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Three public goods and lexicographic preferences: replacement principle*

Lars Ehlers†

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Abstract

We study the problem of locating multiple public goods for a group of agents with single-peaked preferences over an interval. An alternative specifies for each public good a location. In Miyagawa (1998) each agent consumes only his most preferred public good without rivalry. We extend preferences lexicographically and characterize the class of rules satisfying *Pareto-optimality* and *replacement-domination*. We show that for three public goods, this results in a very similar characterization to Miyagawa (2001a): only the two rules which either always chooses the left-most *Pareto-optimal* alternative or always chooses the right-most *Pareto-optimal* alternative satisfy these properties. This is in contrast to Ehlers (2002) who showed that for two goods the corresponding characterization is substantially different to Miyagawa (2001a).

JEL classification: D71, D81, H41

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1 Introduction

We consider the problem of choosing multiple locations in an interval for an exogenously given number of identical public facilities. Each agent has a "single-peaked" preference relation over the interval and is allowed to select which public facility to use. An agent's preference relation is "single-peaked" if up to a certain point, his "peak", his welfare is strictly increasing, and it is strictly decreasing beyond that point. For example, a certain number of bus stops have to be located along a street. Other examples are gymnasiums, libraries, schools, telephone booths, and broadcasting news during a day. The planner faces the problem of choosing for each preference profile a list of locations.[1]

An economy is completely described by the set of feasible locations, the set of agents, their preferences, and the number of public goods. A solution is a systematic way to assign to each economy and each public facility a location. Moulin (1980) introduces this problem in the special case of one good chosen from a one-dimensional continuum and considers strategic issues.

The literature that follows extends his work in several directions.[2] However, until Miyagawa (1998) there was no axiomatic study of the problem of locating multiple public goods. Hotelling (1929) considers two competing businesses choosing where to locate on a street. He assumes that the businesses are identical and each individual patronizes only the one that is closest to where he lives. In Miyagawa (1998) an alternative specifies for each of the two public goods a location. Such a list is an *option set* and each agent compares two option sets by comparing their best elements

[1] In the latter example the planner has to choose a time schedule.

[2] Danilov (1994) and Schummer and Vohra (2002) allow the set of feasible locations to be a tree, and Border and Jordan (1983), and Zhou (1991) model it as a multi-dimensional continuum.

according to his preference relation over locations. We call this extension of single-peaked preferences from the set of possible locations to the set of alternatives its *max-extension*.

We contrast the max-extension with a different extension of preferences. The town government has to locate two public facilities, say two libraries, on a street. The libraries, though identical, only have one copy of each book. Then a certain book may have been already lent out and an individual that wants to borrow this book may have to drive to his second choice library. In most cases each individual visits his most preferred library, but sometimes both. Another example is the provision of telephone booths. Given two alternatives, first an agent compares the most preferred locations of each of the two alternatives, and if there is a tie, then he compares the other locations. We call this extension of single-peaked preferences the *lexicographic-extension*. Primarily, each agent uses the facility at his most preferred location, but he might be forced to consume the facility at his second choice location because the other facility is out of use.

As a basic requirement we impose *Pareto-optimality* meaning that the rule chooses for each preference profile an efficient alternative. In our model, *Pareto-optimality* is weaker than in Miyagawa (1998). If the smallest and the greatest peak of a preference profile are distinct, then each alternative which is efficient for the max-extension is also efficient for the lexicographic-extension.

We study the following notion of fairness. If the environment of an economy changes, then the welfare of all agents who are not responsible for this change are affected in the same direction: either all weakly gain or all weakly lose. As a variable parameter of an economy which may change over time, we consider preferences. Solidarity applied to such situations says that when the preference relation of an agent changes, then the welfare of all other agents are affected in the same direction. This replacement principle is called *welfare-domination under preference-replacement*, or simply *replacement-domination*. Moulin (1987) introduces *replacement-domination* in the context of binary choice with quasi-linear preferences. He calls it "agreement".[3]

For two pure public goods and the max-extension, Miyagawa (2001) shows that there are only two rules satisfying *Pareto-optimality* and *replacement-domination*. Ehlers (2002) shows for two public goods that each rule satisfying *Pareto-optimality* and *replacement-domination* is described by means of a continuous and single-peaked binary relation over the set of locations. For each preference profile such a rule chooses one location to be a most preferred peak in the peak profile according to the fixed single-peaked relation. The second location is indifferent to this peak according to the fixed single-peaked relation such that, if *Pareto-optimality* is not violated, the locations belong to opposite sides of the peak of the fixed relation. These rules are called single-peaked preference rules and are characterized by *Pareto-optimality* and *replacement-domination*.

Our main result shows that for the provision of three public goods, the result of Ehlers (2002) does not extend. For three public goods and the lexicographic extension, only two rules satisfy *Pareto-optimality* and *replacement-domination*, the *smallest-peak rule* and the *greatest-peak rule*. The smallest-peak rule chooses all three locations to be the smallest reported peak. The greatest-peak rule chooses all three locations to be the greatest reported peak. By locating one more facility, the single-peaked preference rules are restricted to only two rules. This result is similar to that of Miyagawa (2001a).

For different models of public good economies the rules satisfying *Pareto-optimality* and *replacement-domination* have been identified. For the provision of one public good, Thomson (1993), and Ehlers and Klaus (2001) characterize the class of rules satisfying these properties on closed intervals. Each of these rules is determined by a unique point, called the target point. A target rule chooses for each preference profile the efficient location that is closest to the target point. Vohra (1999) and Klaus (2001) characterize the same class of rules on tree networks. Klaus and Protopapas

[3]A review of the literature is by Thomson (1999).

(2020) characterize target correspondences for multi-valued location of one public good in an interval. For two pure public goods and the max-extension, Umezawa (2012) shows the incompatibility of *Pareto-optimality* and *replacement-domination* for tree networks. Gordon (2007a,b) shows an incompatibility for the location of one public good on a cycle. Harless (2015) considers small changes for one public good economies.

Another solidarity property is *population-monotonicity* which requires that the welfares of all agents are affected in the same direction when the population changes. For one public good, Ching and Thomson (1999) and Klaus (2001) characterize the class of target rules by *Pareto-optimality* and *population-monotonicity*. Gordon (2007a,b) consider cycles and a more general framework. For the location of two public goods and the max-extension of preferences (and the lexicographic extension, respectively), Miyagawa (2001b) (and Ehlers (2003), respectively) identifies a certain class of rules and shows that these rules are characterized by *Pareto-optimality* and *population-monotonicity*. Ehlers (2001) and Heo (2012) consider other properties for the location of two public goods and the max-extension.

The implications of other properties and settings of multiple public goods have been studied. Barberà and Beviá (2002, 2006) and Ju (2008) consider the location of public facilities when the agents are not free to choose the good they consume. Jackson and Nicoló (2004) and Bogomolnaia and Nicoló (2005) and Jackson and Nicoló (2005) allow for congestion. Bochet and Gordon (2012) and Bochet, Gordon and Saran (2013) consider the location multiple pure goods without congestion. Reffgen and Svensson (2012) and Alcalde-Unzu and Vorsatz (2018) consider other issues.

The organization of the paper is as follows. Section 2 introduces the general model and the axioms. Section 3 defines the max-extension and presents the main result of Miyagawa (2001a). Section 4 shows our main result for three public goods and the lexicographic-extension. Section 5 compares our result with Ehlers (2002). The Appendix contains the proof of our main result.

