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Université de Montréal
Pavillon Lionel-Groulx, CIREQ
C.P. 6128, succursale Centre-ville
Montréal QC H3C 3J7
Téléphone : (514) 343-6557
Télécopieur : (514) 343-7221
cireq@umontreal.ca
http://www.cireqmontreal.com

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Optimal Debt Management in a Liquidity Trap*

Hafedh Bouakez† Rigas Oikonomou‡ Romanos Priftis§

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Abstract

We study optimal debt management in the face of shocks that can drive the economy into a liquidity trap and call for an increase in public spending in order to mitigate the resulting recession. Our approach follows the literature of macroeconomic models of debt management, which we extend to the case where the zero lower bound on the short-term interest rate may bind. We wish to identify the conditions under which removing long-maturity government debt from the secondary market can be an optimal policy outcome. We show that the optimal debt-management strategy is to issue short-term debt if the government faces a sizable exogenous increase in public spending and if its initial liability is not very large. In this case, our results run against the standard prescription of the debt-management literature. In contrast, if the initial debt level is high, then issuing long term government bonds is optimal. Finally, we find a role for revisions in the debt management strategy during LT episodes, whereby the government actively manages the maturity structure, in some cases removing long bonds from the secondary market.

JEL Classification : E43, E62, H63

Keywords: Debt Management, Debt Maturity, Fiscal Policy, Liquidity Trap, Monetary Policy, Tax Smoothing, Portfolio Rebalancing.

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† Department of Applied Economics, HEC Montréal, 3000 chemin de la Côte-Sainte-Catherine, Montréal, Québec, Canada H3T 2A7 and CIREQ. E-mail: hafedh.bouakez@hec.ca.

‡ Université Catholique de Louvain. IRES, Collège L. H. Dupriez, 3 Place Montesquieu 1348 Louvain la Neuve, Belgium. Email: rigas.oikonomou@uclouvain.be.

§ Modelling Unit, Directorate General for Economic and Financial Affairs, European Commission, Rue de la Loi 170, Brussels 1049, Belgium. Email: romanos.priftis@ec.europa.eu.

1 Introduction

A prominent characteristic of the latest recession in the US was the sharp decline in short-term nominal interest rates, which reached their zero lower bound (ZLB), a situation referred to as the liquidity trap (LT). At the same time, large rises in spending levels led to rapid increases in government debt liabilities, bringing to surface concerns over the solvency of the government's budget. The maturity structure of debt held by the public also went through a dramatic shift; the buybacks of long term bonds from the secondary market conducted by the Federal Reserve contributed to a sharp shortening of the maturity of debt in the hands of the private sector.

In this paper, we study optimal debt management in a LT. We follow the trail of a literature that analyzes government portfolios in macroeconomic models in which the optimal choice of the maturity structure of public debt enables the government to smooth taxes through time (e.g., Angeletos (2002) and Buera and Nicolini (2004), hereafter ABN, Faraglia, Marcet and Scott (2010) among others). The optimal portfolios that emerge from these models feature two key properties: first, issuing only long term debt is optimal, since this allows the government to take full advantage of the negative covariance between long bond prices and deficits, and 'complete the markets'; second, government portfolios are constant over time, meaning that governments do not have to actively manage the maturity of debt.

The above mentioned papers abstract from shocks that can drive the economy into a LT, focusing mainly on disturbances in the level of spending. We wish to determine whether this plays a role in producing optimal policies which are so far from observed practices during the recent downturn.

Our model features monopolistic competition and sticky prices, and assumes that monetary and fiscal policies are coordinated; a benevolent planner with full commitment controls inflation and distortionary taxes. We model LTs assuming shocks to preferences which raise the discount factor and induce agents to postpone consumption (e.g., Eggertsson and Woodford (2003, 2006) and Christiano, Eichenbaum and Rebelo (2011)). Realistically, we consider cases where these shocks are accompanied by simultaneous increases in spending levels.

Using our model we ask: What types of debt should the government issue in LTs, and also is there a role for active debt management during these episodes? Our findings are as follows: First, issuing only long term bonds is not (always) the optimal policy. Depending on the magnitude of the spending shocks and the initial debt level of the government, it may be that short term financing is optimal; when the initial liability is low and the increase in public spending that occurs when the economy enters the LT is substantial, the government prefers to issue short bonds. Second, the bond positions that emerge from our model are not constant through time. We find a role for revisions in the debt management strategy during LT episodes, whereby the government actively manages the maturity structure, in some cases removing long bonds from the secondary market.

To understand the first finding, that long bonds are not always optimal, note that when preference shocks hit, long bond prices increase so that a government that issues long debt experiences an increase in its liability. Whether or not is optimal to incur this loss in the value of debt depends on the response of current and future primary surpluses to preference and spending shocks. There are two effects in the model. First, spending shocks put the intertemporal budget of the government into deficit. Second, preference shocks lower future interest rates and so tend to increase the present value of the future surpluses to which the government must commit in order to repay a high initial debt level. If the first effect prevails, when spending shocks are substantial and initial debt levels are not, then the covariance between long bond prices and government deficits becomes positive and short term financing is optimal. If the second effect is the dominant one, the covariance is negative, and this restores the optimality of long term debt.

Our second finding, that optimal portfolios may be rebalanced during LT episodes, is due to the fact that in the presence of the ZLB constraint, the Ramsey equilibrium outcome is 'history dependent' even though we assume complete financial markets. This property is well known; in the presence of constraints involving forward expectations of future endogenous variables (in our case consumption and inflation) the history of shocks that hit the economy matters for allocations. Since in our model this involves changes in taxes, consumption and inflation through time, it impacts asset prices and government deficits, making it optimal to rebalance portfolios.

Section 2 of our paper describes the model economy and the Ramsey policy equilibrium. In Section 3, we consider a version of our model – with quasi-linear preferences – that delivers analytical results for the optimal portfolio. Since in this case real interest rates are not impacted by consumption growth (and therefore taxes), the government can only rely on committing to a higher level of future inflation to satisfy the ZLB constraint. As we show, in this model, the impact of the history of shocks on the optimal allocation is limited. This property is summarized by the fact that the Lagrange multiplier on the ZLB constraint remains constant through time. This turns out to be a special case of our model which implies that the optimal portfolio remains constant throughout the LT. Portfolio rebalancing during LTs obtains only under curved utility, when policy can influence future consumption levels through taxation. Section 4 relies on numerical simulations to solve the model under more general preferences, and Section 5 concludes.

This paper is closely related to two further strands of the literature. First, there is is a long stream of papers which characterize optimal fiscal and monetary policies under the ZLB and in a variety of policy environments, featuring either full commitment or discretion. Prominent examples are Eggertsson and Woodford (2003, 2006), Eggertsson (2006), Adam and Billi (2006), Werning (2011), Schmidt (2013), Jung et al (2013), Nakata (2015), and Bouakez, Guillard, and Roulleau-Pasdeloup (2017). The findings of these papers are obviously relevant to our analysis; for example the Ramsey policy outcome in our model ought to be similar to Eggertson and Woodford (2006). Since our focus is on debt management, we spend little time discussing the properties of taxes and inflation, since these are known from the previous literature. The only case where we insist on these properties is in Section 3, where we present analytical results under quasi-linear utility. To our knowledge, the expressions and results contained in Section 3 are new and complement previous findings in the literature.

Second, a recent literature studies the impact of long term asset purchases by the Federal Reserve (Quantitative Easing) during the recent downturn (see e.g., Chen, Curdia and Ferrero (2012), De Graeve and Theodoridis (2017) among others). In these models government bond markets are segmented, due to transaction costs and investors with 'preferred habitats'- trading only in an subset of the assets available.[1] These models are therefore suitable to analyze if long bond purchases had an impact on long term interest rates and thus on consumption and investment during the recession. We abstract from these features; our analysis builds on the presumption that markets are complete. But with our frictionless benchmark we offer an alternative view on why debt management during LTs is important, namely that it can stabilize the government's budget and mitigate the required tax adjustments in response to spending shocks and mounting debt levels. Our paper is therefore complementary to the recent literature on Quantitative Easing.

2 Model

2.1 Agents

2.1.1 Preferences

We consider an infinite horizon economy, populated by a representative household with preferences defined by

$$(1) \qquad E_0\left[\sum_{t=0}^{\infty}\beta^t\left(\widetilde{u}(c_t,\xi_t)+\widetilde{v}(h_t,\xi_t)\right)\right],$$

where c_t denotes consumption, h_t denotes hours, and β is the discount factor. The term ξ_t represents a shock to preferences that we will model as in Eggertsson and Woodford (2003, 2006). We assume that preferences are such that $\widetilde{u}(c_t,\xi_t)=\xi_t u(c_t)$ and $\widetilde{v}(h_t,\xi_t)=\xi_t v(h_t)$. A drop in ξ_t relative to $\xi_{t+j}, j=1,2...$ implies that the household wants to postpone consumption (and leisure) to the future.

2.1.2 Firms

The consumption good is produced by a representative, perfectly competitive, final-good producer using a Dixit-Stigltiz aggregator of a continuum of differentiated intermediate products. The production of a generic intermediate product i is carried out by a monopolistically competitive firm using a linear technology $y_{i,t}=h_{i,t}$. The demand for product i is given by $Y_t d(P_{i,t}/P_t)$, where $P_{i,t}$ is the price of intermediate product i, P_t is the price of the composite final good, and Y_t is output in the final-good sector. The demand function, d, satisfies additional assumptions that guarantee the existence of a symmetric equilibrium, namely, $d(1)=1$ and $d'(1)\equiv\eta<-1$.

We assume that the prices of intermediate goods are costly to adjust. Following Rotemberg (1982), adjustment costs are given by $\frac{\theta}{2}(\frac{P_{i,t}}{P_{i,t-1}}-1)^2$, where $\theta\geq 0$ governs the degree of price stickiness. When $\theta=0$, prices are fully flexible. When $\theta\to\infty$, prices will remain constant through time.

[1] See also Vayanos and Vila (2009), Gibaud, Nosbusch and Vayanos (2013) and Greenwood et al (2015) for alternative microfoundations of bond market clienteles and preferred habitat.

