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NONRENEWABLE RESOURCE OLIGOPOLIES AND THE CARTEL-FRINGE GAME

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Nonrenewable resource oligopolies and the cartel-fringe game*

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Abstract

We specify and solve a closed-loop dominant firm nonrenewable resource game, with a price-taking fringe. We show that (i) the outcomes of the closed-loop and the open-loop dominant firm nonrenewable resource game (à la Salant 1976) coincide and (ii) when the number of fringe firms becomes arbitrarily large, the equilibrium outcome of the closed-loop oligopoly game does not coincide with the equilibrium outcome of the closed-loop dominant firm nonrenewable resource game. Thus, the interpretation of the dominant firm model, where the fringe is assumed from the outset to be price-taker, as a limit case of an asymmetric oligopoly where the number of fringe firms tends to infinity, does not extend to the case where firms can use closed-loop strategies. *JEL* Classification: D43, Q30, C73, C61

Key words: nonrenewable resources, cartel-fringe, Nash equilibrium, open-loop, closed-loop, feedback.

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1 Introduction

The cartel-fringe model, also called the dominant firm model, of the oil market describes the pricing of oil in a situation where supply comes from a coherent cartel and a large group of fringe members. The model was introduced by Salant (1976), who considered the case of zero extraction costs and a continuum of price taking fringe members. He employed the open-loop Nash equilibrium (OLNE) as the equilibrium concept. The model was later analyzed by Ulph and Folie (1980), again with a continuum of fringe members and the OLNE equilibrium concept, but for positive constant marginal extraction costs, possibly differing between the cartel and the fringe. The cartel takes as given the production path of the fringe and chooses a price path whereas the fringe firms are price takers and determine their production paths. The cartel and the fringe simultaneously choose their respective strategy. Because each firm's strategy is in the form of a path we call this game the open-loop dominant firm nonrenewable resource game. An important contribution of Salant (1976) is to provide microfoundations of this model by showing that it is a limiting case of an asymmetric oligopoly model where fringe firms don't act as price takers. More precisely, consider the asymmetric oligopoly game with one dominant firm (e.g., with a low cost of extraction and/or larger reserves) and a finite number of fringe firms who compete à la Cournot in the natural resource market. Salant (1976) shows that when the number of fringe firms becomes arbitrarily large the equilibrium outcome of the open-loop game coincides with the equilibrium outcome of the open-loop dominant-firm nonrenewable resource game.

Open-loop strategies are acceptable in environments where firms can commit over the whole time horizon to a production path or a price path, for instance under the assumption of a perfect futures' market. However, this may not be an acceptable way to model firms' strategies in environments where firms have information about stocks at future dates and have the flexibility to change their course of actions during the game: the equilibrium obtained with open-loop strategies may not be subgame perfect. In the latter case, we consider the set of closed-loop strategies where a firm chooses states' (i.e., stocks) dependent strategies.

In this paper we specify and solve a closed-loop dominant firm nonrenewable resource game. We show that (i) the outcomes of the closed-loop and the open-loop dominant firm nonrenewable resource games coincide and (ii) when the number of fringe firms becomes

arbitrarily large, the equilibrium outcome of the closed-loop oligopoly game does not coincide with the equilibrium outcome of the closed-loop dominant-firm nonrenewable resource game. While the first result shows the robustness of the open-loop cartel-fringe outcome derived in Salant (1976), our second result contrasts with the case where firms use open-loop strategies.

More specifically, we consider an oligopoly where each firm exploits a private exhaustible resource and where one firm (the cartel) has a cost advantage over the other firms (fringe firms). All firms compete à la Cournot in the resource market. Assume the cartel chooses a strategy that specifies the extraction rate at each moment as a function of the state, described by the vector of stocks of all firms, at that moment. While the cartel takes the strategy of each fringe firm as given, its extraction rate depends on the its own stock as well as all fringe firms' stocks. When weighing the impact of an extra unit of extraction at a given moment it takes into account three effects (i) the additional revenue, (ii) the reduction of its available stock and (iii) the impact of this change in its own stock on the extraction of its competitors. This latter effect that we refer to as the feedback effect is absent when firms use open-loop strategies. We show that the equilibrium outcome of the open-loop game cannot be supported as the outcome of an equilibrium of the closed-loop game. This is due to the presence of the feedback effect. More surprisingly, we show that this remains true even in the limit case where the number of fringe firms is let to tend to infinity, while keeping the aggregate resource stock unchanged: the feedback effect does not vanish as the market power of each fringe firm is diluted by the increase in the total number of fringe firms.

In deriving our conclusions we exploit the analysis in Benckroun et al. (2008) which provides a full characterization of the open-loop Nash equilibrium of an asymmetric nonrenewable resource game with a finite as well as an infinite number of fringe players, for all possible constant marginal extraction costs. Benckroun et al. (2008) is closely related to Lewis and Schmalensee (1980) and Loury (1986) which have studied the case of a finite number of oligopolists. The former authors were mainly interested in the order of exploitation and their analysis mainly concerns the case of two players. Loury studies the case of equal costs. All these papers focus on the case where firms use open-loop strategies.

Polasky (1990) shows in a discrete time model with a finite number of players that the open-loop equilibrium is not subgame perfect if the exhaustion dates of firms differ. He

then considers a duopoly model with linear demand and equal and constant marginal extraction costs. He also postulates an exogenous instant of time T , after which the extracted commodity is worthless. He then claims that if the per period profit function is quadratic in extraction and depends only on current extraction (and not on existing stocks) and if no firm exhausts before T , open-loop and feedback equilibria coincide. But then he proves that in the duopoly model with equal initial stocks and equal constant marginal extraction costs and in the absence of an exogenous T , the open-loop and the feedback equilibrium do not coincide because one firm can and will manipulate its own exhaustion time in a profitable way. The present paper uses a continuous time formulation of a nonrenewable resource oligopoly, allows for asymmetries between firms (in terms of costs, stocks and number of firms in each category) and includes the cartel-fringe framework.

Our methodology is related to the work done by Groot et al. (1992, 2003) who studied the case of the cartel being a Stackelberg leader and the fringe being a price taker. The cartel-fringe model with Stackelberg leadership was first introduced by Gilbert (1978). It is well-known that in this model the open-loop Stackelberg equilibrium concept suffers from time inconsistency for plausible parameter values, and is therefore not a feedback equilibrium (see Newbery (1981) and Ulph (1982)). But open-loop and closed-loop equilibrium outcomes do coincide for at least some parameter values. In this paper we consider the case where the cartel and fringe firms *simultaneously* choose their respective strategies.

To our knowledge this paper is a first to specify a closed-loop formulation for a dominant firm dynamic game. The difficulty lies in reconciling the intrinsic *myopic* behavior of a fringe firm assumed through price taking and the rather sophisticated (or farsighted) behavior assumed by the use of closed-loop strategies. We propose the following scenario for the closed-loop dominant firm model: each fringe firm takes the price path as given and determines its extraction strategy which is allowed depend on its own stock only; the cartel takes each fringe firm's strategy as given and determines a pricing strategy (or alternatively a production strategy) that depends on its own stock and all fringe's stocks. The outcome of this simultaneous move is an equilibrium if the market of the resource is in equilibrium at each moment.

We present the model as well as the open-loop Nash equilibrium with a finite number of fringe firms in the next section. In section 3, we compare the equilibrium outcomes of

the open-loop oligopoly game and the closed-loop oligopoly game. The crux of the paper is in section 4 where we consider the closed-loop dominant-firm nonrenewable resource game.

