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Walter BOSSERT and Kotaro SUZUMURA

CIREQ, Université de Montréal
C.P. 6128, succursale Centre-ville
Montréal (Québec) H3C 3J7
Canada

téléphone : (514) 343-6557
télécopieur : (514) 343-5831
cireq@umontreal.ca
http://www.cireq.umontreal.ca

Université de Montréal McGill Concordia UNIVERSITY

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A Characterization of Consistent Collective Choice Rules*

Walter Bossert
Département de Sciences Economiques and CIREQ
Université de Montréal
C.P. 6128, succursale Centre-ville
Montréal QC H3C 3J7
Canada
FAX: (+1 514) 343 7221
e-mail: walter.bossert@umontreal.ca

and

Kotaro Suzumura
Institute of Economic Research
Hitotsubashi University
Kunitachi, Tokyo 186-8603
Japan
FAX: (+81 42) 580 8353
e-mail: suzumura@ier.hit-u.ac.jp

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Corresponding Author: Walter Bossert

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Abstract. We characterize a class of collective choice rules such that collective preference relations are *consistent.* Consistency is a weakening of transitivity and a strengthening of acyclicity requiring that there be no cycles with at least one strict preference. The properties used in our characterization are unrestricted domain, strong Pareto, anonymity and neutrality. If there are at most as many individuals as there are alternatives, the axioms provide an alternative characterization of the Pareto rule. If there are more individuals than alternatives, however, further rules become available. *Journal of Economic Literature* Classification No.: D71.

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1 Introduction

Arrow's (1951; 1963) theorem regarding the impossibility of defining a collective choice rule possessing some seemingly innocuous properties is one of the most fundamental results in the theory of collective decision-making. There have been numerous attempts to modify his framework in order to avoid impossibilities, such as weakening some of his original properties or departing from the stringent informational assumption that only ordinally measurable and interpersonally non-comparable information on individual well-being is available.

The route of escape from the negative conclusion of Arrow's theorem that we follow in this paper consists of relaxing the requirement that the social ranking be an ordering for all preference profiles under consideration. In this spirit, Sen (1969; 1970, Theorem 5*3) characterized the Pareto extension rule under the assumption that social preferences are quasi-transitive but not necessarily transitive while retaining the completeness assumption. Weymark (1984, Theorem 3) allowed social preferences to be incomplete but imposed full transitivity and, as a result, obtained a characterization of the Pareto rule.

An interesting question that emerges in this context is what happens if transitivity is weakened to *consistency.* Consistency, a property introduced by Suzumura (1976), is intermediate in strength between transitivity and acyclicity and coincides with transitivity in the presence of reflexivity and completeness. It is logically independent of quasi-transitivity and requires that there be no preference cycles with at least one strict preference.

Consistency is of importance because, as Suzumura (1976) demonstrated, it is *necessary and sufficient* for the existence of an *ordering extension;* that is, a binary relation R can be extended to an ordering respecting all (weak and strict) preferences according to R if and only if R is consistent. This fundamental insight represents a significant strengthening of the classical extension theorem and its variants due to Szpilrajn (1930), establishing that the transitivity of an incomplete relation is *sufficient* for the existence of an ordering extension. Because consistency constitutes the weakest possible coherence property that needs to be satisfied if we do not want to give up all hope of compatibility with an ordering, consistency appears to be *the* natural weakening of the transitivity requirement, particularly in the absence of completeness.

In spite of its importance and significance, consistency has received relatively little attention in the past (see Bossert, 2006, for an overview of its application, such as the analysis of rational choice due to Bossert, Sprumont and Suzumura, 2005). In this paper, we examine the consequences of weakening transitivity to consistency in the context of Arrow's theorem. It turns out that, in some circumstances, consistency permits a much larger class of possible collective choice rules as compared to those that become available if completeness is dropped as a requirement on a social relation but the full force of transitivity is retained.

