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Forest Management: are Double or Mixed Rotations Desirable?

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Abstract
In this paper, we study a particular uneven-aged forest stand management pattern that is often advocated in practice. The forest structure under consideration is similar to a normalized forest à la Faustmann, with the following difference: rather than being single aged, each forest tract contains trees of two age classes so that it is submitted to a form of selective cutting. Each harvest involves all of the older trees and only a fraction of the younger ones; hence the name mixed rotation. Trees left standing at harvest help stimulate natural regeneration and improve various environmental and amenity characteristics of the forest. We model this effect by using a cost function that varies with respect to the harvest rate of younger trees. We derive the properties that this cost function must exhibit in order some form of mixed rotation to be superior to the conventional single rotation à la Faustmann; we also characterize the mixed rotation in terms of duration and the harvest rate of younger trees, and we compare its properties with Faustman's rule.

key words: forest management; Faustmann's rule; normal forest; synchronized forest; uneven-aged lots; amenity value; mixed rotation; selective cutting.

J.E.L. classification: Q00; Q23; D29.

Résumé
Nous étudions un cas particulier d'aménagement forestier inéquien qui est recommandé dans la pratique actuelle. La structure de la forêt est similaire à une forêt normalisée à la Faustmann avec la différence suivante: au lieu d'être équien, chaque lot comporte deux classes d'âge; il est soumis à une forme de coupe sélective. A chaque récolte, on coupe tous les arbres les plus vieux ainsi qu'une fraction des arbres les plus jeunes; d'où le nom de rotation mixte. Les arbres non coupés aident la régénération naturelle et améliorent diverses caractéristiques environnementales et esthétiques de la forêt. Nous modélisons cet effet en utilisant une fonction de coût qui varie avec le taux de récolte des arbres jeunes. Nous dérivons les propriétés que cette fonction de coût doit satisfaire pour que la rotation mixte soit préférable à la rotation standard à la Faustmann; nous caractérisons la rotation mixte en termes de durée et de taux de récolte des jeunes arbres, que nous comparons avec le cas de Faustmann.

mots-clés: aménagement forestier; règle de Faustmann; forêt normalisée; forêt synchronisée; forêt inéquienne; aménités; rotation mixte; coupe sélective.

Classification J.E.L.: Q00; Q23; D29.

1 Introduction

One traditional question of forestry management is the optimal harvest time. Faustmann (1849) addressed the question by defining the optimal age at which trees should be cut as the age when the net cumulated discounted value of an infinite sequence of harvests is maximum if all harvests occur at the same age.

It is cheaper to clear cut a patch of forest than to deal with each tree individually. Consequently Faustmann's rule, although it can be determined for any individual tree, is generally understood to apply to whole cohorts. This leads to the notion of a normal, or "synchronized", forest which is a forest divided into as many equal size even-aged lots as there are time periods between the time a new crop is established and the time it is harvested. More generally a forest territory should be divided into several such normal forests. In a normal forest, each lot is periodically clear cut. The number of lots is equal to the optimal rotation age, which is, according to Faustmann's rule, the age at which growth has slowed down to such a level that the increase in the timber value of standing trees over an additional period is equal to the sum of the interest that would be earned on the crop and on the site value if these amounts were invested for the same period.

Faustmann's analysis and the normal forest have been central to forestry research and practice, although a general proof that the normal forest is indeed the optimal form of organization in the long run has been elusive (Mitra and Wan, 1986; Heaps, 1984; Salo and Tahvonen, 2002, 2003; Uusivuori and Kuuluvainen, 2005). In his analysis, Faustmann assumes that the unit price of timber and the unit cost of harvesting are known and constant. The above authors generalize the model by assuming that the net instantaneous utility from harvest is strictly concave. However while Heaps specified continuity in both time (the harvest age) and space (plot size) and was not able to prove that the normal forest was optimal, Mitra and Wan, as well as Salo-Tahvonen and Uusivuori-Kuuluvainen studied models with continuous space and discrete time. They all found that cyclical stationary solutions may arise, that is to say optimal harvest schemes that converge to a steady state where the forest is characterized by an age-class structure that is not a normal forest. However Salo and Tahvonen showed that the non-cyclical stationary solution (the normal forest) is optimal when the time unit is allowed to become very small. In this paper we make use of that result to focus on the normal forest.

Despite the attention devoted to the optimality of the normal forest as a mode of management and organization, the literature is silent about the size of even-aged tracts in a normal forest. For example in the above papers if x_{it} represents the land area covered with trees of age i at time t, then a forest is normal if and only if $x_{it} = x_{jt}\ \forall\ i,\ j,\ t$, with $i,\ j \in \left\{0, 1, 2, ..., T^F\right\}$, where T^F is Faustmann rotation and trees are necessarily harvested when and only when they reach age T^F. This formalization is compatible with tract surfaces of several hectares or may correspond to a single tree, or fraction of tree for that matter, per tract. It is also compatible with x_{it} being composed of spatially discontinuous areas. Yet questioning the normal forest as a management practice stems more from the fact that it involves sizeable stands of even-aged trees and clear cutting, than from the fact that all trees are cut at the same age, which is compatible with selective cutting. Economies of scale at the level of the harvest unit play a major role in shaping the normal forest and should not be assumed away in considering alternative forest management practices. In this paper the focus is not on scale but on the costs and benefits of departing from the normal forest by not cutting all trees at the same age. However the model is formulated in such a way that forest tracts are scale efficient.

Faustmann also assumes that the forest and the forest land do not produce any other form of value than revenues from timber harvest. Following Hartman. (1976) and culminating in the present with Uusivuori and Kuuluvainen (2005), a rich theoretical literature focuses on forests that provide both timber and amenities.

For Bowes and Krutilla (1989) the amenity values provided by a forest depend on the mix of stand ages. As noted by Uusivuori and Kuuluvainen (2005), this makes optimal decisions dependent on the entire forest structure, a complexity that imposes severe limits on both analytical and numerical analysis. In a model where each age class contributes to amenity additively and separately, Uusivuori and Kuuluvainen show that a noncyclical steady state solution may imply old-growth preservation combined with timber harvesting. However modelling the environmental and amenity contributions of age classes fails to account for a major cost of timber exploitation in a normal forest: the environmental, amenity, and silvicultural cost of clear-cutting entire, usually even-aged, stands. As already mentioned, age-class models are silent about the spatial distribution of trees; all trees in a particular age class may be felled without any clear cutting to take place. Although it is often implicitly assumed that age classes coincide with tree stands (see, e.g. note 2 in Uusivuori and Kuuluvainen), the size of the stand is immaterial in such models. Consequently they are powerless when it comes to evaluate the environmental, amenity, and silvicultural costs or benefits of mitigating the impact of clear-cutting in an otherwise normal forest. To our knowledge there is no theoretical paper that has addressed that issue.