2 Model and the Open-loop Nash equilibrium

There are two types of mines c and f , distinguished by their marginal extraction costs. There is one c -type mine, owned by a cartel, and there are n mines of the f -type. The owner of an f -mine is called a fringe member. Marginal extraction costs are constant: k^c and k^f . The cartel's initial stock is S_0^c . Fringe firm i ($i = 1, 2, \dots, n$) is endowed with an initial stock S_{0i}^f . Demand for the resource is stationary and linear with a choke price \bar{p} : $p(t) = \bar{p} - d(t)$, where $p(t)$ is the price at time t , $d(t)$ is the quantity demanded at time t and $\bar{p} > \max\{k^c, k^f\}$. We work in continuous time, which starts at time 0. Extraction rates at time $t \geq 0$ are denoted by $q^c(t) \geq 0$ and $q_i^f(t) \geq 0$. Define $q^f(t) = \sum_{i=1}^n q_i^f(t)$ and $S_0^f = \sum_{i=1}^n S_{0i}^f$ as aggregate supply and initial aggregate stocks of the fringe firms. In an equilibrium at each moment $t \geq 0$ the price of the resource is given by $p(t) = \bar{p} - q^c(t) - q^f(t)$. For the time being all fringe firms are assumed identical with regard to their stocks: $S_{0i}^f = S_0^f/n$. Any feasible extraction path for a firm is subject to the condition that total extraction over time equals the initial stock. This is called the resource constraint. It reads

$$\int_0^{\infty} q^c(s) ds = S_0^c$$

for the cartel and

$$\int_0^{\infty} q_i^f(s) ds = S_{0i}^f$$

and for fringe member i . We formulate the resource constraints as an equality because in any equilibrium all resource stocks will get exhausted in view of the assumption that $\bar{p} > \max\{k^c, k^f\}$. In the oligopoly game, firms compete à la Cournot in the resource market and the objective of each firm is to maximize the discounted sum of its profits with an equal and constant discount rate r .

Definition: Open-loop Nash Cournot equilibrium (OLNE)

A vector $q(\cdot) \equiv (q^c(\cdot), q_1^f(\cdot), \dots, q_n^f(\cdot))$ with $q(t) \geq 0$ for all $t \geq 0$ is an open-loop Nash-Cournot equilibrium if

- i. all resource constraints are satisfied
- ii.

$$\begin{aligned} & \int_0^{\infty} e^{-rs} [\text{Max} \{ \bar{p} - q^c(s) - q^f(s), 0 \} - k^c] q^c(s) ds \\ & \geq \int_0^{\infty} e^{-rs} [\text{Max} \{ \bar{p} - \hat{q}^c(s) - q^f(s), 0 \} - k^c] \hat{q}^c(s) ds \end{aligned}$$

for all feasible \hat{q}^c .

- iii. for all $i = 1, 2, \dots, n$

$$\begin{aligned} & \int_0^{\infty} e^{-rs} [\text{Max} \{ \bar{p} - q^c(s) - q^f(s), 0 \} - k^f] q_i^f(s) ds \\ & \geq \int_0^{\infty} e^{-rs} [\text{Max} \{ \bar{p} - q^c(s) - \sum_{j \neq i} q_j^f(s) - \hat{q}_i^f(s), 0 \} - k^f] \hat{q}_i^f(s) ds \end{aligned}$$

for all feasible \hat{q}_i^f .

Benchekroun et al. (2008) characterize the OLNE of this nonrenewable resource oligopoly game. They allow for an arbitrary number of firms that have the c -type mines. For our present purpose this is less relevant, as will be made clear in due course. By S , C and F we denote intervals of time with simultaneous supply, sole supply by the cartel and sole supply by the fringe, respectively. Benchekroun et al. have established the following proposition.

Proposition 1

- i. *Suppose*

$$\frac{1}{2}(\bar{p} + k^c) < k^f$$

For a given S_0^f , there exists $\tilde{S}_0^c > 0$ such that the OLNE sequence reads $C \rightarrow S \rightarrow F$ if $S_0^c > \tilde{S}_0^c$ and $S \rightarrow F$ if $S_0^c \leq \tilde{S}_0^c$.

- ii. *Suppose*

$$\frac{1}{2}(\bar{p} + k^c) = k^f$$

Then the OLNE yields $S \rightarrow F$

iii. Suppose

$$\frac{1}{2}(\bar{p} + k^c) > k^f$$

Let $\sigma \equiv \frac{\bar{p} + nk^f - (n+1)k^c}{n(\bar{p} + k^c - 2k^f)}$. The OLNE sequence depends on the initial stocks as displayed below

Stocks	$S_0^c/S_0^f < \sigma$	$S_0^c/S_0^f = \sigma$	$S_0^c/S_0^f > \sigma$
OLNE	$S \rightarrow F$	S	$S \rightarrow C$

3 Open-loop versus closed-loop: the case of a finite number of players

A closed-loop strategy for a firm is a decision rule that gives the extraction rate at t as a function of t and the vector of stocks at time t , $S(t) = (S^c(t), S_1^f(t), S_2^f(t), \dots, S_n^f(t))$. The definition of a closed-loop Nash equilibrium CLNE reads as follows¹.

Definition: Closed-loop Nash-Cournot equilibrium (CLNE)

A vector of closed-loop strategies $\phi \equiv (\phi^c, \phi_1^f, \dots, \phi_n^f)$ is a closed-loop Nash-Cournot equilibrium if

- i. the resource constraint is satisfied for all firms, where $q^c(t) = \phi^c(t, S(t))$ and $q_i^f(t) = \phi_i^f(t, S(t))$ ($i = 1, 2, \dots, n$)
- ii.

$$\begin{aligned} & \int_0^{\infty} e^{-rs} [\text{Max}\{\bar{p} - \sum_{i=1}^n \phi_i^f(s, S(s)) - \phi^c(t, S(t)), 0\} - k^c] \phi^c(t, S(t)) ds \\ & \geq \int_0^{\infty} e^{-rs} [\text{Max}\{\bar{p} - \sum_{i=1}^n \phi_i^f(s, S(s)) - \hat{\phi}^c(t, S(t)), 0\} - k^c] \hat{\phi}^c(t, S(t)) ds \end{aligned}$$

for all feasible strategies $\hat{\phi}^c$.

- iii. for all $i = 1, 2, \dots, n$

¹For both the OLNE and CLNE we give an ad-hoc definition for this resource game, for a more formal treatment we refer to Dockner et al. (2000) or Başar and Olsder (1995).

$$\begin{aligned}
& \int_0^{\infty} e^{-rs} [\text{Max}\{\bar{p} - \sum_{j=1}^n \phi_j^f(s, S(s)) - \phi^c(t, S(t)), 0\} - k^f] \phi_i^f(s, S(s)) ds \\
\geq & \int_0^{\infty} e^{-rs} [\text{Max}\{\bar{p} - \sum_{j \neq i}^n \phi_j^f(s, S(s)) - \hat{\phi}_i^f(s, S(s)) - \phi^c(t, S(t)), 0\} - k^f] \hat{\phi}_i^f(s, S(s)) ds
\end{aligned}$$

for all feasible strategies $\hat{\phi}_i^f$.

In this section we determine whether the OLNE outcome can coincide with the outcome of a CLNE.

The case $S \rightarrow F$

Proposition 1 provides conditions for the OLNE equilibrium to contain the sequence $S \rightarrow F$. We seek to determine if there exists a CLNE, that is therefore subgame-perfect, that replicates the exploitation path of the OLNE, given a vector of initial stocks. The cartel takes the closed-loop strategy of the fringe $\phi^f(S, t)$ as given and chooses a closed-loop strategy $\phi^c(S, t)$ that maximizes its discounted sum of profits

$$\int_t^{\infty} e^{-rs} (\text{Max}\{\bar{p} - q^c(s) - \phi^f(S(s), s), 0\} - k^c) q^c(s) ds \quad (1)$$

subject to

$$\int_t^{\infty} q^c(s) ds \leq S^c \quad (2)$$

and

$$\int_t^{\infty} \phi_i^f(S(s), s) ds \leq S_i^f, \quad i = 1, 2, \dots, n \quad (3)$$

for all non-negative couples (S, t) , with $q^c(s) = \phi^c(S(s), s)$.