The axioms we impose are unrestricted domain, strong Pareto, anonymity and neutrality. If there are at least as many alternatives as there are agents, an alternative characterization of the Pareto rule is obtained. The difference between this characterization and Weymark's is that we weaken the transitivity requirement imposed on the social ordering to consistency and strengthen independence of irrelevant alternatives to neutrality. However, if there are fewer alternatives than agents, additional rules satisfy the above axioms. We characterize all of them and obtain the above-mentioned new axiomatization of the Pareto rule as a special case. Especially in applications where there are many voters and relatively few candidates (this is the case for political elections, to name a prominent and important example), our result shows that it is possible to go considerably beyond the limitations of unanimity imposed by the Pareto rule. This is achieved at little cost because consistency still ensures the existence of an ordering consistent with the social relation and, therefore, this paper opens up substantial new possibilities in the design of collective choice mechanisms.

In addition, this paper develops a new approach to the analysis of collective choice rules in the sense that it does not rely on previously applied proof techniques. In particular, tools such as Sen's (1979) *field expansion lemmas* which allow one to extend "local" observations to arbitrary collections of alternatives crucially rely on transitivity (or quasi-transitivity), and consistency is not sufficient to obtain these types of results. Therefore, a novel approach to identifying the class of collective choice rules compatible with standard axioms is called for when working with consistency, and we believe that the techniques developed here will prove useful in numerous other applications.

The following section introduces our basic definitions along with a preliminary observation. Section 3 contains the statement and proof of our characterization result and Section 4 concludes with some examples designed to illustrate some important features of our characterization.

2 Preliminaries

Suppose there is a set of alternatives X containing at least three elements, that is, $|X| \geq 3$ where $|X|$ denotes the cardinality of X. The population is $N = \{1, \ldots, |N|\}$ with $|N| \in \mathbb{N} \setminus \{1\}$, where $\mathbb{N}$ denotes the set of all natural numbers. Let $R \subseteq X \times X$ be a (binary) relation. For simplicity, we write xRy instead of $(x, y) \in R$ and $\neg\, xRy$ instead of $(x, y) \notin R$. The *asymmetric factor* P of R is defined by

$$xPy \;\Leftrightarrow\; [xRy \text{ and } \neg\, yRx]$$

for all $x, y \in X$. The *symmetric factor* I of R is defined by

$$xIy \;\Leftrightarrow\; [xRy \text{ and } yRx]$$

for all $x, y \in X$. If R is interpreted as a *weak preference relation,* that is, xRy means that x is considered at least as good as y, then P and I are the *strict preference relation* and the *indifference relation* corresponding to R.

A relation R is *reflexive* if and only if, for all $x \in X$,

$$xRx$$

and R is *complete* if and only if, for all $x, y \in X$ such that $x \neq y$,

$$xRy \text{ or } yRx.$$

Furthermore, R is *transitive* if and only if, for all $x, y, z \in X$,

$$[xRy \text{ and } yRz] \;\Rightarrow\; xRz$$

and R is *quasi-transitive* if and only if P is transitive. R is *consistent* if and only if, for all $M \in \mathbb{N} \setminus \{1, 2\}$ and for all $x^1, \ldots, x^M \in X$,

$$x^{m-1} R x^m \;\forall m \in \{2, \ldots, M\} \;\Rightarrow\; \neg\, x^M P x^1$$

and, finally, R is *acyclical* if and only if, for all $M \in \mathbb{N} \setminus \{1, 2\}$ and for all $x^1, \ldots, x^M \in X$,

$$x^{m-1} P x^m \;\forall m \in \{2, \ldots, M\} \;\Rightarrow\; \neg\, x^M P x^1.$$

Transitivity implies consistency which, in turn, implies acyclicity but the reverse implications are not true in general. However, if R is reflexive and complete, transitivity and consistency are equivalent.

An *ordering* is a reflexive, complete and transitive relation. If R is an ordering, there is no ambiguity in using chains of individual preferences involving more than two alternatives; for instance, $xPyPz$ means that x is better than y which, in turn, is better than z and, by the transitivity of R, x is better than z.