Yet forest managers advocate new practices that take such aspects of forestry into account. Instead of clear cutting, various forms of selective harvesting and tree retention are often recommended. This implies forests that may called normalized in the sense that they are composed of tracts of identical surface exhibiting identical age structures at harvest time, but differ from the conventional normalized forest in that trees are not necessarily cut at the same age determined by Faustmann's rule and in that tracts may not be even-aged.

In this paper, we study a particular uneven-aged stand model that is often advocated in practice. Under that management rule the forest structure is similar to a normalized forest, with the following difference: rather than being even-aged, each forest tract combines trees of at most two age classes. Each time a harvest takes place on the tract, it involves all of the older trees and only a fraction of the younger ones; hence the name of mixed rotation. We derive the harvest rate and rotation cycle that are optimal for that particular structure.

Besides its empirical relevance, this structure has rich but manageable theoretical implications. In fact Wan (1994), as well as Salo and Tahvonen (2002), also used a forest management structure involving two age classes. However, the age of the trees in each class was not a choice variable: by assumption trees could live at most two periods and the decision maker had to decide how many trees of the first age class were to be cut and how many were allowed to reach the second age class before being cut. Here the age at which trees are cut can be chosen provided the constraint on the age structure is met. The focus of Wan and of Salo and Tahvonen was also different: they were interested in the existence of optimal cycles and the optimality of reaching the long-run normal forest, as discussed above.

After each harvest, the remaining uncut trees help stimulate the natural regeneration of new trees, which is less expensive than seedling plantation, and they protect newly established seedlings, whether from natural origin or planted. Besides helping regeneration, leaving a certain proportion of grown trees standing at harvest time until the next harvest has several beneficial effects. It reduces soil damage thus improving sustainability; it improves water retention, reduces the trauma of timber exploitation to wild life, and, generally increases amenity value and the social acceptability of timber exploitation. More generally, it is also argued that this forest structure is environmentally and aesthetically more acceptable than the normal forest and its ugly clear-cut patches. The proponents of such tree retention practices claim that uneven-aged forest tracts are better adapted to environmental and social objectives, and that this form of selective harvesting also preserves genetic resources and the diverse composition of forests (Bergeron et al., 1999), as well as it permits greater carbon sequestration.

We model these effects by introducing a function that allows net revenues to depend on various forest management costs, and on various benefits, not necessarily commercial, that are also part of forest value. Analyses in the tradition of Faustmann often involve fixed costs occurring at harvest time (regeneration costs for example); they are called fixed because they do not depend on harvest quantity. The function introduced in our model to take account of benefits or costs that do not depend on the harvest rate will be called net cost function in order to facilitate comparisons with traditional models, although it may be negative if benefits exceed costs, or positive in the opposite case. Although it enters the objective function in the same way as fixed costs do in Faustmann's analysis, the net cost function introduced in this paper depends on the forest management practice under study. Precisely we model the net cost as a function of the proportion of trees left standing when a stand is harvested. The higher the proportion of trees left standing, the lower regeneration costs and the higher the various amenity benefits just mentioned. That proportion will be controlled by the decision maker in the model described below.

Given the tree volume growth function, we derive the properties that the net cost function must exhibit in order some form of mixed rotation or double rotation to be superior to the conventional single rotation à la Faustmann. Mixed and double rotations involve choosing the proportion of younger trees left standing at each harvest; and choosing the periodicity of harvests, the rotation.

We find that the normal forest à la Faustmann is the optimal form of management when costs and amenity values do not depend on the proportion of trees allowed to grow over a second rotation. Also, while the basic intuition explaining Faustmann's rotation is preserved, where the opportunity cost of bare land is a component of the cost of waiting, we find that the opportunity cost of bare land is augmented by the opportunity cost of the trees left standing at harvest when the proportion of trees cut at each harvest is below 100%.

In general, the higher the discount rate, the further the second-order condition differs from strict concavity of the growth function at harvest age. This is because the cost of delaying harvest reduces the benefit from growth even if marginal tree growth is not diminishing. When some trees are left standing at harvest to be cut down at the next harvest, this consideration must be incorporated in choosing the harvest age of younger trees as well as the implied harvest age of older trees.

The optimal proportion of trees cut at harvest instead of being allowed to grow further until next harvest is shown to depend on the relative weight of the beneficial externality brought by older trees. It is minimum at zero when the beneficial externality weights high. This implies a radical departure from clear cutting as only half the trees on any given forest tract are cut down at harvest. In contrast when the beneficial externality does not weigh much, clear cutting and management à la Faustmann are optimal.

Section 2 describes the structure of the theoretical model and establishes notations. The problem is solved and results are given in Section 3. We conclude with Section 4.

2 Theoretical Model

2.1 Problem Structure and Notation

We consider a forest composed of many identical territories, each made up of T tracts of equal size. Each such tract is covered with a forest stand composed of two age classes of trees. The first class contains n_1 younger trees; the second class is made up of n_2 trees that are older than the first ones by T years. On each tract harvest occurs every T years and is partial: all older trees are harvested while only a proportion m, $0 \leq m \leq 1$, of the younger age class is cut. Thus at harvest the n_1 younger trees are T years old and the n_2 older trees are $2T$ years old. The proportion $(1 - m)$ of younger trees left standing makes up the n_2 trees that are allowed to grow further to be cut at the next harvest, when they are $2T$ years old. The total number n of trees on any tract is given and such that $n = n_1 + n_2$. The forest is organized in such a way that one harvest takes place on one of the T identical stands in each territory every year.

We further assume that the same proportion m is maintained on all tracts. It follows that the forest territory under consideration is in a steady state and produces a steady flow of wood with a rotation of T. However, this is not a normal forest in the usual sense because not all trees are cut at the same age, except in the special case where $m = 1$.

The assumption that the forest is big enough to be composed of many identical T tracts territories is a convenient way to abstain from a discussion of overall forest size while retaining the notion of optimal establishment size. Suppose unit harvest costs depend on size; then tract size, hence territory size, affect profitability. Loosely speaking, the steady state forest should then be analogous to an industry composed of firms of optimum size: in the long run it is divided into identical territories of optimum size. We will focus on one such territory. Moreover, since all T tracts in the territory are identical except for tree age (hence harvest date), the analysis will highlight one particular tract, mimicking Faustmann's approach in that sense. Note that the assumption of optimal territory size and the analogy with long run industry equilibrium does not imply that land rents are zero. A good analysis of endogenous commercial forest size when the land area is effectively infinite is provided by Sahashi (2002).