The Hamiltonian associated with the cartel's problem is given by

$$H^c(q^c, S, \mu_c^c, \mu_{f_i}^c, t) = e^{-rt} \left(\text{Max}\{\bar{p} - q^c - \sum_{i=1}^n \phi_i^f(S, t), 0\} - k^c \right) q^c - \mu_c^c q^c - \sum_{i=1}^n \mu_{f_i}^c \phi_i^f(S, t)$$

where μ_c^c is the costate variable associated with S^c and $\mu_{f_i}^c$ is the costate variable associated with S_i^f . Applying the maximum principle gives the following set of necessary conditions for an interior solution at time t :

$$e^{-rt} (\bar{p} - 2q^c(t) - \phi^f(S(t), t) - k^c) - \mu_c^c(t) = 0 \quad (4)$$

$$\dot{\mu}_c^c(t) = -\frac{\partial H^c}{\partial S^c} = \sum_{i=1}^n (e^{-rt} q^c(t) + \mu_{f_i}^c(t)) \frac{\partial \phi_i^f(S(t), t)}{\partial S^c} \quad (5)$$

$$\dot{\mu}_{f_i}^c(t) = -\frac{\partial H^c}{\partial S_i^f} = \sum_{i=1}^n (e^{-rt} q^c(t) + \mu_{f_i}^c(t)) \frac{\partial \phi_i^f(S(t), t)}{\partial S_i^f} \quad (6)$$

where

$$\phi^f(S(t), t) = \sum_{i=1}^n \phi_i^f(S(t), t)$$

Appendix A provides a further characterization of the OLNE in this case, based on Benckroun et al. (2008). There it is shown that along the phase of simultaneous supply, taken to be from time 0 till time t_1 , the production paths of the fringe and the cartel along the OLNE are given by

$$(n+2)q^c(t) = \bar{p} + n(k^f + \lambda^f e^{rt}) - (n+1)(k^c + \lambda^c e^{rt}) \quad (7)$$

$$\frac{2+n}{n}q^f(t) = \bar{p} + (k^c + \lambda^c e^{rt}) - 2(k^f + \lambda^f e^{rt}) \quad (8)$$

where λ^c and λ^f are the constant shadow prices of the resource stocks of the cartel and the fringe members respectively. Hence, in view of (4) and (7), for a CLNE to result in the extraction path of the OLNE, we must have $\mu_c^c(s) = \lambda^c$, for all instants $s \geq t$ for all $t \geq 0$. From necessary condition (5) it follows that then

$$\sum_{i=1}^n (e^{-rt} q^c(t) + \mu_{f_i}^c(t)) \frac{\partial \phi_i^f(S(t), t)}{\partial S^c} = 0$$

Given the symmetry of fringe firms we must have either $e^{-rs} q^c + \mu_{f_i}^c = 0$ where q^c is the OLNE equilibrium path of the cartel, and therefore $\mu_{f_i}^c(t) = -e^{-rt} q^c(t)$, or $\partial \phi_i^f(S(t), t) / \partial S^c = 0$. The first possibility is in contradiction with the necessary conditions since it implies from (6) that $\dot{\mu}_{f_i}^c(t) = 0$, but $e^{-rs} q^c(t)$ is not constant along the OLNE. Thus we have established that for the OLNE outcome to coincide with the outcome of a CLNE it is necessary that along the equilibrium path

$$\frac{\partial \phi_i^f(S(t), t)}{\partial S^c} = 0 \text{ for all } t \geq 0, \text{ for every fringe firm } i.$$

Given the symmetry of fringe firms we have

$$\frac{\partial \phi_i^f(S(t), t)}{\partial S^c} = \frac{1}{n} \frac{\partial \phi^f(S(t), t)}{\partial S^c} = \frac{1}{n} \frac{\partial q^f(t)}{\partial S^c}$$

which gives

$$\frac{2+n}{n} \frac{\partial \phi^f(S(t), t)}{\partial S^c} = \frac{\partial (\lambda^c - 2\lambda^f) e^{rt}}{\partial S^c}$$

where

$$\lambda^f = e^{-rT} (\bar{p} - k^f) \quad \text{and} \quad \lambda^c = \frac{n}{n+1} \lambda^f + \frac{e^{-rt_1} (\bar{p} + nk^f - (n+1)k^c)}{n+1}$$

As explained in appendix A the first of these two latter equations states that the market price at the instant of exhaustion of the resource equals the choke price; the second equation follows from the requirement that the price path is continuous. The two equations yield

$$\lambda^c - 2\lambda^f = \left(\frac{n}{n+1} - 2 \right) e^{-rT} (\bar{p} - k^f) + \frac{1}{n+1} e^{-rt_1} (\bar{p} + nk^f - (n+1)k^c) \quad (9)$$

The time of transition t_1 and the final time T satisfy (see appendix A):

$$(2+n)rS_0^c = (\bar{p} + nk^f - (n+1)k^c) (rt_1 - 1 + e^{-rt_1}) \quad (10)$$

$$r \left(S_0^f + \frac{n}{n+1} S_0^c \right) = \frac{n}{n+1} (\bar{p} - k^f) (rT - 1 + e^{-rT}) \quad (11)$$

From (9) we have

$$\frac{\partial (\lambda^c - 2\lambda^f)}{\partial S^c} = -r \left(\frac{n}{n+1} - 2 \right) e^{-rT} \frac{\partial T}{\partial S^c} (\bar{p} - k^f) - r e^{-rt_1} \frac{\partial t_1}{\partial S^c} \frac{(\bar{p} + nk^f - (n+1)k^c)}{n+1} \quad (12)$$

We derive $\partial T / \partial S^c$ and $\partial t_1 / \partial S^c$ from (10) and (11) and substitute them into (12) to obtain

$$\frac{\partial (\lambda^c - 2\lambda^f)}{\partial S^c} = r \left(\frac{n+2}{n+1} \right) \left(\frac{1}{(Te^{rT} - e^{rt})} - \frac{1}{(t_1 e^{rt_1} - e^{rt})} \right)$$

For any $t \geq 0$, we have that

$$f(X) = \frac{1}{(Xe^{rX} - e^{rt})}$$

is strictly decreasing in X and therefore $\frac{\partial (\lambda^c - 2\lambda^f)}{\partial S^c} \neq 0$ since $T > t_1$. Hence for any equilibrium that reads $C \rightarrow S \rightarrow F$ or $S \rightarrow F$ a necessary condition for the CLNE to yield the OLNE extraction path is not met. Note that our result holds true even in the limit case where $n = \infty$ since for $n \rightarrow \infty$ we have

$$\frac{\partial (\lambda^c - 2\lambda^f)}{\partial S^c} = r \left(\frac{1}{(Te^{rT} - e^{rt})} - \frac{1}{(t_1 e^{rt_1} - e^{rt})} \right) \neq 0$$

The argument also goes through for any cost constellation that yields this equilibrium sequence. We have thus shown the following.

Proposition 2

Suppose the OLNE yields the sequence $S \rightarrow F$ then the OLNE extraction path cannot be obtained as the extraction path of a CLNE. This is true even when $n \rightarrow \infty$.

To conclude our analysis we note that for a vector of strategies to qualify as a non-degenerate CLNE it must specify extraction rates for all possible values of the initial stocks. Since there always exists a range of initial stocks such that the OLNE yields the sequence $S \rightarrow F$ we conclude from proposition 2 that there exists no CLNE that will replicate the OLNE equilibrium outcome for *all* values of the vector of stocks.

It turns out that this result is robust to restrictions on the state space. Suppose we consider a less restrictive condition where we require a CLNE to replicate an OLNE outcome only for a subset of positive measure of the state space.

The case $S \rightarrow C$

For initial values of the stocks such that the OLNE sequence is $S \rightarrow C$, Proposition 2 does not rule out the possibility that there exists a CLNE to replicate an OLNE outcome. We know from Proposition 1 that if $k^f < \frac{1}{2}[\bar{p} + k^f]$ the equilibrium reads $S \rightarrow C$ if the initial resource stock of the fringe is not too large. We seek to determine whether there exists a feedback Nash equilibrium, that is therefore subgame-perfect, that replicates the exploitation path of the OLNE, given a vector of initial stocks. Along the phase of simultaneous supply equations (7) and (8) hold, where, in the case at hand

$$\lambda^c = e^{-rT}(\bar{p} - k^c) \quad \text{and} \quad \frac{1}{2}(\bar{p} + k^c + e^{rt_1}\lambda^c) = k^f + e^{rt_1}\lambda^f$$

and where the transition date t_1 and the exhaustion date T are respectively given by

$$\frac{2+n}{n}rS_0^f = (\bar{p} + k^c - 2k^f)(rt_1 - 1 + e^{-rt_1})$$

and

$$r\left(S_0^f + \frac{1}{2}S_0^f\right) = (\bar{p} - k^c)(rT - 1 + e^{-rT}).$$

It readily follows that $q^f(t)$ is independent of S^c . Contrary to the previous case we will henceforth concentrate on the fringe. The problem is that we cannot repeat the

steps taken in the previous case, since we have to be clear about what to mean by a marginal change in the stock of one of the fringe members, keeping the other stocks fixed. This poses a difficulty because it has been assumed that all fringe members are equal, and the OLNE has been derived under that assumption. However, it is not difficult to conceptualize what will happen if one fringe member is given an addition to its reserve. All other fringe members will exhaust their resource before this fringe member under consideration does, as is formally demonstrated in Appendix B. Hence it is left with the cartel as sole competitor. We are therefore done if we can show that the OLNE and the CLNE do not coincide for the case of a single cartel and a single fringe member. Due to symmetry this is straightforward since we can repeat the steps taken in the previous case, *ceteris paribus*, and obtain the same negative result. For the sake of completeness the proof is given in detail in appendix C.