2 The Model

We follow Ehlers (2002). Let $N \equiv \{1, \ldots, n\}$, $n \in \mathbb{N}$, be the set of agents. Each agent $i \in N$ is equipped with a single-peaked and continuous preference relation R_i over $[0, 1]$. By I_i we denote the indifference relation associated with R_i, and by P_i the corresponding strict preference relation. *Single-peakedness* means that there exists a location, called the *peak* of R_i and denoted by $p(R_i)$, such that for all $x, y \in [0, 1]$, if $x < y \leq p(R_i)$ or $x > y \geq p(R_i)$, then yP_ix. By $\mathcal{R}$ we denote the set of all single-peaked preferences over $[0, 1]$, and by $\mathcal{R}^N$ the set of *(preference) profiles* $R \equiv (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. Given $S \subseteq N$, R_S denotes the restriction $(R_i)_{i \in S}$ of $R \in \mathcal{R}^N$ to S. Given $R \in \mathcal{R}^N$, $\underline{p}(R)$ denotes the smallest peak in the profile $(p(R_i))_{i \in N}$, and $\overline{p}(R)$ the greatest peak in the profile $(p(R_i))_{i \in N}$.

There is a fixed number, denoted by $m \in \mathbb{N}$, of identical public facilities. This number is exogenously given. Note that we do not exclude the case $m = 1$. For each of the facilities we have to select a location in the interval [0,1]. Let $M \equiv \{1, \ldots, m\}$. The order in which we locate the facilities is irrelevant as each agent has the freedom to choose the public good he prefers. An *alternative* is a m-tuple $x = (x_1, \ldots, x_m)$ such that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_m \leq 1$. Let $[0, 1]^M$ denote the set of all alternatives. We allow the possibility that for some $k, k' \in M$, $x_k = x_{k'}$.

A *(decision) rule* is a mapping φ that associates with each $R \in \mathcal{R}^N$ an alternative, denoted by $\varphi(R) = (\varphi_1(R), \ldots, \varphi_m(R))$. In other words a rule selects for each profile and each facility a location.

We extend preferences from the set of locations to the set of alternatives lexicographically. Each agent compares two alternatives via the *lexicographic preference relation* over $[0, 1]^M$ induced by his single-peaked preference relation over [0,1].

Lexicographic-Extension of Preferences: Let $i \in N$ and $R_i \in \mathcal{R}$. Given two alternatives $x, y \in [0, 1]^M$ and two permutations τ, ρ of M such that $x_{\tau(1)}R_ix_{\tau(2)}R_i \ldots R_ix_{\tau(m)}$ and $y_{\rho(1)}R_iy_{\rho(2)}R_i \ldots R_iy_{\rho(m)}$, *$x$ is lexicographically strictly preferred to y* if and only if there exists $h \in M$ such that for all $k < h$, $x_{\tau(k)} I_i y_{\rho(k)}$ and $x_{\tau(h)} P_i y_{\rho(h)}$. Furthermore, *$x$ is lexicographically indifferent to y* if and only if for all $k \in M$, $x_{\tau(k)} I_i y_{\rho(k)}$.

Abusing notation, we use the same symbols to denote preferences over possible locations and lexicographic preferences over alternatives. When we extend preferences lexicographically, weak upper contour sets are neither closed nor open, and non-convex. Furthermore, indifference sets only contain a finite number of alternatives.

Pareto-optimality says that for each preference profile the chosen alternative cannot be changed in such a way that no agent is worse off and some agent is better off relative to his lexicographic preference relation. Given $S \subseteq N$ and $R \in \mathcal{R}^N$, let $E(R_S)$ denote the set of *Pareto-optimal* (or efficient) alternatives for R_S. Formally, $E(R_S) \equiv \{y \in [0,1]^M \mid \text{for all } x \in [0,1]^M, \text{ if for some } i \in S, xP_iy, \text{ then for some } j \in S, yP_jx\}$.

Pareto-Optimality: For all $R \in \mathcal{R}^N$: $\varphi(R) \in E(R)$.

When $m \geq 2$, for *Pareto-optimality* to hold it is not sufficient that for each public good the selected location belongs to $[\underline{p}(R), \overline{p}(R)]$. For every chosen alternative it is necessary that each closed interval having as two endpoints two selected locations contains at least one peak. The straightforward proof is left to the reader.

Lemma 2.1 *Let φ be a rule. Then φ satisfies Pareto-optimality if and only if for all $R \in \mathcal{R}^N$ the following holds: (i) for all $k \in M$, $\varphi_k(R) \in [\underline{p}(R), \overline{p}(R)]$, and (ii) for all $k \in \{1, \ldots, m-1\}$, there exists $i \in N$ such that $p(R_i) \in [\varphi_k(R), \varphi_{k+1}(R)]$.*

By Lemma 2.1, the set of efficient alternatives depends only on the peaks of the profile. Finally, we prove that for any two efficient alternatives, if all agents are indifferent between them, then the two alternatives are the same.

Lemma 2.2 *For all $S \subseteq N$, all $R \in \mathcal{R}^N$, and all $x, y \in E(R_S)$, if for all $i \in S$, xI_iy, then $x = y$.*

Proof. Let $j \in S$ be such that $p(R_j) = \min_{i \in S} p(R_i)$. Since $x, y \in E(R_S)$, $p(R_j) \leq x_1 \leq x_2 \leq \cdots \leq x_m$ and $p(R_j) \leq y_1 \leq y_2 \leq \cdots \leq y_m$. Because xI_jy, it follows that for all $k \in M$, $x_k = y_k$. Hence, $x = y$. $\square$

2.1 Replacement-Domination

The solidarity property we discuss is *welfare-domination under preference-replacement*, or for short *replacement-domination*, introduced by Moulin (1987). It requires that when the preference relation of some agent changes, the welfare of all other agents are affected in the same direction.

Replacement-Domination: For all $j \in N$ and all $R, \bar{R} \in \mathcal{R}^N$ such that $R_{N\setminus\{j\}} = \bar{R}_{N\setminus\{j\}}$: either [for all $i \in N\setminus\{j\}$, $\varphi(R)R_i\varphi(\bar{R})$] or [for all $i \in N\setminus\{j\}$, $\varphi(\bar{R})R_i\varphi(R)$].

We state some implications of *Pareto-optimality* and *replacement-domination*. These implications are very general and not specific to our model. They hold also in the public good models of Thomson (1993) and Vohra (1999), and when we use the max-extension (defined below) in our model.

First, if the preference relation of some agent changes and the choices of the rule at the initial and new profile are *Pareto-optimal* for the profile consisting of the remaining agents' preferences, then the same alternative is chosen for both profiles.

Lemma 2.3 *Let φ be a rule satisfying Pareto-optimality and replacement-domination. Let $j \in N$ and $R, \bar{R} \in \mathcal{R}^N$ be such that $R_{N\setminus\{j\}} = \bar{R}_{N\setminus\{j\}}$. If $\varphi(R), \varphi(\bar{R}) \in E(R_{N\setminus\{j\}})$, then $\varphi(R) = \varphi(\bar{R})$.*

Proof. By *replacement-domination*, [for all $i \in N\setminus\{j\}$, $\varphi(R)R_i\varphi(\bar{R})$] or [for all $i \in N\setminus\{j\}$, $\varphi(\bar{R})R_i\varphi(R)$]. Since $\varphi(R), \varphi(\bar{R}) \in E(R_{N\setminus\{j\}})$, it follows that for all $i \in N\setminus\{j\}$, $\varphi(R)I_i\varphi(\bar{R})$. Hence, by Lemma 2.2, $\varphi(R) = \varphi(\bar{R})$. $\square$

Second, if the preference relation of some agent changes and all *Pareto-optimal* alternatives at the new profile are also efficient for the profile consisting of the remaining agents' preferences, then all these agents weakly prefer the alternative chosen by the rule for the new profile to the initially chosen alternative.