The set of all orderings on X is denoted by $\mathcal{R}$ and its $|N|$-fold Cartesian product is $\mathcal{R}^{|N|}$. The set of all reflexive and transitive relations on X is $\mathcal{T}$, and the set of all reflexive and consistent relations on X is denoted by $\mathcal{C}$. The set of all binary relations on X is $\mathcal{B}$. A *profile* is a $|N|$-tuple $\mathbf{R} = (R_1, \ldots, R_{|N|}) \in \mathcal{R}^{|N|}$.

A *collective choice rule* is a mapping $f\colon \mathcal{D} \to \mathcal{B}$ where $\mathcal{D} \subseteq \mathcal{R}^{|N|}$ is the *domain* of this function, assumed to be non-empty. A *consistent collective choice rule* is a collective choice rule f such that $f(\mathbf{R}) \in \mathcal{C}$ for all $\mathbf{R} \in \mathcal{D}$, and a *transitive collective choice rule* is a collective choice rule f such that $f(\mathbf{R}) \in \mathcal{T}$ for all $\mathbf{R} \in \mathcal{D}$. Note that, because $\mathcal{D} \subseteq \mathcal{R}^{|N|}$, we retain the assumption that all admissible profiles are composed of individual preferences which are orderings. On the other hand, we allow social preferences to be incomplete and we permit violations of transitivity as long as consistency is satisfied. For each profile $\mathbf{R} \in \mathcal{D}$, $R = f(\mathbf{R})$ is the social preference corresponding to $\mathbf{R}$, and P and I are the strict preference relation and the indifference relation corresponding to R. We use $B(x, y; \mathbf{R})$ to denote the set of individuals such that $x \in X$ is better than $y \in X$ in the profile $\mathbf{R} \in \mathcal{R}^{|N|}$, that is, for all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^{|N|}$, $B(x, y; \mathbf{R}) = \{i \in N \mid xP_i y\}$.

An example for a transitive (and, thus, consistent) collective choice rule is the *Pareto rule* $f^p\colon \mathcal{R}^{|N|} \to \mathcal{B}$ defined by $R^p = f^p(\mathbf{R})$, where

$$xR^p y \;\Leftrightarrow\; [xR_i y \;\forall i \in N]$$

for all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^{|N|}$.

The following lemma, which will be of use in the proof of our main result, establishes that the cardinalities of these sets satisfy a *triangle inequality.*

Lemma 1 *For all $x, y, z \in X$ and for all $\mathbf{R} \in \mathcal{R}^{|N|}$,*

$$|B(x, z; \mathbf{R})| \leq |B(x, y; \mathbf{R})| + |B(y, z; \mathbf{R})|.$$

Proof. Let $x, y, z \in X$ and $\mathbf{R} \in \mathcal{R}^{|N|}$. First, we prove that

$$B(x, z; \mathbf{R}) \subseteq B(x, y; \mathbf{R}) \cup B(y, z; \mathbf{R}). \tag{1}$$

Suppose $i \notin B(x,y;\mathbf{R}) \cup B(y,z;\mathbf{R})$. Because individual preferences are complete, this implies yR_ix and zR_iy. By transitivity, zR_ix and, thus, $i \notin B(x,z;\mathbf{R})$, which proves (1).

Clearly, (1) implies

$$|B(x,z;\mathbf{R})| \leq |B(x,y;\mathbf{R}) \cup B(y,z;\mathbf{R})|.$$

Furthermore, we obviously must have

$$|B(x,y;\mathbf{R}) \cup B(y,z;\mathbf{R})| \leq |B(x,y;\mathbf{R})| + |B(y,z;\mathbf{R})|.$$

Combining the last two inequalities yields the desired result. ■

The following axioms are standard in the literature on social choice.

Unrestricted domain. $\mathcal{D} = \mathcal{R}^{|N|}$.

Strong Pareto. For all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{D}$,

(i) $xR_iy \ \forall i \in N \ \Rightarrow \ xRy$;

(ii) $[xR_iy \ \forall i \in N$ and $\exists j \in N$ such that $xP_jy] \ \Rightarrow \ xPy$.