Let $v(t)$ be the wood volume of a tree of age t. For most species that function has the following properties, which we assume here:

Assumption 1: The tree volume function.

a. $v(t) = 0$ *for* $0 \leq t \leq t_0$.

b. $v(t)$ *is non negative and continuously differentiable on* R^+.

c. *There are two positive values* t_1 *and* t_2 *with* $t_0 \leq t_1 < t_2$, *such that the marginal rate of growth of* $v(t)$ *is positive and increasing for* $t_0 \leq t \leq t_1$ *and is positive and decreasing for* $t_1 < t \leq t_2$. *For* $t > t_2$, *the marginal rate is negative or null; at* t_2, *the tree reaches its maximum timber volume.*

The existence of steady states defined in terms of T and m is easy to establish by construction. While this does not mean that convergence toward the appropriate steady state is a property of the optimal dynamic management policy for any initial situation, the analysis of Salo and Tahvonen (2002) indicates this is likely to be the case when t is treated as a continuous variable. Thus we focus on the steady state.

The total stand timber volume, nS, is the sum of the volumes in each age class. When the younger trees have age t, $t \leq T$, so that older ones are $t + T$ old,

$$nS = n_1 v(t) + n_2 v(t + T).$$

At harvest the forest operator cuts all trees aged $2T$ and only a fraction m of trees aged T. Younger trees left standing at harvest become the older trees to be cut at the next harvest. Therefore, $n_2 = n_1(1 - m)$ so $n_2 \leq n_1$. Since a certain proportion of trees are left standing at harvest, the harvest volume is smaller than the total timber volume on the stand:

$$nH = mn_1 v(T) + n_1(1 - m)v(2T). \tag{1}$$

where H is the average harvest volume, that is to say the ratio of the total harvest volume on the typical forest tract, over the total number of trees on that tract.[1] The number of trees cut (and regenerated) at each rotation is $mn_1 + (1-m)\,n_1 = n_1$ for all values of m. But since $n_1 + n_2 = n$ and $n_2 = n_1(1-m)$,

$$n_1\,(m;n) = \frac{1}{2-m}n\ ;\ n_2\,(m;n) = \frac{1-m}{2-m}n. \tag{2}$$

Thus choosing the proportion m determines the steady state proportion of trees belonging to each age class and the proportion of trees harvested and left standing at each harvest. Accordingly, for any tract size measured by n, we can rewrite H and S as functions of the rotation age and the proportion of younger trees cut at each harvest:

$$H\,(T,m;n) = \frac{m}{2-m}v(T) + \frac{1-m}{2-m}v(2T). \tag{3}$$

$$S\,(T,m;n) = \frac{1}{2-m}v(T) + \frac{1-m}{2-m}v(2T) \tag{4}$$

2.2 Forest Costs and Values

Forests benefits other than harvest revenues are often reduced by wood exploitation. We model them as part of forestry costs. Thus forest exploitation involves a variety of costs: administration, harvest, regeneration, transportation, forest maintenance, environmental costs, amenity costs, etc....[2] We assume that these costs are determined by three elements. The first one is scale; assuming that tree density is not a choice variable, the number of trees per tract defines territory size. As the focus of our paper is mixed or multiple rotations, not optimal scale, we use a model under which scale is optimized, either because unit cost is independent of scale, or because the number of trees per tract is set at its cost minimizing level n^*, an optimum level which is independent of the proportion of younger and older trees (see below).

Second the harvest age, T or $2T$. We do not assume that T affects costs directly, but

[1] Trees left standing at harvest are nonetheless included in establishing the ratio.

[2] Some of these costs are incurred irrespective of forest management practices. Many administrative costs fall into that category. They are usually ignored as fixed costs in the marginal analysis of forest management decisions and we do so here.

since costs are incurred at harvest time only[3], the choice of T determines the frequency at which they must be born. Traditionally planting and regeneration costs have been modelled that way and have been called fixed cost because they do not depend on volume.

Third, as we have explained, one key argument in favor of mixed/double rotations is that such practices help regeneration and reduce environmental damage as well as amenity loss, thus reducing the costs associated with each harvest relative to costs when clear cutting. We assume that the total cost incurred at each harvest is higher, the higher m, that is the smaller the proportion of trees allowed to stay uncut at harvest.

According to the above hypotheses the total cost incurred at each steady state harvest of $nH(T, m; n)$ on a typical tract may be written as $ng(n) C(m)$, where $g(n) C(m)$ is the unit cost. The function $g(n)$ is an index describing the effect of scale on unit costs: it reaches a minimum at n^*. At any scale, $C(m) > 0$ and $C'(m) \geq 0$. We assume long-run cost efficiency, whether due to perfect competition or good planning. This requires that the long-run size of the typical forest tract minimizes total industry cost: $n = \arg\min_n \frac{N}{nT} ng(n) C(m)$ where N is the exogenous total number of trees in the whole forest, so that $\frac{N}{nT}$ is the number of tracts harvested at any given date (one tract in each territory). This implies $n = n^*$.

Thus the total cost incurred at each steady state harvest of $n^* H(T, m; n^*)$ on a typical forest tract is $n^* g(n^*) C(m)$. Let us further normalize g such that $g(n^*) = 1$ adjusting C accordingly; then $C(m)$ can be interpreted as the unit cost[4] associated with each typical harvest on an optimum size forest tract.

[3]Many costs and benefits occur as flows over the rotation period. In particular leaving a certain proportion of trees uncut at each harvest affect the stand and the whole forest over successive periods until the next harvest. In a non cyclical steady state, the benefits and costs occuring between harvests can be cumulated and treated as if they occured at harvest time.

[4]By this definition, the unit cost is the total cost incurred at harvest divided by the number trees on the forest tract, whatever the proportion of these trees that is cut at harvest.