Lemma 2.4 *Let φ be a rule satisfying Pareto-optimality and replacement-domination. Let $j \in N$ and $R, \bar{R} \in \mathcal{R}^N$ be such that $R_{N\backslash\{j\}} = \bar{R}_{N\backslash\{j\}}$. If $E(\bar{R}) = E(R_{N\backslash\{j\}})$, then for all $i \in N\backslash\{j\}$, $\varphi(\bar{R})R_i\varphi(R)$.*

Proof. By *replacement-domination*, either [for all $i \in N\backslash\{j\}$, $\varphi(\bar{R})R_i\varphi(R)$] or [for all $i \in N\backslash\{j\}$, $\varphi(R)R_i\varphi(\bar{R})$]. Suppose that the assertion of Lemma 2.4 does not hold. Thus, for all $i \in N\backslash\{j\}$, $\varphi(R)R_i\varphi(\bar{R})$, and for some $t \in N\backslash\{j\}$, $\varphi(R)P_t\varphi(\bar{R})$. Because $E(\bar{R}) = E(R_{N\backslash\{j\}})$, $\varphi(\bar{R}) \in E(R_{N\backslash\{j\}})$. The previous two facts constitute a contradiction. $\square$

3 The Max-Extension

Preferences are defined over [0,1]. When $m \geq 2$ the set of alternatives and the set of locations are distinct. Agents consume the pure public goods without rivalry and each agent uses only the good located at his best point. An example is the location of bus stops. Using this motivation, Miyagawa (1998) extends preferences from locations to alternatives as follows. Given two alternatives, an agent strictly prefers the first alternative to the second if he strictly prefers the best location of the first alternative to the best location of the second relative to his single-peaked preference relation over locations. Each alternative is an *option set* from which agents can freely select, and assume that each agent compares two alternatives by comparing the locations that he prefers.

Max-Extension of Preferences, R_i^{max}: Let $i \in N$ and $R_i \in \mathcal{R}$. Given two alternatives $x, y \in [0,1]^M$, *x is maximally strictly preferred to y*, $xP_i^{max}y$, if and only if for some $\bar{k} \in M$ and all $k \in M$, $x_{\bar{k}} P_i y_k$. Furthermore, x *is maximally indifferent to* y, $x I_i^{max} y$, if and only if for some $\bar{k}, \tilde{k} \in M$ and all $k \in M$, $x_{\bar{k}} R_i y_k$ and $y_{\tilde{k}} R_i x_k$.

When we extend preferences maximally, weak upper contour sets[4] are closed but not convex.

The first requirement says that for each preference profile the chosen alternative cannot be changed in such a way that no agent is worse off and some agent is better off. Given $N \in \mathcal{P}$ and $R \in \mathcal{R}^N$, let $E(R^{max})$ denote the set of *Pareto-optimal* (or efficient) alternatives for R when preferences are extended maximally. Formally, $E(R^{max}) \equiv \{y \in [0,1]^M \,|\, \text{for all } x \in [0,1]^M, \text{ if for some } i \in N,\ x P_i^{max} y, \text{ then for some } j \in N,\ y P_j^{max} x\}$.

Pareto-Optimality: For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, $\varphi(R) \in E(R^{max})$.

When $m \geq 2$, for *Pareto-optimality* it is necessary but not sufficient that for each public good the selected location belongs to $[\underline{p}(R), \overline{p}(R)]$. When $m = 1$ for *Pareto-optimality* it is also sufficient that for each profile the chosen location belongs to the peaks interval of that profile.

Remark 3.1 For all $R \in \mathcal{R}^N$, let $E(R^{max})$ denote the set of *Pareto-optimal* alternatives in $[0,1]^M$ when we extend preferences maximally. It is easy to see that for all $x \in [0,1]^M$, $x \in E(R^{max})$ if and only if (i) $x_1, x_2 \in [\underline{p}(R), \overline{p}(R)]$ and (ii) for some $i, j \in N$, $p(R_i), p(R_j) \in [x_1, x_2]$, $x_1 P_i x_2$, and $x_2 P_j x_1$. For the lexicographic-extension of preferences, *Pareto-optimality* is weaker than for the max-extension. For all $R \in \mathcal{R}^N$ such that $\underline{p}(R) < \overline{p}(R)$, $E(R^{max}) \subset E(R)$. Generally the set $E(R)$ is considerably larger than $E(R^{max})$. For example, let $R \in \mathcal{R}^N$ be such that $\{p(R_i) \,|\, i \in N\} = \{0, 1\}$. Then $E(R^{max}) = \{(0,1)\} \subset ([0,1] \times \{1\}) \cup (\{0\} \times [0,1]) = E(R)$. $\triangleleft$

[4]Given $i \in N$, $R_i \in \mathcal{R}$, and $x \in [0,1]^M$, the set $B(x, R_i) = \{y \in [0,1]^M \,|\, y R_i^{max} x\}$ is the *weak upper contour set of* x *at* R_i^{max}.

Below let $M \equiv \{1, 2\}$. The following two rules satisfy *replacement-domination*. One is the rule that chooses for each profile the two smallest distinct peaks, and the other is the rule that chooses for each profile the two greatest distinct peaks.

Left-Peaks Rule, φ^L: For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, if $\underline{p}(R) = \overline{p}(R)$, then $\varphi^L(R) \equiv (\underline{p}(R), \underline{p}(R))$, and otherwise, $\varphi^L(R) \equiv (\underline{p}(R), \min\{p(R_j) \,|\, j \in N \text{ and } \underline{p}(R) < p(R_j)\})$.

Right-Peaks Rule, φ^G: For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, if $\underline{p}(R) = \overline{p}(R)$, then $\varphi^G(R) \equiv (\overline{p}(R), \overline{p}(R))$, and otherwise, $\varphi^G(R) \equiv (\max\{p(R_j) \,|\, j \in N \text{ and } p(R_j) < \overline{p}(R)\}, \overline{p}(R))$.

It turns out that for populations with at least four agents these two rules are the only ones satisfying in addition *Pareto-optimality*. Let $\mathcal{V}(N) \subseteq \mathcal{R}^N$ denote the set of all preference profiles $R \in \mathcal{R}^N$ such that $\underline{p}(R) < \overline{p}(R)$.

Theorem 3.2 (Miyagawa, 2001a) *Let $M = \{1, 2\}$ and $|N| \geq 4$. On the domain $\mathcal{V}(N)$ the left-peaks rule and the right-peaks rule are the only rules satisfying Pareto-optimality and replacement-domination for the max-extension of preferences.*

Finally we discuss the location of three public facilities. The above result of Miyagawa (2001a) generalizes to these cases as follows:[5] If $|N| \geq 5$, then a rule satisfies *Pareto-optimality* and *replacement-domination* with respect to the max-extension if and only if either for all profiles the three different smallest peaks are chosen, or for all profiles the three different greatest peaks are chosen.

[5]Personal communication with E. Miyagawa at the *Fourth International Meeting of the Society for Social Choice and Welfare*, 1998, Vancouver, BC, Can.

4 Lexicographic Extension: Three Public Goods

Below we consider the provision of three public goods, i.e., $M = \{1, 2, 3\}$.

In our model the left-peaks rule and the right-peaks rule "correspond" to the smallest peak rule and the greatest-peak rule. The smallest-peak rule always locates all public goods at the smallest peak, and the greatest-peak rule always at the greatest peak. Both of these rules satisfy *Pareto-optimality* and *replacement-domination*.

Smallest-Peak Rule, $\underline{\phi}$: For all $R \in \mathcal{R}^N$ and all $k \in M$, $\underline{\phi}_k(R) \equiv \underline{p}(R)$.

Greatest-Peak Rule, $\overline{\phi}$: For all $R \in \mathcal{R}^N$ and all $k \in M$, $\overline{\phi}_k(R) \equiv \overline{p}(R)$.

Theorem 4.1 *Let $M = \{1, 2, 3\}$ and $|N| \geq 5$. Then the smallest-peak rule and the greatest-peak rule are the only rules satisfying Pareto-optimality and replacement-domination for the lexicographic extension of preferences.*

Remark 4.2 Our two rules and Miyagawa's rules have the following features in common. For all preference profiles, the left-peaks rule chooses the unique left-most *Pareto-optimal* alternative relative to the max-extension, and the smallest-peak rule chooses the unique left-most *Pareto-optimal* alternative relative to the lexicographic-extension. By this we mean that for all $R \in \mathcal{R}^N$, for all $x \in E(R^{max}) \setminus \{\varphi^L(R)\}$, $L_1(R) \leq x_1$ and $L_2(R) < x_2$, and for all $y \in E(R) \setminus \{\underline{\phi}(R)\}$, $\underline{\phi}_1(R) \leq y_1$ and $\underline{\phi}_2(R) < y_2$. Similarly, the right-peaks rule and the greatest-peak rule choose always the unique right-most *Pareto-optimal* alternative relative to each extension. All these rules choose only the extreme *Pareto-optimal* alternatives. $\triangleleft$

The Appendix contains the proof of Theorem 4.1.[6]

In Theorem 4.1 it is not possible to weaken *Pareto-optimality* to *unanimity*. The latter property requires that when all agents have the same preference, the rule locates

[6]It is an open question whether in Theorem 4.1 the restriction $n \geq 5$ is tight or not.

all three public goods at the common peak. The following example establishes the previous fact.

