed (except for when he is the dictator) and this has some flavor as Gibbard (1977, 1978), who characterized convex combinations of dictatorships and duples, and Barberà and Jackson (1995) of trading along fixed prices.

Note that Theorem 2 does not extend to the full domain since, as we will show in the following subsection, every strategy-proof and envyfree mechanism will be wasteful at some profiles in $\mathcal{R}^N$.

Example 1

R_1	R_2	R_3
a	b	c
b		

Here agent 1 *will be assigned object* a *for sure under RCD – either as a dictator who gets to choose her most preferred object, or as the only one who finds object* a *acceptable after* 2 *or* 3 *have chosen as dictators. However, she will split object* b *with* 2 *after* 3 *has chosen as a dictator. Hence* 2 *is assigned* b *with probability less than one in which case both* 2 *and* b *remain unassigned – a violation of ex-post weak non-wastefulness.*

5.2 Ex-Post Weak Non-Wastefulness and Weak Unanimity

First, we consider the domain where being unassigned is not necessarily ranked at the bottom of an agent's preference relation. Obviously, Theorem 1 remains true on the domain $\mathcal{R}^N$ (as this is a superdomain of $\underline{\mathcal{R}}^N$). As we show below, on this domain we may replace unanimity with ex-post weak non-wastefulness without altering the conclusion of Theorem 1.

Corollary 1 *On the domain $\mathcal{R}^N$ for $|N| \geq 3$, there exists no mechanism which is ex-post weakly non-wasteful, strategy-proof and envyfree.*

Proof. By Theorem 1, it suffices to show that ex-post weak non-wastefulness and strategy-proofness imply unanimity.

Let φ satisfy ex-post weak non-wastefulness and strategy-proofness. Suppose that φ violates unanimity. Then for some profile R and $\mu \in \mathcal{M}$, we have $\mu_i R_i \mu'_i$ for all $i \in N$ and all $\mu' \in \mathcal{M}$ and yet φ does not attach probability one to μ. If i finds no object acceptable ex-post weak non-wastefulness implies $\varphi_{ii}(R) = 1$. So for there to be a violation of unanimity, there exists $j \in N$ such that $R_j : \mu_j \, R_j \, \ldots$ with $\mu_j \in O$ and $\varphi_{j\mu_j}(R) < 1$. Let $R'_j : \mu_j \, R_j \, j \, \ldots$ and $R' = (R'_j, R_{-j})$. By strategy-proofness, $\varphi_{j\mu_j}(R') = \varphi_{j\mu_j}(R) < 1$. Thus, by ex-post weak non-wastefulness, there exists $k \in N \backslash \{j\}$ with $\varphi_{k\mu_j}(R') > 0$. Hence, by ex-post weak non-wastefulness, $R_k : \mu_k \, R_k \ldots \mu_j \, R_k \ldots k$ and $\varphi_{k\mu_k}(R') < 1$. Then k in R' is in the role of j in R and we move to R'' where k is ranked second by k and k receives her most preferred object μ_k with probability less than one. Again this requires another agent l receiving μ_k with positive probability – continuing in this way, we arrive at a profile where all agents rank a different alternative first, consider at most one object acceptable and yet there is at least one agent who remains unassigned with positive probability – a violation of ex-post weak non-wastefulness. ■

The above corollary is the first impossibility result not involving any efficiency requirement and only an extremely weak ex-post non-wastefulness notion.

Now on the domain $\mathcal{R}^N$ one may even further weaken the notion of unanimity: a mechanism φ satisfies weak unanimity if for all R where there exists $\mu \in \mathcal{M}$ such that for all $i \in N$, both $\mu_i R_i \mu'_i$ and $i R_i \mu'_i$ for all $\mu' \in \mathcal{M}$ with $\mu'_i \neq \mu_i$, then $\varphi(R)$ attaches probability one to μ. In other words, if every agent finds at most one object acceptable and no two agents find the same object acceptable, then with probability one each agent who finds no object acceptable receives none while all others receive their only acceptable object.

Most importantly, this "efficiency" requirement is compatible with strategy-proofness and envyfreeness.