3 The mixed rotation Problem

We want to compare steady state forest management practices characterized by two parameters, m and T. According to (2) such steady states imply well defined proportions of trees of each age categories; these proportions depend on m only. Comparing alternative management practices requires not only the comparison of net cumulative revenues under alternative practices but requires also accounting for differences in the opportunity cost of the stands implied by each alternative. In the traditional Faustmann analysis the stand is entirely clear cut at each harvest so that there are no differences in stand opportunity cost at different rotation values. Here steady state forest stands differ according to m and these differences must be accounted for in choosing the optimal rotation. This requires evaluating the present discounted value of the stream of net harvest revenues over an infinite number of rotations, given the steady state levels of $n_1(m)$ and $n_2(m)$, minus the cost of the steady state stand $n_1(m;n)\,v(T)\,P + n_2(m;n)\,v(2T)\,P$ where P is the price of wood, assumed constant. Thus the objective function is

$$\sum_{k=0}^{\infty} e^{-krT}\left[Pn^*H(T,m;n^*) - n^*C(m)\right] - n_1(m;n^*)\,v(T)\,P - n_2(m;n^*)\,v(2T)\,P \qquad (5)$$

where k is the harvest index, $(k = 0, ..., \infty)$ and r the real interest rate. Using (1), (2), (4) and dividing by n^*, the objective function can be written as

$$W(T,m) = \frac{1}{1-e^{-rT}}\left\{P\left[\frac{m-1+e^{-rT}}{2-m}v(T) + \frac{1-m}{2-m}e^{-rT}v(2T)\right] - C(m)\right\} \qquad (6)$$

or

$$W(T,m) = \frac{1}{1-e^{-rT}}\left\{PH(T,m) - C(m)\right\} - S(T,m)\,P \qquad (7)$$

According to this formulation of the objective function, revenues are generated, and net costs are paid, at the beginning of each rotation. Values are discounted to date zero. At date zero the steady state stand is acquired and the first steady state harvest follows immediately $(k = 0)$. The next harvest occurs after a period of T time units, and so on. The objective function gives, on a per tree basis, the sum of the discounted values of all future harvests, net of the costs associated with each one, and net of the value of the stand just before the first harvest. Thus it can be interpreted as the value of bare land for the surface unit corresponding to one tree. In the Faustmann tradition, it does not account for the transition from one steady state to another made necessary in practice by a change in any variable. The special case of Faustman's problem obtains for $m = 1$ where it can be verified using (3) and (4) that

$$W(T, 1) = \frac{e^{-rT}}{1 - e^{-rT}} \{Pv(T) - C(1)\} - C(1)$$

This expression corresponds to the formulation of Faustmann's problem where the fixed cost is interpreted as a planting cost to be born T periods prior to the first harvest.

In order to maximize the objective, the forest operator chooses the optimal rotation age T^* and the optimal fraction m^* of younger trees harvested.

3.1 The Optimal Rotation

Let us start with the choice of T, given any exogenous value of m in the interval $[0, 1]$. We assume that $W(T, m)$ is non negative for some admissible pair (T, m), with $T > 0$ which means that the forest is worth exploiting and the problem is not trivial. The question whether management à la Faustmann is preferable $(m = 1)$ or not will be addressed further below.

Provided second-order conditions are satisfied the first-order condition defines the optimal rotation cycle T^* as the harvest age at which $\frac{\partial W(T,m)}{\partial T} = 0$. In Section A of the appendix, we show that the optimal rotation T^* must in consequence be such that

$$\begin{aligned} & r\frac{e^{-rT}}{1 - e^{-rT}} \left\{ P\left[\frac{m}{2-m}v(T) + \frac{1-m}{2-m}v(2T)\right] - C(m) \right\} \\ = \;& P\left(\frac{m}{2-m}v'(T) + \frac{2(1-m)}{2-m}v'(2T)\right) - P\left(\frac{1}{2-m}v'(T) + \frac{2(1-m)}{2-m}v'(2T)\right)(1 - e^{-rT}) \end{aligned} \tag{8}$$

which can also be written as

$$rW(T, m) + r\left[PS(T, m) - (PH(T, m) - C(m))\right] = P\frac{\partial H(T, m)}{\partial T} - P\frac{\partial S(T, m)}{\partial T}\left(1 - e^{-rT}\right) \tag{9}$$

Both (8) and (9) are extensions of the original Faustmann's formula and have similar interpretations. They express non arbitrage conditions equalizing marginal revenue flows on the right-hand side with opportunity costs on the left-hand side. Expression (8) is formulated in terms of known functions exclusively and provides a self contained implicit form for the optimum rotation.

In expression (9) the value function has been substituted in again, making its interpretation easier and comparable with the standard Faustmann rule. It is a modified golden rule of forestry stating that the rotation is optimal when two components are equalized. The first component, on the left-hand side, is the opportunity cost of waiting, that is the interest on capital. It consists of the interest on the land (first term on the left-hand side) whose value is W , and the interest on that part of the stand which is not harvested as expressed in the second term on the left-hand side: the value of the stand just before harvest PS net of the harvest value $PH - C$.

The second component, on the right-hand side of (9), is the benefit from allowing the rotation to increase. The benefit from allowing the trees to continue growing is composed of two elements. The first term on the right-hand side is the value of the marginal increase in harvest. The second term accounts for the change in the steady-state stand value associated with a marginal change in rotation. It enters negatively because the steady-state stand can be viewed as an input for the production of the harvest; a more valuable stand means more costly production, which is a negative contribution to benefit. However the cost of holding the steady state stand is shared by all future harvests; in order to consider its contribution to the current harvest only, the term in $\frac{\partial S(T,m)}{\partial T}$ is thus multiplied by $\left(1 - e^{-rT}\right)$. In fact it can be verified that, if the fraction of the change in stand value allocated to any single harvest is indeed the last term in (9), then the total value of the marginal change in stand value will be exactly allocated over an infinity of harvests: $\sum_{k=0}^{\infty} e^{-rT} P \frac{\partial S(T^*,m)}{\partial T} \left(1 - e^{-rT}\right) = P \frac{\partial S(T^*,m)}{\partial T}$.

To sum up, when the proportion of each age group is exogenous, the determination of the optimal rotation in the mixed rotation forest is determined by a non arbitrage condition reminiscent of Faustmann's formula. However it differs notably from it because the capital carried over from one rotation to the next does not consist of bare land only, but also includes the trees that are allowed to grow till the next harvest. This is stated in the proposition below.