Example 4.3 Let $c \in [0,1]$. For all $R \in \mathcal{R}^N$ and all $m \in M$, $\tilde{\varphi}_m(R) \equiv med(\underline{p}(R), c, \overline{p}(R))$. The rule $\tilde{\varphi}$ satisfies *unanimity* and *replacement-domination*, but not *Pareto-optimality*. ◦

The above example is simply the same "target rule" (Thomson, 1993; Vohra, 1999) applied to all three locations of the public goods.

5 Discussion

For two public goods and the lexicographic extension, Ehlers (2002) showed that each rule satisfying *Pareto-optimality* and *replacement-domination* is described by a continuous and single-peaked binary relation over $[0,1]$. Before we formally define these rules, we introduce an equivalent representation of a single-peaked preference relation over $[0,1]$.

Let $R_0 \in \mathcal{R}$. Then $0R_0 1$ or $1P_0 0$. Suppose that $0R_0 1$. Since R_0 is continuous, for some $b \in [p(R_0),1]$, $0I_0 b$. For all $x \in [0,b]$, let $f(x) \in [0,b]$ be such that $xI_0 f(x)$ and the following holds: (i) when $x \leq p(R_0)$, $f(x) \geq p(R_0)$, and (ii) when $x \geq p(R_0)$, $f(x) \leq p(R_0)$. Because R_0 is continuous, it follows that f is continuous. Therefore, with R_0 we associate a unique function $f : [0,b] \to [0,b]$ such that f is continuous, $f = f^{-1}$ (this follows from R_0 being a preference relation), and f is strictly decreasing (this follows from single-peakedness of R_0). In particular, f possesses as a unique fixed point $p(R_0)$, i.e. $f(p(R_0)) = p(R_0)$.[7] Furthermore, associated with such a function is a unique single-peaked preference relation on [0,1].

Let $f : [0,b] \to [0,b]$ (or, alternatively, $f : [b,1] \to [b,1]$) be a continuous strictly decreasing function such that $f(0) = b$ ($f(b) = 1$) and $f = f^{-1}$ (f is symmetric). Denote by a its unique fixed point and by $\mathcal{F}$ the set of all such functions.

[7]If $1P_0 0$, then we associate with R_0 a function $f : [b,1] \to [b,1]$ where $bI_0 1$.

Single-Peaked Preference Rules, ϕ^f: Given $f \in \mathcal{F}$ (where $f(p(R_0)) = p(R_0)$), the *single-peaked preference rule* ϕ^f based on f is defined as follows. For all $R \in \mathcal{R}^N$ such that $p(R_{i_1}) \leq \cdots \leq p(R_{i_n})$,

- if $p(R_0) \notin [\underline{p}(R), \overline{p}(R)]$, then

$$\phi^f(R) \equiv \begin{cases} (\underline{p}(R), \underline{p}(R)) & \text{when } p(R_0) < \underline{p}(R), \\ (\overline{p}(R), \overline{p}(R)) & \text{when } \overline{p}(R) < p(R_0). \end{cases}$$

- if $p(R_{i_l}) \leq p(R_0) \leq p(R_{i_{l+1}})$, then

$$\phi^f(R) \equiv \begin{cases} (p(R_{i_l}), f(p(R_{i_l}))) & \text{when } f(p(R_{i_l})) \leq p(R_{i_{l+1}}), \\ (f(p(R_{i_{l+1}})), p(R_{i_{l+1}})) & \text{otherwise.} \end{cases}$$

The main result of Ehlers (2002) shows that for two public goods, if N contains at least 3 agents, then every decision rule satisfying *Pareto-optimality* and *replacement-domination* is a single-peaked preference rule.

Theorem 5.1 (Ehlers, 2002, Theorem 3.3) *Let $M = \{1, 2\}$ and $|N| \geq 3$. Then the single-peaked preference rules are the only rules satisfying Pareto-optimality and replacement-domination for the lexicographic-extension of preferences.*

For two public goods, $\underline{\phi}$ is the single-peaked preference rule when the peak of the social planner is at 0, and $\overline{\phi}$ is the rule when the peak of the social planner is at 1. The corresponding functions $\underline{f}, \overline{f} \in \mathcal{F}$ are given by $\underline{f} : [0, 0] \to [0, 0]$ and $\overline{f} : [1, 1] \to [1, 1]$. Then for two public goods, $\underline{\phi} = \phi^{\underline{f}}$ and $\overline{\phi} = \phi^{\overline{f}}$.

As we have shown the above theorem does not generalize to the provision of three public goods. In that case, if N contains at least five agents, then *Pareto-optimality* and *replacement-domination* admit only the smallest peak rule and the greatest-peak rule. By providing one more good, the single-peaked preference rules are restricted to only two rules.

6 Appendix

Throughout let $n \geq 5$, $m = 3$, and φ be a rule satisfying *Pareto-optimality* and *replacement-domination*.

The following lemma follows from successive applications of Lemma 2.3.

Lemma 6.1 *Let $R \in \mathcal{R}^N$ be such that $|\{p(R_i) \,|\, i \in N\}| \leq 4$. For all $\bar{R} \in \mathcal{R}^N$, if $\{p(\bar{R}_i) \,|\, i \in N\} = \{p(R_i) \,|\, i \in N\}$, then $\varphi(\bar{R}) = \varphi(R)$.*

We show in two lemmas that for any preference profile, the open interval having as endpoints the smallest and the greatest assigned location contains no peak.

Lemma 6.2 *For all $R \in \mathcal{R}^N$ and all $i \in N$, $p(R_i) \notin]\varphi_1(R), \varphi_2(R)[\cup]\varphi_2(R), \varphi_3(R)[$.*

Proof. Suppose that for some $R \in \mathcal{R}^N$ and some $j \in N$, $p(R_j) \in]\varphi_1(R), \varphi_2(R)[\cup]\varphi_2(R), \varphi_3(R)[$. Without loss of generality, we suppose that

$$p(R_j) \in]\varphi_1(R), \varphi_2(R)[. \tag{1}$$

Because $n \geq 5$, there exists $t \in N \backslash \{j\}$ such that $\underline{p}(R_{N\backslash\{t\}}) = \underline{p}(R)$, $\overline{p}(R_{N\backslash\{t\}}) = \overline{p}(R)$, and $\varphi(R) \in E(R_{N\backslash\{t\}})$. Let $R' \in \mathcal{R}^N$ be such that $R'_{N\backslash\{t\}} = R_{N\backslash\{t\}}$ and $p(R'_t) = \varphi_3(R)$. We show that $\varphi(R') = \varphi(R)$. By *replacement-domination*, there are two cases.

If for all $i \in N\backslash\{t\}$, $\varphi(R) R_i \varphi(R')$, then either [$\varphi_3(R') < \varphi_3(R)$ and $\varphi(R') \in E(R_{N\backslash\{t\}})$ which implies together with Lemma 2.3, $\varphi(R') = \varphi(R)$,] or [$\varphi_3(R') = \varphi_3(R)$, $\varphi_2(R') \leq \varphi_2(R)$ and $\varphi(R') \in E(R_{N\backslash\{t\}})$ which implies together with Lemma 2.3, $\varphi(R') = \varphi(R)$].