Proposition 2b

Suppose the OLNE yields the sequence $S \rightarrow C$, i.e.

$$\frac{1}{2}(\bar{p} + k^c) > k^f \text{ and } \frac{S_0^c}{S_0^f} > \sigma$$

then the OLNE extraction path cannot be the outcome of a CLNE extraction path. This is true even when $n \rightarrow \infty$.

4 Open-loop versus closed-loop: the cartel-fringe game

The open-loop Nash cartel-fringe nonrenewable resource game is specified in Salant (1976) and unfolds as follows. There is a coherent cartel and a number of fringe firms each possessing a stock of the nonrenewable resource. Each fringe firm takes the price path as given and chooses a path of extraction, whereas the cartel takes the extraction path of the fringe as given and determines a price path. All firms choose their respective strategies simultaneously. The outcome of this game is an equilibrium if the market equilibrium holds at every moment. We denote the open-loop equilibrium of cartel-fringe game by OL-CFE.

It can be shown that the limit case of the OLNE outcome when the number of fringe firms tends to infinity yields the outcome of an OL-CFE (Salant (1976) Appendix B treats the case where extractions costs are zero).

Proposition 3

The OL-CFE, with price taking behavior of the fringe members, is characterized as follows:

- i. If $\frac{1}{2}(\bar{p} + k^c) < k^f$, then the equilibrium sequence is $C \rightarrow S \rightarrow F$, with the F phase collapsing if S_0^c is 'small'.
- ii. If $\frac{1}{2}(\bar{p} + k^c) = k^f$, then the equilibrium sequence is $S \rightarrow F$
- iii. If $\frac{1}{2}(\bar{p} + k^c) > k^f$, let $\sigma_{CFE} \equiv \frac{k^f - k^c}{\bar{p} + k^c - 2k^f}$ then the equilibrium sequence is

Stocks	$S_0^c/S_0^f < \sigma_{CFE}$	$S_0^c/S_0^f = \sigma_{CFE}$	$S_0^c/S_0^f > \sigma_{CFE}$
OL-CFE	$S \rightarrow F$	S	$S \rightarrow C$

While the open-loop formulation of the cartel-fringe model is widely used and analyzed in the literature, there exists, to our knowledge, no analysis of a closed-loop formulation of the cartel-fringe game. This paper is a first attempt to specify a closed-loop formulation for a dominant firm dynamic game. The difficulty lies in reconciling the intrinsic *myopic* behavior of a fringe firm assumed through price taking and the rather sophisticated (or farsighted) behavior assumed by the use of closed-loop strategies.

We propose the following scenario for the closed-loop dominant firm model: each fringe firm takes the price path as given and determines its extraction strategy which is allowed to depend on its own stock only; the cartel takes the closed-loop representation of the fringe's production path as given and determines a pricing strategy (or alternatively a production strategy) that depends on its own stock and all fringe firms' stocks. The outcome of this simultaneous move is an equilibrium if the market of the resource is in equilibrium at each moment. We denote the closed-loop equilibrium of cartel-fringe game by CL-CFE. Formally

Definition: Closed-loop Cartel-Fringe equilibrium (CL-CFE)

A vector $(\pi, \phi^c, \phi^f, \phi_1^f, \dots, \phi_n^f)$ with a price path $\pi = \pi(t)$ and closed-loop extraction rules $\phi^c = \phi^c(t, S)$, $\phi^f = \phi^f(t, S)$ and $\phi_i^f = \phi_i^f(t, S_i^f)$ ($i = 1, 2, \dots, n$) is a closed-loop Cartel-Fringe equilibrium (CL-CFE) if

- i. the resource constraint is satisfied for all firms, where $q^c(t) = \phi^c(t, S(t))$ and $q_i^f(t) = \phi_i^f(t, S_i^f(t))$ ($i = 1, 2, \dots, n$)

ii. given ϕ^f ,

$$\begin{aligned} & \int_0^{\infty} e^{-rs} [Max \{ \bar{p} - \phi^f(s, S(s)) - \phi^c(t, S(t)), 0 \} - k^c] \phi^c(t, S(t)) ds \\ & \geq \int_0^{\infty} e^{-rs} [Max \{ \bar{p} - \phi^f(s, S(s)) - \hat{\phi}^c(t, S(t)), 0 \} - k^c] \hat{\phi}^c(t, S(t)) ds \end{aligned}$$

for all feasible strategies $\hat{\phi}^c$.

iii. for all $i = 1, 2, \dots, n$, given π

$$\int_0^{\infty} e^{-rs} [\pi(s) - k^f] \phi_i^f(s, S_i^f(s)) ds \geq \int_0^{\infty} e^{-rs} [\pi(s) - k^f] \hat{\phi}_i^f(s, S_i^f(s)) ds$$

for all feasible strategies $\hat{\phi}_i^f$.

iv. for all $t \geq 0$: $\phi^f(t, S^f(t)) = \sum_{i=1}^n \phi_i^f(t, S_i^f(t))$

v. for all $t \geq 0$: $\pi(t) = Max \{ \bar{p} - \phi^f(t, S^f(t)) - \phi^c(t, S(t)), 0 \}$

The function $\phi^f(t, S^f)$ corresponds to the aggregate extraction of the fringe written in a closed-loop form. It is not a strategy per se, it arises from the individual optimal choice of each fringe firm of a production path, and gives the *behavior* of the fringe as a function of the vector of stocks. The cartel takes the fringe's behavior, $\phi^f(t, S^c, S^f)$, as given and determines its pricing (or production) strategy which is allowed to depend on its stock and the fringe's stock. Condition v states that, for any $t \geq 0$, given a vector of stocks, the realization of $\bar{p} - \phi^f(t, S^f(t)) - \phi^c(t, S(t))$ yields the price $\pi(t)$ taken as given in the fringe's problem stated in iii.

The assumption about the fringe firms' behavior is important and is a modelling choice. One could follow alternate assumptions regarding the fringe firm's degree of sophistication. For instance the fringe firm could be allowed to consider the price rule as given but not the price path; in which case the fringe firm can still influence the price path through its influence on its own stock. This latter behavior of the fringe firm did not appeal to us because it assumes that a fringe firm, while determining its best response to a strategy of the cartel, is aware of the impact of its own stock on the market price but is not aware of the impact of its own quantity sold on the same market price. This implication appears rather contradictory. Thus, and in keeping with the typically assumed *myopic* behavior of a fringe firm, we retain the assumption that each fringe

firm takes the price path as given and that it may condition its extraction rate on its own stock only².

We argue that with a price taking fringe, there exists a CL-CFE that yields the same outcome as the OL-CFE outcome, for any composition of the initial stocks. The proof consists of three steps. First we build a closed-loop representation of each fringe firm's production path under the open-loop cartel-fringe equilibrium (Lemma 1 below). Then we show that for the cartel, the closed-loop representation of its open-loop equilibrium price is a best response to the fringe firms closed-loop strategy (built in the first step) (Lemma 3 below). We complete the proof by noting that for each fringe firm, the closed-loop representation of its open-loop equilibrium strategy (built in the first step), is a best response to the open-loop cartel-fringe (OL-CFE) price path.

We only present the details of the proof for the case where the sequence of the OL-CFE is $S \rightarrow C$, i.e., when $\frac{1}{2}(\bar{p} + k^c) > k^f$ and for S^c, S^f such that $\frac{S^c}{S^f} > \sigma_{CFE}$. A similar treatment and the same conclusion regarding the existence of a CL-CFE that yields the same outcome as the OL-CFE outcome holds when the OL-CFE sequence is $S \rightarrow F$. From here on, we are assuming that $\frac{1}{2}(\bar{p} + k^c) > k^f$.

To write closed-loop representations of the open-loop equilibrium paths it will be useful to define the following function

$$h(z) = \ln\left(\frac{1}{z}\right) + z - 1.$$

with domain³ $(0, 1]$. It can easily be checked that the function h is strictly decreasing over $(0, 1]$ with $\lim_{z \rightarrow 0} h(z) = \infty$ and $\lim_{z \rightarrow \infty} h(z) = 0$. Therefore, for any $A \geq 0$ there exists a unique solution in $(0, 1]$ to $h(z) = A$.