Proposition 1 *(Mixed Rotation) When the proportion m of younger trees allowed to reach age $2T$ is given, the optimum rotation T^* satisfies the non arbitrage condition* (9)*. This condition requires the opportunity cost of the land plus the opportunity cost of timber left standing at harvest, exactly to offset the marginal change in current harvest value minus the change in steady state stand value allocated to the current harvest.*

The second-order condition is $\frac{\partial^2 W(T,m)}{\partial T^2} \leq 0$, or, as shown in the appendix,

$$\frac{r}{1-e^{-rT}} \frac{\partial H(T,m)}{\partial T} + \frac{1}{1-e^{-rT}} \frac{\partial^2 H(T,m)}{\partial T^2} \leq \frac{r\left(1+e^{-rT}\right)}{1-e^{-rT}} \frac{\partial S(T,m)}{\partial T} + \frac{\partial^2 S(T,m)}{\partial T^2} \quad (10)$$

Considering (3) and (4) this condition implies a restriction on the curvature of the volume function. In the special case of Faustmann's analysis, it is well known that it is milder than concavity of v at T. This can be verified here by setting $m = 1$ before substituting (3) and (4) into (10) to get the second-order condition for Faustmann rotation: $-rv'(T) + v''(T) \leq 0$. The higher the discount rate, the further the second-order condition is from strict concavity of v at T. This is because the cost of delaying harvest reduces the benefit from growth even if marginal tree growth is not diminishing. In general, when $m < 1$, this is also true. However two extra considerations enter the second-order condition: first the marginal effect on harvest value is different because some trees are cut at age T while some others are cut at age $2T$; second, despite the fact that the mix of age T and age $2T$ trees in the steady state stand is exogenous, the relative contribution to value of these age categories is affected by harvest age. Both elements complicate the curvature restriction by requiring simultaneous consideration of the growth function at age T and age $2T$ as can be verified by substituting (3) and (4) into (10).

Nevertheless, since the volume function is twice continuously differentiable, concave beyond some age t_1, and rising over $[0, t_2]$, it is certain that, if a maximum $T^* \neq 0$ exists as assumed, it is interior and defined by conditions (9) and (10).[5]

[5]Ruling out multiple local maxima would require additional restrictions on the non concave part of v but would not add to the understanding of the decision.

3.2 The Optimal Harvest Rate of Younger Trees

Sofar the proportion m of younger trees cut at harvest has been treated as exogenous and the optimum rotation was established accordingly. Let us now treat m as a variable and the optimum rotation as a function $T^*(m)$; what is the optimal level of m within the interval $[0,1]$? As discussed earlier one expects the answer to depend on properties of the net cost function and on parameters P and r. The solution may be interior, $m \in (0,1)$, or it may be one of two corner solutions. The first possible corner solution $(m = 1)$ corresponds to the original Faustmann's rule: all trees are cut at each harvest. The second possible corner solution $(m = 0)$ generates double rotations, a form of stand management involving overlapping generations of trees where all trees aged $2T$ are cut at each harvest, while all trees aged T are left standing. The interior solution also involves overlapping generations of trees to be cut at age $2T$, but, unlike the case of double rotations where trees are not cut until they reach age $2T$, each harvest involves both trees aged T and trees aged $2T$; we call this mixed rotations.

Since $T^* = T^*(m)$ corresponds to an interior maximum of $W(T,m)$ with respect to T, the envelope theorem implies

$$\begin{aligned}\frac{dW(T^*(m),m)}{dm} &= \frac{\partial W(T^*(m),m)}{\partial m} \\ &= \frac{1}{1-e^{-rT^*(m)}}\frac{P}{(2-m)^2}\left\{\left(1+e^{-rT^*(m)}\right)v(T^*(m)) - e^{-rT^*(m)}v(2T^*(m)\right. \\ &\left. -\frac{(2-m)^2}{P}C'(m)\right\} \end{aligned} \quad (11)$$

Consider the situation where costs do not vary with m: $C(m) = c$. Equation (11) implies

$$sign\frac{dW(T^*(m),m)}{dm} = sign\left\{\left(1+e^{-rT^*(m)}\right)v(T^*(m)) - e^{-rT^*(m)}v(2T^*(m))\right\} \quad (12)$$

If the term on the right-hand side of (12) is positive for any $m \in [0,1]$, then $\frac{dW(T^*(m),m)}{dm} > 0$ for any $m \in [0,1]$ so that the optimal value of m is $m^* = 1$. Vice versa if the unique optimum choice is $m^* = 1$, then it is necessary that the sign in (12) be non negative at $m = 1$ and be strictly positive on part of $[0,1]$; a sufficient condition for $m^* = 1$ would be $\left(1+e^{-rT^*(m)}\right)v(T^*(m)) - e^{-rT^*(m)}v(2T^*(m)) \geq 0$ for any $m \in [0,1]$; this condition is difficult to check as it involves $T^*(m)$. A less demanding sufficient condition would be $\left(1+e^{-rT}\right)v(T) - e^{-rT}v(2T) \geq 0$ for any T; however that condition is obviously violated for growth functions that are convex at low values of T.

Nonetheless, it is possible to show that the optimal value of m is $m^* = 1$ when $C'(m) = 0$. Given a tract of forest land on which a maximum of n^* trees can be grown, suppose a decision maker had the possibility to choose a hypothetical management formula consisting in the repetition of two indefinite sequences: harvesting n_1 trees at a cost of c per tree at age T_1, and harvesting, at the same cost per tree, n_2 trees at age T_2, with $n_1 \geq 0$, $n_2 \geq 0$ and $n_1 + n_2 \leq n^*$. Suppose further that the objective of that decision maker was the same as in the mixed rotation problem, that is to maximize the present discounted value of the stream of net harvest revenues over an infinite number of rotations minus the cost of the initial stand:[6]

$$\max_{n_1,n_2,T_1,T_2} \left[\sum_{k=0}^{\infty} e^{-krT_1}\left(Pn_1v\left(T_1\right) - n_1c\right) + \sum_{k=0}^{\infty} e^{-krT_2}\left(Pn_2v\left(T_2\right) - n_2c\right)\right] - P\left(n_1v\left(T_1\right) + n_2v\left(T_2\right)\right) \tag{LC}$$

Problem (LC) is identical to the mixed rotation problem, the maximization of 5 by choice of m and T under constraints (2) with $C(m) = c$; but it is less constrained. Indeed, as in the mixed rotation problem, n_1 and n_2 must be non negative and such that $n_1 + n_2 \leq n^*$, but they do not need to satisfy condition (2). Also, while the cost per tree cut is the same in both problems, trees cut at age T_1 do not need to be cut simultaneously with trees cut at age T_2 in the (LC) problem. In fact, without loss of generality one may assume $T_1 \leq T_2$ and interpret trees cut at T_1 as the younger trees of the mixed rotation model and trees cut at T_2 as the older trees; then, as a third difference, the mixed rotation problem is subject to the constraint $T_2 = 2T_1$ which does not apply in the (LC) problem.