If for all $i \in N\backslash\{t\}$, $\varphi(R') R_i \varphi(R)$, then $\varphi_1(R') \leq \varphi_1(R)$ and $\varphi_3(R) \leq \varphi_3(R')$. Suppose that $\varphi(R') \neq \varphi(R)$. Then $\varphi_1(R') < \varphi_1(R)$ or $\varphi_3(R) < \varphi_3(R')$. By (1), $\varphi_1(R') < \varphi_1(R)$ implies $\varphi_2(R') < \varphi_2(R)$ and $\varphi_3(R) < \varphi_3(R')$. By our choice of t, $\varphi(R) \in E(R_{N\backslash\{t\}})$. Thus, for some $l \in N\backslash\{t\}$, $p(R_l) \in [\varphi_2(R), \varphi_3(R)]$ and

$\varphi(R)P_l\varphi(R')$, which contradicts *replacement-domination*. Thus, this case cannot occur. When $\varphi_3(R) < \varphi_3(R')$, similar arguments yield a contradiction to *replacement-domination*.

Without loss of generality, we suppose that $p(R'_1) = \underline{p}(R')$, $p(R'_2) = \varphi_3(R')$. Let $l \notin \{1,2,j\}$. Let $R'' \in \mathcal{R}^N$ be such that $R''_{N\setminus\{l\}} = R'_{N\setminus\{l\}}$ and $p(R''_l) = \varphi_1(R')$. Using the above arguments it follows that $\varphi(R'') = \varphi(R')$.

Let $\bar{R} \in \mathcal{R}^N$ be such that $\bar{R}_1 = R''_l$, $\bar{R}_2 = R''_2$, and for all $i \in N\setminus\{1,2\}$, $\bar{R}_i = R''_j$. Successive applications of Lemma 2.3 imply $\varphi(\bar{R}) = \varphi(R'')$. Let $x \in]\varphi_1(\bar{R}), p(\bar{R}_3)[$ be such that $x\bar{P}_3\varphi_2(\bar{R})$. Let $\bar{R}^x \in \mathcal{R}^N$ be such that $\bar{R}^x_{N\setminus\{4\}} = \bar{R}_{N\setminus\{4\}}$ and $p(\bar{R}^x_4) = x$. We show that $\varphi(\bar{R}^x) = \varphi(\bar{R})$.

If $\varphi(\bar{R}^x) \in E(\bar{R}_{N\setminus\{4\}})$, then Lemma 2.3 implies $\varphi(\bar{R}^x) = \varphi(\bar{R})$. Suppose that $\varphi(\bar{R}^x) \notin E(\bar{R}_{N\setminus\{4\}})$. Hence, by *Pareto-optimality*, $p(\bar{R}^x_4) \in [\varphi_1(\bar{R}^x), \varphi_2(\bar{R}^x)] \subseteq]p(\bar{R}_1), p(\bar{R}_3)[$ or $p(\bar{R}^x_4) \in [\varphi_2(\bar{R}^x), \varphi_3(\bar{R}^x)] \subseteq]p(\bar{R}_1), p(\bar{R}_3)[$. In both cases, by our choice of $p(\bar{R}^x_4)$, $\varphi(\bar{R}^x)\bar{P}_3\varphi(\bar{R})$. Then by *replacement-domination*, for all $i \in N\setminus\{4\}$, $\varphi(\bar{R}^x)\bar{R}_i\varphi(\bar{R})$. Hence, $\varphi_3(\bar{R}^x) = \varphi_3(\bar{R})$ and $\varphi_2(\bar{R}^x) < p(\bar{R}_3) < \varphi_2(\bar{R})$. Thus, $\varphi(\bar{R})\bar{P}_2\varphi(\bar{R}^x)$, which contradicts *replacement-domination*.

Hence, $\varphi(\bar{R}^x) = \varphi(\bar{R})$. To summarize,

$$\varphi_1(\bar{R}^x) = \underline{p}(\bar{R}), p(\bar{R}_3) \in]\varphi_1(\bar{R}^x), \varphi_2(\bar{R}^x)[, \text{ and } \varphi_3(\bar{R}^x) = \overline{p}(\bar{R}). \tag{2}$$

Note that $p(\bar{R}^x_3) = p(\bar{R}^x_5)$ and $|\{p(\bar{R}^x_i) \,|\, i \in N\}| \leq 4$. By Lemma 6.1, we may assume that

$$\varphi_2(\bar{R}^x)\bar{P}^x_5 p(\bar{R}^x_4). \tag{3}$$

Let $\tilde{R}^x \in \mathcal{R}^N$ be such that $\tilde{R}^x_{N\setminus\{1\}} = \bar{R}^x_{N\setminus\{1\}}$ and $\tilde{R}^x_1 = \bar{R}^x_3$. By Lemma 2.4, for all $i \in N\setminus\{1\}$,

$$\varphi(\tilde{R}^x)\bar{R}^x_i\varphi(\bar{R}^x). \tag{4}$$

Thus, $\varphi_3(\tilde{R}^x) = \varphi_3(\bar{R}^x)$. We show that the following.

Claim 0: $\varphi_1(\tilde{R}^x) = \underline{p}(\tilde{R}^x) = x$.

Proof of Claim 0: Suppose that

$$x < \varphi_1(\tilde{R}^x). \tag{5}$$

Let $R'_4 \in \mathcal{R}$ be such that $\varphi_1(\bar{R}^x)P'_4\varphi_1(\tilde{R}^x)$, $\varphi_1(\bar{R}^x)P'_4\varphi_2(\bar{R}^x)$ and $p(R'_4) = x$. By Lemma 6.1, we have both $\varphi(R'_4, \bar{R}^x_{N\setminus\{4\}}) = \varphi(\bar{R}^x)$ and $\varphi(R'_4, \tilde{R}^x_{N\setminus\{4\}}) = \varphi(\tilde{R}^x)$. By construction, $\varphi(R'_4, \bar{R}^x_{N\setminus\{4\}})P'_4\varphi(R'_4, \tilde{R}^x_{N\setminus\{4\}})$. Since $E(\tilde{R}^x) = E(\tilde{R}^x_{N\setminus\{1\}})$ and both $E(R'_4, \bar{R}^x_{N\setminus\{4\}}) = E(\bar{R}^x)$ and $E(R'_4, \tilde{R}^x_{N\setminus\{4\}}) = E(\tilde{R}^x)$, this is a contradiction to Lemma 2.4. $\diamond$

Since $\varphi_1(\tilde{R}^x) = \underline{p}(\tilde{R}^x) = x = p(\bar{R}^x_4)$, (3) and (4) imply, $\varphi_2(\tilde{R}^x) = \varphi_2(\bar{R}^x)$. To summarize,

$$\varphi_1(\tilde{R}^x) = x,\ \varphi_2(\tilde{R}^x) = \varphi_2(\bar{R}^x),\ \text{and } \varphi_3(\tilde{R}^x) = \overline{p}(\bar{R}). \tag{6}$$

Let $\hat{R}^x \in \mathcal{R}^N$ be such that $\hat{R}^x_{N\setminus\{2\}} = \bar{R}^x_{N\setminus\{2\}}$ and $\hat{R}^x_2 = \bar{R}^x_3$. By Lemma 2.4, for all $i \in N\setminus\{2\}$, $\varphi(\hat{R}^x)\bar{R}^x_i\varphi(\bar{R}^x)$. Thus, $\varphi_1(\hat{R}^x) = \underline{p}(\bar{R})$. By (2) and using the same argument as in Claim 0, we obtain $\varphi_3(\hat{R}^x) = \overline{p}(\hat{R}^x) = p(\bar{R}_3)$. By *Pareto-optimality*, $\varphi_2(\hat{R}^x) \in [\underline{p}(\hat{R}^x), \overline{p}(\hat{R}^x)]$. We show that without loss of generality we may assume that

$$\varphi_2(\hat{R}^x) \neq x. \tag{7}$$

If $\varphi_2(\hat{R}^x) = x$, then

$$\varphi_1(\hat{R}^x) = \underline{p}(\bar{R}), \varphi_2(\hat{R}^x) = x,\ \text{and } \varphi_3(\hat{R}^x) = p(\bar{R}_3). \tag{8}$$

Let $y = \frac{1}{2}(x + p(\bar{R}_3))$ and consider the profiles $\bar{R}^y$ and $\hat{R}^y$ which we define in the same way as $\bar{R}^x$ and $\hat{R}^x$. Using the same arguments as for $\bar{R}^x$ and $\hat{R}^x$, (2) holds for $\bar{R}^y$, and the first and last equality of (8) hold for $\hat{R}^y$. If $\varphi_2(\hat{R}^y) = y$, then the previous fact and (8) together with Lemma 6.1 contradict *replacement-domination*. Hence, $\varphi_2(\hat{R}^y) \neq y$ and instead of x we use y for the definitions of $\bar{R}^x$, $\tilde{R}^x$, and $\hat{R}^x$.