Intermediate-good producers seek to maximize

$$(2) \qquad E_t \sum_{k=0}^{\infty} \beta^k \frac{u_c(c_{t+k})\xi_{t+k}}{u_c(c_t)\xi_t} \left\{ \frac{P_{i,t+k}}{P_{t+k}} Y_{t+k} d\left(\frac{P_{i,t+k}}{P_{t+k}}\right) - w_{t+s} h_{i,t+k} - \frac{\theta}{2}\left(\frac{P_{i,t+k}}{P_{i,t+k-1}} - 1\right)^2 \right\},$$

subject to the constraint $h_{i,t+k} = Y_{t+k} d(P_{i,t+k}/P_{t+k})$. The first-order condition with respect to $P_{i,t}$ is given by

$$\frac{1}{P_t} Y_t d\left(\frac{P_{i,t}}{P_t}\right) + \frac{P_{i,t}}{P_t^2} Y_t d'\left(\frac{P_{i,t}}{P_t}\right) - w_t Y_t d'\left(\frac{P_{i,t}}{P_t}\right)\frac{1}{P_t} - \theta\left(\frac{P_{i,t}}{P_{i,t-1}} - 1\right)\frac{1}{P_{i,t-1}}$$
$$+\beta E_t \frac{u_{c,t+1}\xi_{t+1}}{u_{c,t}\xi_t} \theta \left(\frac{P_{i,t+1}}{P_{i,t}} - 1\right) \frac{P_{i,t+1}}{P_{i,t}^2} = 0,$$

where $u_{c,t} \equiv u'(c_t)$.

This equation forms the Phillips curve, describing the inflation output trade-off in our model. Imposing a symmetric equilibrium (all firms set the same price) gives

$$(3) \qquad (\pi_t - 1)\pi_t = \frac{\eta}{\theta}\left(\frac{1+\eta}{\eta} - w_t\right) Y_t + \beta E_t \frac{u_{c,t+1}\xi_{t+1}}{u_{c,t}\xi_t} (\pi_{t+1} - 1)\pi_{t+1},$$

where $\pi_t \equiv \frac{P_t}{P_{t-1}}$ denotes gross inflation.

2.1.3 Government and markets

The government engages in two activities — it levies taxes on the household's labor income and trades with the household in bond markets to finance a spending process $\{g_t\}_0^\infty$. We denote labor-income taxes by τ_t. Moreover, let B_t^j be the quantity of a bond issued in period t that promises to pay a unit of income in $t+j$. Let the price of such a bond be q_t^j. For simplicity, denote by $\mathcal{J}$ the set of available maturities.[2]

Following ABN, we study an economy in which government bonds can complete the market. That is, there exists a portfolio of bonds that perfectly insures the government's budget at any period and for every realization of the shocks. We characterize the optimal portfolio assuming that the government in every period buys back the entire stock of debt and then reissues, following ABN. Clearly under complete markets, it becomes immaterial whether governments repurchase long bonds or redeem them at maturity, but also the assumption of long debt buybacks seems a reasonable approximation of debt management policy during the recent downturn.[3]

[2]Notice that even though we can have $\mathcal{J} \equiv \{1, 2, ..., \bar{j}\}$, that is, allow the government to trade with any maturity ranging from one period to some maximum length $\bar{j}$, in practice, we will not need all of these trades to be realized. Since there will be only two states of shocks, it suffices to have $\mathcal{J} = \{1, N\}$ (a one-period bond and an N-period bond). This structure is standard in the macro debt-management literature.

[3]Recall that our model features coordinated monetary and fiscal policies and so it features a consolidated budget constraint. Thus from the point of view of the model, it is immaterial whether the FED or the Treasury carries out buyback operations.

The government budget constraint can be written as

$$\sum_{j\in\mathcal{J}} q_t^j B_t^j = \sum_{j\in\mathcal{J}} q_t^{j-1} B_{t-1}^j + P_t(g_t - \tau_t w_t h_t). \tag{4}$$

The initial debt level of the government will be denoted by $\overline{B}_{-1}$.

2.1.4 Household optimization

Given the policies described above, the household maximizes utility (1) subject to the sequence of budget constraints

$$\sum_{j\in\mathcal{J}} q_t^j B_t^{j,H} = \sum_{j\in\mathcal{J}} q_t^{j-1} B_{t-1}^{j,H} + P_t(1-\tau_t) w_t h_t - P_t c_t + \widetilde{\Pi}_t, \tag{5}$$

where $B_t^{j,H}$ represents the quantity of debt of maturity j demanded by the household in t and $\widetilde{\Pi}_t$ denotes profits of the firms operated by the household. From the household's optimization, we can derive the following optimality conditions:

$$(1-\tau_t)w_t = -\frac{v_{h,t}}{u_{c,t}},$$

which gives the tax rate as a function of the marginal utility ratio and w_t and

$$q_t^j = \beta^j E_t \frac{u_{c,t+j}\xi_{t+j} P_t}{u_{c,t}\xi_t P_{t+j}},$$

which equates the bond price with the marginal utility growth divided by the price ratio. Moreover, it holds that $q_t^0 = 1$. Notice further that since we assume a representative household, the prices q_t^j are also the prices of bonds in the *secondary market*. When the government buys back outstanding debt in the market it has to pay q_t^{j-1} for the $j-1$ maturity as is evident from equations (4) and (5).

2.2 Uncertainty

Our goal is to determine debt-management policies both before and during the LT. We therefore assume that, in period 1, the economy can be hit by a shock that lowers ξ_1 to a value $\underline{\xi} < \overline{\xi} = \xi_0$. This shock occurs with probability equal to ω. If the shock is realized then the value of the preference parameter remains equal to $\underline{\xi}$ with probability ϕ in each period until another shock arrives (at rate $1-\phi$) and thereafter $\xi_t = \overline{\xi}$ for all t. In other words, following a shock that lowers the value of ξ_1 the preference parameter is a first order Markov process with the transition matrix

$$P_\xi \equiv \begin{bmatrix} 1 & 0 \\ (1-\phi) & \phi \end{bmatrix}.$$

In the case where the shock to preferences does not occur in period 1 (probability $1-\omega$) we set $\xi_t = \overline{\xi}$ for all t.

Allowing the value of ξ_t to drop in period 1 is not sufficient for the economy to fall in a LT. It must be that the drop is large enough so that the zero lower bound is violated and policy reacts to satisfy the constraint. Because we will use a variety of different setups, it is not possible to find sufficient conditions under which the difference between $\underline{\xi}$ and $\overline{\xi}$ is such that the constraint binds. For each of the versions of the model we will consider, we will report the required value $\underline{\xi}$ that gives persistent LTs according to the process defined previously.

Turning to the process of government spending, which is assumed exogenous to the model, we let $g_t = \overline{g}$ (the steady-state value) in periods where the ZLB is not binding. When the economy is in the LT, we will set $g_t = \underline{g} \geq \overline{g}$. Preference shocks will be, in some parameterizations of the model, accompanied by increases in spending levels that will last for as long as the economy remains in the LT.

2.3 The Ramsey problem

We assume that the government maximizes the household's utility under full commitment as in ABN. We follow the primal approach; we eliminate the tax rate and the bond prices from the program using the equilibrium expressions for these objects.

Let $\mathbf{c}$ denote the sequence of consumptions $[c_0, c_1, ...]$ and similarly for the other variables of the model. A competitive equilibrium is a feasible allocation $\mathbf{c}, \mathbf{h}, \mathbf{g}$, a price system $\mathbf{w}, \mathbf{q}, \pi$ and a government policy $\mathbf{g}, \tau, \pi, \mathbf{b}$ such that given prices and the policy, $\mathbf{c}, \mathbf{h}, \pi$ solve the firms' and household's maximization problem and satisfy the sequence of government budget constraints.

The Ramsey program chooses $\tau, \pi, \mathbf{b}$ selecting the competitive equilibrium that maximizes (1). As is well known, under complete markets, this is equivalent to choosing $\mathbf{c}, \mathbf{h}, \pi, \mathbf{w}$ to maximize the household's utility subject to the following date 0 implementability constraint (e.g., Chari and Kehoe (1999), Faraglia et al (2010)):[4]

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}\xi_t}{u_{c,0}\xi_0} \left[-g_t + \left(1 + \frac{v_{h,t}}{u_{c,t} w_t} \right) w_t h_t \right] = \overline{B}_{-1}, \tag{6}$$

together with the Phillips curve (3) and the resource constraint,

$$c_t + g_t + \frac{\theta}{2}(\pi_t - 1)^2 = h_t. \tag{7}$$

Notice, however, that in our economy maximizing welfare subject to (3), (6) and (7) does not suffice to find an equilibrium policy that also satisfies the ZLB on the nominal interest rate. Therefore, the optimal allocation also needs to satisfy

$$q_t^1 \equiv \beta E_t \frac{u_{c,t+1}\xi_{t+1}}{u_{c,t}\xi_t \pi_{t+1}} \leq 1, \tag{8}$$

[4] (6) is the intertemporal budget of the government; it be easily derived by iterating forward on equation (4).

for all t, to ensure that the competitive equilibrium selected by the planner gives a sequence of short term rates $i_t = \frac{1}{q_t^1} - 1$ bounded from below by zero.