For any $S^f \geq 0$, let x be the unique solution in $(0, 1]$ to

$$h(x) = \frac{rS^f}{\bar{p} + k^c - 2k^f} \tag{13}$$

and for any $S^c, S^f \geq 0$, let y be the unique solution in $(0, 1]$ to

$$h(y) = r \frac{2S^c + S^f}{\bar{p} - k^c}. \tag{14}$$

Lemma 1

²Given a price path, the only payoff relevant information for a fringe firm is its own available stock.

³The reason why we focus on this domain is transparent in Lemma 1 and its proof, see e.g. (15).

For any $S^c, S^f \geq 0$ such that the OL-CFE sequence is $S \rightarrow C$, the OL-CFE outcome coincides with the outcome of the following closed-loop strategies :

$$\phi^f(S^f) = q^f(t, x) = (\bar{p} + k^c - 2k^f)(1 - x) \quad (15)$$

and

$$\phi^c(S^c, S^f) = q^c(t, x, y) = \frac{1}{2}(\bar{p} + k^c - 2k^f)(x - 1) - \frac{1}{2}(\bar{p} - k^c)(y - 1) \quad (16)$$

where x and y are respectively the unique solutions in $(0, 1]$ to (13) and (14).

Proof: see Appendix D.

Note that when $S^f = 0$ we have $x = 1$ and when $S^f = S^c = 0$ we have $y = 1$. Therefore, the closed-loop strategies given in (15) and (16) also represent the open-loop extraction paths during the last phase C , where the cartel is the sole supplier, with

$$\phi^f(0) = q^f(t, 1, y) = 0$$

and

$$\phi^c(S^c, 0) = q^c(t, 1, y) = \frac{1}{2}(\bar{p} - k^c)(1 - y).$$

We also remark that the strategies are feedback strategies (they do not depend on time explicitly); this is due to the fact that the problem of each firm is autonomous.

The closed-loop representation of the production paths allows to get the cartel's discounted sum of profits in a closed-loop form.

Lemma 2

For any $S^c, S^f \geq 0$ such that the OL-CFE sequence is $S \rightarrow C$, a closed-loop representation of the cartel's discounted sum of profits at an initial date t with stocks (S^f, S^c) is

$$\begin{aligned} \Pi^c(t, x, y) = & \frac{e^{-rt}}{4r} \{ 4(k^f - k^c)^2(1 - x) + 4(k^f - k^c)(\bar{p} + k^c - 2k^f)x \ln\left(\frac{1}{x}\right) \\ & + (\bar{p} + k^c - 2k^f)^2(x - x^2) + (\bar{p} - k^c)^2(y^2 + x - 2y) \} \end{aligned} \quad (17)$$

where x and y are respectively the unique solutions in $(0, 1]$ to (13) and (14).

Proof: see Appendix E.

We are now able to state the following.

Lemma 3

For any $S^c, S^f \geq 0$ such that the OL-CFE sequence is $S \rightarrow C$, the cartel's closed-loop strategy (16) (representation of the cartel's open-loop equilibrium production path) is a best response to the fringe's closed-loop behaviour (15) (representation of the fringe's open-loop equilibrium production path).

Proof: see Appendix F.

Given the price path of the OL-CFE, and using the symmetry among the fringe firms it is straightforward to show that the following strategy

$$\phi_i^f(S_i^f) = q_i^f(t, x) = \frac{1}{n} (\bar{p} + k^c - 2k^f) (1 - x_i) \quad (18)$$

where for any $S^f \geq 0$, x_i is the unique solution in $(0, 1]$ to

$$h(x_i) = \frac{nrS_i^f}{\bar{p} + k^c - 2k^f}, \quad (19)$$

is a closed-loop representation of the best response of the fringe firm to the OL-CFE price path.

The resource market clearing condition is obviously satisfied since it is satisfied under the OL-CFE and the closed-loop strategies replicate the output path and therefore price path of that equilibrium⁴.

Proposition 3

For any $S^c, S^f \geq 0$ such that the OL-CFE sequence is $S \rightarrow C$, there exists a CL-CFE that yields the same outcome as the OL-CFE's outcome.

Remark: The same treatment and result holds for the case where the OL-CFE's sequence is $S \rightarrow F$. Given the similarity (in the approach and length) of the proof with the case presented in Proposition 3, it is omitted.

Proposition 3 combined with Proposition 2b allows us to draw an important conclusion regarding the microfoundation of the cartel-fringe model.

Corollary

The closed-loop cartel-fringe equilibrium outcome does not coincide with the outcome of the limit case of the asymmetric oligopoly CLNE where the number of fringe firms tends to infinity.

This is in sharp contrast with Salant (1976) where price taking behaviour of the fringe is justified as the limit case of an asymmetric oligopoly where the number of

⁴For $\phi^c(S^c, S^f)$ and $\phi^f(S^f)$ and given initial stocks, the realizations of $p(t; \phi^c(S^c, S^f), \phi^f(S^f))$ yields the OL-CFE price path.

fringe firms is arbitrarily large. The difference is due to the presence of the additional level of interaction in the game with closed-loop strategies. In the case of a closed-loop oligopoly, when deriving its best response to the competitors' strategies, each firm (large and small) can still impact the extraction rates of its competitors (even though it takes their strategies as given). This additional layer of interaction in a CLNE makes the OLNE and the CLNE differ and does not vanish as the market power of fringe firms goes to zero. When firms can use closed-loop strategies, the outcome of the game where the fringe is assumed from the outset to be price taker is not useful to predict the outcome of the limit case where the market power of the fringe firms becomes arbitrarily small.

5 Conclusions

We have considered the exploitation of a nonrenewable resource under imperfect competition and where firms are asymmetric. In the case of an asymmetric oligopoly model we have shown that the outcome of the OLNE cannot be obtained as the outcome of a CLNE even in the limit case where the number of high cost firms tends to infinity. In the case of the benchmark cartel-fringe model, we specified and solved a closed-loop dominant firm nonrenewable resource game, with a price taking fringe. We have shown that the outcomes of the closed-loop and the open-loop dominant firm nonrenewable resource game (à la Salant 1976) coincide. Moreover, we have shown that the interpretation of the dominant firm model, where the fringe is assumed from the outset to be price taker, as a limit case of an asymmetric oligopoly where the number of fringe firms tends to infinity does not extend to the case where firms can use closed-loop strategies. Indeed, when the number of fringe firms becomes arbitrarily large, the equilibrium outcome of the closed-loop oligopoly game does not coincide with the equilibrium outcome of the closed-loop dominant firm nonrenewable resource game.

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Appendix A

Here we summarize the findings on the open-loop Nash equilibrium with a finite number of players. There is one cartel and there are n fringe members. Each fringe firm i takes the strategy profile of its n competitors as given and maximizes its present value profits subject to the resource constraint. The corresponding Hamiltonian reads

$$H_i^f(q_i^f, q^c, q^f, \lambda_i^f, t) = e^{-rt} (\bar{p} - q^c - q^f - k^f) q_i^f + \lambda_i^f (-q_i^f)$$

where q^f and q^c denote the aggregate supply by the fringe and the supply by the cartel, respectively. For the cartel the Hamiltonian reads

$$H^c(q^c, \lambda^c, q^f, t) = e^{-rt} (\bar{p} - q^c - q^f - k^f) q^c + \lambda^c (-q^c)$$

Among the necessary conditions we have that the co-state variables are constant since stocks are absent from the Hamiltonians. In addition, the Hamiltonians are maximized with respect to the own supply of the agent. We will use the symmetry among the fringe players, i.e. $q_i^f = q^f/n$ and $\lambda_i^f = \lambda^f$ for all i . Then we arrive at the following necessary conditions.

Along an F interval:

$$e^{-rt} \left(\bar{p} - q^f(t) - \frac{1}{n} q^f(t) - k^f \right) = \lambda^f$$

$$p(t) = \frac{1}{n+1} (\bar{p} + n(k^f + \lambda^f e^{rt})) \leq k^c + e^{rt} \lambda^c.$$

The first condition follows from the maximization of the Hamiltonian of player i . The second condition is necessary in order for the cartel not to supply.