We assume that (7) holds. Then

$$\varphi_1(\hat{R}^x) = \underline{p}(\bar{R}), \varphi_2(\hat{R}^x) \neq x,\ \text{and } \varphi_3(\hat{R}^x) = p(\bar{R}_3). \tag{9}$$

Let $\sigma : N \to N$ be the permutation such that for all $i \in N\backslash\{1,2\}$, $\sigma(i) = i$, $\sigma(1) = 2$, and $\sigma(2) = 1$. By Lemma 6.1, $\varphi(\sigma(\tilde{R}^x)) = \varphi(\tilde{R}^x)$.[8] By the previous fact, (6), and (9), $\varphi(\sigma(\tilde{R}^x))\hat{P}_4^x\varphi(\hat{R}^x)$ and $\varphi(\hat{R}^x)\hat{P}_3^x\varphi(\sigma(\tilde{R}^x))$. Since $\sigma(\tilde{R}^x)_{N\backslash\{1\}} = \hat{R}^x_{N\backslash\{1\}}$, the previous relations contradict *replacement-domination*. □

Lemma 6.3 *For all $R \in \mathcal{R}^N$ and all $i \in N$, $p(R_i) \notin]\varphi_1(R), \varphi_3(R)[$.*

Proof. Suppose that for some $R \in \mathcal{R}^N$ and some $j \in N$, $p(R_j) \in]\varphi_1(R), \varphi_3(R)[$. By Lemma 6.2, $p(R_j) = \varphi_2(R)$. Without loss of generality, we suppose that $p(R_1) = \underline{p}(R)$ and $p(R_2) = \overline{p}(R)$. Let $\bar{R} \in \mathcal{R}^N$ be such that $\bar{R}_{\{1,2,j\}} = R_{\{1,2,j\}}$ and for all $i \in N\backslash\{1,2,j\}$, $\bar{R}_i = R_j$. Successive applications of Lemma 2.4 yield $\varphi(\bar{R}) = \varphi(R)$. Let $\bar{R}^1 \in \mathcal{R}^N$ be such that $\bar{R}^1_{N\backslash\{2\}} = \bar{R}_{N\backslash\{2\}}$ and $\bar{R}^1_2 = \bar{R}_j$, and $\bar{R}^2 \in \mathcal{R}^N$ be such that $\bar{R}^2_{N\backslash\{1\}} = \bar{R}^1_{N\backslash\{1\}}$ and $\bar{R}^2_1 = \bar{R}_2$. Using the same arguments as in the proof of Claim 0 we can show that

$$\varphi_1(\bar{R}^1) < \varphi_2(\bar{R}^1) = \varphi_3(\bar{R}^1) = p(\bar{R}^j),$$

and

$$p(\bar{R}_j) = \varphi_1(\bar{R}^2) = \varphi_2(\bar{R}^2) < \varphi_3(\bar{R}^2) = p(\bar{R}^j).$$

By Lemma 6.1 and the two previous facts, the same argument as at the end of the proof of Lemma 6.2 yields a contradiction to *replacement-domination*. □

Next, we show that for any preference profile the assigned locations of all public goods are equal.

Lemma 6.4 *For all $R \in \mathcal{R}^N$, $|\{\varphi_k(R) \,|\, k \in M\}| = 1$.*

Proof. Suppose that for some $R \in \mathcal{R}^N$, $|\{\varphi_k(R) \,|\, k \in M\}| \geq 2$. Then $\varphi_1(R) < \varphi_3(R)$ and by *Pareto-optimality*, for some $j \in N$, $p(R_j) \in [\varphi_1(R), \varphi_3(R)]$. By Lemma 6.3, $p(R_j) = \varphi_1(R)$ or $p(R_j) = \varphi_3(R)$. Without loss of generality, we suppose that $p(R_1) = \underline{p}(R)$ and $p(R_2) = \overline{p}(R)$. We consider five cases.

[8] For any profile $R \in \mathcal{R}^N$, let $\sigma(R) \equiv (R_{\sigma(i)})_{i \in N}$ denote the profile R renamed according to σ.

Case 1: For some $j \in N$, $\underline{p}(R) < \varphi_1(R) < \varphi_2(R) = \varphi_3(R) = p(R_j) < \overline{p}(R)$ and for all $i \in N$, $p(R_i) \neq \varphi_1(R)$.

Let $\bar{R} \in \mathcal{R}^N$ be such that $\bar{R}_{\{1,2,j\}} = R_{\{1,2,j\}}$ and for all $i \in N \backslash \{1, 2, j\}$, $\bar{R}_i = R_1$. Successive applications of Lemma 2.3 yield $\varphi(\bar{R}) = \varphi(R)$. Given $x \in]\varphi_1(R), \varphi_2(R)[$, let $\bar{R}^x \in \mathcal{R}^N$ be such that $\bar{R}^x_{N\backslash\{j\}} = \bar{R}_{N\backslash\{j\}}$ and $p(\bar{R}^x_j) = x$. We prove the following claim.

Claim 1: For all $x \in]\varphi_1(\bar{R}), \varphi_2(\bar{R})[$, $|\{k \in M \,|\, \varphi_k(\bar{R}^x) = x\}| \geq 2$.
Proof of Claim 1: Let $x \in]\varphi_1(\bar{R}), \varphi_2(\bar{R})[$. By Lemmas 6.2 and 6.3, $\varphi_3(\bar{R}^x) \leq x$ or $\varphi_1(\bar{R}^x) \geq x$.

If $\varphi_3(\bar{R}^x) \leq x$, then $\varphi(\bar{R})\bar{P}_2\varphi(\bar{R}^x)$. Thus, by *replacement-domination*, $\varphi_1(\bar{R}^x) > \varphi_1(\bar{R})$. Hence, by *Pareto-optimality*, $|\{k \in M \,|\, \varphi_k(\bar{R}^x) = x\}| \geq 2$.

If $\varphi_1(\bar{R}^x) \geq x$, then $\varphi(\bar{R})\bar{P}_1\varphi(\bar{R}^x)$. Thus, by *replacement-domination*, $\varphi_3(\bar{R}^x) \leq \varphi_3(\bar{R})$. Hence, by *Pareto-optimality*, $|\{k \in M \,|\, \varphi_k(\bar{R}^x) = x\}| \geq 2$. ◇

Claim 1 and *replacement-domination* imply that for all $x \in]\varphi_1(\bar{R}), \varphi_2(\bar{R})[$,

$$\varphi_1(\bar{R}^x), \varphi_2(\bar{R}^x), \varphi_3(\bar{R}^x) \in]\varphi_1(\bar{R}), \varphi_3(\bar{R})]. \tag{10}$$