Let $(\lambda_s, \lambda_{p,t}, \lambda_{f,t}, \lambda_{ZLB,t})$ be the vector of Lagrange multipliers; we formulate the Lagrangian as:

$$\mathcal{L} = E_0 \sum_t \beta^t \Big\{ u(c_t)\xi_t + v(h_t)\xi_t + \lambda_{f,t}\left(h_t - c_t - g_t - \frac{\theta}{2}(\pi_t - 1)^2\right) - \lambda_{ZLB,t}\left(u_{c,t}\xi_t - \beta E_t \frac{u_{c,t+1}\xi_{t+1}}{\pi_{t+1}}\right)$$

$$-\lambda_{p,t}\left(u_{c,t}\xi_t\pi_t(\pi_t - 1) - \frac{\eta}{\theta}h_t u_{c,t}\xi_t\left(\frac{1+\eta}{\eta} - w_t\right) - \beta E_t u_{c,t+1}\xi_{t+1}\pi_{t+1}(\pi_{t+1} - 1)\right)\Big\}$$

$$-\lambda_s\left[E_0 \sum_{t=0}^{\infty} \beta^t u_{c,t}\xi_t\left[-g_t + (1 + \frac{v_{h,t}}{u_{c,t}w_t})w_t h_t\right] - u_{c,0}\xi_0\overline{B}_{-1}\right]. \tag{9}$$

2.3.1 Optimality

Denoting by $s_t \equiv -g_t + (1 + \frac{v_{h,t}}{u_{c,t}w_t})w_t h_t$ the per-period government's surplus the first-order conditions for the optimum can be written as

$$u_{c,t}\xi_t - \lambda_{f,t} + \lambda_{p,t}\frac{\eta}{\theta}h_t u_{cc,t}\xi_t\left(\frac{1+\eta}{\eta} - w_t\right) - u_{cc,t}\xi_t\left(\lambda_{ZLB,t} - \lambda_{ZLB,t-1}\frac{1}{\pi_t}\right) - \lambda_s\xi_t\left(u_{cc,t}s_t + u_{c,t}s_{c,t}\right)$$

$$+\lambda_s u_{c,0}\xi_0\overline{B}_{-1}\mathcal{I}_{t=0} = 0, \tag{10}$$

$$v_{h,t}\xi_t + \lambda_{f,t} - \lambda_s u_{c,t}\xi_t s_{h,t} + \lambda_{p,t}\frac{\eta}{\theta}u_{c,t}\xi_t\left(\frac{1+\eta}{\eta} - w_t\right) = 0, \tag{11}$$

$$-\theta\lambda_{f,t}(\pi_t - 1) - \lambda_{ZLB,t-1}\frac{u_{c,t}\xi_t}{\pi_t^2} = 0, \tag{12}$$

$$-\lambda_s s_{w,t} - \lambda_{p,t}\frac{\eta}{\theta}h_t = 0, \tag{13}$$

where $s_{c,t} \equiv -\frac{v_{h,t}}{u_{c,t}^2}u_{cc,t}h_t$ and $s_{h,t} = \frac{v_{hh,t}}{u_{c,t}}h_t + \left(1 + \frac{v_{h,t}}{u_{c,t}w_t}\right)w_t$, and $s_{w,t} = h_t$.

(10) is the first order condition for consumption in t. $\mathcal{I}$ denotes the indicator function, and therefore the last term in this equation applies solely in period 0 and if the outstanding liability of the government $\overline{B}_{-1}$ is non-zero.[5] (11) and (12) are the first order conditions with respect to hours and inflation. From (13), the first order condition with respect to wages, we get that $\lambda_{p,t} = -\lambda_s\frac{\theta}{\eta}$ (constant).

Finally, from complementary slackness, we have

$$\lambda_{ZLB,t}\left(u_{c,t}\xi_t - \beta E_t\frac{u_{c,t+1}\xi_{t+1}}{\pi_{t+1}}\right) = 0.$$

Together with the constraints from the planner's program, these equations yield the system that needs to be solved to obtain the optimal allocation.

[5] In the case of positive initial debt levels this term captures the incentive of the government to manipulate interest rates through changes in taxes (see Faraglia et al (2016)).

2.4 Optimal debt management

Given the solution to the planner's program we follow the approach of ABN to recover the optimal portfolio of the government. Since in our model there are two states of the world in each period the optimal portfolio can be found as a linear combination of two maturities provided that these assets give us a matrix of returns that is of full rank. As is standard in the literature, we consider a short bond, of one period maturity, and a long bond of maturity equal to N periods.

Let $x^t = ((\xi_0, g_0), ..., (\xi_t, g_t))$ denote the history of shocks until t and let (b_t^1, b_t^N) denote the optimal portfolio of the *real* value of short and long bonds issued in t. (b_t^1, b_t^N) can be found as a solution to the following system of equations:

$$\begin{bmatrix} \frac{1}{\pi\left(x^t,(\overline{\xi},\overline{g})\right)} & \frac{q_{t+1}^{N-1}\left(x^t,(\overline{\xi},\overline{g})\right)}{\pi\left(x^t,(\overline{\xi},\overline{g})\right)} \\ \frac{1}{\pi\left(x^t,(\underline{\xi},\underline{g})\right)} & \frac{q_{t+1}^{N-1}\left(x^t,(\underline{\xi},\underline{g})\right)}{\pi(\left(x^t,(\underline{\xi},\underline{g})\right)} \end{bmatrix} \cdot \begin{bmatrix} b_t^1 \\ b_t^N \end{bmatrix} = \begin{bmatrix} S_{t+1}\left(x^t,(\overline{\xi},\overline{g})\right) \\ S_{t+1}\left(x^t,(\underline{\xi},\underline{g})\right) \end{bmatrix}. \tag{14}$$

where $S_{t+1}\left(x^t,(\underline{\xi},\underline{g})\right)$ is the present discounted value of the government surplus when the economy remains in the LT, meaning that following history x^t we have $(\xi_{t+1}, g_{t+1}) = (\underline{\xi},\underline{g})$. Analogously, $S_{t+1}\left(x^t,(\overline{\xi},\overline{g})\right)$ is the surplus when the economy experiences in $t+1$ an increase in the value of ξ_t and permanently escapes the LT.

According to (14) the government issues a portfolio in period t that ensures that the intertemporal constraint is satisfied with equality in both states in $t+1$ given the future policies for taxes and inflation. System (14) has a unique solution if and only if $q_{t+1}^{N-1}\left(x^t,(\overline{\xi},\overline{g})\right) \neq q_{t+1}^{N-1}\left(x^t,(\underline{\xi},\underline{g})\right)$.

2.5 Long-run distortion smoothing

Under optimal debt management, the government exploits variations in bond prices that help reduce the real payout of debt in periods of large fiscal needs; tax adjustments do not have to be substantial. In Sections 3 and 4 we will use (14) to characterize the optimal portfolios that support the tax smoothing objective. Here, we provide an analytical insight on the importance of debt management in smoothing tax distortions, focusing on the properties of the economy after it has escaped from the LT, and permanently settled to the new long run equilibrium. We show that the values of government debt, taxes and inflation at the end of the episode are independent of the duration of the episode and the magnitude of the shocks that hit the economy during the LT. This important model property is summarized in the following Proposition.

Proposition 1. *Consider an arbitrary realization of the shocks whereby the economy enters the LT in period 1, and period $T \geq 1$ is the last period in which $\xi_t = \underline{\xi}$. Thereafter, $\xi_t = \overline{\xi}$. After period $T+1$ i) the optimal inflation rate is 0. ii) the optimal tax rate equals $\overline{\tau}$ and the debt level equals $\overline{B}$ (constants). iii) $\overline{\tau}$ and $\overline{B}$ are independent of x^T and T as well as the magnitude of the shocks which occur from period 1 to T.*

Proof. See Appendix.

Notice that Proposition 1 tells us essentially that long-run allocations are history independent; following a sequence of preference and spending shocks $\{x_t\}_1^T = \{(\underline{\xi}, \underline{g}), ..., \underline{\xi}, \underline{g})\}$, the long run values of consumption, hours, inflation and taxes are independent of the random variable T. History independence is, however, only a long run property in our model with the ZLB constraint. In the short run, the presence of the constraint makes the history of shocks matter for the allocation; this can be easily seen by noticing the presence of the lagged value of the multiplier $\lambda_{ZLB,t-1}$ in the first order conditions. This property differentiates our model from the existing literature on optimal debt management under complete markets. We discuss the impact of the history of shocks on allocations and optimal portfolios in the following Sections.

2.6 Functional forms

For the remaining sections we assume preferences of the form:

$$u(c_t) + v(h_t) = \frac{c_t^{1-\gamma_c} - 1}{1 - \gamma_c} - \psi \frac{h_t^{1+\gamma_h}}{1 + \gamma_h}, \tag{15}$$

where γ_c represents the relative risk aversion coefficient and γ_h is the inverse of the Frisch elasticity of labor supply. In Section 3 we will consider the special case $\gamma_c = 0$ (quasi-linear preferences). This case is convenient for two reasons: First, as we show below, the multiplier $\lambda_{ZLB,t}$ is constant, which greatly attenuates the influence of history dependence on the optimal allocation. Second, under quasi-linear preferences, the model solution is tractable enough to allow for useful analytical insights on optimal debt management when the shocks are not persistent. In Section 4, we study the (more plausible) scenario $\gamma_c > 0$.

3 A Special Case: Quasi-Linear Preferences

Under quasi-linear preferences ($\gamma_c = 0$), the system of first order conditions becomes:

$$\xi_t = \lambda_{f,t}, \tag{16}$$

$$v_h(h_t)\xi_t + \xi_t - \lambda_s \xi_t \left[v_{hh}(h_t)h_t + \left(1 + v_h(h_t)\frac{1}{w_t}\right) w_t \right] + \lambda_p \frac{\eta}{\theta} \xi_t \left(\frac{1+\eta}{\eta} - w_t \right) = 0, \tag{17}$$

$$-\theta \xi_t (\pi_t - 1) - \lambda_{ZLB,t-1} \frac{\xi_t}{\pi_t^2} = 0, \tag{18}$$

The following Proposition shows the properties of inflation, hours and taxes in the model.

Proposition 2: Inflation, Hours and Taxes with Quasi-Linear Utility *Assume that preferences are of the form (15) with $\gamma_c = 0$. Assume $\xi_t = \underline{\xi}$ for $t = 1, 2, .., T$ and $\xi_t = \overline{\xi}$ for $t \geq T + 1$.*

i) The optimal path of inflation is given by:

$$\pi_t = \begin{cases} 1 & t = 0,1 \text{ and } t = T+2, T+3, ... \\ \\ \beta \frac{\phi \underline{\xi} + (1-\phi)\overline{\xi}}{\underline{\xi}} \equiv \underline{\pi} & t = 2,3,...,T+1 \end{cases} \tag{19}$$

ii) Hours are constant over time so that $h_t = \overline{h}$. The optimal tax rate satisfies:

$$\tau_t = \begin{cases} \omega_{\lambda_s}\left(1 - \lambda_s \gamma_h - \widetilde{\eta}\right) & t = 0, t \geq T+2 \\ \\ \omega_{\lambda_s}\left(1 - \lambda_s \gamma_h - \frac{1}{\frac{1+\eta}{\eta} + \kappa \underline{\pi}^2(\underline{\pi}-1)}\left[1 + \lambda_s \kappa \underline{\pi}^2(\underline{\pi}-1)\right]\right) & t = 1 \\ \\ \omega_{\lambda_s}\left(1 - \lambda_s \gamma_h - \frac{1}{\frac{1+\eta}{\eta} + \kappa \underline{\pi}(\underline{\pi}-1)^2}\left[1 + \lambda_s \kappa \underline{\pi}(\underline{\pi}-1)^2\right]\right) & t = 2,3,....T \\ \\ \omega_{\lambda_s}\left(1 - \lambda_s \gamma_h - \frac{1}{\frac{1+\eta}{\eta} - \kappa \underline{\pi}(\underline{\pi}-1)}\left[1 - \lambda_s \kappa \underline{\pi}(\underline{\pi}-1)\right]\right) & t = T+1 \end{cases} \tag{20}$$

where $\omega_{\lambda_s} \equiv \frac{1}{1-\lambda_s(1+\gamma_h)}$ and $\widetilde{\eta} = \frac{\eta}{1+\eta}$. Moreover, $\kappa = \frac{\theta}{\eta \overline{h}}$.