Anonymity. For all bijections $\rho\colon N \to N$ and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{D}$,

$$R_i = R'_{\rho(i)} \ \forall i \in N \ \Rightarrow \ R = R'.$$

Independence of irrelevant alternatives. For all $x, y \in X$ and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{D}$,

$$[xR_iy \Leftrightarrow xR'_iy \text{ and } yR_ix \Leftrightarrow yR'_ix] \ \forall i \in N \ \Rightarrow \ [xRy \Leftrightarrow xR'y \text{ and } yRx \Leftrightarrow yR'x].$$

Neutrality. For all $x, y, z, w \in X$ and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{D}$,

$$[xR_iy \Leftrightarrow zR'_iw \text{ and } yR_ix \Leftrightarrow wR'_iz] \ \forall i \in N \ \Rightarrow \ [xRy \Leftrightarrow zR'w \text{ and } yRx \Leftrightarrow wR'z].$$

As is straightforward to verify, the Pareto rule satisfies all of the axioms introduced above.

3 Consistent Collective Choice Rules

Weymark (1984, Theorem 3) has shown that the Pareto rule is the only transitive collective choice rule satisfying unrestricted domain, strong Pareto, anonymity and independence of irrelevant alternatives. As a corollary to our main result, we will obtain an alternative characterization that is obtained by strengthening independence of irrelevant alternatives to neutrality and weakening transitivity to consistency. To describe all collective choice rules satisfying our requirements, we introduce some additional definitions. Let

$$\mathcal{S} = \{(w, \ell) \in \{0, \ldots, |N|\}^2 \mid 0 \leq |X|\ell < w + \ell \leq |N|\} \cup \{(0, 0)\}$$

and, furthermore, define

$$\Sigma = \{S \subseteq \mathcal{S} \mid (w, 0) \in S \; \forall w \in \{0, \ldots, |N|\}\}.$$

For $S \in \Sigma$, define the *S-rule* $f^S : \mathcal{R}^{|N|} \to \mathcal{B}$ by $R^S = f^S(\mathbf{R})$, where

$$xR^S y \;\Leftrightarrow\; [\exists (w, \ell) \in S \text{ such that } |B(x, y; \mathbf{R})| = w \text{ and } |B(y, x; \mathbf{R})| = \ell]$$

for all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^{|N|}$. The set S specifies the pairs of numbers of agents who have to consider an alternative x better (respectively worse) than an alternative y in order to obtain a weak preference of x over y according to the profile under consideration. Clearly, because only the number of individuals matters and not their identities, the resulting rule is anonymous. Analogously, neutrality is satisfied because these numbers do not depend on the alternatives to be ranked. Strong Pareto follows from the requirement that the pairs $(w, 0)$ be in S in the definition of Σ. Reflexivity of the social relation follows from the reflexivity of the individual preferences and the observation that $(0, 0) \in S$ for all $S \in \Sigma$. As will be shown in the proof of our characterization result, the social relation R^S is consistent due to the restrictions imposed on the pairs (w, ℓ) in the definition of $\mathcal{S}$. Conversely, the S-rules are the only rules satisfying our axioms. Thus, we obtain the following theorem.

Theorem 1 *A consistent collective choice rule f satisfies unrestricted domain, strong Pareto, anonymity and neutrality if and only if there exists $S \in \Sigma$ such that $f = f^S$.*

Proof. 'If.' As mentioned before the theorem statement, that the S-rules satisfy unrestricted domain, strong Pareto, anonymity and neutrality is straightforward to verify. Because reflexivity is obvious, it remains to establish that $R^S = f^S(\mathbf{R})$ is consistent for all $S \in \Sigma$ and for all $\mathbf{R} \in \mathcal{R}^{|N|}$. Let $S \in \Sigma$ and suppose, by way of contradiction, that there exist $\mathbf{R} \in \mathcal{R}^{|N|}$, $M \in \mathbb{N} \setminus \{1, 2\}$ and $x^1, \ldots, x^M \in X$ such that $x^{m-1} R^S x^m$ for all $m \in \{2, \ldots, M\}$ and $x^M P^S x^1$. By definition of R^S, there exist $(w_1, \ell_1), \ldots, (w_M, \ell_M) \in S$ such that $|B(x^{m-1}, x^m; \mathbf{R})| = w_{m-1}$ and $|B(x^m, x^{m-1}; \mathbf{R})| = \ell_{m-1}$ for all $m \in \{2, \ldots, M\}$.