Claim 2: For some $y \in]\varphi_1(\bar{R}), \varphi_2(\bar{R})[$, $\varphi_1(\bar{R}^y) < \varphi_2(\bar{R}^y) = \varphi_3(\bar{R}^y) = y$.
Proof of Claim 2: Suppose that for all $x \in]\varphi_1(\bar{R}), \varphi_2(\bar{R})[$,

$$\varphi_1(\bar{R}^x) \geq x. \tag{11}$$

By Claim 1 and *replacement-domination*, for some $\tilde{x} \in]\varphi_1(\bar{R}), \varphi_2(\bar{R})[$, $\varphi_3(\bar{R}^{\tilde{x}}) < \varphi_2(\bar{R})$. Let $\bar{x} = \frac{1}{2}(\varphi_3(\bar{R}^{\tilde{x}}) + \varphi_2(\bar{R}))$. By (11), $\varphi(\bar{R}^{\tilde{x}})\bar{P}_1\varphi(\bar{R}^{\bar{x}})$ and $\varphi(\bar{R}^{\bar{x}})\bar{P}_2\varphi(\bar{R}^{\tilde{x}})$, which contradicts *replacement-domination*. ◇

Claim 2 guarantees the existence of y. Let $z = \varphi_1(\bar{R}^y)$. By (10), $z > \varphi_1(\bar{R})$. Consider $\varphi(\bar{R}^y)$ and $\varphi(\bar{R}^z)$. By Claim 1, $|\{k \in M \,|\, \varphi_k(\bar{R}^z) = z\}| \geq 2$. Thus, $\varphi(\bar{R}^z)\bar{P}_1\varphi(\bar{R}^y)$. By *replacement-domination*, $\varphi(\bar{R}^z)\bar{R}_2\varphi(\bar{R}^y)$. Hence,

$$\varphi_1(\bar{R}^z) = \varphi_2(\bar{R}^z) = z < y < \varphi_3(\bar{R}^z). \tag{12}$$

Let $w = \frac{1}{2}(y+\varphi_3(\bar{R}^z))$. Consider $\varphi(\bar{R}^y)$ and $\varphi(\bar{R}^w)$. By Claim 1, $|\{k \in M \mid \varphi_k(\bar{R}^w) = w\}| \geq 2$. Thus, $\varphi(\bar{R}^w)\bar{P}_2\varphi(\bar{R}^y)$. By *replacement-domination*, $\varphi(\bar{R}^w)\bar{R}_1\varphi(\bar{R}^y)$. Hence,

$$\varphi_1(\bar{R}^w) < z < \varphi_2(\bar{R}^w) = \varphi_3(\bar{R}^w) = w. \tag{13}$$

By our choice of w, (12), and (13), it follows that $\varphi(\bar{R}^w)\bar{P}_1\varphi(\bar{R}^z)$ and $\varphi(\bar{R}^z)\bar{P}_2\varphi(\bar{R}^w)$, which contradicts *replacement-domination*. Hence, Case 1 cannot occur.

Case 2: For some $j \in N$, $\underline{p}(R) < \varphi_1(R) = \varphi_2(R) = p(R_j) < \varphi_3(R) < \overline{p}(R)$ and for all $i \in N$, $p(R_i) \neq \varphi_3(R)$.

Case 2 is analogous to Case 1.

Case 3: $\varphi_2(R) \in]\varphi_1(R), \varphi_3(R)[$.

By Lemmas 6.2 and 6.3, for all $i \in N$, $p(R_i) \in [\underline{p}(R), \varphi_1(R)] \cup [\varphi_3(R), \overline{p}(R)]$. Thus, by *Pareto-optimality*, for some $j, l \in N$, $p(R_j) = \varphi_1(R)$ and $p(R_l) = \varphi_3(R)$. Let $\bar{R} \in \mathcal{R}^N$ be such that $\bar{R}_1 = R_j$, $\bar{R}_2 = R_l$, and for all $i \in N\backslash\{1,2\}$, $\bar{R}_i = R_l$. Successive applications of Lemmas 2.3 and 6.1 yield $\varphi(\bar{R}) = \varphi(R)$. Let $\bar{R}' \in \mathcal{R}^N$ be such that $\bar{R}'_{N\backslash\{3\}} = \bar{R}_{N\backslash\{3\}}$ and $p(\bar{R}'_3) = \varphi_2(\bar{R})$. By Lemmas 6.2 and 6.3, $\varphi_1(\bar{R}') \geq p(\bar{R}'_3)$ or $\varphi_3(\bar{R}') \leq p(\bar{R}'_3)$. Without loss of generality, we suppose that $\varphi_3(\bar{R}') \leq p(\bar{R}'_3)$. By *replacement-domination*, $\varphi_1(\bar{R}') > \underline{p}(\bar{R}')$. Since Cases 1 and 2 cannot occur, we obtain

$$\varphi_1(\bar{R}') = \varphi_2(\bar{R}') = \varphi_3(\bar{R}') = p(\bar{R}'_3). \tag{14}$$

Let $\tilde{R} \in \mathcal{R}^N$ be such that $\tilde{R}_{N\backslash\{3\}} = \bar{R}_{N\backslash\{3\}}$ and $p(\tilde{R}_3) = \frac{1}{2}(\varphi_2(\bar{R}) + \varphi_3(\bar{R}))$. By Lemmas 6.2 and 6.3, $\varphi_1(\tilde{R}) \geq p(\tilde{R}_3)$ or $\varphi_3(\tilde{R}) \leq p(\tilde{R}_3)$. By *replacement-domination* and (14), $\varphi_3(\tilde{R}) \leq p(\tilde{R}_3)$ and $\varphi_1(\tilde{R}) < \varphi_2(\bar{R})$. Since Cases 1 and 2 cannot occur, *Pareto-optimality* and *replacement-domination* imply

$$\varphi_1(\tilde{R}) = \underline{p}(\tilde{R}) < \varphi_2(\tilde{R}) = \varphi_3(\tilde{R}) = p(\tilde{R}_3). \tag{15}$$

Let $\hat{R} \in \mathcal{R}^N$ be such that $\hat{R}_{N\backslash\{3\}} = \bar{R}_{N\backslash\{3\}}$ and $p(\hat{R}_3) = \frac{1}{2}(\varphi_1(\bar{R})+\varphi_2(\bar{R}))$. Symmetric arguments as above yield

$$p(\hat{R}_3) = \varphi_1(\hat{R}) = \varphi_2(\hat{R}) < \varphi_3(\hat{R}) = \overline{p}(\hat{R}). \tag{16}$$

By (15) and (16), $\varphi(\tilde{R})\bar{P}_1\varphi(\hat{R})$ and $\varphi(\hat{R})\bar{P}_2\varphi(\tilde{R})$, which contradicts *replacement-domination*. Hence, Case 3 cannot occur.

Case 4: $|\{\varphi_k(R)\,|\,k \in M\}| = 2$ and for some $j, l \in N$, $p(R_j) = \varphi_1(R)$ and $p(R_l) = \varphi_3(R)$.

Without loss of generality, we suppose that $p(R_j) = \varphi_1(R) = \varphi_2(R)$ and $p(R_l) = \varphi_3(R)$. Let $\bar{R} \in \mathcal{R}^N$ be such that $\bar{R}_1 = R_j$, $\bar{R}_2 = R_l$, and for all $i \in N\backslash\{1,2\}$, $\bar{R}_i = R_l$. Successive applications of Lemmas 2.3 and 6.1 yield $\varphi(\bar{R}) = \varphi(R)$.