Proposition 2 *On the domain $\mathcal{R}^N$ for $|N| \geq 3$, the random careless dictator mechanism satisfies weak unanimity, strategy-proofness and envyfreeness.*

The above follows from the definition of the mechanism on that domain.

5.3 Increasing the Bound for q-Unanimity via Waste

Second, we return to the domain $\underline{\mathcal{R}}^N$ with three agents.

Instead, one may allow for wasting objects with positive probability, i.e. sometimes leaving an object unassigned. If this is possible, then mechanism ϕ below satisfies strategyproofness, envyfreeness, and q-unanimity for $q = \frac{5}{6} = Q_\phi$. Moreover, this mechanism is anonymous and neutral (meaning that ex-post weak non-wastefulness is indispensable in Proposition 1).

Example 2 *Let $N = \{1, 2, 3\}$ and $O = \{a, b, c\}$. For all $R \in \underline{\mathcal{R}}^N$, $\phi(R)$ is defined as follows (where $Q_\phi = \frac{5}{6}$ so that any agent is always unassigned with probability $\frac{1}{6}$):*

(1) If under R all agents rank different objects at the top, then each agent receives his top ranked object with probability $\frac{5}{6}$.

(2) If under R all agents rank the same object at the top, then each agent receives his top ranked object with probability $\frac{1}{3}$, his second ranked object with probability $\frac{1}{3}$ and his third ranked object with probability $\frac{1}{6}$.

(3) Otherwise two agents rank the same object at the top, say 1 and 2 rank a at the top, and agent 3 ranks b at the top; then agent 3 receives his top ranked object with probability $\frac{1}{2}$, his second ranked object with probability $\frac{1}{6}$ and his third ranked object with probability $\frac{1}{6}$; any of the agents ranking a first and ranking c second receives a with probability $\frac{5}{12}$ and c with probability $\frac{5}{12}$; and any agent ranking a first and ranking b second receives his top ranked object with probability $\frac{5}{12}$, his second ranked object with probability $\frac{1}{4}$ and his third ranked object with probability $\frac{1}{6}$.

It is straightforward that ϕ satisfies $\frac{5}{6}$-unanimity, envyfreeness, anonymity and neutrality. We check that ϕ satisfies satisfies strategyproofness: if R is in (1), then this is obvious; if R is in (3), then

- *agent 3 can only deviate to another profile in (3) (reversing a and c with no change or ranking c first with a worse outcome) or to a profile in (2) by ranking a first – but then he gets weakly less of his most preferred object (at most $\frac{1}{3}$ instead of $\frac{1}{2}$) and of his two most preferred objects (at most $\frac{2}{3}$ instead of $\frac{2}{3}$);*

- *agent 1 (or 2) who ranks c second can deviate to another profile in (3) by reversing the order of b and c (worsening his outcome) or by ranking b first (but then he gets only $\frac{5}{12}$ of his two most preferred objects a and c), or to a profile in (1) by ranking c first (but then he gets none of his most preferred object and still $\frac{5}{6}$ of his two most preferred objects);*

- *agent 1 (or 2) who ranks b second can deviate to another profile in (3) by reversing the order of b and c (worsening his outcome) or by ranking b first (but then gets $\frac{1}{4}$ or none of his most preferred object a and still at most $\frac{2}{3}$ of his two most preferred objects a and b), or to a profile in (1) by ranking by ranking c first – but then he gets none of his two most preferred objects a and b;*

if R is in (2), then an agent can only deviate to another profile in (2) by reversing the ranking of his second- and third-most preferred object (worsening his outcome) or to a profile in (3) (but then he receives the most preferred object with probability $\frac{1}{6} < \frac{1}{3}$ and the least preferred object with a probability of at least $\frac{1}{6}$).

Note that if we blew up all probabilities by factor $\frac{6}{5}$ in the above example, then every agent would be assigned with probability 1 – though at least at some profiles and for some objects the assignment probabilities would sum to more than 1. Thus, another way of looking at it is the following: a certain excess capacity allows to escape the impossibility of envyfreeness, strategy-proofness and unanimity – how much excess capacity is needed exactly? This is an open question left for further research.

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