Furthermore, we must have $|B(x^M, x^1; \mathbf{R})| = w_M$ and $|B(x^1, x^M; \mathbf{R})| = \ell_M$ with w_M positive; if $w_M = 0$, we have $(w_M, \ell_M) = (0, 0)$ by definition of $\mathcal{S}$ and it follows that $x^1 I^S x^M$, contrary to our hypothesis $x^M P^S x^1$.

If max $\{\ell_1, \ldots, \ell_M\} = 0$, (repeated if necessary) application of Lemma 1 yields

$$\begin{aligned} |B(x^3, x^1; \mathbf{R})| &\leq |B(x^3, x^2; \mathbf{R})| + |B(x^2, x^1; \mathbf{R})|, \\ &\vdots \\ |B(x^M, x^1; \mathbf{R})| &\leq |B(x^M, x^{M-1}; \mathbf{R})| + \ldots + |B(x^2, x^1; \mathbf{R})| = 0. \end{aligned}$$

But this contradicts our earlier observation that $|B(x^M, x^1; \mathbf{R})| = w_M > 0$.

If max $\{\ell_1, \ldots, \ell_M\} > 0$, suppose this maximum is achieved at ℓ_m for some $m \in \{1, \ldots, M\}$. By definition of $\mathcal{S}$, $|X| \geq 3 > 0$ and $w_m + \ell_m > |X|\ell_m$ together rule out the possibility that $w_m + \ell_m > |X|w_m$ and, therefore, we must have $(\ell_m, w_m) \notin S$ and the preference corresponding to the m^{th} element in the chain is strict. This, in turn, allows us to assume, without loss of generality, that $m = M$; this can be achieved with a simple relabeling of the elements in our chain if required. Invoking Lemma 1 again and using the maximality of ℓ_M, we obtain

$$\begin{aligned} |B(x^3, x^1; \mathbf{R})| &\leq |B(x^3, x^2; \mathbf{R})| + |B(x^2, x^1; \mathbf{R})| \leq 2\ell_M \\ &\vdots \\ |B(x^M, x^1; \mathbf{R})| &\leq |B(x^M, x^{M-1}; \mathbf{R})| + \ldots + |B(x^2, x^1; \mathbf{R})| \leq (M-1)\ell_M. \end{aligned}$$

Because $M \leq |X|$, this implies

$$|B(x^M, x^1; \mathbf{R})| \leq (|X| - 1)\ell_M. \tag{2}$$

By assumption and by the definition of $\mathcal{S}$, we have $|B(x^M, x^1; \mathbf{R})| = w_M > (|X| - 1)\ell_M$, a contradiction to (2).

'Only if.' Suppose f is a consistent collective choice rule satisfying the axioms of the theorem statement. Let

$$\begin{aligned} S \;=\; \{(w, \ell) \mid \exists\; x, y \in X \text{ and } \mathbf{R} \in \mathcal{R}^{|N|} \text{ such that} \\ |B(x, y; \mathbf{R})| = w, |B(y, x; \mathbf{R})| = \ell \text{ and } xRy\}. \end{aligned}$$

By anonymity and neutrality, S is such that the relation R is equal to R^S. It remains to show that $S \in \Sigma$. That $(w, 0) \in S$ for all $w \in \{0, \ldots, |N|\}$ follows from strong Pareto. Clearly, for all $(w, \ell) \in S$, $|X|\ell \geq 0$ and $w + \ell \leq |N|$.