Proof. See the Appendix.

Several comments are in order: Notice first that under quasi-linear utility it suffices to pin down hours, inflation and taxes in order to compute the equilibrium. Since $u(c_t) = c_t$ consumption levels have no influence on the nominal interest rate and therefore the value of c_t does not need to be considered when finding the equilibrium in this model; c_t is a residual that is set to satisfy the resource constraint.[6] Second, with $u_{cc,t} = 0$ the government essentially has only one instrument, inflation, to satisfy the ZLB constraint which now is $\beta E_t \frac{\xi_{t+1}}{\xi_t \pi_{t+1}} \leq 1$. Since from (18) inflation in $t+1$ depends only on the value of $\lambda_{ZLB,t}$ which is predetermined in $t+1$, the optimal inflation rate is independent of whether the economy remains in the LT in $t+1$ or exits from the trap in $t+1$. Third, even though taxes are not useful to satisfy the ZLB, from (20) we see that the optimal tax rate changes over time. This reflects the fact that inflation impacts wages, i.e. from equation (3). To maintain constant hours, which is the optimum under complete markets, the planner has to compensate for the movements in wages with changes in the tax schedule.

Given the path of taxes, wages and inflation we can derive the government's surplus in period t as

$$s_t \equiv \tau_t w_t \overline{h} = \overline{h}\left(w_t - \frac{1+\eta}{\eta}\right) + \overline{h}\frac{1+\eta}{\eta}\left(1 - \psi \overline{h}^{\gamma_h} \frac{\eta}{1+\eta}\right) \equiv \overline{h}\left(w_t - \frac{1+\eta}{\eta}\right) + \overline{s}. \tag{21}$$

[6] Practically, the problem is not defined if we assume that $\theta \to \infty$ since in this case utility diverges to minus infinity. Large values of θ could give us that $c_t < 0$ for some t. Here, we have not imposed non-negativity constraints on consumption; we assume that θ is such that the model always gives us a consumption level exceeding zero.

In other words, the surplus in t is equal to the long run value $\overline{s}$ plus the term $\overline{h}\left(w_t - \frac{1+\eta}{\eta}\right)$, which measures the deviation of wages in t from their long run value $\frac{1+\eta}{\eta}$. (21) can be written as:

$$
s_t = \begin{cases} \overline{s}, & t = 0, t \geq T+2 \\ \\ \underline{\underline{s}} \equiv \overline{s} + \frac{\theta}{\eta}\underline{\pi}^2(\underline{\pi}-1) - (\underline{g} - \overline{g}), & t = 1 \\ \\ \underline{s} \equiv \overline{s} + \frac{\theta}{\eta}\underline{\pi}(\underline{\pi}-1)^2 - (\underline{g} - \overline{g}), & t = 2, 3, ...T \\ \\ \overline{\overline{s}} \equiv \overline{s} - \frac{\theta}{\eta}\underline{\pi}(\underline{\pi}-1), & t = T+1 \end{cases} \tag{22}
$$

which, consistent with previous derivations, defines four different values of s_t, for when the economy enters the LT, for when it remains in the LT, for the period it leaves the trap and in the long run equilibrium. We will later use (22) to derive analytically the properties of optimal debt management.

3.1 Optimal debt management under quasi-linear preferences: Analytical results

Our previous derivations may have given to the reader the impression that this model admits a closed form solution. In practice, however, it does not; to calculate the values of s_t, we have to recover the value of λ_s which satisfies the intertemporal budget constraint of the government. Since λ_s enters in a nonlinear fashion in the expressions derived above, the model needs to be solved with numerical methods. In the Online Appendix we present a simple numerical algorithm to do this.

In this section, we derive the qualitative features of optimal portfolios analytically and for this, we make further simplifying assumptions. We set $\phi = 0$, so that shocks to preferences and spending are purely transitory, which in turn implies that the LT episode lasts for only one period. In this way we can essentially do away with $\underline{s}$ defined in (22), since deterministically $T = 1$. This simplifies the algebra considerably and is without loss of generality.

When $\phi = 0$ we can easily show that $\pi_2 = \underline{\pi} = \beta\frac{\overline{\xi}}{\underline{\xi}}$. Moreover, the long bond prices in period 1 are given by: $q_1^{N-1}\left(x^0, (\underline{\xi}, \underline{g})\right) = \frac{\beta^{N-1}}{\underline{\pi}}\frac{\overline{\xi}}{\underline{\xi}} = \beta^{N-2}$, if the economy falls in the LT, and $q_1^{N-1}\left(x^0, (\overline{\xi}, \overline{g})\right) = \beta^{N-1}$, if the preference shock does not occur in $t = 1$. We can derive the present discounted value of the government's surplus in each of the two possible states in $t = 1$ as:

$$
S_1\left(x^0, x_1\right) = \begin{cases} \frac{\overline{s}}{1-\beta} & x_1 = (\overline{\xi}, \overline{g}) \\ \\ \underline{\underline{s}} + \beta\frac{\overline{\xi}}{\underline{\xi}}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right) & x_1 = (\underline{\xi}, \underline{g}) \end{cases} \tag{23}
$$

From (14) the optimal portfolio is given by:

$$b_0^1 = \frac{\beta^{N-2} S_1\left(x^0,(\overline{\xi},\overline{g})\right) - \beta^{N-1} S_1\left(x^0,(\underline{\xi},\underline{g})\right)}{\beta^{N-2}(1-\beta)} \quad \text{and} \quad b_0^N = \frac{-S_1\left(x^0,(\overline{\xi},\overline{g})\right) + S_1\left(x^0,(\underline{\xi},\underline{g})\right)}{\beta^{N-2}(1-\beta)}, \tag{24}$$

showing that the signs of the bond positions (b_0^1, b_0^N) hinge crucially on the relative magnitudes of $S_1\left(x^0,(\overline{\xi},\overline{g})\right)$ and $S_1\left(x^0,(\underline{\xi},\underline{g})\right)$. As can be easily seen from (24), whenever the present value of the surplus increases during LT episodes, the government will issue long bonds and the standard results of the debt management literature will apply in our model. However, the opposite holds if during LTs fiscal revenues decrease and the intertemporal budget goes into deficit.

To gain insights on how S_1 is impacted by the preference shock note that combining (22) and (23) yields

$$S_1\left(x^0,(\underline{\xi},\underline{g})\right) = \overline{s} + \beta\frac{\overline{\xi}}{\underline{\xi}}\left(\frac{\overline{s}}{1-\beta}\right) - (\underline{g}-\overline{g}). \tag{25}$$

Let us use (25) to study the following scenarios:

1. Assume that $\overline{B}_{-1} = 0$ and $\underline{g} = \overline{g}$. It then holds that $\overline{s} = 0$.[7] (24) gives us $b_0^1 = b_0^N = 0$.

   Why is it optimal to not issue any long or short debt at all? Suppose that the government had chosen $b_0^N > 0$ and $b_0^1 < 0$. Since $q_1^{N-1}\left(x^0,(\underline{\xi},\underline{g})\right) = \beta^{N-2} > q_1^{N-1}x\left({}^0,(\overline{\xi},\overline{g})\right) = \beta^{N-1}$, the real payout of government debt in $t=1$ would increase in state $(\overline{\xi},\overline{g})$. Then the government would have to commit to increase taxes (permanently) to finance the debt. The opposite holds $b_0^N < 0$ and $b_0^1 > 0$. Portfolios different from $(0,0)$ give rise to unnecessary fluctuations in the tax schedule.

2. Assume that $\overline{B}_{-1} = 0$ and $\underline{g} > \overline{g}$. Assume further that the shock to spending is sufficiently small so that $\overline{s} \approx 0$. (24) gives:[8]

   $$b_0^1 \approx \frac{\beta(\underline{g}-\overline{g})}{1-\beta} > 0 \quad \text{and} \quad b_0^N \approx -\frac{(\underline{g}-\overline{g})}{\beta^{N-2}(1-\beta)} < 0. \tag{26}$$

   Therefore, the government optimally issues short term debt (financed through long term savings) in period 0.

   Since bond prices rise during LTs, taxes will be smoother if the value of debt depreciates when the government's budget goes into deficit. By holding long private bonds and issuing short term debt, the government can benefit from the capital gain.

3. Finally, consider the case where $\overline{B}_{-1} > 0$ and $\underline{g} = \overline{g}$. We now have $\overline{s} > 0$ to finance the initial debt level. Moreover, $S_1\left(x^0,(\underline{\xi},\underline{g})\right) > S_1\left(x^0,(\overline{\xi},\overline{g})\right)$. By (24), we obtain $b_0^N > 0$.

[7] The present value of the surplus at date 0 is $\overline{s} + \beta\left[\omega S_1\left(x^0,(\underline{\xi},\underline{g})\right) + (1-\omega)S_1\left(x^0,(\overline{\xi},\overline{g})\right)\right]$. It must be that $\overline{s} = 0$ to satisfy the intertemporal budget at date zero. The value of λ_s will be pinned down accordingly.

[8] Given the previous derivations a shock in spending in period 1 requires to set $\overline{s} > 0$ in order to balance the budget. The deficit during the LT needs to be compensated with a (small) positive surplus in other periods.