It is immediate to show that the solution of problem (LC) is such that $T_1 = T_2 = T^F(c)$, where $T^F(c)$ is the Faustmann rotation, and such that $n_1 + n_2 = n^*$. Then its

[6] There is no need to impose the existence of a steady state; the choices of T_1 and T_2 may imply various cycles.

optimized value is the same for any admissible value of n_1 and n_2, and coincides with Faustman's forest value for a fixed cost of c, $W^F(T;c)$, with $T = T^F(c)$. Since the mixed rotation problem is more constrained, its optimand cannot exceed that value:

$$W(T^*(m^*), m^*; c) \leq W^F\left(T^F(c); c\right)$$

However $(T^*(1), 1)$ is admissible in the mixed rotation problem, with $T^*(1) = T^F(c)$ and it can be verified that $W(T^F(c), 1; c) = W^F\left(T^F(c); c\right)$. It follows that $m^* = 1$ solves the mixed rotation problem when $C(m) = c$. This is stated in Proposition 2.

Proposition 2 *(Cost does not depend on m) When $C(m) = c$, the optimal proportion of younger trees cut at each harvest is $m^* = 1$ and the optimal age at which they are cut is Faustman's rotation $T^F(c)$.*

Consider now the case where net costs depend on m, so that $C'(m) > 0$. A necessary and sufficient condition for the optimal value of m to be lower than unity is then

$$\frac{dW(T^*(1), 1)}{dm} \leq 0$$

or, by (11), since $T^*(1) = T^F$

$$\left(1 + e^{-rT^F}\right) Pv(T^F) - e^{-rT^F} Pv(2T^F) \leq C'(1) \tag{13}$$

The expression $\left(1 + e^{-rT^F}\right) Pv(T^F) - e^{-rT^F} Pv(2T^F)$ represents the discounted gain in timber revenues, net of the change in the opportunity cost of the trees left standing at each harvest, caused by shifting production from the age $2T$ class to the age T class, as a result of marginally increasing m. When $m = 1$ as in expression (13) this proportion cannot be increased; the condition applies to a marginal reduction in m from its level of unity. Condition (13) states that the marginal loss in net discounted revenues caused by a reduction in m must be smaller than the corresponding marginal cost saving.

Similarly, a necessary and sufficient condition for the optimum value of m to be higher than zero is

$$\frac{dW(T^*(0), 0)}{dm} > 0$$

or

$$\left(1+e^{-rT^*(0)}\right)\frac{P}{4}v(T^*(0))-e^{-rT^*(0)}\frac{P}{4}v(2T^*(0))\geq C'(0) \tag{14}$$

$T^*(0)$ is the optimal rotation age when $m=0$ and only trees aged $2T\,(0)$ are harvested. When they hold together, conditions (13) and (14) are necessary and sufficient for the optimum value of m to be interior. Although it is not clear whether $\left(1+e^{-rT^F}\right)v(T^F)-e^{-rT^F}v(2T^F)$ is higher or lower than $\left(1+e^{-rT^*(0)}\right)v(T^*(0))-e^{-rT^*(0)}v(2T^*(0))$, the fact that the latter is weighted by $P/4$ in (14) while the former is weighted by P in (13) suggests that the cost function must be very convex in order to meet both conditions.[7]

Thus it appears that the conditions for an interior value of m are relatively stringent. It is easy to construct a corner solution at $m=0$ by observing that condition (14), as well as the first-order condition for an interior solution in $[0,1]$ obtained by setting (11) equal to zero, are sure to be violated at low values of P while (13) is sure to be satisfied. Similarly a corner solution at $m=1$ arises at high values of P, when condition (13) and the first-order condition for an interior solution are sure to be violated while (**??**) is sure to be satisfied.[8] This is intuitively easy to grasp. When P is high, the weight of timber revenues relative to forestry and amenity costs is high in the objective function: in that case it is optimal to manage the forest à la Faustmann ($m=1$) which was shown in Proposition 2 to be best when costs do not depend on m. On the contrary, when P is low, costs and amenity considerations weigh high in the objective function and the cost minimizing management option ($m=0$) is selected. However no cost gain can be realized if $C'=0$, the case falling under Proposition 2.

These results are spelled out in the following proposition.

Proposition 3 *(Cost depends on m) When $C'(m)>0$,*

[7] At an interior solution, $W\left(T^*\left(m\right),m\right)$ must be concave in m and satisfy the first-order condition $\frac{dW(T^*(m^*),m^*)}{dm^*}=0$, i.e.,

$$\frac{P(1+e^{-rT})}{(2-m^*)^2}v(T^*(m))-\frac{Pe^{-rT}}{(2-m^*)^2}v(2T^*(m))=\frac{dC(m^*)}{dm}. \tag{15}$$

This does not rule out the possibility for $W\left(T^*\left(m\right),m\right)$ to change curvature on $[0,1]$.

[8] Note that the argument relies on C' being strictly positive and applies whatever the (unknown) curvature of $W\left(T^*\left(m\right),m\right)$.

1. *For any tree volume function satisfying Assumption 1 and for any parameters P and r, if the cost function $C(m)$ satisfies condition* (13)*, then the optimal proportion of younger trees included in each harvest is lower than unity. That is: some trees are allowed to grow until age $2T$ before being harvested.*

2. *If furthermore $C(m)$ satisfies condition* (14)*, then the optimal proportion of younger trees included in each harvest is strictly between zero and one. That is: while some trees are allowed to grow until age $2T$ before being harvested, some trees are also cut at age T.*

3. *A corner solution at $m = 0$ ($m = 1$) occurs at low enough (high enough) values of P.*

Item 3 of the proposition indicates that the optimal level of m depends on the price of wood. When the price of wood relative to costs increases, the weight of wood revenues increases relative to the weight of costs and amenities in the objective function. Since Proposition 2 has shown that Faustman's rule and $m = 1$ are optimal when costs cannot be manipulated, it is not surprising that a similar result obtains when costs have little weight in the objective function. Thus when C' is positive, as P increases from a low level to a high level relative to costs and amenities, the optimal proportion of younger trees cut at each harvest increases from zero to unity.