Along a C interval:

$$e^{-rt} (\bar{p} - 2q^c(t) - k^c) = \lambda^c$$

$$p(t) = \frac{1}{2} (\bar{p} + k^c + \lambda^c e^{rt}) \leq k^f + e^{rt} \lambda^f$$

Along an S interval

$$(2+n)q^c(t) = \bar{p} + n(k^f + \lambda^f e^{rt}) - (n+1)(k^c + \lambda^c e^{rt})$$

$$\frac{n+2}{n} q^f(t) = \bar{p} + k^c + \lambda^c e^{rt} - 2(k^f + \lambda^f e^{rt})$$

$$p(t) = \frac{1}{2+n} (\bar{p} + k^c + \lambda^c e^{rt} + n(k^f + \lambda^f e^{rt})).$$

Continuity of the price path at the different possible transitions gives:

- a transition at t from S to C or vice versa requires

$$\frac{1}{2}(\bar{p} + k^l + \lambda^l e^{rt}) = k^f + \lambda^f e^{rt}$$

- a transition at t from S to F or vice versa requires

$$\frac{1}{n+1}(\bar{p} + n(k^f + \lambda^f e^{rt})) = k^c + \lambda^c e^{rt}$$

- a transition at t from F to C or vice versa requires

$$\frac{1}{2}(\bar{p} + k^c + \lambda^c e^{rt}) = \frac{1}{n+1}(\bar{p} + n(k^f + \lambda^f e^{rt}))$$

We also have to take into account that at the moment of exhaustion of all resource stocks, the price must have reached the choke level:

$$p(T) = \bar{p}$$

Consider the sequence $S \rightarrow C$, with C the final phase before exhaustion and where the transition takes place at instant of time t_1 and exhaustion at T . Then it is tedious but straightforward to derive (see Benchekroun et al. (2008))

$$\frac{2+n}{n} r S_0^f = (\bar{p} + k^c - 2k^f) (rt_1 - 1 + e^{-rt_1})$$

$$(2+n)rS_0^c = -\frac{1}{2}n(\bar{p} + k^c - 2k^f)(rt_1 - 1 + e^{-rt_1}) + (1 + \frac{1}{2}n)(\bar{p} - k^c)(rT - 1 + e^{-rT})$$

For the sequence $S \rightarrow F$ we have

$$(2+n)rS_0^c = (\bar{p} + nk^f - (n+1)k^c)(rt_1 - 1 + e^{-rt_1})$$

$$\frac{2+n}{n} r S_0^f = -\frac{1}{n+1}(\bar{p} + nk^f - (n^f + 1)k^c)(rt_1 - 1 + e^{-rt_1}) + \frac{2+n}{n+1}(\bar{p} - k^f)(rT - 1 + e^{-rT})$$

Appendix B

In this appendix we modify the problem discussed in appendix A so as to allow for an additional fringe member with a larger stock than all other n fringe members. We will show that the stocks of all other fringe members will be depleted before the stock of this particular fringe member is. The variables referring to the larger fringe member are denoted by upper bars. Among the necessary conditions for an OLN we have

$$\begin{aligned} e^{-rt} (\bar{p} - 2\bar{q}^f(t) - q^c(t) - q^f(t) - k^f) &\leq \bar{\lambda}^f \\ e^{-rt} (\bar{p} - \bar{q}^f(t) - q^c - \frac{n+1}{n}q^f(t) - k^f) &\leq \lambda^f \\ e^{-rt} (\bar{p} - \bar{q}^f(t) - 2q^c - q^f(t) - k^c) &\leq \lambda^c \end{aligned}$$

with equality holding if $\bar{q}^f(t), q^f(t)$ (aggregate supply of all other fringe members) and $q^c(t)$ are positive, respectively. Since the fringe members only differ with respect to the stocks, the shadow price of the larger stock is smaller than the shadow price of each smaller stock: $\bar{\lambda}^f < \lambda^f$. This fact implies that we cannot have simultaneous supply at the end because that would imply

$$\bar{p} = k^f + e^{rT}\bar{\lambda}^f = k^f + e^{rT}\lambda^f = k^c + e^{rT}\lambda^c$$

which violates the requirement $\bar{\lambda}^f < \lambda^f$. It cannot be the case that the larger stock is exhausted before the smaller stock, because that would require that

$$\begin{aligned} e^{-r\bar{T}} (\bar{p} - q^c(\bar{T}) - q^f(\bar{T}) - k^f) &\leq \bar{\lambda}^f \\ e^{-r\bar{T}} (\bar{p} - q^c - \frac{n+1}{n}q^f(\bar{T}) - k^f) &\leq \lambda^f \end{aligned}$$

at the time \bar{T} of exhaustion of the larger stock, which is infeasible.

Appendix C

Here we prove that the case $S \rightarrow C$ cannot be sustained as a closed-loop equilibrium. As was made clear in the main text as well as in Appendix B, we only have to consider the case of a single fringe member. The cartel takes the closed-loop strategy of the

fringe as given $\phi^f(S, t)$ and chooses a closed-loop strategy $\phi^c(S, t)$ that maximizes its discounted sum of profits

$$\int_t^\infty e^{-rs} (\bar{p} - q^c(s) - \phi^f(S(s), s) - k^c) q^c(s) ds$$

subject to

$$\int_t^\infty q^c(s) ds \leq S^c$$

and

$$\int_t^\infty \phi^f(S(s), s) ds \leq S^f$$

for all non-negative couples (S, t) , with $q^c(s) = \phi^c(S(s), s)$. The Hamiltonian for the cartel reads

$$H^c(q^c, S, \mu_c^c, \mu_f^c) = e^{-rt} (\bar{p} - q^c - \phi^f(S, t) - k^c) q^c - \mu_c^c q^c - \mu_f^c \phi^f(S, t)$$

where μ_c^c is the costate variable associated with S^c and μ_f^c is the costate variable associated with S^f . Applying the Maximum Principle gives the following set of necessary conditions for an interior solution (i.e. $q^f > 0$ and $q^c > 0$):

$$e^{-rt} (\bar{p} - 2q^c(t) - \phi^f(S(t), t) - k^c) - \mu_c^c(t) = 0$$

$$\dot{\mu}_c^c(t) = -\frac{\partial H^c}{\partial S^c} = (e^{-rt} q^c(t) + \mu_f^c(t)) \frac{\partial \phi^f(S(t), t)}{\partial S^c}$$

$$\dot{\mu}_f^c(t) = -\frac{\partial H^c}{\partial S^f} = (e^{-rt} q^c(t) + \mu_f^c(t)) \frac{\partial \phi^f(S(t), t)}{\partial S^f}$$

We consider the case where the OLNE consists of a final phase with $S \rightarrow C$. The Hamiltonian associated with the OLNE problem of firm j ($j = c, f$) reads

$$H^j(q^j, \lambda^j, t) = e^{-rt} (\bar{p} - q^c - q^j - k^j) q^j + \lambda^j (-q^j)$$

Among the necessary conditions we have that the co-state variable λ^j is constant. In addition the Hamiltonian is maximized. This implies that if at time t there is simultaneous supply we have

$$3q^c(t) = \bar{p} + k^f + \lambda^f e^{rt} - 2(k^c + \lambda^c e^{rt})$$

$$3q^f(t) = \bar{p} + (k^c + \lambda^c e^{rt}) - 2(k^f + \lambda^f e^{rt})$$

$$3p(t) = \bar{p} + k^c + \lambda^c e^{rt} + k^f + \lambda^f e^{rt}$$

Along the C interval we have

$$2q^c(t) = \bar{p} - k^c - \lambda^c e^{rt}$$

$$2p(t) = \bar{p} + k^c + \lambda^c e^{rt}$$

In addition, the equilibrium price is continuous at the time of transition t_1 . Moreover, at the final time T the price must be equal to \bar{p} . Taking this into account we can derive the stocks needed to have this equilibrium from some t in the S -phase on. We end up with the following set of equations at such an instant of time.

$$3q^c(t) = \bar{p} + (k^f + \lambda^f e^{rt}) - 2(k^c + \lambda^c e^{rt})$$

$$\lambda^c = e^{-rT} (\bar{p} - k^f)$$

$$\frac{1}{3}(\bar{p} + k^c + \lambda^c e^{rt_1} + k^f + \lambda^f e^{rt_1}) = \frac{1}{2}(\bar{p} + k^f + \lambda^f e^{rt_1})$$

$$3rS^f(t) = (\bar{p} + k^c - 2k^f) (rt_1 - rt - 1 + e^{rt-rt_1})$$

$$\begin{aligned} 3rS^c(t) &= -\frac{1}{2}(\bar{p} + k^c - 2k^f) (rt_1 - rt - 1 + e^{rt-rt_1}) \\ &\quad + \frac{3}{2}(\bar{p} - k^c) (rT - rt - 1 + e^{rt-rT}) \end{aligned}$$