When the initial debt level is positive, the government commits to a sequence of positive surpluses in future periods to repay the debt. Since $\frac{\bar{\xi}}{\underline{\xi}} > 1$, $\bar{s} + \beta\frac{\bar{\xi}}{\underline{\xi}}\left(\frac{\bar{s}}{1-\beta}\right)$ exceeds $\frac{\bar{s}}{1-\beta}$ and therefore, during the LT, the present value of the future surpluses increases. Unless $b_0^N > 0$ the government would have to reduce taxes following the shock in preferences.

To summarize, the optimal portfolio is determined through the balance of the following forces. When the preference shock hits, long bond prices increase. Governments that issue long-term debt in period 0 suffer a capital loss. Whether or not it is optimal to incur this loss depends on the behavior of the present value of the surplus. When $S_1\left(x^0, (\underline{\xi}, \underline{g})\right) < S_1\left(x^0, (\bar{\xi}, \bar{g})\right)$ (as in the case where spending increases and $\overline{B}_{-1} = 0$) the optimal debt management strategy is to hold long term savings. However, when debt is positive there is a second force that makes issuing long debt optimal. Due to the 'discounting impact' of the fall in interest rates, we may have $S_1\left(x^0, (\underline{\xi}, \underline{g})\right) > S_1\left(x^0, (\bar{\xi}, \bar{g})\right)$; to stabilize taxes the government must issue long debt.

3.2 Portfolio rebalancing

Using our previous derivations and focusing on the case where $\phi > 0$, we can investigate another important property of debt management in the LT; whether the optimal portfolio needs to be rebalanced over time, in the sense that an active management of maturity emerges from the solution of (14). The following proposition states the result.

Proposition 3. *Optimal debt management under $\gamma_c = 0$ (quasi-linear preferences) and $\phi > 0$ features portfolio rebalancing only at the time the economy enters the LT (i.e., at $t = 1$). Thereafter, no further rebalancing occurs: the optimal portfolio remains constant until the economy escapes from the trap.*

Proof. See Appendix.

To understand why no further rebalancing takes place once the economy has entered the LT, notice that (i) bond prices are constant during the LT, since from (19) the optimal inflation rate is constant, and (ii) the sequences $S_{t+1}\left(x^t, (\underline{\xi}, \underline{g})\right)$ and $S_{t+1}\left(x^t, (\bar{\xi}, \bar{g})\right)$ are constant for $t = 1, 2, ..T - 1$ (this follows from (22)). Therefore, the solution of (14) delivers exactly the same portfolio in every period during the LT. In the Appendix we prove the above formally.

The limited extent of portfolio rebalancing implied by Proposition 3 reflects the fact that the history of shocks exerts only a limited influence on the optimal allocation under quasi-linear preferences. This is evident for example from equations (20) and (22), which show that the duration of the LT episode T essentially does not impact taxes and the surplus after $t = 2$ and until the economy escapes from the trap. This property of the model with quasi-linear preferences derives directly from the constancy of the Lagrange multiplier λ_{ZLB} when the ZLB is binding.[9] This multiplier is a state variable in the model; it is a essentially a variable that 'remembers' the promises of the government to raise inflation in the future. Since it is constant and therefore independent of T, the duration of the episode exerts

[9]Combining (18) and (19) it is easy to see that the multiplier $\lambda_{ZLB,t-1} = -\theta\underline{\pi}^2(\underline{\pi} - 1)$ remains constant when the ZLB is binding.

only a limited impact on allocations, bond prices and the sequence of government surplus. As we shall see next, however, in the more plausible case $\gamma_c > 0$, where λ_{ZLB} is not constant, optimal allocations and debt management change over time, during the LT.

4 Optimal Debt Management in the General Case

We now consider the general case where $\gamma_c > 0$ and $0 < \phi < 1$. In this case, the model does not have an analytical solution and a numerical algorithm is required to find the optimal allocation. In the Online Appendix we discuss the challenges of solving this model numerically, in the presence of time varying λ_{ZLB}, and propose a solution method based on the Parameterizing Expectations Algorithm (PEA) of den Haan and Marcet (1990). Our algorithm is similar to numerical procedures that have been used in the literature to solve optimal Ramsey policy under incomplete markets, which also feature time varying Lagrange multipliers.[10]

4.1 Calibration

In order to solve the model numerically we need to give values to the structural parameters. Each period represents a quarter, and therefore we set $\beta = 0.99$. We set $\gamma_h = 1$ so that the Frisch elasticity of labor supply is equal to 1, and choose ψ so that in the deterministic steady state, the household spends 20 percent of its unitary time endowment in market work. Notice that the value of ψ depends on the initial debt level assumed. We will study cases where $\overline{B}_{-1} = 0$ as in ABN, but also cases where the initial debt is positive. In each case we adjust the value of ψ to hit the hours target. Finally, we set $\gamma_c = 1$. In the Online Appendix we experiment with higher values of the risk aversion coefficient.

To calibrate η and θ, we follow Schmitt-Grohe and Uribe (2004) and choose values of -6 and 17.5, respectively. In the deterministic steady state, the level of public expenditure is set to 20% of output and therefore we have $\overline{g} = 0.04$. In the numerical experiments we will consider values for $\underline{g} \in \{1, 1.04, 1.08\}\overline{g}$.[11] In the US government spending was at 19.4 percent of output in 2007 (20.4 in 2008), before rising to 21.4 percent in 2009 and 21.2 in 2010.[12] These numbers are in range of the positive spending shocks scenarios we consider here.

To calibrate the preference shock process, we proceed as follows: First, we set, $\phi = 0.8$. This gives us an average duration of LT episodes equal to 5 quarters, well within the range of values considered in the literature. Second, we normalize $\overline{\xi} = 1$. Third, to calibrate $\underline{\xi}$, we take the following steps: For each version of the model that we solve, we compute the optimal allocation in an economy without

[10] See for example Aiyagari et al (2002) and Faraglia et al (2017).

[11] These values are consistent with the relevant literature. For example, Faraglia, Marcet and Scott (2010) assume a two state annual spending process where the states are 7 percent higher and lower than the mean. In Schmitt-Grohe and Uribe (2004) the log of quarterly government spending has conditional standard deviation of 0.03; therefore its two state analogue would have values 3 percent higher and lower than the mean. Assuming values 4 percent above steady state is therefore comparable to Schmitt-Grohe and Uribe (2004). When we consider values 8 percent higher than $\overline{g}$ we are targeting a high volatility scenario.

[12] These numbers are taken from the FRED, 'Shares of gross domestic product: Government consumption expenditures and gross investment' series.

preference shocks. Denote the short bond price that derives from this (history independent) allocation by $q^{HI} = \beta\frac{(1-\phi)u_c^L+\phi u_c^H}{u_c^H}$ where u_c^L (u_c^H) denotes the marginal utility of consumption when spending is at $\overline{g}$ ($\underline{g}$). We then find the value of $\underline{\xi}$ such that $\widetilde{q} \equiv \beta\frac{(1-\phi)u_c^L+\phi u_c^H\underline{\xi}}{\underline{\xi}u_c^H} = 1+\epsilon$, where $\epsilon = 0.0011$.[13]

Finally, we set $\omega = 0.5$ in our benchmark calibration. In the Online Appendix we show that our results are robust to alternative calibrations for ω. Table 1 summarizes the parameter values discussed above.

[Table 1 About Here]

4.2 Zero initial debt

4.2.1 Behavior of endogenous variables

We start by discussing the case in which the initial level of government debt is zero. We first show the response of the economy to the LT shock, showing the behavior of endogenous variables that are useful to subsequently characterize the paths of bond prices, surpluses and government portfolios. Figure 1 traces the behavior of consumption, taxes, inflation and λ_{ZLB} after the preference shock in period 1. Each of the subplots shows simulations over 15 model periods. The solid lines correspond to the case where $\underline{g} = \overline{g}$, the dashed lines, $\underline{g} = 1.04\overline{g}$ and the crossed lines, $\underline{g} = 1.08\overline{g}$.

[Figure 1 About Here]

The properties which emerge from the figure are similar to Eggertsson and Woodford (2003, 2006). When the shocks hit the planner decreases consumption below steady state but commits to gradually increase it over time if the LT persists. The ZLB constraint is satisfied along the optimal path because the planner also increases the consumption level at the 'exit' (period $T+1$, middle panel). The adjustment of consumption, is explained by the behavior of taxes. The middle-bottom panel shows the response of the (annualized) inflation rate in percentage points. Inflation displays very little volatility. This property can be traced to the (high) value of θ we have assumed in the calibration (see for example Schmitt-Grohe and Uribe (2004)).

4.2.2 Optimal portfolios

Table 2 shows optimal government portfolios under the zero initial debt scenario. The table reports separately portfolios when $\underline{g} = \overline{g}$, $\underline{g} = 1.04\overline{g}$ and $\underline{g} = 1.08\overline{g}$. Columns 2-3 show the optimal mixture between short and long bonds expressed as percentages of steady state GDP; Columns 4-5 show the price of long term debt under $x_{t+1} = (\underline{\xi}, \underline{g})$ (Column 4) and $x_{t+1} = (\overline{\xi}, \overline{g})$ (Column 5). Finally, Columns 6-7 report the present discounted value of the surplus.

[13] To clarify the above, note that for different parameterizations of the spending process, we need to assume different values for $\underline{\xi}$ to make the ZLB bind when $\gamma_c > 0$. Since spending shocks will lower bond prices, $\underline{\xi}$ needs to drop with the difference $\underline{g} - \overline{g}$. Moreover, note that preference shocks do not impact consumption and inflation if the ZLB constraint does not bind. This makes finding u_c^L and u_c^H sufficient to pin down $\widetilde{q}$.

Clearly, in the case where $\underline{g} = \overline{g}$, $\widetilde{q}$ equals $\beta\frac{(1-\phi)+\phi\underline{\xi}}{\underline{\xi}}$. We find $\underline{\xi} = 0.9469$ in this case. Moreover, we have $\underline{\xi} = 0.9430$ (0.9391) when $\underline{g} = 1.04\overline{g}$ ($\underline{g} = 1.08\overline{g}$).

[Table 2 About Here]

We report the above quantities over 4 different periods. At $t = 0$ we report the optimal portfolio that is determined prior to the realization of the preference shock. We also report the prices and the surpluses that will prevail in $t = 1$, depending on the state of the economy in that period. The remaining rows report the portfolios in $t = 1, 4, 9$ and the prices and surpluses in $t = 2, 5, 10$ respectively. Our goal is to identify 'portfolio rebalancing', i.e. changes in the optimal debt management policy during LT episodes.