Items 1 and 2 spell out the conditions for an interior solution for m. As the price of wood increases relative to cost, the transition from a corner solution at $m = 0$ to a corner solution at $m = 1$ may be smooth and involve a range of prices over which the optimum value of m is interior, or may occur as a sudden jump from zero to unity. Precisely, let $\underline{P}$ be the highest value of P below which condition (14) is violated; and let $\bar{P}$ be the lowest price above which condition (13) is violated. Whether $\underline{P}$ is smaller or higher than $\bar{P}$ depends on the cost function and the volume growth function. If indeed it is smaller, then there exists an interval $\left[\underline{P}, \bar{P}\right]$ over which Item 2 of Proposition 3 applies and the optimum value of m is interior. In the opposite case, no such interval exists. Instead there exists an interval $\left[\bar{P}, \underline{P}\right]$ over which $m = 0$ and $m = 1$ are both local maxima[9]; at values of P closer to $\bar{P}$ $m = 0$ is the global maximum; at values close enough to $\underline{P}$, the global maximum is $m = 1$. The optimal value of m is increasing in P as in the previous configuration, but the progression is not smooth and consists in a jump from zero to unity when P overtakes some critical value. As discussed earlier, this configuration is not unlikely.

3.3 Comparing Faustmann's Rotation and Rotations with Tree Retention

Suppose that m is exogenous. Then differentiating the first-order condition for T totally implies:

$$\frac{dT}{dm} = -\frac{W_{Tm}(T, m)}{W_{TT}(T, m)}$$

where W_{TT} is negative by the second-order condition. Suppose further that m is at its optimal level m^* so that the first-order condition for m can be used in evaluating W_{Tm}. Then we show in the Appendix that W_{Tm} is proportional to

$$\Delta = -re^{-rT}\left[v(T) - v(2T)\right] + \left(1 + e^{-rT}\right)v'(T) - e^{-rT}2v'(2T) \tag{16}$$

The sign of Δ depends on the levels and slopes of v at T and at $2T$ and consequently on the curvature of v over interval $[T, 2T]$.

When $\Delta > 0$ the optimum harvest age increases as the optimum proportion of trees cut at harvest increases. In other words, the rotation is longer the higher the proportion of younger trees cut at each harvest; it is maximum under Faustmann's formula, when no trees aged $2T$ are cut. In fact departing from Faustmann's formula is a decision to reduce m from an initial level of one in order to allow some trees to reach age $2T$. If T was not reduced by such a change that would unambiguously raise the average age at which trees were cut. The monotonicity of $T^*(m)$ tampers this effect: while some trees are cut at a higher age ($2T$ instead of T) under mixed rotations than under Faustmann's formula, trees cut at age T are cut earlier because T is reduced. Examination of Δ indicates

[9] This property is related to the curvature of $W(T(m), m)$. For example if $W(T(m), m)$ is convex over $[0, 1]$ then $\bar{P} < \underline{P}$.

that this is the likely situation. Indeed if no tree is allowed to grow into the decreasing part of the growth function, the first term is positive. The second term is also positive unless the growth function is very convex in the interval $[T, 2T]$; it is definitely positive in particular if the function is concave over that interval.

In general the harvest age and the proportion of trees cut at age T versus age $2T$ are determined jointly and vary with the price of wood relative to costs and amenities. As determined in the previous section, if the solution is interior, m^* increases as P increases. This covariation is noted $\frac{dP}{dm} > 0$ bellow. The sign of the covariation of T^* and m^* can be studied by differentiating totally the first-order condition for T arising from the maximization of (6) with respect to T and m, where P is treated as parameter:

$$\frac{dT}{dm} = -\frac{W_{Tm}(T, m) + W_{TP}\frac{dP}{dm}}{W_{TT}(T, m)}$$

where, using the first-order condition (8), W_{TP} can be shown to be negative; and where W_{TT} is negative by the second-order condition while W_{Tm} is proportional to Δ and likely to be positive as just discussed.

This implies that the direction of the change in T as m changes is ambiguous, which is not surprising as conflicting effects are at work. On one hand, there is the effect just described, applying when m changes exogenously with P fixed; when Δ is positive, it calls for longer rotations as m increases. On the other hand, when the change in T^* is caused by an increase in P, so that it occurs jointly with an increase in m, costs incurred at each harvest are now offset by higher revenues; then they can be incurred with higher frequency which implies that T^* should be reduced.

When the optimal level of m is interior, changes in T^* associated with marginal changes in P are themselves of the same order of magnitude as changes in P. In contrast allowing a tree to grow from age T^* to age $2T^*$ is a discrete change of a higher order of magnitude. In particular, when m is interior and diminishes from unity (the Faustmann forest) to some value strictly smaller than unity (mixed rotation), the age at which young trees are cut may be higher or lower than Faustman's rotation, but the age at which older trees are cut is definitely higher than Faustman's rotation. A similar conclusion may not hold if the optimal level of m is a corner solution and, as a result of a small change in P, jumps down from unity to zero.

4 Conclusion

Tree retention and forest management practices involving several possible cutting ages are increasingly viewed as preferable to management à la Faustmann and its "normal" even-aged forest lots.

In this paper, we have discussed a particular type of selective harvest involving two age-classes on each lot, and two harvest ages: in each lot harvests take place every T periods; at each harvest all the older trees are cut and only a fraction of younger trees are cut. Besides helping forest regeneration and soil protection, this type of forest management has several advantages in terms of aesthetics, sustainability, biodiversity and social acceptability, which were modeled by allowing costs to depend on the proportion of trees left standing at each harvest.

The paper has established and analyzed the optimum proportion of younger trees cut at each harvest and the optimum rotation. It has determined the conditions on the net cost of harvesting that this forestry practice must induce if it is to dominate the standard normalized forest management à la Faustmann. Depending on the net cost function, different harvest solutions are possible. The whole forest stand should be harvested at each rotation if net costs (including amenities) are not dependent on tree retention. This corresponds to the normal even-aged forest lot à la Faustmann. However, when net costs increase sufficiently with the harvest rate of younger trees, it is optimal to leave some of these trees uncut until they reach an age equal to twice the rotation. This results in stands where two age classes coexist and where land is never left bare. The polar case opposite to Faustmann's normal forest occurs when net costs weight heavily in the objective function and are sensitive to the proportion of trees cut at each harvest. This would be the case if, say, environmental or amenity considerations were important relative to commercial wood revenues. In that case each forest tract contains as many younger trees as older ones and only older trees are cut at harvest, which leaves half the trees standing at each harvest.

In mixed and double rotations, young trees are cut at an age which may be shorter or longer than Faustmann's rotations. This ambiguity occurs because two opposite effects are at play. On the one hand, selective harvesting reduces net costs, so that those costs can be incurred at higher frequency. On the other hand the exogenous change in the relative weight of wood revenues relative to costs and amenities which is causing the change from management à la Faustmann to mixed rotation must be such that the relative weight of net costs is increased, calling for a lower frequency of harvests. While moving away marginally from Faustmann's normalized forest has an ambiguous effect on the age at which younger trees are cut, the adoption of the mixed rotation management practice creates a second age class on the lot; these trees are definitely harvested at a higher age than the original Faustmann age. However the most noticeable effect of adopting mixed rotations, especially in the extreme polar case of double rotations, is that the lot is no longer clear cut; up to half the trees may be left to grow further at each harvest.