Let $\bar{R}' \in \mathcal{R}^N$ be such that $\bar{R}'_{N\backslash\{3\}} = \bar{R}_{N\backslash\{3\}}$ and $p(\bar{R}'_3) = \frac{1}{2}(\varphi_1(\bar{R}) + \varphi_3(\bar{R}))$. By Lemmas 6.2 and 6.3, $\varphi_1(\bar{R}') \geq p(\bar{R}'_3)$ or $\varphi_3(\bar{R}') \leq p(\bar{R}'_3)$. Without loss of generality, we suppose that $\varphi_3(\bar{R}') \leq p(\bar{R}'_3)$. If $|\{\varphi_k(\bar{R}')\,|\,k \in M\}| = 1$, then choose $p(\bar{R}'_3) \in\ ]\varphi_1(\bar{R}), \varphi_3(\bar{R})[\backslash\{\frac{1}{2}(\varphi_1(\bar{R}) + \varphi_3(\bar{R}))\}$. Since Cases 1, 2, and 3 cannot occur, *replacement-domination* implies $\varphi_1(\bar{R}') = \underline{p}(\bar{R}')$ and $\varphi_2(\bar{R}') = \varphi_3(\bar{R}') = p(\bar{R}'_3)$. Let $\bar{R}'' \in \mathcal{R}^N$ be such that $\bar{R}''_{N\backslash\{4\}} = \bar{R}'_{N\backslash\{4\}}$ and $p(\bar{R}''_4) = \frac{1}{2}(\varphi_1(\bar{R}') + \varphi_3(\bar{R}'))$. By *replacement-domination*, for all $i \in N\backslash\{4\}$, $\varphi(\bar{R}')\bar{R}'_i\varphi(\bar{R}'')$. If $|\{\varphi_k(\bar{R}'')\,|\,k \in M\}| = 1$, then choose $p(\bar{R}''_4) \in\ ]\varphi_1(\bar{R}'), \varphi_3(\bar{R}')[\backslash\{\frac{1}{2}(\varphi_1(\bar{R}') + \varphi_3(\bar{R}'))\}$. Because Cases 1, 2, and 3 cannot occur, *replacement-domination* implies $\varphi_1(\bar{R}'') = \varphi_2(\bar{R}'') = p(\bar{R}''_4)$ and $\varphi_3(\bar{R}'') = p(\bar{R}''_3)$. Let $\tilde{R} \in \mathcal{R}^N$ be such that $\tilde{R}_{N\backslash\{3\}} = \bar{R}_{N\backslash\{3\}}$ and $\tilde{R}_3 = \bar{R}''_1$. By Lemma 2.3, $\varphi(\tilde{R}) = \varphi(\bar{R}'')$. Thus, $\underline{p}(\tilde{R}) < p(\tilde{R}_4) = \varphi_1(\tilde{R}) = \varphi_2(\tilde{R}) < \varphi_3(\tilde{R}) < \overline{p}(R)$ and for all $i \in N$, $p(\tilde{R}_i) \neq \varphi_3(\tilde{R})$. By Case 2, this cannot occur. Hence, Case 4 cannot occur.

Case 5: Otherwise.

As we are not in Case 3, we have $|\{\varphi_k(R)\,|\,k \in M\}| = 2$. Thus, $\varphi_1(R) = \varphi_2(R)$ or $\varphi_2(R) = \varphi_3(R)$. Without loss of generality, $\varphi_1(R) = \varphi_2(R)$. By *Pareto-optimality*, for some $j \in N$, $p(R_j) = \varphi_1(R)$.

Because $n \geq 5$, there exists $t \in N\backslash\{j\}$ such that $\underline{p}(R_{N\backslash\{t\}}) = \underline{p}(R)$, $\overline{p}(R_{N\backslash\{t\}}) = \overline{p}(R)$, and $\varphi(R) \in E(R_{N\backslash\{t\}})$. Let $R' \in \mathcal{R}^N$ be such that $R'_{N\backslash\{t\}} = R_{N\backslash\{t\}}$ and $p(R'_t) = \varphi_3(R)$. Using the same arguments as at the beginning of the proof of Lemma 6.2, it follows that $\varphi(R') = \varphi(R)$.

But for profile R' we have $|\{\varphi_k(R') \,|\, k \in M\}| = 2$, $p(R'_j) = \varphi_1(R')$ and $p(R'_t) = \varphi_3(R')$. Then we are in Case 4, a contradiction. $\square$

The final lemma completes the proof of Theorem 4.1.

Lemma 6.5 $\varphi \in \{\underline{\phi}, \overline{\phi}\}$.

Proof. First, suppose that for some $R \in \mathcal{R}^N$ and $k \in M$, $\varphi_k(R) \in]\underline{p}(R), \overline{p}(R)[$. By Lemma 6.4 and *Pareto-optimality*, for some $j \in N$,

$$\varphi_1(R) = \varphi_2(R) = \varphi_3(R) = p(R_j).$$

Without loss of generality, we suppose that $p(R_1) = \underline{p}(R)$ and $p(R_2) = \overline{p}(R)$. Let $\bar{R} \in \mathcal{R}^N$ be such that $\bar{R}_{\{1,2,j\}} = R_{\{1,2,j\}}$ and for all $i \in N\backslash\{1,2,j\}$, $\bar{R}_i = R_1$. Successive applications of Lemma 2.3 yield $\varphi(\bar{R}) = \varphi(R)$. Let $\bar{R}' \in \mathcal{R}^N$ be such that $\bar{R}'_{N\backslash\{j\}} = \bar{R}_{N\backslash\{j\}}$ and $\bar{R}'_j = \bar{R}_2$. By Lemma 6.4, either [for all $k \in M$, $\varphi_k(\bar{R}') = \underline{p}(\bar{R}')$] or [for all $k \in M$, $\varphi_k(\bar{R}') = \overline{p}(\bar{R}')$]. Both cases yield a contradiction to *replacement-domination*. Hence, for all $R \in \mathcal{R}^N$,

$$\{\varphi_k(R) \,|\, k \in M\} \in \{\{\underline{p}(R)\}, \{\overline{p}(R)\}\}. \tag{17}$$

Let $R^o \in \mathcal{R}^N$ be such that $p(R^o_1) = 0$ and for all $i \in N\backslash\{1\}$, $p(R^o_i) = 1$. By (17), without loss of generality, we suppose that for all $k \in M$, $\varphi_k(R^o) = 0$. We show that $\varphi = \underline{\phi}$.[9]

Let $R \in \mathcal{R}^N$. We show that $\varphi(R) = \underline{\phi}(R)$. Let $\bar{R} \in \mathcal{R}^N$ be such that $p(\bar{R}_1) = \underline{p}(R)$ and for all $i \in N\backslash\{1\}$, $p(\bar{R}_i) = \overline{p}(R)$. By (17) and successive applications of Lemmas 2.3 and 6.1 yield

$$\varphi(\bar{R}) = \varphi(R). \tag{18}$$

Let $\tilde{R} \in \mathcal{R}^N$ be such that $\tilde{R}_{N\backslash\{1\}} = \bar{R}_{N\backslash\{1\}}$ and $\tilde{R}_1 = R^o_1$. Successive applications of Lemma 2.3, *replacement-domination*, and (17) yield

$$\varphi(\tilde{R}) = \varphi(R^o). \tag{19}$$

[9] When for all $k \in M$, $\varphi_k(R^o) = 1$ we would show that $\varphi = \overline{\phi}$.

Let $\tilde{R}' \in \mathcal{R}^N$ be such that $\tilde{R}'_{N\setminus\{2\}} = \tilde{R}_{N\setminus\{2\}}$ and $\tilde{R}'_2 = \bar{R}_1$. By *replacement-domination* and (17),

$$\varphi(\tilde{R}') = \varphi(\tilde{R}). \tag{20}$$

If $p(\tilde{R}'_2) = 0$, then Lemma 6.1 implies $\varphi(\tilde{R}') = \varphi(\bar{R})$. Thus, by (18), for all $k \in M$, $\varphi_k(R) = \underline{p}(R)$. Hence, $\varphi(R) = \underline{\phi}(R)$, the desired conclusion.

If $p(\tilde{R}'_2) > 0$, then by Lemma 6.1, we may assume that $0\tilde{P}'_2\overline{p}(\tilde{R}')$. Let $\tilde{R}'' \in \mathcal{R}^N$ be such that $\tilde{R}''_{N\setminus\{1\}} = \tilde{R}'_{N\setminus\{1\}}$ and $\tilde{R}''_1 = \tilde{R}'_3$. By *replacement-domination*, (17), (19), (20), and our choice of $\tilde{R}'_2$, for all $k \in M$, $\varphi_k(\tilde{R}'') = \underline{p}(\tilde{R}'')$. By Lemma 6.1, $\varphi(\tilde{R}'') = \varphi(\bar{R})$. Thus, by (18), for all $k \in M$, $\varphi_k(R) = \underline{p}(R)$. Hence, $\varphi(R) = \underline{\phi}(R)$, the desired conclusion. □

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