Model without spending shocks. Consider first the case $g_t = \overline{g}$. The values of short and long bond positions are close to zero. This holds for all periods reported. These predictions are consistent with our previous theoretical findings (under quasi-linear preferences, in Section 3). As reported in the last two columns of the table, it holds that $S\left(x^t, (\underline{\xi}, \underline{g})\right) \approx S\left(x^t, (\overline{\xi}, \overline{g})\right) \approx 0$ in all periods. It is trivial to show that the solution to (14) will give us long and short term debt close to zero.

A well known prediction of the debt management literature is that governments take positions in the bond market that are several multiples of GDP. This is shown in several papers that study optimal portfolios assuming that government debt is initially zero (e.g., ABN, Faraglia Marcet and Scott (2010), Nosbusch (2008)). The intuition is that government spending shocks change significantly the present value of the surplus, however, they do not impart a large effect on bond prices. In the case of preference shocks, however, the opposite holds. Bond prices show considerable variability;[14] the intertemporal value of the surplus does not.

Model with spending shocks. We now turn to the case where $\underline{g} > \overline{g}$. Table 2 shows that (b_t^1, b_t^N) equals (76.7%, -82.4%) in $t = 0$ when spending increases by 4 percent during the LT. In $t = 1, 4, 9$ we have (75.8%, -81.5%), (74.5%, -80.3%) and (74.0%, -79.8%) respectively. As explained above, governments want to issue short term debt when $cov(q_{t+1}^{N-1}, S_{t+1}) < 0$ (where cov denotes the covariance). The numbers reported in Columns 4-7 show that this property holds in the model when we introduce spending shocks. Bond prices increase in periods when $x_t = (\underline{\xi}, \underline{g})$ (to roughly 0.956) and drop (to around 0.914) when the economy escapes from the LT. Due to the rise in public spending the government runs a deficit during the LT.

Notice further that when we increase the variance of the spending shocks, we find even larger positions of short debt and long savings. When $\underline{g}$ is 8% higher than in the steady state the optimal portfolios are (153.4%, -164.8%), (151.6%, -163.0%), (149.2%, -160.5%) and (148.3%, -159.6%) for $t = 0, 1, 4, 9$ respectively. This may seem surprising if we expect that the larger variability of spending shocks should bring us closer to the ABN benchmark, where long debt is optimal. The reason this does not occur in our simulations is again that while preference shocks generate large swings in asset prices, spending shocks do not. Instead, spending shocks impact considerably the intertemporal budget of the government, and the larger they are the larger the deficit during the LT. Under zero initial debt, in the presence of both preference and spending shocks, governments will always want to issue short debt.

[14] The property that preference shocks generate large variability in interest rates is discussed (for example) in Ravenna and Seppälä (2006).

Portfolio rebalancing (under zero initial debt). A key property that emerges clearly from Table 2 is that the government changes its debt management strategy over time. As discussed previously, this property is attributed to the fact that the model features a state variable, λ_{ZLB}, which is now time varying, and as we previously showed, consumption, inflation and taxes change during the LT. This obviously impacts bond prices and government surpluses and hence it impacts the bond portfolio. The changes in the optimal portfolio are obviously not substantial, they only amount to a slight fanning in of bond positions during LT episodes and in cases where spending shocks are positive. As we shall later show, assuming positive initial debt is sufficient to make revisions of optimal debt management during LTs, large.

4.3 Positive initial debt

We now consider the more empirically plausible scenario of positive government debt. In Table 3, we assume that $\overline{B}_{-1}$ is 60% of GDP at the annual horizon (240% at the quarterly horizon). This value is close to the actual debt to GDP ratio in the US in 2007-8.[15]

[Table 3 About Here]

Notice first that with positive initial government debt, the optimal debt management strategy changes dramatically relative to previous findings. We now find that the government finds optimal to issue long-term debt. For example, when $\underline{g} = \overline{g}$, the portfolio is (-29.3%, 304.6%) at $t = 0$; when $\underline{g}$ is 4% above the steady-state value, we have (b_0^1, b_0^N)=(46.4%, 223.5%); thus mostly long bonds are issued.

Our previous remarks can explain the change in optimal debt management. When initial debt is positive, the government must commit to run positive surpluses on average in order to redeem its initial liability. When the preference shock hits the economy in period 1, the value of S increases when government spending does not rise considerably, due to the discounting effect of interest rates on the future primary surpluses. As the table shows, in all cases we have $S_{t+1}\left(x^t, (\underline{\xi}, \underline{g})\right) > S_{t+1}\left(x^t, (\overline{\xi}, \overline{g})\right)$, and hence $cov(q_{t+1}, S_{t+1}) > 0$. This restores the optimality of long term debt.

The relative positions of short and long bonds hinge crucially on the magnitude of the fiscal shocks with larger shocks leading to more balanced portfolios, since they lower the impact of the preference shocks on $S_{t+1}\left(x^t, (\underline{\xi}, \underline{g})\right)$ and therefore the covariance of q_{t+1} and S_{t+1}. When spending shocks are 8 percent above the steady state, the optimal portfolio becomes $(122.0\%, 142.4\%)$ in $t = 0$. The share of short-term debt in the total market value of government debt is now nearly 50 percent! Obviously, if we consider even larger values for the spending shocks, we can obtain portfolios tilted towards short debt and even portfolios where the government prefers to hold long private bonds.[16]

Portfolio rebalancing (under positive initial debt). We have shown that optimal debt management calls for issuing both short and long bonds under the high debt/high spending scenario. Short

[15] In particular the number are as follows: in 2007 the ratio of public debt to GDP was around 62 percent (64 in 2008). These numbers are taken from the FRED, 'Total Public Debt as Percent of Gross Domestic Product' series.

[16] This however requires to reverse the sign of $cov(q_{t+1}, S_{t+1})$ and our simulations suggest, it requires incredibly large values of spending shocks. For example, even when we increase spending by 10 or 12 percent above steady state values, corresponding to a persistent increase equal to 2 (2.4) percent of output, we continue to find that the government optimally issues some long debt.

debt is a good hedge for the government against fiscal shocks during LTs; long debt offers insurance against the effect of lower rates on the present discounted value of future surpluses. We now turn to the second crucial aspect of debt management in LTs, portfolio rebalancing. Focusing on the cases of positive spending shocks, Columns 2 and 3 of Table 3 show that the government revises considerably the debt management strategy over time. For example, under $\underline{g} = 1.04\overline{g}$, we obtain (b_t^1, b_t^N) equal to $(46.4\%, 223.5\%)$ at $t = 0$ $(54.4\%, 215.0\%)$ at $t = 1$, and $(69.7\%, 198.9\%)$ at $t = 9$. In the case where $\underline{g} = 1.08\overline{g}$ these numbers are $(122.0\%, 142.4\%)$, $(129.4\%, 134.7\%)$ and $(143.4\%, 120.0\%)$ respectively. In other words, the government increases considerably the share of short term debt in the portfolio, removing long term bonds from the hands of the private sector! Moreover, the adjustments are in both cases large fractions of GDP and considerably larger than in the case of zero initial debt we previously studied. This is explained by the fact that at positive debt levels the solution to (14) displays higher sensitivity to changes in $S_{t+1}\left(x^t, (\overline{\xi}, \overline{g})\right)$ during LT episodes.[17]

4.4 Sensitivity analysis

We study the sensitivity of our results to perturbations to the following parameters: the degree of risk aversion, the degree of price rigidity, the persistence of LTs, and their probability of occurrence. To conserve space, we only briefly summarize our main findings here, and refer the reader to the Online Appendix for a full description and a detailed discussion of the results. We find that the degree of risk aversion, the degree of price stickiness, and the likelihood of falling in a LT are of little relevance for the optimal portfolio. The latter is more sensitive to the duration of LTs in absolute levels, but the qualitative features of the solution are preserved. Notice that the above elements (perhaps with the exception of price stickiness) are also the ones that are more difficult to be accurately pinned down from the data. Any value for the risk aversion coefficient within the interval [1,5] is considered plausible in the literature. Moreover, LTs have been quite rare historically, and thus the value ω is difficult to pin down. Finally, evidence on the duration of the episodes is hard to come by. Had our results varied considerably across these parameters, there would be little we could say about optimal debt management. A positive note therefore is that our model's predictions are robust across a number of parameters that are not easy to measure.

4.5 Discussion

Our results show that the complete market approach to optimal debt management can rationalize why governments want to issue portfolios of both short and long bonds and actively manage the maturity structure, in some cases removing long term debt from the hands of the private sector, during LT episodes. Recall that our model brings together monetary and fiscal policies under a single authority and therefore, from the point of view of the model, it is immaterial whether 'active debt management' during LTs is assigned to the FED or the Treasury; what matters is the types of debt that appear on the

[17]Note that this finding is similar to Faraglia, Marcet and Scott (2010) who document that debt management under complete markets features 'excess sensitivity' to small changes in the underlying economic environment. Here, changes derive from the history dependence property of the optimal allocation.

consolidated budget constraint. Taken literally, the model can rationalize 'optimal long debt buybacks', like the ones we observed during the recent downturn.

We must however emphasize that our goal here is not to find an exact value for the optimal portfolio and suggest it to debt managers, or (even) come up with a formula that translates model numbers into portfolios. We are obviously aware of the numerous simplifications our modelling approach brings to the very complex interactions between monetary and fiscal policies and financial markets; the Ramsey outcome we studied here derived from a frictionless market, whereby the costs of debt issuance and portfolio rebalancing are equal to zero.

As discussed in the 'Introduction', a recent literature on 'Quantitative Easing' builds models with segmented markets, and in which groups of investors have preferences over different maturities, giving rise to well defined demand curves for each maturity. In these models, governments desire to manage maturity in order to impact asset prices and thus impact consumption and investment during LTs. These margins are not accounted for in our model. However, the forces that we identify here, and the analogous forces in the recent literature on 'Quantitative Easing' seem to complement each other, at least under the most empirically relevant scenario of high initial debt and spending shocks, we have considered. Whether this preliminary insight carries through to a model where tax smoothing objectives and segmented financial markets are both present, remains to be explored.