From here on the analysis proceeds along the same lines as in the other case treated in the main text. For completeness we write down the full argument. For a CLNE to result in the extraction path of the OLN, we must have $\mu_c^c(s) = \lambda^c$ for all instants $s \geq t$ for all $t \geq 0$. Therefore μ_c^c is constant. It follows that then

$$(e^{-rt} q^c(t) + \mu_c^c) \frac{\partial \phi^f(S(t), t)}{\partial S^c} = 0$$

This implies that either (i) $e^{-rt}q^c(t) + \mu_f^c(t) = 0$ where q^c is the OLNE equilibrium path of the cartel and therefore $\mu_f^c(t) = -e^{-rt}q^c(t)$ or (ii)

$$\frac{\partial \phi^f(S(t), t)}{\partial S^c} = 0.$$

Condition (i) implies that $\dot{\mu}_f^c = 0$, but $e^{-rt}q^c(t)$ is not constant along the OLNE. Hence, for a CLNE to result in the extraction path of the OLNE, we must have

$$\frac{\partial (\phi^f(S(t), t))}{\partial S^c} = 0$$

along the OLNE where there is simultaneous supply. We next show that this condition is not met in the open-loop Nash equilibrium.

Our strategy is to assume that the open-loop equilibrium is subgame perfect. Consequently we represent extraction by the cartel as a function of time and the existing stocks. So, we first write

$$3 \frac{\partial \phi^c}{\partial S^f} = \frac{\partial (\lambda^f - 2\lambda^c) e^{rt}}{\partial S^f}$$

We have

$$\lambda^f - 2\lambda^c = -\frac{3}{2}e^{-rT}(\bar{p} - k^c) + \frac{1}{2}e^{-rt_1}(\bar{p} + k^c - 2k^c)$$

So

$$\frac{\partial (\lambda^f - 2\lambda^c)}{\partial S^f} = r \frac{3e^{-rT}}{2} \frac{\partial T}{\partial S^f} (\bar{p} - k^c) - r \frac{\partial t_1}{\partial S^f} \frac{e^{-rt_1}(\bar{p} + k^c - 2k^c)}{2}$$

The derivatives with respect to the stocks follow from the expressions derived above for these stocks.

$$3r = (\bar{p} + k^c - 2k^f) (rt_1 - re^{rt-rt_1}) \frac{\partial t_1}{\partial S^f}$$

$$0 = -\frac{1}{2}(\bar{p} + k^c - 2k^f) (rt_1 - re^{rt-rt_1}) \frac{\partial t_1}{\partial S^f} + \frac{3}{2}(\bar{p} - k^c) (rT - re^{rt-rT}) \frac{\partial T}{\partial S^f}$$

Therefore

$$0 = -\frac{3}{2}r + \frac{3}{2}(\bar{p} - k^c) (rT - re^{rt-rT}) \frac{\partial T}{\partial S^f}$$

Or

$$\frac{1}{(T - e^{rt-rT})} = (\bar{p} - k^c) \frac{\partial T}{\partial S^f}$$

and

$$\frac{3}{(t_1 - e^{rt-rt_1})} = (\bar{p} + k^c - 2k^f) \frac{\partial t_1}{\partial S^f}$$

Substituting gives

$$\begin{aligned} \frac{\partial (\lambda^f - 2\lambda^c)}{\partial S^f} &= r \frac{3e^{-rT}}{2} \frac{1}{(T - e^{rt-rT})} - r \frac{3}{(t_1 - e^{rt-rt_1})} \frac{e^{-rt_1}}{2} \\ &= \frac{3r}{2} \left(\frac{e^{-rT}}{(T - e^{rt-rT})} - \frac{e^{-rt_1}}{(t_1 - e^{rt-rt_1})} \right) \\ &= \frac{3r}{2} \left(\frac{1}{(Te^{rT} - e^{rt})} - \frac{1}{(t_1 e^{rt_1} - e^{rt})} \right) \end{aligned}$$

For any t we have that

$$f(X) = \frac{1}{(Xe^{rX} - e^{rt})}$$

is strictly decreasing in X and therefore $\frac{\partial(\lambda^f - 2\lambda^c)}{\partial S^c} \neq 0$.

Appendix D

For any $t \in [0, t_1]$ we have

$$q^f = (\bar{p} + k^c - 2k^f) - (2\lambda^f - \lambda^c) e^{rt} \quad (20)$$

and

$$q^c = (k^f - k^c) + (\lambda^f - \lambda^c) e^{rt} \quad (21)$$

Therefore, after substitution into the inverse demand, we have that the price is

$$p = \bar{p} - q^f - q^c = k^f + \lambda^f e^{rt} \quad (22)$$

For any $t \in [t_1, T]$ we have

$$q^f = 0$$

and

$$q^c = \frac{1}{2} (\bar{p} - k^c - \lambda^c e^{rt}) \quad (23)$$

The transition time t_1 is given by

$$e^{rt_1} = \frac{\bar{p} + k^c - 2k^f}{2\lambda^f - \lambda^c} \quad (24)$$

and the terminal time T is given by

$$e^{rT} = \frac{\bar{p} - k^c}{\lambda^c} \quad (25)$$

The costate variables λ^c and λ^f are determined using the resource constraints, which gives

$$\int_t^{t_1} \{(\bar{p} + k^c - 2k^f) - (2\lambda^f - \lambda^c) e^{rs}\} ds = S^f(t)$$

or

$$(\bar{p} + k^c - 2k^f)(t_1 - t) - (2\lambda^f - \lambda^c) \frac{(e^{rt_1} - e^{rt})}{r} = S^f(t) \quad (26)$$

For the cartel we have

$$\int_t^{t_1} \{(k^f - k^c) + (\lambda^f - \lambda^c) e^{rs}\} ds + \int_{t_1}^T \frac{1}{2} (\bar{p} - k^c - \lambda^c e^{rs}) ds = S^c(t)$$

or

$$(k^f - k^c)(t_1 - t) + (\lambda^f - \lambda^c) \frac{(e^{rt_1} - e^{rt})}{r} + \frac{1}{2} (\bar{p} - k^c)(T - t_1) - \frac{1}{2} \lambda^c \frac{(e^{rT} - e^{rt_1})}{r} = S^c(t) \quad (27)$$

Let

$$x \equiv \frac{(2\lambda^f - \lambda^c) e^{rt}}{\bar{p} + k^c - 2k^f} = e^{r(t-t_1)} \text{ and } y \equiv \frac{\lambda^c e^{rt}}{\bar{p} - k^c} = e^{r(t-T)} \quad (28)$$

We now show that x and y can be determined as the unique solutions to respectively (13) and (14).

Substituting t_1 from (24) into (26) yields after algebraic manipulations

$$(\bar{p} + k^c - 2k^f) \left(\frac{1}{r} \ln \left(\frac{\bar{p} + k^c - 2k^f}{2\lambda^f - \lambda^c} \right) - t \right) - (2\lambda^f - \lambda^c) \frac{\left(\frac{\bar{p} + k^c - 2k^f}{2\lambda^f - \lambda^c} - e^{rt} \right)}{r} = S^f(t) \quad (29)$$

which can be simplified into

$$\ln \left(\frac{\bar{p} + k^c - 2k^f}{2\lambda^f - \lambda^c} e^{-rt} \right) - 1 + \frac{(2\lambda^f - \lambda^c)}{(\bar{p} + k^c - 2k^f)} e^{rt} = \frac{rS^f(t)}{(\bar{p} + k^c - 2k^f)} \quad (30)$$

or

$$\ln \left(\frac{1}{x} \right) + x = \frac{rS^f(t)}{\bar{p} + k^c - 2k^f} + 1 \quad (31)$$

Combining (26) and (27) gives after simplification

$$(\bar{p} - k^c)(T - t) - \lambda^c \frac{(e^{rT} - e^{rt})}{r} = 2S^c(t) + S^f(t)$$

Substituting T from (25) gives after manipulations

$$\ln\left(\frac{1}{y}\right) + y = r \frac{2S^c(t) + S^f(t)}{\bar{p} - k^c} + 1 \quad (32)$$

Thus x and y depend on S^f and (S^f, S^c) respectively and combined with (21) and (20) along with (28) gives a closed loop representation of the open-loop paths (16) and (15).