As an auxiliary result, we show that

$$w > \ell \tag{3}$$

for all $(w, \ell) \in S \setminus \{(0,0)\}$. By way of contradiction, suppose that $w \leq \ell$ for some $(w, \ell) \in S \setminus \{(0,0)\}$. Because $(w, \ell) \neq (0,0)$ by assumption, this implies $\ell > 0$ and, by strong Pareto, $w > 0$. By unrestricted domain and the assumption $|X| \geq 3$, we can choose $x, y, z \in X$ and $\mathbf{R} \in \mathcal{R}^{|N|}$ so that

$$xP_iyP_iz \;\; \forall i \in \{1, \ldots, w\}$$

and

$$yP_izP_ix \;\; \forall i \in \{\ell+1, \ldots, \ell+w\}.$$

Furthermore, if $w < \ell$, let

$$yP_ixP_iz \;\; \forall i \in \{w+1, \ldots, \ell\}$$

and if $w + \ell < |N|$, let

$$xI_iyI_iz \;\; \forall i \in \{w+\ell+1, \ldots, |N|\}.$$

Because $|B(z, x; \mathbf{R})| = |B(x, y; \mathbf{R})| = w$ and $|B(x, z; \mathbf{R})| = |B(y, x; \mathbf{R})| = \ell$, we must have zRx and xRy. By strong Pareto, it follows that yPz and we obtain a contradiction to the consistency of R. This establishes (3).

To complete the proof, we have to show that $w + \ell > |X|\ell$ for all $(w, \ell) \in S \setminus \{(0,0)\}$. By way of contradiction, suppose this is not true. Then there exists a pair $(w_0, \ell_0) \in S \setminus \{(0,0)\}$ such that $w_0 + \ell_0 \leq |X|\ell_0$ or, equivalently,

$$w_0 \leq (|X| - 1)\ell_0. \tag{4}$$

Combining (3), which is true for all $(w, \ell) \in S \setminus \{(0,0)\}$ and thus for (w_0, ℓ_0), with (4), we obtain

$$\ell_0 < w_0 \leq (|X| - 1)\ell_0. \tag{5}$$

Clearly, $\ell_0 = 0$ is inconsistent with (5). Thus, $\ell_0 > 0$.

(3) immediately implies that, for any $(w, \ell) \in S \setminus \{(0,0)\}$, $(\ell, w) \notin S$. Thus, in particular, whenever $|B(x, y; \mathbf{R})| = w_0$ and $|B(y, x; \mathbf{R})| = \ell_0$, we must have xPy and not merely xRy.

We now distinguish two cases. The first of these occurs whenever w_0 is a positive multiple of ℓ_0. That is, given (5), there exists $\beta \in \{3, \ldots, |X|\}$ such that $w_0 = (\beta -$

$1)\ell_0$ (and, thus, $w_0 + \ell_0 = \beta\ell_0$). By unrestricted domain, we can choose β alternatives $x^1, \ldots, x^\beta \in X$ and a profile $\mathbf{R} \in \mathcal{R}^{|N|}$ such that

$$\begin{aligned}
x^1 P_i x^2 P_i \ldots P_i x^{\beta-1} P_i x^\beta &\quad \forall i \in \{1, \ldots, \ell_0\},\\
x^2 P_i x^3 P_i \ldots P_i x^\beta P_i x^1 &\quad \forall i \in \{\ell_0 + 1, \ldots, 2\ell_0\},\\
&\vdots\\
x^{\beta-1} P_i x^\beta P_i x^1 P_i \ldots P_i x^{\beta-2} &\quad \forall i \in \{(\beta-2)\ell_0 + 1, \ldots, (\beta-1)\ell_0\},\\
x^\beta P_i x^1 P_i \ldots P_i x^{\beta-2} P_i x^{\beta-1} &\quad \forall i \in \{(\beta-1)\ell_0 + 1, \ldots, \beta\ell_0\}
\end{aligned}$$

and, if $|N| > w_0 + \ell_0 = \beta\ell_0$,

$$x^1 I_i x^2 I_i \ldots I_i x^{\beta-1} I_i x^\beta \;\; \forall i \in \{w_0 + \ell_0 + 1, \ldots, |N|\}.$$