5 Conclusion

A model of optimal government portfolios, under complete financial markets, and with shocks that can drive the economy to a liquidity trap, was presented in this paper. Our findings suggest that it is optimal to finance deficits with short bonds when spending levels rise sharply during LT episodes and government debt levels are low. In contrast, if LTs are characterized by small spending shocks and outstanding government liabilities are substantial, then governments should focus on issuing long term debt. Under empirically plausible scenarios with both high initial debt levels and large fiscal shocks, we find that the optimal portfolio features both short and long bond issuances, and is actively managed during the LT.

Our findings should be viewed as a useful benchmark for the optimal maturity structure of debt during liquidity traps. Recent interventions by central banks in the bond market, have decreased the amounts of long term government in the hands of the private sector when interest rates hit the zero lower bound. This paper identifies the conditions under which such a policy is optimal in a frictionless financial market and under the assumption that fiscal, monetary and debt management authorities coordinate in the Ramsey policy. The main takeaways and insights from this paper, however, are broadly applicable to models with frictions, including the recent literature on Quantitative Easing. Future research should embed the tax smoothing objective of debt management considered in this paper, in models with segmented bond markets, transaction costs, and separate monetary and fiscal policies.

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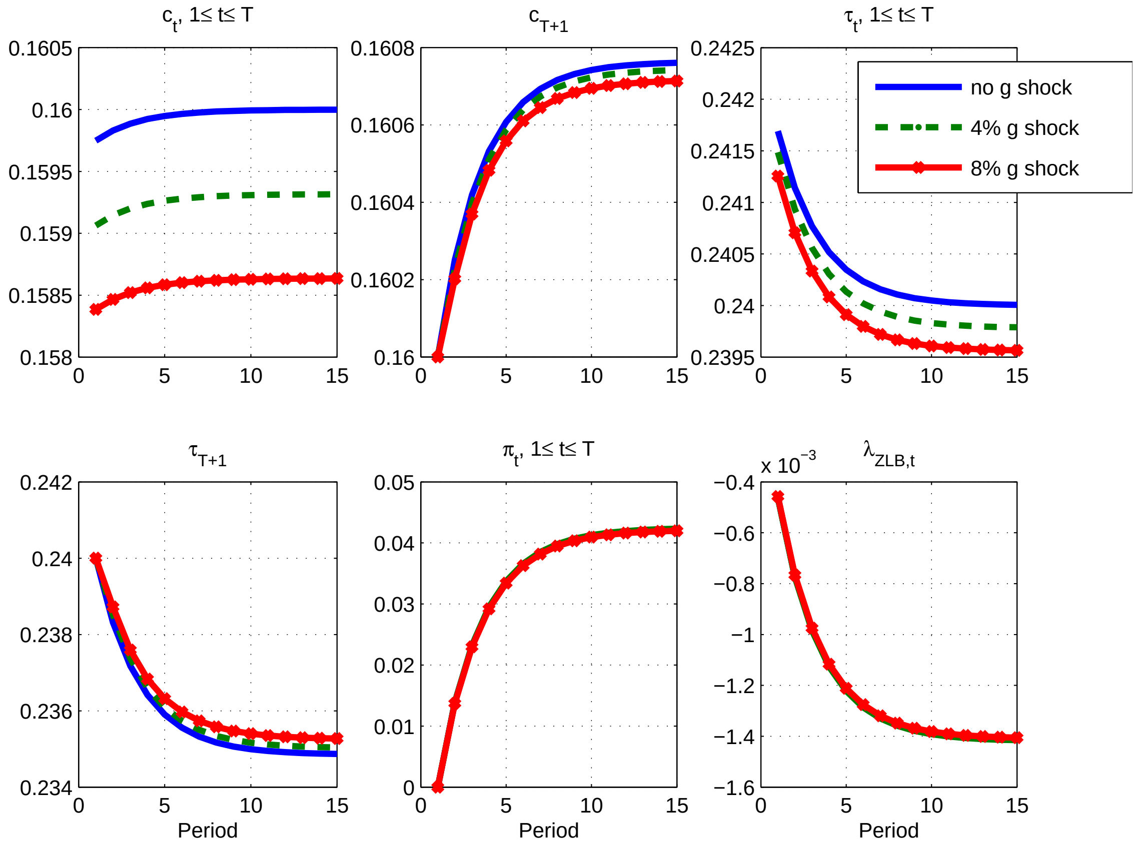
Figure 1: **Responses of endogenous variables to the preference shocks: various values of $\underline{g}$.**



Notes: The figure shows the adjustment of consumption, taxes and inflation to the preference shock under the ZLB constraint. The solid line corresponds to the case $\underline{g} = \overline{g}$, the dashed line to $\underline{g} = 1.04\overline{g}$ and the crossed (red) line to $\underline{g} = 1.08\overline{g}$. The top left panel shows the response of consumption during the LT episode. The middle top, shows the response at the 'exit' from the LT (period $T+1$). The top right and bottom left show responses of taxes (during and right after the LT respectively). The middle bottom panel shows inflation and the bottom right traces the behavior of $\lambda_{ZLB,t}$.

Table 1: Calibrated Parameters

Symbol	Value	Description
β	0.99	Discount factor
γ_c	1	Relative risk aversion
$\frac{\eta}{1+\eta}$	1.2	Gross value added markup
θ	17.5	Degree of price stickiness
γ_h	1	Inverse elasticity of labor supply
ω	0.5	Probability $\xi_1 = \underline{\xi}$
ϕ	0.8	Persistence of preference shock
$\bar{g}$	0.04	Steady state government spending
$\bar{h}$	0.2	Steady state hours worked
$\bar{c}$	0.16	Steady state consumption

Notes: The table reports parameter values under the benchmark calibration of the model. See text for further details.

Table 2: Optimal portfolios under $\gamma_c = 1$ and $\widetilde{B}_{-1} = 0$

Period	b_t^1	b_t^N	$q_{t+1}^{N-1}\left(x^t\left(\underline{\xi}, \underline{g}\right)\right)$	$q_{t+1}^{N-1}\left(x^t\left(\overline{\xi}, \overline{g}\right)\right)$	$S_{t+1}\left(x^t\left(\underline{\xi}, \underline{g}\right)\right)$	$S_{t+1}\left(x^t\left(\overline{\xi}, \overline{g}\right)\right)$
			$g_1 = \overline{g}$			
0	0.000	0.000	0.956	0.914	0.000	0.000
1	-0.001	0.000	0.956	0.914	0.000	0.000
4	-0.003	0.000	0.957	0.913	-0.001	-0.001
9	-0.003	0.000	0.957	0.913	-0.001	-0.001
			$g_1 = \overline{g} \times 1.04$			
0	0.767	-0.824	0.956	0.914	-0.004	0.003
1	0.758	-0.815	0.956	0.914	-0.004	0.003
4	0.745	-0.803	0.957	0.913	-0.005	0.002
9	0.740	-0.798	0.957	0.913	-0.005	0.002
			$g_1 = \overline{g} \times 1.08$			
0	1.534	-1.648	0.956	0.914	-0.008	0.006
1	1.516	-1.630	0.956	0.914	-0.009	0.005
4	1.492	-1.605	0.957	0.913	-0.009	0.005
9	1.483	-1.596	0.957	0.913	-0.009	0.005

Notes: Columns 2-3 show the optimal mixture between short and long bonds; Columns 4-5 show the price of long term debt under $x_{t+1} = \left(\underline{\xi}, \underline{g}\right)$ (Column 4) and $x_{t+1} = \left(\overline{\xi}, \overline{g}\right)$ (Column 5); Columns 6-7 report the present discounted value of the surplus. In the first set of rows, uncertainty is only driven by preference shocks, whilst in the remaining sets of rows a spending shock is added which drives $\underline{g}$ 4% higher than $\overline{g}$ (second set) and $\underline{g}$ 8% higher than $\overline{g}$ (third set). Initial debt is zero.

Table 3: Optimal portfolios under $\gamma_c = 1$ and $\widetilde{B}_{-1} = 60\%$

Period	b_t^1	b_t^N	$q_{t+1}^{N-1}\left(x^t\left(\underline{\xi},\underline{g}\right)\right)$	$q_{t+1}^{N-1}\left(x^t\left(\overline{\xi},\overline{g}\right)\right)$	$S_{t+1}\left(x^t\left(\underline{\xi},\underline{g}\right)\right)$	$S_{t+1}\left(x^t\left(\overline{\xi},\overline{g}\right)\right)$
			$g_1 = \overline{g}$			
0	-0.293	3.046	0.956	0.914	0.524	0.498
1	-0.203	2.951	0.956	0.914	0.524	0.499
4	-0.078	2.819	0.957	0.913	0.524	0.499
9	-0.032	2.771	0.957	0.913	0.524	0.500
			$g_1 = \overline{g} \times 1.04$			
0	0.464	2.235	0.956	0.914	0.520	0.501
1	0.544	2.150	0.956	0.914	0.520	0.502
4	0.656	2.032	0.957	0.913	0.520	0.502
9	0.697	1.989	0.957	0.913	0.520	0.503
			$g_1 = \overline{g} \times 1.08$			
0	1.220	1.424	0.956	0.914	0.516	0.504
1	1.294	1.347	0.956	0.914	0.516	0.505
4	1.396	1.240	0.957	0.913	0.516	0.506
9	1.434	1.200	0.957	0.913	0.517	0.506

Notes: The rows and columns of the table are ordered in the same manner as Table 2. Initial debt is 60% at annual horizon.

A Appendix

A.1 Derivations

Proof of Proposition 1: i) follows from equation (12) and the fact that $\lambda_{ZLB,T+1} = \lambda_{ZLB,T+2} = \lambda_{ZLB,T+3}... = 0$ (from complementary slackness). Then, we have that

$$-\theta\lambda_{f,t}(\pi_t - 1) = 0 \rightarrow \pi_t = 1.$$

To show ii) first note that since inflation equals zero for $t > T+1$, $w_t = \frac{1+\eta}{\eta}$. From (10) and (11), and replacing $g_t = \overline{g}$, we have

$$u_{c,t} + v_{h,t} - \lambda_s u_{c,t}\left[\frac{v_{hh,t}}{u_{c,t}}h_t + \left(1 + \frac{v_{h,t}}{u_{c,t}}\frac{\eta}{1+\eta}\right)\frac{1+\eta}{\eta}\right] - \lambda_s u_{cc}(c_t)\left[-\overline{g} + \left(1 + \frac{v_{h,t}\eta}{u_{c,t}(1+\eta)}\right)\frac{1+\eta}{\eta}h_t\right]$$

$$+\lambda_s u_c(c_t)\left(-\frac{v_{h,t}}{u_{c,t}^2}u_{cc,t}h_t\right) = 0, \tag{27}$$

where $c_t + \overline{g} = h_t$.