For any $t \in [t_1, T]$ we have $S^f = 0$ and $x = 1$. It can easily be checked that substituting $x = 1$ into (15) and (16) yields the extraction path of the cartel when it is a sole supplier $q^f = 0$ and (23) ■

Appendix E

After substitution of (20) and (21) the cartel's profits are given by

$$\begin{aligned} \Pi^c &= \int_t^{t_1} e^{-rs} (k^f - k^c + \lambda^f e^{rs}) (k^f - k^c + (\lambda^f - \lambda^c) e^{rs}) ds \\ &\quad + \int_{t_1}^T e^{-rs} \frac{1}{2} (\bar{p} - k^c - \lambda^c e^{rs}) \frac{1}{2} (\bar{p} - k^c + \lambda^c e^{rs}) ds \end{aligned}$$

or

$$\begin{aligned} r\Pi^c &= (k^f - k^c)^2 (e^{-rt} - e^{-rt_1}) + (k^f - k^c) (2\lambda^f - \lambda^c) (rt_1 - rt) + \lambda^f (\lambda^f - \lambda^c) (e^{rt_1} - e^{rt}) \\ &\quad + \frac{1}{4} (\bar{p} - k^c)^2 (e^{-rt_1} - e^{-rT}) - \frac{1}{4} (\lambda^c)^2 (e^{rT} - e^{rt_1}) \end{aligned} \quad (33)$$

We first λ^f and λ^c as functions of x and y . We use (28) and get

$$\lambda^f e^{rt} = \frac{(\bar{p} + k^c - 2k^f)x + (\bar{p} - k^c)y}{2}$$

and

$$\lambda^c e^{rt} = (\bar{p} - k^c)y$$

We then determine t_1 and T as functions of x and y using (24), (25). Substituting λ^f , λ^c , t_1 and T as functions of x and y into (33) gives after algebraic manipulations (17) ■

Appendix F

To prove this claim we show that (17) satisfies the Hamilton Jacobi Bellman (HJB) equation of the cartel's problem.

Since $x = x(S^f)$, from (31), and $y = y(S^f, S^c)$, from (32), we define

$$V^c(t, S^f, S^c) = \Pi^c(t, x, y)$$

We check now that V satisfies the HJB equation for all (S^f, S^c) (such that the equilibrium sequence is $S \rightarrow C$)

$$-\frac{\partial V^c}{\partial t} = \frac{\partial V^c}{\partial S^f} (-\phi^f) + \text{Max}_{q^c} \left\{ (\bar{p} - k^c - \phi^f - q^c) q^c e^{-rt} + \frac{\partial V^c}{\partial S^c} (-q^c) \right\} \quad (34)$$

with q^f given by (15). This is done in two steps: (i) we first check that q^c given by (16) solves the maximization problem; (ii) we show that when ϕ^f is given by (15) and q^c is given by (16) the function $V^c(t, S^f, S^c) = \Pi^c(t, x, y)$ satisfies the cartel's HJB equation.

(i) The first order condition associated with the maximization problem gives

$$(\bar{p} - k^c - \phi^f - 2q^c) e^{-rt} - \frac{\partial V^c}{\partial S^c} = 0$$

or

$$q^c = \frac{1}{2} \left(\bar{p} - k^c - \phi^f - \frac{\partial V^c}{\partial S^c} e^{rt} \right) \quad (35)$$

We now compute the derivative

$$\frac{\partial V^c}{\partial S^c} = \frac{\partial \Pi^c}{\partial x} \frac{\partial x}{\partial S^c} + \frac{\partial \Pi^c}{\partial y} \frac{\partial y}{\partial S^c}$$

We have $\frac{\partial x}{\partial S^c} = 0$ and $\frac{\partial y}{\partial S^c} = \frac{2r}{\bar{p} - k^c} \frac{y}{y-1}$ and

$$\frac{\partial \Pi^c}{\partial y} = \frac{e^{-rt}}{4r} (2(\bar{p} - k^c)^2 y - 2(\bar{p} - k^c)^2)$$

Hence

$$\begin{aligned} \frac{\partial V^c}{\partial S^c} &= \frac{e^{-rt}}{4r} 2(\bar{p} - k^c)^2 (y - 1) \frac{2r}{\bar{p} - k^c} \frac{y}{y-1} \\ \frac{\partial V^c}{\partial S^c} &= e^{-rt} (\bar{p} - k^c) y = \lambda^c \end{aligned} \quad (36)$$

Substitution of $\frac{\partial V^c}{\partial S^c} e^{rt}$ and of ϕ^f from 15 gives

$$q^c = \frac{1}{2} (\bar{p} - k^c - (\bar{p} + k^c - 2k^f) (1 - x) - (\bar{p} - k^c) y) \quad (37)$$

which (after simplification) is identical to (16).

(ii) We have

$$\frac{\partial V^c}{\partial t} = \frac{\partial \Pi^c}{\partial t} = -r\Pi^c$$

and

$$\frac{\partial V^c}{\partial S^c} = e^{-rt} (\bar{p} - k^c) y = \lambda^c \quad (38)$$

We now turn to $\frac{\partial V^c}{\partial S^f}$. We have

$$\frac{\partial V^c}{\partial S^f} = \frac{\partial \Pi^c}{\partial x} \frac{\partial x}{\partial S^f} + \frac{\partial \Pi^c}{\partial y} \frac{\partial y}{\partial S^f}$$

with

$$\frac{\partial x}{\partial S^f} = \frac{r}{\bar{p} + k^c - 2k^f} \frac{x}{x-1} \quad \text{and} \quad \frac{\partial y}{\partial S^f} = \frac{r}{\bar{p} - k^c} \frac{y}{y-1}$$

After simplification we have

$$\frac{\partial \Pi^c}{\partial x} = \frac{e^{-rt}}{4r} \left[4(k^f - k^c) (\bar{p} + k^c - 2k^f) \ln\left(\frac{1}{x}\right) + 2(\bar{p} + k^c - 2k^f)^2 (1-x) \right]$$

and thus

$$\frac{\partial \Pi^c}{\partial x} \frac{\partial x}{\partial S^f} = \frac{e^{-rt}}{4} \left[4(k^f - k^c) \frac{x}{x-1} \ln\left(\frac{1}{x}\right) - 2(\bar{p} + k^c - 2k^f) x \right]$$

We also have

$$\frac{\partial \Pi^c}{\partial y} = \frac{e^{-rt}}{4r} 2(\bar{p} - k^c)^2 (y-1)$$

and thus

$$\frac{\partial \Pi^c}{\partial y} \frac{\partial y}{\partial S^f} = \frac{e^{-rt}}{4} 2(\bar{p} - k^c) y$$

We can now obtain $\frac{\partial V^c}{\partial S^f}$ as the sum of $\frac{\partial \Pi^c}{\partial y} \frac{\partial y}{\partial S^f}$ and $\frac{\partial \Pi^c}{\partial x} \frac{\partial x}{\partial S^f}$ which gives

$$\frac{\partial V^c}{\partial S^f} = (k^f - k^c) \frac{x}{x-1} e^{-rt} \ln\left(\frac{1}{x}\right) - \frac{1}{2} (\bar{p} + k^c - 2k^f) x e^{-rt} + \frac{1}{2} (\bar{p} - k^c) y e^{-rt}$$

The last step consists of checking that when substituting each term $\frac{\partial V^c}{\partial t}$, $\frac{\partial V^c}{\partial S^f}$, $\frac{\partial V^c}{\partial S^c} q^f$ and q^c into the HJB the equality holds for all $S^f, S^c \geq 0$. This step is skipped. It involves lengthy but straightforward algebraic simplifications only. More specifically it can be shown that each side of the HJB equation

$$-\frac{\partial V^c}{\partial t} = -q^f \frac{\partial V^c}{\partial S^f} - q^c \frac{\partial V^c}{\partial S^c} + (\bar{p} - k^c - q^f - q^c) q^c e^{-rt} \quad (39)$$

reduces to

$$\begin{aligned} & -x \left(\ln \frac{1}{x} \right) (k^c - k^f) (p + k^c - 2k^f) \\ & - \frac{1}{4} (p + k^c - 2k^f)^2 x^2 + \frac{1}{2} (p - k^c) (p + k^c - 2k^f) x + \\ & \frac{1}{4} (p - k^c)^2 y^2 - \frac{1}{2} (p - k^c)^2 y + (k^c - k^f)^2 \blacksquare \end{aligned}$$