We have $|B(x^{m-1}, x^m; \mathbf{R})| = (\beta - 1)\ell_0 = w_0$ and $|B(x^m, x^{m-1}; \mathbf{R})| = \ell_0$ for all $m \in \{2, \ldots, \beta\}$ and, furthermore, $|B(x^\beta, x^1; \mathbf{R})| = (\beta - 1)\ell_0 = w_0$ and $|B(x^1, x^\beta; \mathbf{R})| = \ell_0$. Therefore, $x^{m-1} P x^m$ for all $m \in \{2, \ldots, \beta\}$ and $x^\beta P x^1$, contradicting the consistency of R.

Finally, we consider the case in which w_0 is not a positive multiple of ℓ_0. Clearly, this is only possible if $\ell_0 > 1$. By (5), there exists $\alpha \in \{3, \ldots, |X|\}$ such that

$$(\alpha - 2)\ell_0 < w_0 < (\alpha - 1)\ell_0. \tag{6}$$

By unrestricted domain, we can consider α alternatives $x^1, \ldots, x^\alpha \in X$ and a profile $\mathbf{R} \in \mathcal{R}^{|N|}$ such that

$$\begin{aligned}
x^2 P_i x^3 P_i \ldots P_i x^\alpha P_i x^1 &\quad \forall i \in \{1, \ldots, \ell_0\},\\
&\vdots\\
x^{\alpha-1} P_i x^\alpha P_i x^1 P_i \ldots P_i x^{\alpha-2} &\quad \forall i \in \{(\alpha-3)\ell_0 + 1, \ldots, (\alpha-2)\ell_0\},\\
x^\alpha P_i x^1 P_i \ldots P_i x^{\alpha-2} P_i x^{\alpha-1} &\quad \forall i \in \{(\alpha-2)\ell_0 + 1, \ldots, w_0\},\\
x^1 P_i x^2 P_i \ldots P_i x^{\alpha-1} P_i x^\alpha &\quad \forall i \in \{w_0 + 1, \ldots, 2w_0 - (\alpha-2)\ell_0\},\\
x^1 P_i x^\alpha P_i x^2 P_i \ldots P_i x^{\alpha-1} &\quad \forall i \in \{2w_0 - (\alpha-2)\ell_0 + 1, \ldots, w_0 + \ell_0\}
\end{aligned}$$

and, if $|N| > w_0 + \ell_0$,

$$x^1 I_i x^2 I_i \ldots I_i x^{\alpha-1} I_i x^\alpha \;\; \forall i \in \{w_0 + \ell_0 + 1, \ldots, |N|\}.$$

This profile is well-defined because (6) implies

$$w_0 < 2w_0 - (\alpha - 2)\ell_0 < w_0 + \ell_0.$$

We have $|B(x^{m-1}, x^m; \mathbf{R})| = w_0$ and $|B(x^m, x^{m-1}; \mathbf{R})| = \ell_0$ for all $m \in \{2, \ldots, \alpha\}$ and, furthermore, $|B(x^\alpha, x^1; \mathbf{R})| = w_0$ and $|B(x^1, x^\alpha; \mathbf{R})| = \ell_0$. Therefore, $x^{m-1} P x^m$ for all $m \in \{2, \ldots, \alpha\}$ and $x^\alpha P x^1$, again contradicting the consistency of R. ■

4 Examples

Clearly, the Pareto rule is a special case of the rules characterized in the previous section; it is obtained for $S = \{(w, 0) \mid w \in \{0, \ldots, |N|\}\}$. If $|X| \geq |N|$, this is the only rule satisfying the axioms of the theorem statement. This is the case because only pairs (w, ℓ) where $\ell = 0$ are in S in the presence of this inequality. To see this, suppose, to the contrary, that there exists $(w, \ell) \in S$ such that $\ell > 0$. Because $(w, \ell) \in S$, it follows that $|N| \geq w + \ell > |X|\ell > 0$. Combined with $|X| \geq |N|$, this implies $|N| > |N|\ell$ which is impossible if $\ell > 0$. Thus, if $|X| \geq |N|$, our theorem provides an alternative characterization of the Pareto rule. This axiomatization differs from Weymark's (1984) in that independence of irrelevant alternatives is strengthened to neutrality and transitivity is weakened to consistency. Note that, in this case, transitivity is implied by the conjunction of consistency and the axioms employed in our theorem.