Notice that (27) defines an nonlinear equation in consumption that applies in all periods after $T+1$. The solution defines a constant consumption level $\overline{c}$ and since hours are also constant, taxes are constant. The per-period surplus, s_t, is therefore also constant (equal to $\overline{s}$).

Turning to the debt levels and recalling that once uncertainty is removed from the model, a short bond is sufficient to complete the markets, we have

$$B_{t-1}^1 = \sum_{k=1}^{\infty}\beta^{k-1}\frac{u_{c,t+k}}{u_{c,t}}\left[-\overline{g} + \left(1 + \frac{v_{h,t+k}\eta}{u_{c,t+k}(1+\eta)}\right)\frac{1+\eta}{\eta}h_{t+k}\right] = \frac{\overline{s}}{1-\beta} = \overline{B},$$

for $t = T+2, T+3, ...$

iii) follows easily from the above derivations. ■

Proof of Proposition 2: Towards i): From equation (18), the optimal inflation satisfies $-\theta(\pi_t - 1)\pi_t^2 = \lambda_{ZLB,t-1}$. Inflation differs from zero whenever the ZLB has binded in the previous period. Moreover, $\pi_{t+1}\left(x^t, (\overline{\xi}, \overline{g})\right) = \pi_{t+1}\left(x^t, (\underline{\xi}, \underline{g})\right)$. If the economy has just escaped the LT, the inflation rate remains equal to the level it would be if the economy remained in the trap for a one more period.

From the ZLB constraint we have:

$$\beta E_t \frac{\xi_{t+1}}{\xi_t}\frac{1}{\pi_{t+1}} = 1,$$

for $t = 1, 2, ...T$. Since π_{t+1} is not random (conditional on date t information) we have

$$\pi_{t+1} = \beta E_t \frac{\xi_{t+1}}{\xi_t} = \beta\frac{\phi\underline{\xi} + (1-\phi)\overline{\xi}}{\underline{\xi}},$$

for $t = 1, 2, ...T$. For $t = 0$ we have $-\theta(\pi_1 - 1)\pi_1^2 = \lambda_{ZLB,0} = 0$ and therefore $\pi_1 = 1$.

To show ii) first note that combining (16) and (17) together with $\lambda_s = -\frac{\eta}{\theta}\lambda_p$, we get that

$$v_{h,t} + 1 - \lambda_s \left(v_{hh,t}h_t + v_{h,t}\right) + \lambda_p \frac{1+\eta}{\theta} = 0, \tag{28}$$

which gives $h_t = \overline{h}$. Moreover, noting that under preferences (15) we have $v_{hh}(h_t)h_t = \gamma_h v_h(h_t)$, it follows that:

$$\tau_t(1 - \lambda_s(1+\gamma_h)) = \left(1 - \lambda_s\gamma_h - \frac{1}{w_t}\right) + \frac{\lambda_s}{w_t}\left(\frac{1+\eta}{\eta} - w_t\right). \tag{29}$$

From (3) and making use of the fact that $h_t = \overline{h}$ we have

$$0 = \frac{\eta}{\theta}\left(\frac{1+\eta}{\eta} - w_t\right)\overline{h} + \beta(\underline{\pi} - 1)\underline{\pi}^2, \tag{30}$$

at $t = 1$,

$$(\underline{\pi} - 1)\underline{\pi} = \frac{\eta}{\theta}\left(\frac{1+\eta}{\eta} - w_t\right)\overline{h} + \beta(\underline{\pi} - 1)\underline{\pi}^2, \tag{31}$$

for $1 < t \leq T$ and

$$(\underline{\pi} - 1)\underline{\pi} = \frac{\eta}{\theta}\left(\frac{1+\eta}{\eta} - w_t\right)\overline{h}, \tag{32}$$

for $t = T + 1$. Using the above expressions we can get (20). ■

Proof of Proposition 3: We derive bond prices and the present value of the government's surplus under quasi-linear preferences. From (22) we have 4 distinct values for s_t: $\overline{s}$, $\underline{\underline{s}}$, $\underline{s}$ and $\overline{\overline{s}}$. Consider first period $t > T + 1$. Then, trivially, $S_t = \frac{\overline{s}}{1-\beta}$. Now consider $t = T + 1$. Then we have is $S_t = \overline{\overline{s}} + \beta\frac{\overline{s}}{1-\beta}$.

For $t = 2, 3, ..$ the value of S satisfies following recursive equation:

$$\underline{\xi}\underline{S} = \underline{\xi}\underline{s} + \beta\phi\underline{\xi}\underline{S} + \beta(1-\phi)\overline{\xi}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right),$$

where $\underline{S} = S_t$, $t = 2, 3..., T$ and T is random. The above equation gives:

$$\underline{S} = \frac{1}{1-\beta\phi}\left[\underline{s} + \beta(1-\phi)\frac{\overline{\xi}}{\underline{\xi}}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right)\right].$$

In period 1 if $\xi_1 = \underline{\xi}$, we get:

$$\underline{\xi}S_1 = \underline{\xi}\underline{\underline{s}} + \beta\phi\underline{\xi}\underline{S} + \beta(1-\phi)\overline{\xi}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right),$$

and so

$$S_1 = \underline{\underline{s}} + \frac{\beta\phi}{1-\beta\phi}\underline{s} + \frac{\beta(1-\phi)}{1-\beta\phi}\frac{\overline{\xi}}{\underline{\xi}}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right).$$

Finally, in period zero we have:

$$S_0 = (1-\omega)\frac{\overline{s}}{1-\beta} + \omega\left[\overline{s} + \beta\frac{\overline{\overline{\xi}}}{\underline{\xi}}\left(\underline{\underline{s}} + \frac{\beta\phi}{1-\beta\phi}\underline{s}\right) + \frac{\beta^2(1-\phi)}{1-\beta\phi}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right)\right].$$

To derive the bond prices consider first period 1. If the LT shock does not occur we have that $q^{N-1}\left(x^0, (\overline{\xi}, \overline{g})\right) = \beta^{N-1}$. However, if the preference shock occurs we need to determine the price $q^{N-1}\left(x^0, (\underline{\xi}, \underline{g})\right)$ as the weighted sum of all future possible realizations of the quantity $\beta^{N-1}\frac{\xi_N}{\xi_1}\frac{P_1}{P_N}$, where P_t denotes the price level in t.

With probability $(1-\phi)\phi^{k-1}$ the shock lasts for k periods for $k = 1, 2, ..., N-1$ t and from Proposition 2 we have:

$$\beta^{N-1}\frac{\xi_N}{\xi_1}\frac{P_1}{P_N} = \beta^{N-1}\frac{\overline{\xi}}{\underline{\xi}}\frac{1}{\underline{\pi}^k}$$

Finally, with probability ϕ^{N-1} the shock does not end before period N and therefore we have $\beta^{N-1}\frac{\xi_N}{\xi_1}\frac{P_1}{P_N} = \beta^{N-1}\frac{1}{\underline{\pi}^{N-1}}$.

Put together the price of the long bond in period 1 equals

$$q^{N-1}\left(x^0, (\underline{\xi}, \underline{g})\right) = E_1\beta^{N-1}\frac{\xi_N}{\xi_1}\frac{P_1}{P_N} = \beta^{N-1}\left[\frac{\overline{\xi}}{\underline{\xi}}\sum_{k=1}^{N-1}(1-\phi)\phi^{k-1}\frac{1}{\underline{\pi}^k} + \frac{1}{\underline{\pi}^{N-1}}\right],$$

Moreover, it is easy to show that $q^{N-1}\left(x^t, (\underline{\xi}, \underline{g})\right) = q^{N-1}\left(x^0, (\underline{\xi}, \underline{g})\right)$, or bond prices do not change during the LT episode.

Finally the bond price at the exit (i.e. in $T+1$) satisfies $q^{N-1}\left(x^T, (\overline{\xi}, \overline{g})\right) = \beta^{N-1}$ since inflation is zero in period $T+2$ and thereafter.

Given the above system (14) is

$$(33)\qquad \begin{bmatrix} 1 & \beta^{N-1} \\ 1 & \beta^{N-1}\left(\frac{\overline{\xi}}{\underline{\xi}}\sum_{k=1}^{N-1}(1-\phi)\phi^{k-1}\frac{1}{\underline{\pi}^k} + \frac{1}{\underline{\pi}^{N-1}}\right) \end{bmatrix} \cdot \begin{bmatrix} b_0^1 \\ b_0^N \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\beta}\overline{s} \\ \underline{\underline{s}} + \frac{\beta(1-\phi)}{1-\beta\phi}\frac{\overline{\xi}}{\underline{\xi}}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right) + \frac{\beta\phi}{1-\beta\phi}\underline{s} \end{bmatrix},$$

for the ex ante portfolio in $t = 0$ and

$$(34)\qquad \begin{bmatrix} 1 & \frac{\beta^{N-1}}{\underline{\pi}} \\ 1 & \frac{\beta^{N-1}}{\underline{\pi}}\left(\frac{\overline{\xi}}{\underline{\xi}}\sum_{k=1}^{N-1}(1-\phi)\phi^{k-1}\frac{1}{\underline{\pi}^k} + \frac{1}{\underline{\pi}^{N-1}}\right) \end{bmatrix} \cdot \begin{bmatrix} b_t^1 \\ b_t^N \end{bmatrix} = \begin{bmatrix} \overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s} \\ \frac{\beta(1-\phi)}{1-\beta\phi}\frac{\overline{\xi}}{\underline{\xi}}(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}) + \frac{\underline{s}}{1-\beta\phi} \end{bmatrix},$$

for $t = 1, 2, ..., T$, i.e. during the LT. From (33), (34) and (22) it is straightforward to show that the optimal portfolio is rebalanced only at the entry, in period 1. ■

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