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APPENDIX

A First order condition for T:

The first-order condition (FOC) is

$$\frac{\partial W(T,m)}{\partial T} = 0,$$

where $W(T,m) = \frac{1}{1-e^{-rT}} \{PH(T,m) - C(m)\} - S(T,m)P$. This gives

$$\frac{-re^{-rT}}{(1-e^{-rT})^2} \{PH(T,m) - C(m)\} + \frac{1}{1-e^{-rT}} P \frac{\partial H(T,m)}{\partial T} - P \frac{\partial S(T,m)}{\partial T} = 0$$

$$\frac{re^{-rT}}{1-e^{-rT}} \{PH - C\} = P \frac{\partial H}{\partial T} - \left(1 - e^{-rT}\right) P \frac{\partial S}{\partial T}$$

Substituting (3) and (4), this gives Expression (8) in the text. The above expression can also be written as

$$r \frac{1}{1-e^{-rT}} (PH - C) - r \frac{1-e^{-rT}}{1-e^{-rT}} (PH - C) = \frac{P\partial H}{\partial T} - \left(1 - e^{-rT}\right) \frac{P\partial S}{\partial T}$$

Using $W = \frac{1}{1-e^{-rT}} \{PH - C\} - SP$,

$$r[W(T,m) + S(T,m)P] - r[PH(T,m) - C(m)] = P \frac{\partial H(T,m)}{\partial T} - P \frac{\partial S(T,m)}{\partial T} \left(1 - e^{-rT}\right)$$

from which (9) follows.

B Second order condition for T:

Let us use the following notation:

$$\begin{aligned} W(T,m) &= \frac{1}{1-e^{-rT}} [PH(T,m) - C(m)] - S(T,m)P \\ &= A(T)[PH(T,m) - C(m)] \; - S(T,m)P \text{ with } A(T) = \frac{1}{1-e^{-rT}}. \end{aligned}$$

The second-order condition (SOC) is

$$\frac{\partial^2 W(T,m)}{\partial T^2} \leq 0.$$

Therefore,

$$\frac{\partial^2 A(T)}{\partial T^2} [PH(T,m) - C(m)] + 2P \frac{\partial A(T)}{\partial T} \frac{\partial H(T,m)}{\partial T} + PA(T) \frac{\partial^2 H(T,m)}{\partial T^2} - P \frac{\partial^2 S(T,m)}{\partial T^2} \leq 0$$

Denoting partial derivatives by subscripts,

$$\frac{r^2 e^{-rT}\left(1+e^{-rT}\right)}{\left(1-e^{-rT}\right)^3}(PH-C)+2PA_TH_T+PAH_{TT}-PS_{TT}\leq 0$$

so that

$$\frac{r\left(1+e^{-rT}\right)}{\left(1-e^{-rT}\right)^2}\left[PH_T-PS_T\left(1-e^{-rT}\right)\right]+2PA_TH_T+PAH_{TT}-PS_{TT}\leq 0.$$

In the last expression, we used the fact that $\frac{re^{-rT}}{1-e^{-rT}}[PH-C]=PH_T-PS_T\left(1-e^{-rT}\right)$ by the first order condition.

Replacing A and A_T, the SOC is therefore

$$\frac{r}{1-e^{-rT}}\frac{\partial H(T,m)}{\partial T}+\frac{1}{1-e^{-rT}}\frac{\partial^2 H(T,m)}{\partial T^2}\leq\frac{r\left(1+e^{-rT}\right)}{1-e^{-rT}}\frac{\partial S(T,m)}{\partial T}+\frac{\partial^2 S(T,m)}{\partial T^2}$$

C Proof that W_{Tm} is proportional to Δ

$$\begin{aligned}W_T=&-r\frac{e^{-rT}}{\left(1-e^{-rT}\right)^2}\left[\frac{m}{2-m}v(T)+\frac{1-m}{2-m}v(2T)-\frac{C(m)}{P}\right]\\&+\frac{1}{1-e^{-rT}}\left(\frac{m}{2-m}v'(T)+\frac{1-m}{2-m}2v'(2T)\right)-\frac{1}{2-m}v'(T)-\frac{1-m}{2-m}2v'(2T)\end{aligned}$$

Hence

$$\begin{aligned}W_{Tm}=&-r\frac{e^{-rT}}{\left(1-e^{-rT}\right)^2}\left[\frac{2}{(2-m)^2}v(T)+\frac{-1}{(2-m)^2}v(2T)-\frac{C'(m)}{P}\right]\\&+\frac{1}{1-e^{-rT}}\left(\frac{2}{(2-m)^2}v'(T)+\frac{-1}{(2-m)^2}2v'(2T)\right)-\frac{1}{(2-m)^2}v'(T)-\frac{-1}{(2-m)^2}2v'(2T)\end{aligned}$$

$$\begin{aligned}W_{Tm}=&\frac{1}{\left(1-e^{-rT}\right)}\frac{1}{(2-m)^2}\left\{-r\frac{e^{-rT}}{\left(1-e^{-rT}\right)}\left[2v(T)-v(2T)-\frac{(2-m)^2C'(m)}{P}\right]\right.\\&\left.+\left(1+e^{-rT}\right)v'(T)-e^{-rT}2v'(2T)\right\}\end{aligned}$$

Hence W_{Tm} is proportional to:

$$\Delta=-r\frac{e^{-rT}}{\left(1-e^{-rT}\right)}\left[2v(T)-v(2T)-\frac{(2-m)^2C'(m)}{P}\right]+\left(1+e^{-rT}\right)v'(T)-e^{-rT}2v'(2T)\tag{17}$$

Since it is assumed that $m=m^*$, the first-order condition (11) holds; it implies:

$$\frac{(2-m)^2}{e^{-rT^*(m)}P}C'=\frac{\left(1+e^{-rT^*(m)}\right)}{e^{-rT^*(m)}}v(T^*(m))-v(2T^*(m)$$

Substituting into (17), it follows that:

$$\Delta=-re^{-rT}\left[v(T)-v(2T)\right]+\left(1+e^{-rT}\right)v'(T)-e^{-rT}2v'(2T)\tag{18}$$