However, if $|X| < |N|$, the Pareto rule is not the only rule satisfying the axioms of Theorem 1. For example, consider the collective choice rule f^S corresponding to the set $S = \{(w, 0) \mid w \in \{0, \ldots, |N|\}\} \cup \{(|N| - 1, 1)\}$. For $(w, \ell) = (|N| - 1, 1)$, we have $|N| = |N| - 1 + 1 = w + \ell = |N| \cdot 1 > |X|\ell > 0$ and, thus, the relevant inequalities are satisfied.

Once rules other than the Pareto rule are available, transitivity is no longer guaranteed (but, of course, all S-rules are consistent as established in our theorem). For example, suppose $X = \{x, y, z\}$, $N = \{1, 2, 3, 4\}$, $S = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (3, 1)\}$ and consider the profile $\mathbf{R}$ defined by

$$\begin{aligned} &x P_1 y P_1 z, \\ &x P_2 y P_2 z, \\ &z P_3 x P_3 y, \\ &y P_4 z P_4 x. \end{aligned}$$

According to $R^S = f^S(\mathbf{R})$, we have $x P^S y$ and $y P^S z$ because $|B(x, y; \mathbf{R})| = |B(y, z; \mathbf{R})| = 3$ and $|B(y, x; \mathbf{R})| = |B(z, y; \mathbf{R})| = 1$. But $|B(x, z; \mathbf{R})| = |B(z, x; \mathbf{R})| = 2$ and, thus, $\neg\, x R^S z$ so that R^S is not transitive (not even quasi-transitive).

An interesting feature of the S-rules is that there may be "gaps" in the set of possible values of w or ℓ within a rule. For instance, suppose $X = \{x, y, z\}$, $N = \{1, 2, 3, 4, 5, 6, 7\}$ and $S = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0), (5, 2)\}$. Consider the pair $(w, \ell) = (5, 2)$. We have $|N| = w + \ell = 7 > 6 = 3 \cdot 2 = |X|\ell > 0$ and, thus, f^S is well-defined. In addition to the rankings generated by unanimity, five agents can ensure a superior ranking of an alternative over another against two agents with the opposite preference but, on the other hand, if six agents prefer x to y and one agent prefers y to x, non-comparability results.

The conclusion of Theorem 1 does not hold if merely independence of irrelevant alternatives rather than neutrality is imposed. Suppose $x^0, y^0 \in X$ are two distinct alternatives. Define a collective choice rule by letting

$$xRy \;\Leftrightarrow\; [xR^p y \;\text{ or } (\neg\, xR^p y \text{ and } \neg\, yR^p x \text{ and } \{x, y\} = \{x^0, y^0\})]$$

for all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^{|N|}$. This is a consistent collective choice rule satisfying unrestricted domain, strong Pareto, anonymity and independence of irrelevant alternatives. However, neutrality clearly is violated.

Consistency cannot be weakened to acyclicity in our characterization result. The collective choice rule defined by letting

$$xRy \;\Leftrightarrow\; [xR^p y \;\text{ or } |B(x, y; \mathbf{R})| = |B(y, x; \mathbf{R})| = 1]$$

for all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^{|N|}$ produces acyclical social preferences and satisfies the axioms of Theorem 1. However, social preferences are not always consistent. For example, suppose $X = \{x, y, z\}$ and $N = \{1, 2, 3\}$, and consider the profile $\mathbf{R}$ defined by

$$xP_1yP_1z,$$
$$zP_2xP_2y,$$
$$xI_3yI_3z.$$

According to $R = f(\mathbf{R})$, we obtain xPy, yIz and zIx, a social preference relation that is not consistent.

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